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Workshop on TFT's at Northwestern
Notes, errors, and incompleteness by Alex Hoffnung

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DRAFT VERSION ONLY

1 Lecture 4

What does it mean for $SO(2)$ to act on an ∞-groupoid X? Recall we can think of X as a space.

$$SO(2) \times X \to X$$

$x \in X$

$$SO(2) \times x \to X$$

a loop in X.
Partial answer:
Every object $x \in X$ has a canonical automorphism.

Let $\mathcal{C}$ be a $\otimes(\infty, 2)$-category.

X a fully dualizable object. This means there exists $X^\vee$

$$e \colon X \otimes X^\vee \to 1$$

$$c \colon 1 \to X^\vee \otimes X$$

e has a left adjoint

$$e^L, c \colon 1 \to X \otimes X^\vee \cong X^\vee \otimes X$$

Question:
How are e^L and C related?
e^L "encodes" a map $S_X \colon X \to X$ (where S is supposed to remind us of the Serre functor)
If c has a left adjoint

$$c^L \colon X^\vee \otimes X \to 1$$

"adjoint" to some map $T_X \colon X \to X$.
Exercise: Axiomatics force $S_X = T_X^{-1}$.

Claim:
X is fully dualizable if it has aa dual $X^\vee$ and e, c have left adjoints.

Question:
How do we know S and T are related to $SO(2)$ action?
Answer:
Check in the universal case.

$e = 2Bord^{fr}$, $x = *^{+}$, $x = *^{-}$
FIGURES OF EVAL AND COEVAL MAPS

Fix a field k and consider a $\otimes$-2-category $\mathcal{C}$ of k-algebras, bimodules and bimodule maps.
Question:
When is an object of e fully dualizable?
Claim:
Every object is dualizable.

$$k \xrightarrow{A} A \otimes A^{op} \xrightarrow{A} k$$

These should be eval and coeval.

When is A fully dualizable?

- A is dualizable as a k-module
- A is dualizable as an $A \otimes A^{op}$-module

These conditions are equivalent to requiring A to be semisimple over k.
Fix:
Use fancier rings

To get more examples we go to an $(\infty, 2)$-category of k-dg algebras, dg bimodules, maps of bimodulesm chain homotopies,...

The above says A must be "smooth" and "proper".

Suppose A satisfies only the first condition above. Then A has a dual as an $A \otimes A^{op}$-module. A^* may not be an invertible bimodule.

Definition 1. *$\mathcal{C}$ a $\otimes - (\infty, 2)$-category, $X \in \mathcal{C}$ is* 1.5-**dualizable** *if X is dualizable, evaluation map*

$$X \otimes X^{\vee} \to 1$$

has a left adjoint and morphism $S_X\colon X \to X$ is invertible.

2 claims:
Claim 1: $SO(2)$ acts on the 1.5-dualizable objects

Claim 2: (Costello):

$$Fun^{\otimes}(2Bord^{n.c}, \mathcal{C}) \cong \left(\mathcal{C}^{1.5-dual,\sim}\right)^{SO(2)}$$

$2Bord^{n.c.}$ an $(\infty, 2)$-category defined like $2Bord$ but has fewer 2-morphisms...

Application (Costello):
String topology
Let M be a compact connected manifold, choose a basepoint $* \in M$, and let

$$A = C_*(\Omega M, k)$$

Claim:
A is 1.5-dualizable and $SO(2)$-fixed.

We need

$$k \to A^* \otimes_{A \otimes A^{op}} A \cong A \otimes_{A \otimes A^{op}} A \cong HH_*(A)$$

If $A = C_*(\Omega M)$, then $HH_* A = C_*(LM)$, where $LM = Maps(S^1, M)$.

$$\begin{tikzcd} k \arrow[rr] \arrow[dr, "f"'] & & C_*(LM) \\ & C_*(M) \arrow[ur] & \end{tikzcd}$$

where f is "take the fundamental class [M]". Poincare duality gives the desired result here.

We get

$$Z \colon 2Bord^{n.c.} \to \mathcal{C}$$

$$Z(S^1) \cong C_*(LM)$$

Conclusion:
$C_*(LM)$ has lots of operations.

These are the Chas-Sullivan string topology operations.

Corollary:
These are defined for any homology manifold.

The four present lectures rest on three foundations:

- Baez-Dolan
- Costello
- Galatius, Madsen, Tillman, Weiss

We want to spend the rest of the time talking about the third foundation:

Let Σ be a surface with genus g.

$$\begin{tikzcd} \Sigma \arrow[r] & E \arrow[d] \\ & BDiff(\Sigma) \end{tikzcd}$$

: Note:
On E, we have a cononical rank 2-vector bundle $(T_{E/BDiff(\Sigma)}) \in H^2(E)$.

$$e(T)^{n+1} \in H^{2n+2}(E)$$

$$\pi_* e(T)^{n+1} = K_n \in H^{2n}(Bdiff(\Sigma))$$

Mumford Conjecture:(Madsen-Weiss)

$$\mathbb{Q}[K_1, K_2, \ldots] \to H^*(BDiff(\Sigma), \mathbb{Q})$$

This map is an isomorphism in a range of degrees going to ∞ as g grows.
Strategy:

1. Define a space X and a map

$$BDiff(\Sigma) \to X$$

2. Prove that tis map is an isomorphis in cohomology in a range of degrees.

3. Identify X with another space Y

4. Compute $H^*(Y) \cong \mathbb{Q}[K_1, K_2, \ldots$

The second step is the hardest. We discuss 1 and 3 here.
$BDiff(\Sigma) \subset \hom_{2Bord}(\emptyset, \emptyset)$
Note:
If $\mathcal{C}$ is any higher category, we can make an ∞-groupoid (or space) $\mid \mathcal{C} \mid$ by inverting all morphisms in $\mathcal{C}$.

$$BDiff^+(\Sigma) \to \Omega^2 \mid 2Bord \mid \stackrel{def}{=} X$$

This was step 1.
Claim:
X is an "E_∞-space", so X is an infinite loop space. I.e., $X \cong \Omega X_1, X_1 \cong \Omega X_2, \ldots$

What infinite loop space is it? Say X' is some other infinite loop space.

$$\hom_{infloopspace}(X, X') \cong Fun^{\otimes}(\mid 2Bord \mid, X')$$

$$\mid 2Bord^{fr} \mid QS^0 \cong \lim_{\to n} \Omega^n S^n$$

with an action of $SO(2)$ on the RHS via the J-homomorphism.
$\mid 2Bord \mid \cong (QS^0)_{hSO(2)}$
$\mid 2Bord \mid \cong \Omega^{\infty-2} MTSO$

Theorem 2. *(G-M-T-W)*

$$X \cong \Omega^\infty MTSO(2)$$

Now do step 4 which is an easy calculation in homotopy theory.