The cyclic bar construction.	The circle action and fixed points.	Elementary examples. 00	Cyclic spectra and THH. 00

A visual introduction to cyclic sets and cyclotomic spectra

Cary Malkiewich (UIUC)

July 7, 2015 Young Topologists Meeting Lausanne, Switzerland

The cyclic bar construction.	The circle action and fixed points.	Elementary examples.	Cyclic spectra and THH.

Goal: the cyclic bar construction and topological Hochschild homology (*THH*) in pictures.

Key idea: "cyclotomic" structure.

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Goal: the cyclic bar construction and topological Hochschild homology (*THH*) in pictures.

Key idea: "cyclotomic" structure.

Useful for algebraic K-theory. And fun!

The cyclic bar construction.	The circle action and fixed points.	Elementary examples.	Cyclic spectra and THH.
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Review of the bar construction.			



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$$B(X, G, Y) = |[k] \mapsto X \times G^{\times k} \times Y|$$
$$X \underbrace{\times}_{d_0} G \underbrace{\times}_{d_1} G \underbrace{\times}_{d_2} G \underbrace{\times}_{d_3} Y$$

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Recipe for a space: one Δ^k for each (x, g_1, \ldots, g_k, y) .

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Recipe for a space: one Δ^k for each (x, g_1, \ldots, g_k, y) .

$$EG = B(*, G, G), \qquad BG = B(*, G, *)$$

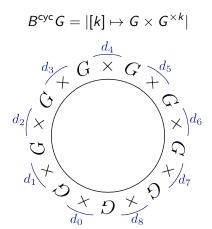
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The cyclic bar construction.	The circle action and fixed points.	Elementary examples.	Cyclic spectra and THH.
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Review of the bar construction.			

Also works for:

- based spaces with smash product
- abelian groups with tensor product
- spectra with the smash product
- diagrams ("G has many objects")

The cyclic bar construction. ○○●○○○	The circle action and fixed points.	Elementary examples. 00	Cyclic spectra and <i>THH</i> .
The cyclic bar construction.			



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The terms $G^{\times k+1}$ form a *cyclic* space.

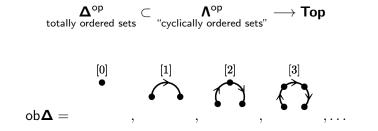
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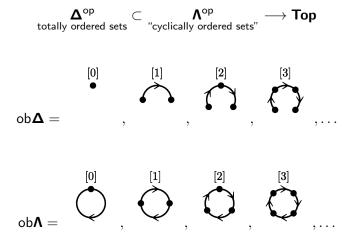
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The morphisms are "degree 1" functors.

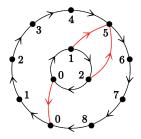
The circle action and fixed points 00000000000 Elementary examples

Cyclic spectra and THH.

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The cyclic bar construction.

Here's a morphism $f : [2] \rightarrow [8]$ in Λ



The circle action and fixed points.

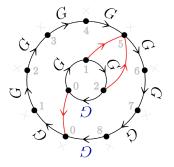
Elementary examples

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It sends G^9 to G^3 like this:

$$\mathbf{G} \times \mathbf{G}^8 \rightarrow \mathbf{G} \times \mathbf{G}^2$$

The circle action and fixed points.

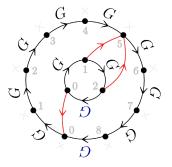
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 $g_0, g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8 \mapsto g_6 g_7 g_8 g_0, g_1 g_2 g_3 g_4 g_5, 1$

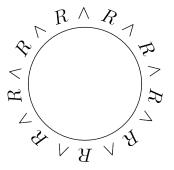
The cyclic bar construction.	The circle action and fixed points.	Elementary examples.	Cyclic spectra and THH.
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Topological Hochschild homology			

To make *Topological Hochschild homology*, just form B^{cyc} in the category of spectra.

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The circle action.			

Theorem

If $X_{\bullet} : \Lambda^{\text{op}} \longrightarrow$ Top is a cyclic space, the realization $|X_{\bullet}|$ has a natural action by the circle group S^1 .



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Proof: X_{\bullet} always a colimit of cyclic sets $\Lambda(-, [n])$ for varying n.

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Just need the circle action on $\Lambda^n := |\mathbf{\Lambda}(-, [n])|$.

Elementary examples

Cyclic spectra and THH. 00

The circle action.

Simplices in $\Lambda^n \leftrightarrow maps [k] \longrightarrow [n]$.

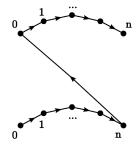
Elementary examples

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The circle action.

Simplices in $\Lambda^n \leftrightarrow \text{maps } [k] \longrightarrow [n]$. Lift to the "universal cover" of [n]:

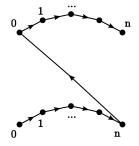


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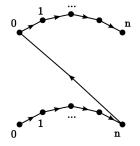
 \leftrightarrow an increasing function

 $f: \{0,\ldots,k\} \longrightarrow \{(0,0), (0,1), \ldots, (0,n), (1,0), (1,1), \ldots, (1,n)\}.$

Elementary examples. 00 Cyclic spectra and THH.

The circle action.

Simplices in $\Lambda^n \leftrightarrow \text{maps } [k] \longrightarrow [n]$. Lift to the "universal cover" of [n]:



 \leftrightarrow an increasing function

 $f: \{0, \dots, k\} \longrightarrow \{(0, 0), (0, 1), \dots, (0, n), (1, 0), (1, 1), \dots, (1, n)\}.$ Unique, unless $f(k) \le (0, n)$ or $f(0) \ge (1, 0)$.

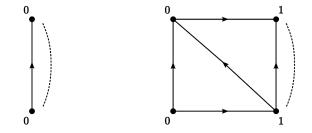
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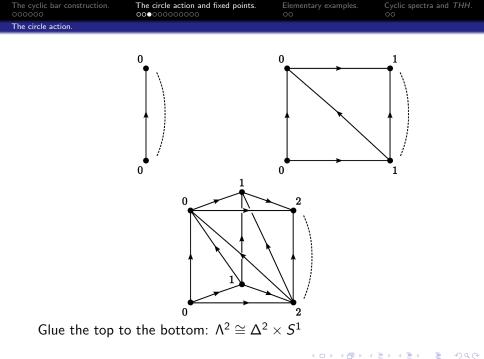
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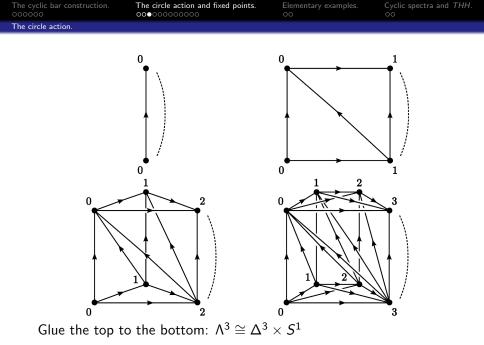
Glue the top to the bottom: $\Lambda^0 \cong \Delta^0 \times S^1$

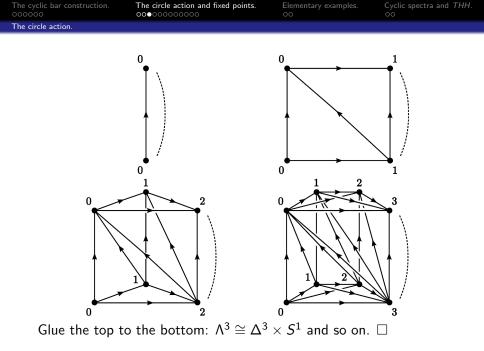
The cyclic bar construction.	The circle action and fixed points.	Elementary examples. 00	Cyclic spectra and <i>THH</i> . 00
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Glue the top to the bottom: $\Lambda^1\cong\Delta^1\times S^1$







Fixed points.			
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The cyclic bar construction.	The circle action and fixed points.	Elementary examples.	Cyclic spectra and THH.

$C_n \leq S^1$ cyclic subgroup – what are its fixed points?

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 $C_n \leq S^1$ cyclic subgroup – what are its fixed points? Simplicial level 0: get one copy of Λ^0 for each $g \in G$



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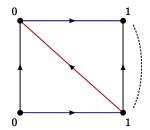
 $C_n \leq S^1$ cyclic subgroup – what are its fixed points? Simplicial level 0: get one copy of Λ^0 for each $g \in G$



Degenerate if g = 1, nondegenerate otherwise.

The cyclic bar construction.	The circle action and fixed points.	Elementary examples.	Cyclic spectra and <i>THH</i> .
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Fixed points.			

Simplicial level 1: we get a $\Lambda^1 = \Delta^1 \times S^1$ for each pair (g_1, g_2) .

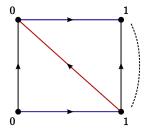


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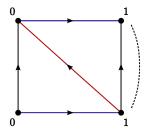
Simplicial level 1: we get a $\Lambda^1 = \Delta^1 \times S^1$ for each pair (g_1, g_2) .



The bottom triangle for (g_1, g_2) is glued to top triangle for (g_2, g_1) and vice-versa.

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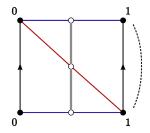


The bottom triangle for (g_1, g_2) is glued to top triangle for (g_2, g_1) and vice-versa. Are any blue points fixed by some nontrivial element of S^1 ?

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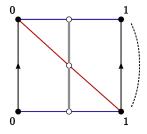
Answer: only the midpoint, and only if $g_1 = g_2$:



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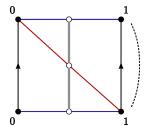
Answer: only the midpoint, and only if $g_1 = g_2$:



The given point must hit itself on the red line again, and only the midpoint does this.

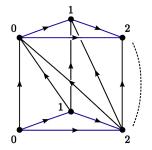
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Answer: only the midpoint, and only if $g_1 = g_2$:



The given point must hit itself on the red line again, and only the midpoint does this. We get a $G \times \Lambda^0$ in the C_2 -fixed points.

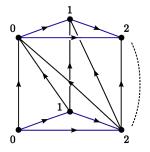
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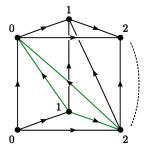
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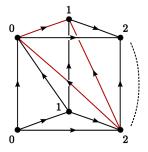
glued by rotating the triple (g_1, g_2, g_3) and rotating the three 3-simplices in the figure.

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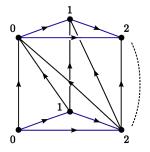
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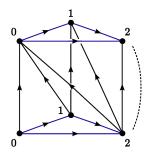
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Fixed points.			

Which points in the blue simplex are fixed?



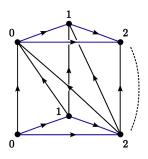
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Fixed points.			
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The cyclic bar construction.	The circle action and fixed points.	Elementary examples.	Cyclic spectra and THH.

Which points in the blue simplex are fixed? Triple must be (g_1, g_1, g_1) , point must be fixed under rotation of

vertices of Δ^2

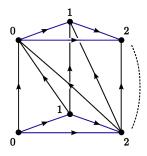


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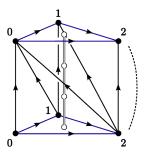
Triple must be (g_1, g_1, g_1) , point must be fixed under rotation of vertices of $\Delta^2 \rightsquigarrow$ only the barycenter.



The cyclic bar construction.	The circle action and fixed points.	Elementary examples. 00	Cyclic spectra and <i>THH</i> .
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Which points in the blue simplex are fixed?

Triple must be (g_1, g_1, g_1) , point must be fixed under rotation of vertices of $\Delta^2 \rightsquigarrow$ only the barycenter. We get C_3 -fixed points:



Get another $G \times \Lambda^0$ in the C_3 -fixed points.

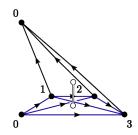
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Fixed points.			

Simplicial level 3: look for fixed points in $G^4 \times \Delta^3$.



The cyclic bar construction.	The circle action and fixed points.	Elementary examples.	Cyclic spectra and <i>THH</i> .
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Fixed points.			

Simplicial level 3: look for fixed points in $G^4 \times \Delta^3$. First chance to get mapped to yourself, by C_4 :

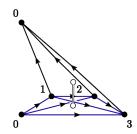


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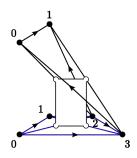
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We get a $G \times \Lambda^0$ in the C_4 -fixed points.

Fixed points.			
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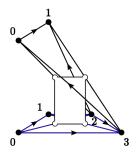
Next chance to get mapped to yourself, by C_2 :



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The cyclic bar construction. 000000	The circle action and fixed points.	Elementary examples. 00	Cyclic spectra and THH.

Next chance to get mapped to yourself, by C_2 :



More fixed points! C_2 acts on Δ^3 by rotating the coordinates twice:

$$(t_0, t_1, t_2, t_3) \mapsto (t_2, t_3, t_0, t_1)$$

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The fixed points form a line Δ^1 .

The cyclic bar construction.

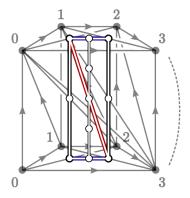
Elementary examples

Cyclic spectra and THH.

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Fixed points.

So, get a copy of $G^2 \times \Lambda^1$ in the C_2 -fixed points.



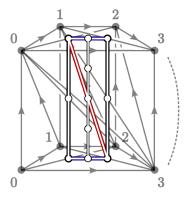
The cyclic bar construction.

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Fixed points.

So, get a copy of $G^2 \times \Lambda^1$ in the C_2 -fixed points.



Can easily formalize now: if $r \mid n$, the piece $G^n \times \Lambda^{n-1}$ has C_r -fixed points $G^{n/r} \times \Lambda^{n/r-1}$.

The cyclic bar construction.	The circle action and fixed points.	Elementary examples.	Cyclic spectra and <i>THH</i> .
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Fixed points.			

Collect it all together:

simp. level	S^1	C_1	C_2	<i>C</i> ₃	<i>C</i> ₄
0	$\{1\} imes \Delta^0$	$G imes \Lambda^0$			
1		$G^2 imes \Lambda^1$	$G imes \Lambda^0$		
2		$G^3 imes \Lambda^2$		$G imes \Lambda^0$	
3		$G^4 imes \Lambda^3$	$G^2 imes \Lambda^1$		$G imes \Lambda^0$
4		$G^5 imes\Lambda^4$			
5		$G^6 imes\Lambda^5$	$G^3 imes \Lambda^2$	$G^2 imes \Lambda^1$	
:	:	:	:	:	:
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Notice anything?

The cyclic bar construction.	The circle action and fixed points.	Elementary examples.	Cyclic spectra and <i>THH</i> .
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Fixed points.			

Collect it all together:

simp. level	S^1	C_1	C_2	<i>C</i> ₃	<i>C</i> ₄
0	$\{1\} imes\Delta^0$	$G imes \Lambda^0$			
1		$G^2 imes \Lambda^1$	$G imes \Lambda^0$		
2		$G^3 imes \Lambda^2$		$G imes \Lambda^0$	
3		$G^4 imes \Lambda^3$	$G^2 imes \Lambda^1$		$G imes \Lambda^0$
4		$G^5 imes\Lambda^4$			
5		$G^6 imes\Lambda^5$	$G^3 imes \Lambda^2$	$G^2 imes \Lambda^1$	
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Notice anything?

$$(B^{\operatorname{cyc}}G)^{\mathcal{C}_n}\cong (B^{\operatorname{cyc}}G)^{\mathcal{C}_1}=B^{\operatorname{cyc}}G$$

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Collect it all together:

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4		$G^5 imes\Lambda^4$			
5		$G^6 imes\Lambda^5$	$G^3 imes \Lambda^2$	$G^2 imes \Lambda^1$	
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4	÷	$G^4 imes \Lambda^3$ $G^5 imes \Lambda^4$			$G imes \Lambda^0$:

Notice anything?

$$(B^{\operatorname{cyc}}G)^{C_n} \cong (B^{\operatorname{cyc}}G)^{C_1} = B^{\operatorname{cyc}}G$$

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An S^1 -space with this property is *cyclotomic*.

The cyclic bar construction.	The circle action and fixed points.	Elementary examples.	Cyclic spectra and <i>THH</i> .
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The free loop space.			

X any unbased space, the free loop space is $LX = Map(S^1, X)$.



The cyclic bar construction.	The circle action and fixed points.	Elementary examples. ●○	Cyclic spectra and <i>THH</i> .
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 C_n -fixed loops must follow the same path n times:

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In fact

Proposition $B^{\text{cyc}}G \simeq L(BG)$

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Square zero extensions and tense	or algebras.		
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The cyclic bar construction.	The circle action and fixed points.	Elementary examples.	Cyclic spectra and THH.



Square zero extensions and tenso	or algebras.		
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The cyclic bar construction.	The circle action and fixed points.	Elementary examples.	Cyclic spectra and THH.

X a based space, $S^0 \lor X$ the "square zero extension" of S^0 by X.

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The cyclic bar construction.	The circle action and fixed points.	Elementary examples.	Cyclic spectra and THH.
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Square zero extensions and tenso	or algebras.		

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Proposition

 $B^{\text{cyc}}(S^0 \vee X) \cong S^0 \vee (\Lambda^0 / \partial \wedge X) \vee (\Lambda^1 / \partial \wedge_{C_2} X \wedge X) \vee (\Lambda^2 / \partial \wedge_{C_3} X^{\wedge 3}) \vee \dots$

The cyclic bar construction.	The circle action and fixed points.	Elementary examples.	Cyclic spectra and <i>THH</i> .
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$$T(X) = S^0 \lor X \lor X^{\land 2} \lor X^{\land 3} \lor \dots$$

$$B^{\mathsf{cyc}}(\mathcal{T}(X)) \cong S^0 \vee (\Lambda^0_+ \wedge X) \vee (\Lambda^1_+ \wedge_{C_2} X \wedge X) \vee (\Lambda^2_+ \wedge_{C_3} X^{\wedge 3}) \vee \dots$$

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Proposition

Switch to based spaces and smash product. X a based space, $S^0 \lor X$ the "square zero extension" of S^0 by X.

The cyclic bar construction. The circle action and fixed points. Elementary examples. Cyclic spectra and THH. 000000 0000000000 0● 00 Square zero extensions and tensor algebras. 0000000000 0000000000

Square zero extensions and tensor	r algebras.		
The cyclic bar construction.	The circle action and fixed points.	Elementary examples.	Cyclic spectra and <i>THH</i> .
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Very similar!

The cyclic bar construction. 000000	The circle action and fixed points.	Elementary examples. ○●	Cyclic spectra and THH.
Square zero extensions and tenso	r algebras.		

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Very similar! (Koszul duality)

The cyclic bar construction.	The circle action and fixed points.	Elementary examples.	Cyclic spectra and <i>THH</i> .
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Lifting the arguments to spectra.			

Apply B^{cyc} to a ring spectrum R, result is THH(R).



The cyclic bar construction.	The circle action and fixed points.	Elementary examples. 00	Cyclic spectra and <i>THH</i> . ●○
Lifting the arguments to spectra.			

Apply B^{cyc} to a ring spectrum R, result is THH(R). Above arguments apply verbatim, if we use *orthogonal spectra* and *geometric fixed points*:

 $\Phi^{C_n} THH(R) \cong THH(R)$



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Earlier model (Bökstedt): extra coherence machinery



Apply B^{cyc} to a ring spectrum R, result is THH(R). Above arguments apply verbatim, if we use *orthogonal spectra* and *geometric fixed points*:

 $\Phi^{C_n}THH(R)\cong THH(R)$

Earlier model (Bökstedt): extra coherence machinery Applications: THH(DX) and its dual, mapping spectra between cyclotomic spectra, bivariant algebraic *K*-theory.

The cyclic bar construction.

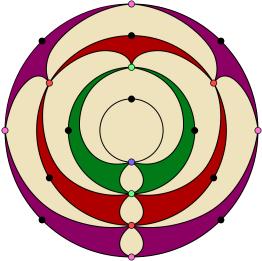
The circle action and fixed points.

Elementary examples.

Cyclic spectra and *THH*. $\odot \bullet$

Thank you!

Takeaway: *THH* is cool!



The face maps of the cyclic bar construction, superimposed on the objects of Λ .