DOLBEAULT COHOMOLOGY

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1. INTRODUCTION

Dolbeault cohomology is a cohomology theory defined on complex manifolds. Although it may appear rather analogous to the de Rham cohomology of a differential manifold, Dolbeault cohomology generally depends upon not just the topological data, but upon the complex structure of the manifold as well. Using the Dolbeault theorem and the Hodge decomposition theorem, we explore two ways of working with and computing Dolbeault cohomology: the first, in the algebraic language of sheaf cohomology, and the second, in the context of Hodge theory.

We begin by introducing the basics notions of complex geometry and of sheaf cohomology. Then, using a degenerate spectral sequence, we shall prove the main result of this paper, the Dolbeault theorem:

Theorem 1.0.1 (Dolbeault theorem)**.** *For a complex manifold X, let $\mathfrak{U} : X = \bigcup U_i$ be a good open cover of X, let Ω^p denote the holomorphic p-forms, let $\check{H}^q(\mathfrak{U}, \Omega^p)$ denote the Čech cohomology, and let $H^{p,q}(X)$ denote the Dolbeault cohomology. Then*

$$\check{H}^q(\mathfrak{U}, \Omega^p) \cong H^{p,q}(X).$$

As an application, we use the Dolbeault theorem to combinatorially compute the Dolbeault cohomology of $\mathbb{P}^1_{\mathbb{C}}$.

In the second half of the paper, we survey the main results from Hodge theory, referring the reader to other sources for the more difficult proofs. The main mileage we will obtain occurs in the Kähler case, where we have strong relations between the Betti and Hodge numbers of the manifold. We will use this relationship to compute the Dolbeault cohomology of $\mathbb{P}^n_{\mathbb{C}}$ for all $n \geq 1$.

2. BASIC CONSTRUCTIONS IN COMPLEX GEOMETRY

2.1. Complex manifolds and complex vector bundles.

We begin by recalling some basic definitions from complex geometry. These definitions follow the treatment found in [1, pp. 14-16]. A more general, and clearer, development that includes the theory of almost complex structures may be found in [3, pp. 57-67], but it is slightly more involved, and the details would distract us from current goals.

Definition 2.1.1. A *complex manifold* is a differentiable manifold X that admits an open cover $\{U_\alpha\}$ and charts $\{(U_\alpha, \phi_\alpha)\}$ where the maps $\phi_\alpha : U_\alpha \to \mathbb{C}^n$ are smooth (when $\mathbb{C}^n$ is identified with $\mathbb{R}^{2n}$) and the compositions $\phi_\alpha \circ \phi_\beta^{-1}$ are holomorphic on $\phi_\beta(U_\alpha \cap U_\beta) \subseteq \mathbb{C}^n$ for all α, β.

Definition 2.1.2. Given a $2n$-(real)-dimensional complex manifold X, the *complexified tangent space* to M at p is defined to be the complexification of the (real) tangent space at p. Explicitly,

$$T_{\mathbb{C},p}(X) := T_{\mathbb{R},p}(X) \otimes \mathbb{C}.$$

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These form the fibres of a complex vector bundle $T_{\mathbb{C}}(X)$ with (complex) rank $2n$. Similarly, we may define the *complexified cotangent space* to be the complexification of the (real) cotangent space at p. Explicitly,

$$T^*_{\mathbb{C},p}(X) := T^*_{\mathbb{R},p}(X) \otimes \mathbb{C}.$$

These form the fibres of a rank $2n$ complex vector bundle $T^*_{\mathbb{C}}(X)$.

Notice that $T^*_{\mathbb{C},p}(X) \cong \operatorname{Hom}(T_{\mathbb{C},p}(X), \mathbb{C})$, and hence the complexified cotangent space is dual to the complexified tangent space. Fix a point $p \in X$, and let $\phi = (z_1, \dots, z_n) : U \to \mathbb{C}^n$ be holomorphic coordinates near p. As a real vector space, $\mathbb{C}^n$ is canonically identified with $\mathbb{R}^{2n}$ sending $(a_1 + ib_1, \dots, a_n + ib_n) \mapsto (a_1, \dots, a_n, b_1, \dots, b_n)$. Thus, by composing ϕ with this map to $\mathbb{R}^{2n}$, we obtain C^∞ coordinates $(x_1, \dots, x_n, y_1, \dots, y_n)$ with $z_i = x_i + iy_i$. Thus, $T_{\mathbb{R},p}(X) = \operatorname{span}_{\mathbb{R}}\langle \partial_{x_i}, \partial_{y_i}\rangle_{1 \le i \le n}$. Letting $\partial_{z_i} := \frac{1}{2}(\partial_{x_i} - i\partial_{y_i})$ and $\partial_{\bar{z}_i} := \frac{1}{2}(\partial_{x_i} + i\partial_{y_i})$, we see that $T_{\mathbb{C},p}(X) = \operatorname{span}_{\mathbb{C}}\langle \partial_{z_i}, \partial_{\bar{z}_i}\rangle_{1 \le i \le n}$. An important remark is that a smooth function $f \in C^\infty(X)$ is holomorphic (resp. antiholomorphic) if and only if $\partial_{\bar{z}_i} f = 0$ (resp. $\partial_{z_i} f = 0$) for all $1 \le i \le n$. This quickly follows from the Cauchy-Riemann equations.

In particular, each complexified tangent space $T_{\mathbb{C},p}(X)$ splits into $T_{\mathbb{C},p}(X) = T_p^{1,0}(X) \oplus T_p^{0,1}(X)$, where $T_p^{1,0}(X) := \operatorname{span}_{\mathbb{C}}\langle \partial_{z_i}\rangle$ and $T_p^{0,1}(X) := \operatorname{span}_{\mathbb{C}}\langle \partial_{\bar{z}_i}\rangle$. Since homomorphic changes of variables preserve the splittings, these fibre-wise splittings induce one of vector bundles $T_{\mathbb{C}}(X) = T^{1,0}(X) \oplus T^{0,1}(X)$. We call $T^{1,0}(X)$ the *holomorphic tangent bundle* and $T^{0,1}(X)$ the *antiholomorphic tangent bundle*.

2.2. **The complexified de Rham complex.** Let dz_i and $d\bar{z}_i$ be the duals of ∂_{z_i} and $\partial_{\bar{z}_i}$ respectively. In a similar way, we obtain a splitting of vector bundles $T^*_{\mathbb{C}}(X) = \Lambda^{1,0}(X) \oplus \Lambda^{0,1}(X)$, where on fibres, $\Lambda_p^{1,0}(X) = \operatorname{span}_{\mathbb{C}}\langle dz_i\rangle$ and $\Lambda_p^{0,1}(X) = \operatorname{span}_{\mathbb{C}}\langle d\bar{z}_i\rangle$. Define $\Lambda^{p,0}(X)$ to be the pth exterior power

$$\Lambda^{p,0}(X) := (\Lambda^{1,0}(X))^{\wedge p}$$

and similarly

$$\Lambda^{0,q}(X) := (\Lambda^{0,1}(X))^{\wedge q}.$$

Define

$$\Lambda^{p,q}(X) := \Lambda^{p,0}(X) \otimes \Lambda^{0,q}(X).$$

We refer to sections of the bundle $\Lambda^{p,q}(X)$ as *(p,q)-forms* or *forms of type (p,q)* on X.

If we let $\Lambda^k_{\mathbb{C}}(X)$ denote the complexified k-forms (on fibres, $\Lambda^k_{\mathbb{C},p}(X) := \Lambda^k_{\mathbb{R},p}(X) \otimes \mathbb{C}$), then one quickly checks that

$$\Lambda^k_{\mathbb{C}}(X) = \bigoplus_{p+q=k} \Lambda^{p,q}(X).$$

Let $d : \Lambda^k(X) \to \Lambda^{k+1}(X)$ denote the de Rham differential. We may extend this to a $\mathbb{C}$-linear differential on the complexified k-forms $d : \Lambda^k_{\mathbb{C}}(X) \to \Lambda^{k+1}_{\mathbb{C}}(X)$. Furthermore, one may check that in local coordinates $(z_1, \dots, z_n)$, one has

$$d(f \cdot dz_I \wedge d\bar{z}_J) = \sum_{i=1}^n \partial_{z_i}(f) dz_i \wedge dz_I \wedge d\bar{z}_J + \sum_{j=1}^n \partial_{\bar{z}_j}(f) d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J \tag{1}$$

where I and J are the multi-indices $I = (i_1, \dots, i_p)$, $J = (j_1, \dots, j_q)$, and $dz_I := dz_{i_1} \wedge \dots \wedge dz_{i_p}$ and $d\bar{z}_J := d\bar{z}_{j_1} \wedge \dots \wedge d\bar{z}_{j_q}$. From this formula, we see that

$$d(\Lambda^{p+q}(X)) \subseteq \Lambda^{p+1,q}(X) \oplus \Lambda^{p,q+1}(X).$$

Hence, we may decompose d uniquely as $d = \partial + \bar{\partial}$ where $\partial(\Lambda^{p,q}(X)) \subseteq \Lambda^{p+1,q}(X)$ and $\bar{\partial}(\Lambda^{p,q}(X)) \subseteq \Lambda^{p,q+1}(X)$. Furthermore, it is clear from (1) that

$$\partial(f \cdot dz_I \wedge d\bar{z}_J) = \sum_{i=1}^{n} \partial_{z_i}(f) dz_i \wedge dz_I \wedge d\bar{z}_J$$

and

$$\bar{\partial}(f \cdot dz_I \wedge d\bar{z}_J) = \sum_{j=1}^{n} \partial_{\bar{z}_j}(f) d\bar{z}_j \wedge dz_I \wedge d\bar{z}_J.$$

In differential topology, one calculates easily that $d^2 = 0$, and hence the same is true for our complexified de Rham differential. Recall that $\mathbb{R}$-valued de Rham cohomology is defined as:

Definition 2.2.1. Let X be a differential manifold. For each $k \in \mathbb{Z}$, let $d_k : \Lambda^k_{\mathbb{R}}(X) \to \Lambda^{k+1}_{\mathbb{R}}(X)$ be the exterior derivative. The *kth de Rham cohomology group* (with coefficients in $\mathbb{R}$) is defined to be

$$H^k_{DR}(X; \mathbb{R}) := \frac{\ker d_k}{\operatorname{Im} d_{k-1}}.$$

We define the *Betti numbers* to be

$$b^k(X) := \dim_{\mathbb{R}} H^k_{DR}(X).$$

We may make the analogous definition for coefficients in $\mathbb{C}$.

Definition 2.2.2. Let X be a complex manifold. For each $k \in \mathbb{Z}$, let $d_{\mathbb{C},k} : \Lambda^k_{\mathbb{C}}(X) \to \Lambda^{k+1}_{\mathbb{C}}(X)$ be the complexified exterior derivative. The *kth de Rham cohomology group* (with coefficients in $\mathbb{C}$) is defined to be

$$H^k_{DR}(X; \mathbb{C}) := \frac{\ker d_{\mathbb{C},k}}{\operatorname{Im} d_{\mathbb{C},k-1}}.$$

Lemma 2.2.3. *The real and complex de Rham cohomology groups satisfy*

$$H^*_{DR}(X; \mathbb{C}) = H^*_{DR}(X; \mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}.$$

Proof. Since $\mathbb{C}$ is a free $\mathbb{R}$-module, it is flat, and hence the functor $- \otimes_{\mathbb{R}} \mathbb{C}$ is exact, thus commuting with kernels and cokernels. In particular, it commutes with taking the homology of a complex. ■

2.3. **Dolbeault cohomology.** From the decomposition $d = \partial + \bar{\partial}$, we see that

$$0 = \partial^2 + \partial \circ \bar{\partial} + \bar{\partial} \circ \partial + \bar{\partial}^2.$$

By the direct sum decomposition of the image $\Lambda^k_{\mathbb{C}}(X) = \bigoplus_{p+q=k} \Lambda^{p,q}(X)$, we immediately see that $\partial^2 = \bar{\partial}^2 = 0$, and hence $\partial \circ \bar{\partial} = -\bar{\partial} \circ \partial$. Since $\bar{\partial}^2 = 0$, for fixed p, the global sections $\Lambda^{p,\bullet}(X)$ become a complex with differential $\bar{\partial}$.

Definition 2.3.1. For fixed p, let $\bar{\partial}_q$ denote the differential $\Lambda^{p,q}(X) \to \Lambda^{p,q+1}(X)$. We define the *Dolbeault cohomology groups* to be

$$H^{p,q}(X) := \frac{\ker \bar{\partial}_q}{\operatorname{Im} \bar{\partial}_{q-1}}.$$

We define the *Hodge numbers* to be

$$h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X).$$

We see that the definition of Dolbeault cohomology depends on the splitting $T^*_{\mathbb{C}}(X) = \Lambda^{1,0}(X) \oplus \Lambda^{0,1}(X)$, and hence on the complex structure of the manifold X, not just on its underlying real structure as we saw with complexified de Rham cohomology.

3. Sheaf cohomology and the Dolbeault theorem

One method of calculating Dolbeault cohomology is by the Dolbeault theorem and using sheaf cohomology. The Dolbeault theorem relates the cohomology of the Dolbeault complex $H^{p,q}(X)$ to the Čech cohomology $\check{H}^q(\mathfrak{U}, \Omega^p)$ of the sheaf of holomorphic forms Ω^p:

$$\check{H}^q(\mathfrak{U}, \Omega^p) \cong H^{p,q}(X).$$

In this section, we first set up the formal language of sheaf cohomology, and use it to prove the Dolbeault theorem. We then show how one can use this to make an explicit computation for 1-dimensional projective space.

3.1. Sheaf cohomology.

Definition 3.1.1. Given a topological space X, a *presheaf of abelian groups* $\mathscr{F}$ on X is an assignment of an abelian group $\mathscr{F}(U)$ to every open subset $U \subseteq X$ along with restriction mappings $\rho_V^U : \mathscr{F}(U) \to \mathscr{F}(V)$ for every pair of opens $V \subseteq U \subseteq X$, satisfying

$$\rho_W^V \circ \rho_V^U = \rho_W^U.$$

We call elements $s \in \mathscr{F}(U)$ *sections* over U, and given $V \subseteq U$, we often write $s|_V := \rho_V^U(s)$ for convenience. Furthermore, we say that a presheaf $\mathscr{F}$ is a *sheaf* provided that for every open cover $U = \bigcup_i U_i$ and sections $s_i \in \mathscr{F}(U_i)$ satisfying $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j (i.e. sections that agree on overlaps), then there exists a unique section $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$. A *morphism* of (pre)sheaves over X $\mathscr{F} \to \mathscr{G}$ is a collection of abelian group homomorphisms $\phi = \{\phi_U\}$, indexed by the open sets $U \subseteq X$, that commute with restriction. Explicitly, the diagram

$$\begin{array}{ccc} \mathscr{F}(U) & \xrightarrow{\phi_U} & \mathscr{G}(U) \\ {\scriptstyle \rho_V^U}\downarrow & & \downarrow{\scriptstyle \rho_V^U} \\ \mathscr{F}(V) & \xrightarrow{\phi_V} & \mathscr{G}(V) \end{array} \tag{2}$$

commutes, where the restriction maps of $\mathscr{F}$ and $\mathscr{G}$ are both abusively denoted ρ_V^U.

Example 3.1.2. We will now describe the sheaves we will be concerned with in this paper. Let $\mathcal{C}$ denote some reasonable class of functions (e.g. smooth, holomorphic, continuous, etc). Let $\pi : E \to X$ be a $\mathcal{C}$ vector bundle over a space X (with $\mathcal{C}$ structure). Define $\mathscr{F}(U)$ to be the $\mathcal{C}$-sections to π over the open set $U \subseteq X$. Then $\mathscr{F}$ is a sheaf of abelian groups over X. Notice that if we let $E \to X$ be the trivial line bundle, then $\mathscr{F}$ is simply the sheaf of $\mathcal{C}$-functions on X.

Definition 3.1.3. Let X be a complex manifold. For each open $U \subseteq X$, let $\Omega^p(U)$ denote the kernel of the map $\Lambda^{p,0}(U) \xrightarrow{\bar{\partial}} \Lambda^{p,1}(U)$. We call $\Omega^p(U)$ the *holomorphic p-forms* on U. Notice that Ω^p is a subsheaf of $\Lambda^{p,0}$.

Remark 3.1.4. *The category of presheaves of abelian groups over X and the category of sheaves of abelian groups over X are both abelian categories. Furthermore, the sheaves of abelian groups over X form a full subcategory of the category of presheaves of abelian groups over X. On the other hand, the notions of cokernel are distinct, and so in particular, if $\mathscr{F}'$, $\mathscr{F}$, and $\mathscr{F}''$ are sheaves, then one can (and often does) have that*

$$0 \to \mathscr{F}' \to \mathscr{F} \to \mathscr{F}'' \to 0$$

is a short exact sequence of sheaves, but is not a short exact sequence of presheaves. We remark that a sequence of sheaves is exact if and only if it is so locally on germs of functions.

Example 3.1.5. The canonical geometric example illustrating the points made in Remark 3.1.4 is as follows. Let $X = \mathbb{C}$ be the complex plane, let $\mathbb{Z}$ denote the sheaf of locally constant integer-valued functions, let $\mathcal{O}_X$ denote the sheaf of holomorphic functions, and let $\mathcal{O}_X^*$ denote the sheaf of nowhere vanishing holomorphic functions on X. Then the sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_X \stackrel{\exp}{\to} \mathcal{O}_X^* \to 0$$

is a short exact sequence of sheaves since locally every nowhere vanishing holomorphic function is the exponential of some other holomorphic function, but this is not true globally since there is no global logarithm function.

Definition 3.1.6. Given a presheaf $\mathscr{F}$ over a topological space X along with some open set U and an open cover $\mathfrak{U} : U = \bigcup_i U_i$, we define the *Čech chains* of $\mathscr{F}$ with respect to the cover $\mathfrak{U}$ to be the complex of abelian groups $\check{C}^*(\mathfrak{U}, \mathscr{F})$ where

$$\check{C}^k(\mathfrak{U}, \mathscr{F}) := \prod_{i_0 < \ldots < i_k} \mathscr{F}(U_{i_1 \cdots i_k}),$$

with the convention that $U_{i_0 \cdots i_k} = U_{i_0} \cap \cdots \cap U_{i_k}$. The differential $\delta : \check{C}^k(\mathfrak{U}, \mathscr{F}) \to \check{C}^{k+1}(\mathfrak{U}, \mathscr{F})$ is defined by the rule

$$\delta(x)_{i_0 \cdots i_k} := \sum_{j=0}^{k} (-1)^j x_{i_0 \cdots \hat{i}_j \cdots i_k}|_{U_{i_0 \cdots i_k}}.$$

An easy check shows that $\delta^2 = 0$, and one defines the *Čech cohomology* to be the cohomology of this complex with respect to this differential δ. That is,

$$\check{H}^k(\mathfrak{U}, \mathscr{F}) := \frac{\ker \check{C}^k(\mathfrak{U}, \mathscr{F}) \stackrel{\delta}{\to} \check{C}^{k+1}(\mathfrak{U}, \mathscr{F})}{\operatorname{Im} \check{C}^{k-1}(\mathfrak{U}, \mathscr{F}) \stackrel{\delta}{\to} \check{C}^k(\mathfrak{U}, \mathscr{F})}.$$

Remark 3.1.7. *Čech cohomology is a cohomology theory for* presheaves*, and not sheaves. That is, for a short exact sequence of presheaves, which, as noted in Remark 3.1.4, is different than a short exact sequence of sheaves, there is an associated long exact sequence of Čech cohomology groups. The long exact sequence of Čech cohomology groups will not make an appearance in this paper.*

Given a C^∞ vector bundle $E \to X$ over a smooth differential manifold X, let $\mathscr{F}$ denote its sheaf of sections. In C^∞-geometry, one can always "kill off" a local section $s_U \in \mathscr{F}(U)$ to make a global section s supported in U by multiplying s_U by a bump function supported in U. This tool will allow us to show that the higher cohomology of $\mathscr{F}$ vanish. First, we generalize this notion of bump function to an arbitrary sheaf, and then prove the vanishing of cohomology in this generality.

Definition 3.1.8. A sheaf $\mathscr{F}$ on a space X is called *fine* provided that for any opens $U \subseteq X$ and an open cover $\mathfrak{U} : U = \bigcup_\alpha U_\alpha$, there exist group homomorphisms $\eta_\alpha : \mathscr{F}(U_\alpha \cap V) \to \mathscr{F}(V)$ for all open $V \subseteq U$ such that

(1) the maps commute with restriction (i.e. for all open $W \subseteq V$ the following diagram commutes):

$$\begin{array}{ccc} \mathscr{F}(V \cap U_\alpha) & \xrightarrow{\eta_\alpha} & \mathscr{F}(V) \\ \downarrow \rho^{V \cap U_\alpha}_{W \cap U_\alpha} & & \downarrow \rho^V_W \\ \mathscr{F}(W \cap U_\alpha) & \xrightarrow{\eta_\alpha} & \mathscr{F}(W) \end{array}$$

(2) for all $\sigma \in \mathscr{F}(V)$, the support of $\eta_\alpha(\sigma) \subseteq U_\alpha \cap V$;
(3) for each $x \in U$, there exists at most finitely many α such that there exists an open $W \subseteq U$ and $\sigma \in \mathscr{F}(W \cap U_\alpha)$ so that the germ of $\eta_\alpha(\sigma)$ is nonzero (i.e. x is contained in at most finitely many of the supports of the partition of unity);
(4) for each $\sigma \in \mathscr{F}(V)$, the sum $\sum_\alpha \eta_\alpha(\sigma|_{U_\alpha \cap V}) = \sigma$. (Note, this sum makes sense by (3).)

We call the maps $\{\eta_\alpha\}$ a *(sheaf-theoretic) partition of unity* for $\mathscr{F}$ subordinate to the cover $\mathfrak{U}$. The *support* of a section $s \in \mathscr{F}(U)$ consists of all the points $x \in U$ such that the germ of s at x is nonzero (i.e. the image of $s \in \mathrm{colim}_{U \ni x} \mathscr{F}(U)$ is nonzero).

Remark 3.1.9. *The support of a section $s \in \mathscr{F}(U)$ is always closed in U. To see the complement is open, notice that the germ of s at x is identically zero if and only if s is zero on a small neighborhood of x.*

Example 3.1.10. The abstract definition of sheaf-theoretic partition of unity may seem a little awkward at first. For the sake of the paper, it will be enough to just work with following example. As discussed above, if we let $E \to X$ be a C^∞ vector bundle over a smooth differential manifold and $U = \bigcup_\alpha U_\alpha$, then there exists a (standard) partition of unity ρ_α subordinate to this cover. Let $\mathscr{F}$ be the sheaf of sections. Then, the map $\eta_\alpha : \mathscr{F}(U_\alpha \cap V) \to \mathscr{F}(V)$ defined by $s \mapsto \rho_\alpha \cdot s_V \in \mathscr{F}(U)$ for $s \in \mathscr{F}(U_\alpha \cap V)$ gives a map of abelian groups that commute with restriction. Clearly the germs of the function $\eta_\alpha(s)$ are supported inside the set $U_\alpha \cap V$, and by construction, any x is contained in at most finitely many of the supports of the ρ_α, and hence the maps η_α satisfy condition (3) of Definition 3.1.8. Thus, the collection of maps $\{\eta_\alpha\}$ form a sheaf-theoretic partition of unity, thus showing that $\mathscr{F}$ is a fine sheaf.

Lemma 3.1.11. *If $\mathscr{F}$ is a fine sheaf on a space X, then its higher Čech cohomology groups with respect to any open cover $\mathfrak{U} : U = \bigcup_i U_i$ vanish :*

$$\check{H}^p(\mathfrak{U}, \mathscr{F}) = 0, \quad p > 0.$$

Proof. Let $p > 0$, and let $\mathfrak{U}$ be an open cover as in the statement. $\mathscr{F}$ is fine, so choose $\{\eta_i\}_{i \in I}$ a sheaf-theoretic partition of unity for $\mathscr{F}$ subordinate to $\mathfrak{U}$. We will show that every cycle $\sigma \in \check{C}^p(\mathfrak{U}, \mathscr{F})$ is actually a boundary. Define

$$\tau_{i_0 \cdots i_{p-1}} := \sum_{j \in I} \eta_j(\sigma_{j, i_0 \cdots i_{p-1}}).$$

Each term $\eta_j(\sigma_{j, i_0 \cdots i_{p-1}})$ is viewed as a section of $U_{i_0} \cap \cdots \cap U_{i_{p-1}}$ since η_j is a map $\eta_j : \mathscr{F}(U_j \cap V) \to V$ where $V := U_{i_0} \cap \cdots \cap U_{i_{p-1}}$. Hence $\tau_{i_0 \cdots i_{p-1}} \in \mathscr{F}(U_{i_0 \cdots i_{p-1}})$, and τ is an element of $\check{C}^{p-1}(\mathfrak{U}, \mathscr{F})$. Using the fact that $\sum_j \eta_j$ is the identity map, a quick calculation shows that $\delta\tau = \sigma$. ■

Remark 3.1.12. *We note that for any sheaf $\mathscr{F}$ over X with cover $\mathfrak{U} : U = \bigcup U_i$, the group $\check{H}^0(\mathfrak{U}, \mathscr{F}) = \mathscr{F}(U)$. This is equivalent to the condition that the presheaf $\mathscr{F}$ is actually sheaf.*

3.2. **Dolbeault theorem.**

In order to prove the Dolbeault theorem, we will need one technical analytic result, analogous to the Poincaré lemma that shows that a open ball in $\mathbb{R}^n$ has trivial de Rham cohomology.

Lemma 3.2.1 ($\bar{\partial}$-Poincaré Lemma)**.** *Given an open polydisc $D := D_{\delta_1, \ldots, \delta_n} = \{(z_1, \ldots, z_n) : |z_i| < \delta_i\} \subseteq \mathbb{C}^n$ for $\delta_i > 0$ (possibly infinite), the higher Dolbeault cohomology vanishes:*

$$H^{p,q}(B) = 0, \quad q \geq 1.$$

Proof. See [1, pp. 25]. ■

Remark 3.2.2. *There is a stronger version of the $\bar{\partial}$-lemma that shows that the higher Dolbeault cohomology groups vanish on arbitrary finite products of polydiscs with punctured polydiscs (i.e. polydiscs minus the origin). We will need this stronger form for later computations.*

Corollary 3.2.3. *If X is a complex manifold, then for all fixed p, the following sequence of sheaves is exact*

$$0 \to \Omega^p \to \Lambda^{p,0} \to \Lambda^{p,1} \to \Lambda^{p,2} \to \cdots,$$

where Ω^p denotes the sheaf of holomorphic p-forms on X.

Proof. A sequence of sheaves is exact if it is so locally, as noted in Remark 3.1.4. But X can be covered by open sets biholomorphic to polydiscs D in $\mathbb{C}^n$. Thus, by Lemma 3.2.1, this sequence is exact away from the group Ω^p. At Ω^p, the exactness follows from the definition of the sheaf Ω^p as the kernel of the map $\Lambda^{p,0} \to \Lambda^{p,1}$. ■

We make one last definition before moving on to state and prove the Dolbeault theorem. To relate Dolbeault cohomology with Čech cohomology of presheaves, we will need to work with certain acyclic open covers $\mathfrak{U}$.

Definition 3.2.4. We say an cover $\mathfrak{U} : U = \bigcup U_i$ is a *good cover* provided that any intersection $U_{i_0 \cdots i_k}$ has vanishing higher Dolbeault cohomology.

Example 3.2.5. By the $\bar{\partial}$-Lemma and Remark 3.2.2, any open cover with all $U_{i_0 \cdots i_k}$ biholomorphic to finite products of polydiscs and punctured polydiscs is a good cover.

Finally, we are prepared to prove the main result of this section, Dolbeault theorem.

Theorem 3.2.6 (Dolbeault theorem)**.** *For X a complex manifold with some good cover $\mathfrak{U} : X = \bigcup U_i$,*

$$\check{H}^q(\mathfrak{U}, \Omega^p) \cong H^{p,q}(X).$$

Remark 3.2.7. *There is a version of the Dolbeault theorem that does not rely upon the existence of a good cover. This version is stated in terms of* sheafified *Čech cohomology, which is the direct limit over all open covers of the presheaf Čech cohomology groups defined in this paper. Furthermore, one may be concerned that perhaps for some complex manifolds there do not exist good covers. These fears are laid to rest by the following lemma.*

Lemma 3.2.8. *Every complex manifold X admits a good cover.*

Proof. We will give a brief outline of the proof for the informed reader. Cover X by open sets $\{U_i\}$ biholomorphic to geometrically convex sets in $\mathbb{C}^n$. By [2, Cor 2.5.6], such sets are "domains of holomorphy," and so too will be intersections of such sets. These are examples of *Stein manifolds* [2, pp. 116], and any coherent analytic sheaf (e.g. Ω^p) will have trivial cohomology [2, Thm 7.4.3]. Since the sheaves $\Lambda^{p,q}$ are fine, they form an acyclic resolution of Ω^p, and Leray's acyclicity lemma, the cohomology of Ω^p may then be calculated from this acyclic resolution, and so the higher Dolbeault cohomology groups vanish. Thus, $\{U_i\}$ is a good cover. ■

Proof of Dolbeault theorem. Fix $p \geq 0$. Consider the double complex $\check{C}^i(\mathfrak{U}, \Lambda^{p,j})$ where we let $\Lambda^{p,j}$ be the sheaf of sections of the vector bundle $\Lambda^{p,j}(X)$. For convenience, we will denote this $\check{C}^i(\Lambda^{p,j})$ with the good cover $\mathfrak{U}$ understood. Let the i-differential be the Čech differential δ, and let the j-differential be $\bar{\partial}$. By the basic algebraic theory of spectral sequences, this double complex forms the E_0-page of two distinct first quadrant spectral sequences that both converge to the cohomology of the total complex. Furthermore, the first two sets of differentials are just induced by the differentials of the bicomplex. The first spectral sequence arises by taking the differentials on the E_0-page to be the $\bar{\partial}$ differential:

4	$\check{C}^0(\Lambda^{p,4})$	$\check{C}^1(\Lambda^{p,4})$	$\check{C}^2(\Lambda^{p,4})$	$\check{C}^3(\Lambda^{p,4})$	$\check{C}^4(\Lambda^{p,4})$
	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
	$\check{C}^0(\Lambda^{p,3})$	$\check{C}^1(\Lambda^{p,3})$	$\check{C}^2(\Lambda^{p,3})$	$\check{C}^3(\Lambda^{p,3})$	$\check{C}^4(\Lambda^{p,3})$
	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
2	$\check{C}^0(\Lambda^{p,2})$	$\check{C}^1(\Lambda^{p,2})$	$\check{C}^2(\Lambda^{p,2})$	$\check{C}^3(\Lambda^{p,2})$	$\check{C}^4(\Lambda^{p,2})$
	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
	$\check{C}^0(\Lambda^{p,1})$	$\check{C}^1(\Lambda^{p,1})$	$\check{C}^2(\Lambda^{p,1})$	$\check{C}^3(\Lambda^{p,1})$	$\check{C}^4(\Lambda^{p,1})$
	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
0	$\check{C}^0(\Lambda^{p,0})$	$\check{C}^1(\Lambda^{p,0})$	$\check{C}^2(\Lambda^{p,0})$	$\check{C}^3(\Lambda^{p,0})$	$\check{C}^4(\Lambda^{p,0})$
	0		2		4

Since $\mathfrak{U}$ is a good cover, we see by the $\bar{\partial}$-Lemma that when we take homology to obtain the E_1-page of the spectral sequence, the result is:

2	0	0	0	0	0
	0	0	0	0	0
0	$\check{C}^0(\Omega^p)\to$	$\check{C}^1(\Omega^p)\to$	$\check{C}^2(\Omega^p)\to$	$\check{C}^3(\Omega^p)\to$	$\check{C}^4(\Omega^p)$
	0		2		4

with differentials equal to the Čech differential δ. Taking homology to arrive at the E_2-page results in

2	0	0	0	0	0
	0	0	0	0	0
0	$\check{H}^0(\Omega^p)$	$\check{H}^1(\Omega^p)$	$\check{H}^2(\Omega^p)$	$\check{H}^3(\Omega^p)$	$\check{H}^4(\Omega^p)$
	0		2		4

All further differentials are zero, so the E_∞ page is equal to the E_2-page.

If instead we were to begin by taking the horizontal differentials on the E_0-page, we would obtain, by Lemma 3.1.11 and Remark 3.1.12, an E_1-page of

4	$\Lambda^{p,4}(X)$	0	0
	$\Lambda^{p,3}(X)$ ↑	0	0
2	$\Lambda^{p,2}(X)$ ↑	0	0
	$\Lambda^{p,1}(X)$ ↑	0	0
0	$\Lambda^{p,0}(X)$ ↑	0	0
	0		2

with the vertical differentials equal to $\bar{\partial}$. Thus, the E_2-page once again equals to E_∞-page and is equal to:

4	$H^{p,4}(X)$	0	0
	$H^{p,3}(X)$	0	0
2	$H^{p,2}(X)$	0	0
	$H^{p,1}(X)$	0	0
0	$H^{p,0}(X)$	0	0
	0		2

By the theory of spectral sequences, we have that both of these E_∞-pages filter the same total cohomology. In each degree, each of these sequences only has one group. Therefore, we must have $H^{p,q}(X) \cong \check{H}^q(\mathfrak{U}, \Omega^p)$ for all p, q, thus completing our proof. ■

3.3. **Calculation with Čech cohomology.** Let's see how the Dolbeault theorem allows us to make explicit computations of Dolbeault cohomology. We will demonstrate the combinatorial method by calculating the Hodge numbers for $\mathbb{P}^1_{\mathbb{C}}$. Although the same method is theoretically applicable to $\mathbb{P}^n_{\mathbb{C}}$ for $n > 1$, it requires vastly more complicated bookkeeping. The Hodge numbers of $\mathbb{P}^n_{\mathbb{C}}$ are most easily computed by using Hodge theory, and for this reason we defer this calcuation until §4.3.

Example 3.3.1. Let $X = \mathbb{P}^1_{\mathbb{C}}$ be complex 1-dimensional projective space with homogeneous coordinates x, y. Let $\mathfrak{U} : U \cup V$ be the standard open cover $U = (x \neq 0)$ and $V = (y \neq 0)$. For $p > 1$, clearly $\Omega^p = 0$, so $H^{p,q} = 0$ for all $q \geq 0$, $p > 1$.

For $p = 0$, the sheaf Ω^0 is just the sheaf of holomorphic functions. By Remark 3.1.12, the group $\check{H}(\mathfrak{U}, \Omega^0)$ is just the group of global sections. Since $\mathbb{P}^1_{\mathbb{C}}$ is compact, by the maximum principle, any global section must be constant, so $\check{H}^0(\mathfrak{U}, \Omega^0) = \mathbb{C}$. Since the cover contains only 2 open sets, $\check{H}^q(\mathfrak{U}, \Omega^p) = 0$ for all $p \geq 0$, $q > 1$. Next, we wish to show that the map $\check{C}^0(\mathfrak{U}, \Omega^0) \to \check{C}^1(\mathfrak{U}, \Omega^0)$ is surjective so that the group $\check{H}^1(U, \Omega^0) = 0$. Any holomorphic function on $U \cap V \cong \mathbb{C} - \{0\}$ is of the form $f = \sum_{i \in \mathbb{Z}} a_i (\frac{x}{y})^i = \sum_{i \geq 0} a_i (\frac{x}{y})^i + \sum_{i>0} a_{-i} (\frac{y}{x})^i$. Therefore, we see that

$$\delta\left(\left(-\sum_{i\geq 0} a_i \left(\frac{x}{y}\right)^i, \sum_{i>0} a_{-i}\left(\frac{y}{x}\right)^i\right)\right) = f \in \check{C}^1(\mathfrak{U}, \Omega^0).$$

Therefore, the map is surjective.

Now let $p = 1$. Let $u := \frac{x}{y}$ and $v := \frac{y}{x}$, and choose

$$(\omega, \eta) \in \Omega^1(U) \times \Omega^1(V) = \check{C}^0(\mathfrak{U}, \Omega^1)$$

with $\omega = \sum_{n=0}^{\infty} a_n u^n du$ and $\eta = \sum_{n=0}^{\infty} b_n v^n dv$. Since $dv = -u^2 du$, we see that $\eta = -\sum_{n=0}^{\infty} b_n u^{-n-2} du$. Therefore,

$$\delta(\omega, \eta) = \sum_{n=0}^{\infty} a_n u^n du + \sum_{n<-1} b_{n+2} \cdot u^n du.$$

From this, we see that $\delta(\omega, \eta) = 0$ if and only if $\omega = \eta = 0$, and that a form $\mu = \sum_{m \in \mathbb{Z}} c_m u^m du \in \Omega^1(U \cap V)$ is in the image of δ if and only if $c_{-1} = 0$. Thus, $\check{H}^i(\mathfrak{U}, \Omega^1) = 0$ unless $i = 1$, in which case it equals $\mathbb{C}$.

In conclusion, the Hodge numbers for $X = \mathbb{P}^1_{\mathbb{C}}$ are $h^{1,1} = h^{0,0} = 1$, and $h^{p,q} = 0$ otherwise.

4. Hodge Theory

Hodge theory and the study of harmonic forms are widely used in both the computation and theory of Dolbeault cohomology. To make these calculations, we must develop a fair amount of Hodge theory. As this relies on fairly involved analysis, we will refer the reader to [1, pp. 80-126] for the more difficult proofs, and provide a brief survey of the results.

4.1. **Laplace operators.**

Definition 4.1.1. A *Hermitian metric* on a complex vector bundle $E \to X$ over any differential manifold X is a global section h of the smooth vector bundle $E^* \otimes E^*$ that on the fibre over any point p is a Hermitian form on the space E. If furthermore $E = T_{\mathbb{C}}X$ for a complex manifold X, then we say that X is a *(complex) Hermitian manifold*. In this case, the real part $g = \frac{1}{2}(h + \bar{h})$ is a Riemannian metric on X and we call $\omega := -1/2 \cdot \operatorname{Im} h$ the *associated* $(1,1)$*-form*. If ω is a closed form (i.e. $d\omega = 0$), then we say that ω is a *symplectic form*, (X, h) is a *Kähler manifold*, and h a *Kähler metric* on X.

Definition 4.1.2. Let E and F be Hermitian vector bundles over a differential manifold X with volume form dv and Hermitian metrics $\langle \cdot, \cdot \rangle_E$ and $\langle \cdot, \cdot \rangle_F$ on E, F respectively. Let $D : C^\infty(E) \to C^\infty(F)$ and $D' : C^\infty(F) \to C^\infty(E)$ be $\mathbb{C}$-linear derivations (i.e. $\mathbb{C}$-linear maps that satisfy the Leibnitz rule). Then D' is called a *formal adjoint of* D provided that

$$\int_M \langle D\alpha, \beta \rangle_F dv = \int_M \langle \alpha, D'\beta \rangle_E dv,$$

for all compactly supported smooth sections $\alpha \in C_0^\infty(E)$ and $\beta \in C_0^\infty(F)$.

Remark 4.1.3. *A $\mathbb{C}$-linear derivation has at most one formal adjoint. The proof of this is routine, and we will not make use of this fact here.*

Lemma 4.1.4. *On a Hermitian manifold (X, h), there exists a Hermitian metric $H(\cdot, \cdot)$ on the complex vector bundle $\Lambda^k_{\mathbb{C}}$ induced by the metric h and with local orthonormal basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \le i_1 < \cdots < i_k \le n\}$ where $\{e_1, \ldots, e_n\}$ denotes a local orthonormal frame for $\Lambda^1_{\mathbb{C}}$.*

Proof. See [3, pp. 100]. ■

Lemma 4.1.5. *The exterior derivative $d : \Lambda^k_{\mathbb{C}}(X) \to \Lambda^{k+1}_{\mathbb{C}}(X)$ has a formal adjoint $\delta : \Lambda^{k+1}_{\mathbb{C}}(X) \to \Lambda^k_{\mathbb{C}}(X)$ with respect to the Hermitian metrics (both denoted $H(\cdot, \cdot)$) on $\Lambda^k_{\mathbb{C}} X$ and $\Lambda^{k+1}_{\mathbb{C}} X$.*

Proof. See [3, pp. 101]. ■

Definition 4.1.6. The *Laplace operator* $\Delta : \Lambda^k_{\mathbb{C}}(X) \to \Lambda^k_{\mathbb{C}}(X)$ is defined by

$$\Delta := d\delta + \delta d.$$

From the definition of formal adjoint, we see that if D has formal adjoint D^* and C has formal adjoint C^*, then $C \circ D$ has formal adjoint $D^* \circ C^*$. Thus, $d\delta$ has formal adjoint δd, and vice versa. Therefore, Δ is formally self-adjoint.

Lemma 4.1.7. *If X is a Hermitian manifold, then the operator δ decomposes into the sum $\delta = \partial^* + \bar{\partial}^*$ where ∂^* and $\bar{\partial}*$ are the formal adjoints of ∂ and $\bar{\partial}$ respectively.*

Proof. See [3, pp. 102]. ■

Definition 4.1.8. If X is a Hermitian manifold, then we define two additional Laplace operators $\Delta^{\partial}, \Delta^{\bar{\partial}} : \Lambda^{p,q}(X) \to \Lambda^{p,q}(X)$ defined by

$$\Delta^{\partial} := \partial\partial^* + \partial^*\partial \quad \text{and} \quad \Delta^{\bar{\partial}} := \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

For the same (formal) reason that the standard Laplace operator Δ is self-adjoint, the operators Δ^{∂} and $\Delta^{\bar{\partial}}$ are self-adjoint.

Now generally these three operators are all distinct. This is to be expected; the first Laplacian Δ is the sum $d\delta + \delta d$, where d was the complexified de Rham differential and δ its formal adjoint, and is independent of the complex structure. On the other hand, the latter two Laplacians Δ^{∂} and $\Delta^{\bar{\partial}}$ depend upon the decomposition of k-forms into (p, q)-forms for $p + q = k$, and therefore directly depends on the complex structure. When (X, h) is a Kähler manifold, there are many ways in which the complex and real structures are compatible (e.g. the Chern connection corresponds to the complexified Levi-Civita connection). This leads us to ask whether, in the Kähler case, there is some nice relationship between these three Laplacian operators. The answer is about as pleasant as one could desire.

Theorem 4.1.9. *On a Kähler manifold, the Laplacian operators satisfy $\Delta = 2\Delta^{\partial} = 2\Delta^{\bar{\partial}}$.*

Proof. See [3, pp. 103]. ■

4.2. **Harmonic decompositions.**

Definition 4.2.1. Let X be a compact complex manifold. The *harmonic k-forms* are defined to be

$$\mathcal{H}^k(X; \mathbb{C}) := \{\omega \in \Lambda^k_{\mathbb{C}}(X) : \Delta\omega = 0\}.$$

If furthermore (X, h) is a complex Hermitian manifold, then the *$\bar{\partial}$-harmonic (p, q)-forms* are defined to be

$$\mathcal{H}^{p,q}(X) := \{\omega \in \Lambda^{p,q}M : \Delta^{\bar{\partial}}\omega = 0\}.$$

Lemma 4.2.2. *Let X be a compact Hermitian manifold. A k-form is harmonic if and only if it is d-closed and δ-closed.*

Proof. If ω is both d-closed and δ-closed, then $\Delta\omega = (d\delta + \delta d)(\omega) = 0$. Conversely, since M is compact and d and δ are formally adjoint, we have that for all k-forms η, $0 = \int_M H(\delta\omega, \eta)dv = 0$. If we let $\eta = \omega$, then we see that

$$0 = \int_M H(\Delta\omega, \omega)dv = \int_M H(d\delta\omega + \delta d\omega, \omega)dv = \int_M |\delta\omega|^2 + |d\omega|^2 dv.$$

Therefore, $d\omega = 0$ and $\delta\omega = 0$. ■

By a completely analogous proof, we get the following lemma:

Lemma 4.2.3. *Let X be a compact Hermitian manifold. A (p, q)-form is $\bar{\partial}$-harmonic if and only if $\bar{\partial}\omega = 0$ and $\bar{\partial}^*\omega = 0$.*

Now we state the main theorems of Hodge theory. Its proof is involved, and we refer the reader to [1, pp. 84-100] for the proof.

Theorem 4.2.4 (Hodge decomposition)**.** *Let X be a compact Hermitian manifold. The space of k-forms decomposes as the direct sum*

$$\Lambda^k_{\mathbb{C}}(X) = \mathcal{H}^k(X;\mathbb{C}) \oplus \delta\Lambda^{k+1}_{\mathbb{C}}X \oplus d\Lambda^{k-1}X.$$

Theorem 4.2.5 (Dolbeault decomposition)**.** *Let X be a compact Hermitian manifold. The space of (p,q)-forms decomposes as the direct sum*

$$\Lambda^{p,q}X = \mathcal{H}^{p,q} \oplus \bar{\partial}^*\Lambda^{p,q+1}(X) \oplus \bar{\partial}\Lambda^{p,q-1}(X).$$

Corollary 4.2.6. *On a compact Hermitian manifold, these direct sum decompositions are orthogonal with respect to the global Hermitian product on $\Lambda^k_{\mathbb{C}}X$ defined by*

$$(\omega,\eta) := \int_X H(\omega,\eta)dv.$$

Proof. We will demonstrate the proof in the case of k-forms, and the proof for (p,q)-forms will be identical. Since d and δ are formally adjoint with $d^2 = \delta^2 = 0$, it is clear that $\delta\Lambda^{k+1}_{\mathbb{C}}(X)$ is orthogonal to $d\Lambda^{k-1}_{\mathbb{C}}(X)$. By Lemma 4.2.2, it also follows that each of these spaces is orthogonal to $\mathcal{H}^k(X;\mathbb{C})$. For example, to see $\delta\Lambda^{k+1}_{\mathbb{C}}$ is orthogonal to the k-harmonic forms, let $\omega \in \mathcal{H}^k$ and calculate

$$\int_X H(\omega,\delta\eta)dv = \int_X H(d\omega,\eta)dv = 0.$$

■

These decompositions yield powerful tools for calculating both de Rham and Dolbeault cohomology. Since the harmonic k-forms (resp. (p,q)-forms) are closed (resp. $\bar{\partial}$-closed), there exists a natural map $\mathcal{H}^k(X;\mathbb{C}) \to H^k_{DR}(X;\mathbb{C})$ (resp. $\mathcal{H}^{p,q}(X) \to H^{p,q}(X)$) given by $\omega \mapsto [\omega]$.

Proposition 4.2.7 (Hodge and Dolbeault isomorphisms)**.** *Let X be a compact Hermitian manifold. The natural map $\mathcal{H}^k(X;\mathbb{C}) \to H^k_{DR}(X;\mathbb{C})$ is an isomorphism. If X is a compact Hermitian manifold, then the natural map $\mathcal{H}^{p,q}(X) \to H^{p,q}(X)$ is an isomorphism.*

Proof. We prove this for (p,q)-forms, and the proof for k-forms is identical after replacing $\bar{\partial}$ and $\bar{\partial}^*$ by d and δ.

Any element $\omega \in \Lambda^{p,q}(X)$ is of the form

$$\omega = \omega_0 + \bar{\partial}\eta + \bar{\partial}^*\theta,$$

for $\omega_0 \in \mathcal{H}^{p,q}(X)$. If ω is a closed form, I claim that $\bar{\partial}^*\theta = 0$. Clearly, $\bar{\partial}\omega_0 = \bar{\partial}^2\eta = 0$. Thus, if ω is closed, then $\bar{\partial}\bar{\partial}^*\theta = 0$. Hence

$$0 = \int_X H(\bar{\partial}\bar{\partial}^*\theta,\theta)dv = \int_X H(\bar{\partial}^*\theta,\bar{\partial}^*\theta)dv = \int_X |\bar{\partial}^*\theta|^2 dv,$$

and so $\bar{\partial}^*\theta = 0$. Thus, if ω is a closed form, its cohomology class $[\omega] = [\omega_0 + \bar{\partial}\eta] = [\omega_0]$. Thus, the natural map is surjective. To see injectivity, if ω_0 is a harmonic (p,q)-form, $[\omega_0] = 0$ implies ω_0 is d-exact, and hence 0 by the Dolbeault decomposition (Theorem 4.2.5). ■

Analogous to the Poincaré duality for compact differential manifolds, a Serre duality exists for harmonic (p,q)-forms. We state this without proof.

Proposition 4.2.8 (Serre duality)**.** *Let (X, h) be a complex Hermitian manifold. The spaces $\mathcal{H}^{p,q}X$ and $\mathcal{H}^{m-p,m-q}$ are dual, and hence isomorphic (via the Hodge star operator). In particular,*

$$h^{p,q} = h^{m-p,m-q}.$$

Proof. See [3, pp. 108]. ∎

Finally, we may use this to obtain the results we desire:

Proposition 4.2.9. *If X is a compact Kähler manifold, then*

$$\mathcal{H}^k(X) = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(X). \tag{3}$$

Furthermore, the Hodge numbers and Betti numbers satisfy the relations

(a) $b^k = \sum_{p+q=k} h^{p,q}$;
(b) $h^{p,q} = h^{q,p}$;
(c) $h^{p,p} \geq 1$ *for all* $0 \leq p \leq m$.

In particular, the Betti number b_k is even when k is odd, and b_k is nonzero when k is even.

Proof. If X is Kähler, by Theorem 4.1.9 we have $\Delta = 2\Delta^{\partial} = 2\Delta^{\bar{\partial}}$. This implies that $\mathcal{H}^{p,q}(X) \subseteq \mathcal{H}^{p+q}(X)$. To prove equation (3), it suffices to show that Δ preserves the degree (p, q). If this is true, then the projection of any harmonic k-form onto its (p, q) direct summand will again be harmonic. Since $\Delta = 2\Delta^{\bar{\partial}}$, it will be (p, q)-harmonic as well. Thus, we will have shown (3). But by definition, the operator $\Delta^{\bar{\partial}} = \bar{\partial} + \bar{\partial}^* + \bar{\partial}^*\bar{\partial}$, and $\bar{\partial}$ raises the q-degree by one while $\bar{\partial}^*$ lowers the q-degree by one. Thus, $\Delta^{\bar{\partial}}$, and hence Δ, respects degrees.

Notice that (a) follows direction from (3) and the Hodge and Dolbeault isomorphisms (Proposition 4.2.7.

To show (b), it will be enough to show that conjugation gives a $\mathbb{C}$-anti-linear isomorphism between $\mathcal{H}^{p,q}(X)$ and $\mathcal{H}^{q,p}(X)$. We know from the definition of $\Lambda^{p,q}(X)$ that conjugation gives a well-defined $\mathbb{C}$-anti-linear isomorphism from $\Lambda^{p,q}(X)$ to $\Lambda^{q,p}(X)$. Thus, it suffices to show that $\Delta^{\bar{\partial}}$ commutes with conjugation. From the definitions of ∂ and $\bar{\partial}$, it is easy to see that $\overline{\partial\bar{\alpha}} = \bar{\partial}\alpha$. An analogous relation holds for the formal adjoints; that is, $\overline{\partial^*\bar{\beta}} = \bar{\partial}^*\beta$. To see this, notice that on a compact manifold X, two forms ω, ω' are equal if and only if for all forms α of the same degree we have

$$\int_X H(\omega, \alpha)dv = \int_X H(\omega', \alpha)dv. \tag{4}$$

For, if equation (4) is satisfied, then in particular we have

$$0 = \int_X H(\omega - \omega', \omega - \omega')dv = \int_X |\omega - \omega'|^2 dv.$$

To show $\overline{\partial^* \bar{\beta}} = \bar{\partial}^* \beta$, we compute:

$$\begin{aligned}\int_X H(\bar{\partial}^* \beta, \alpha) dv &= \int_X H(\beta, \bar{\partial}\alpha) dv \\ &= \int_X \overline{H(\bar{\beta}, \overline{\bar{\partial}\alpha})} dv \\ &= \int_X \overline{H(\bar{\beta}, \partial\bar{\alpha})} dv \\ &= \int_X \overline{H(\partial^* \bar{\beta}, \bar{\alpha})} dv \\ &= \int_X H(\overline{\partial^* \bar{\beta}}, \alpha) dv.\end{aligned}$$

Using these formulas for the interactions of $\bar{\partial}$ and $\bar{\partial}^*$ with conjugation, a formal calculation shows that $\Delta^{\bar{\partial}}(\bar{\omega}) = \overline{\Delta^{\bar{\partial}}(\omega)}$.

Finally, to see (c), it is a fact that the symplectic $(1,1)$-form ω (and all its wedge powers ω^k) are harmonic and have the property that ω^n is a multiple of the volume form when $n = \dim X$. Hence, all higher powers of ω are harmonic and nonzero. ■

Remark 4.2.10. *Proposition 4.2.9 puts strong restraints on the compact complex manifolds that may exhibit a Kähler metric.*

4.3. **Calculation using Hodge theory.** Hodge theory is a powerful tool that may be used to compute Dolbeault cohomology. We now repeat our calculation of the Hodge numbers of $\mathbb{P}^n_{\mathbb{C}}$, generalizing to the case of arbitrary n. As we will see, this will be much simpler than our previous calculation using the Dolbeault theorem.

Example 4.3.1. Let $X = \mathbb{P}^n_{\mathbb{C}}$. It is well-known that X with its usual complex structure has a Kähler metric (e.g. Fubini-Study metric). X is compact, and thus we may use Proposition 4.2.9. Furthermore, from elementary algebraic topology, the Betti numbers of X are $b^i = 0$ if i is odd while $b^i = 1$ if i is even. Thus, the conditions of Proposition 4.2.9 determine the Hodge numbers precisely: $h^{k,k} = 1$ for $0 \leq k \leq n$, and $h^{p,q} = 0$ otherwise.

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