A visual introduction to cyclic sets and cyclotomic spectra

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Young Topologists Meeting

Lausanne, Switzerland

Goal: the cyclic bar construction and
topological Hochschild homology (*THH*) in pictures.

Key idea: "cyclotomic" structure.

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Useful for algebraic K-theory. And fun!

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Recipe for a space: one Δ^k for each $(x, g_1, \ldots, g_k, y)$.

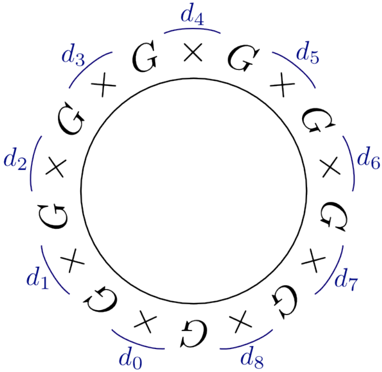
$$EG = B(*, G, G), \qquad BG = B(*, G, *)$$

Also works for:

- based spaces with smash product
- abelian groups with tensor product
- spectra with the smash product
- diagrams ("G has many objects")

The cyclic bar construction.

$$B^{\mathrm{cyc}}G = |[k] \mapsto G \times G^{\times k}|$$



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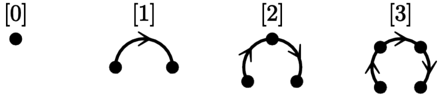

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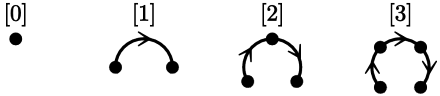

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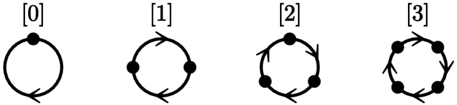

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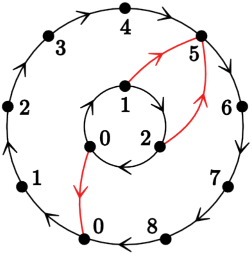
The morphisms are “degree 1” functors.

Here's a morphism $f : [2] \to [8]$ in $\mathbf{\Lambda}$



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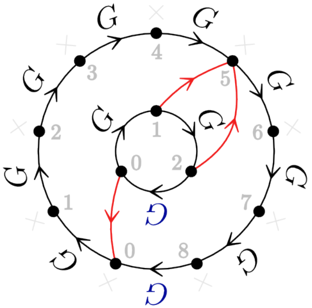

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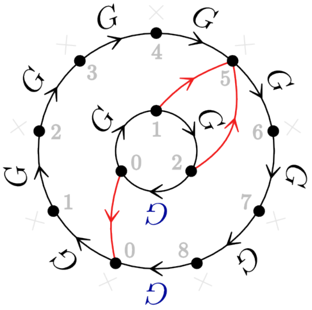

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$$g_0, g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8 \mapsto g_6 g_7 g_8 g_0, g_1 g_2 g_3 g_4 g_5, 1$$

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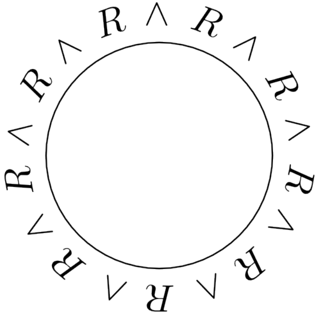

R a ring spectrum.

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Proof: $X_\bullet$ always a colimit of cyclic sets $\mathbf{\Lambda}(-,[n])$ for varying n.

Just need the circle action on $\Lambda^n := |\mathbf{\Lambda}(-,[n])|$.

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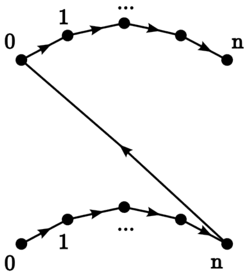
Lift to the "universal cover" of $[n]$:



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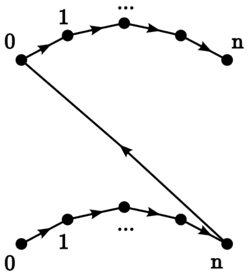

$\leftrightarrow$ an increasing function

$$f : \{0, \dots, k\} \longrightarrow \{(0,0), (0,1), \dots, (0,n), (1,0), (1,1), \dots, (1,n)\}.$$

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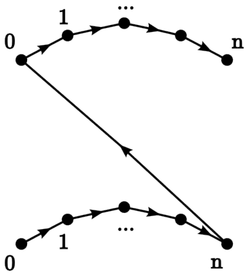

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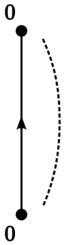
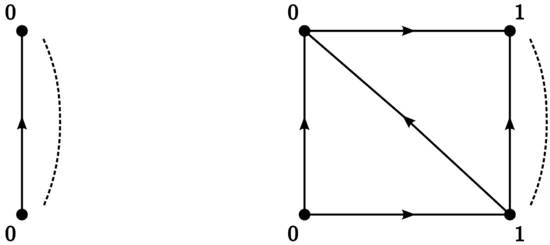
Unique, unless $f(k) \leq (0,n)$ or $f(0) \geq (1,0)$.



Glue the top to the bottom: $\Lambda^0 \cong \Delta^0 \times S^1$



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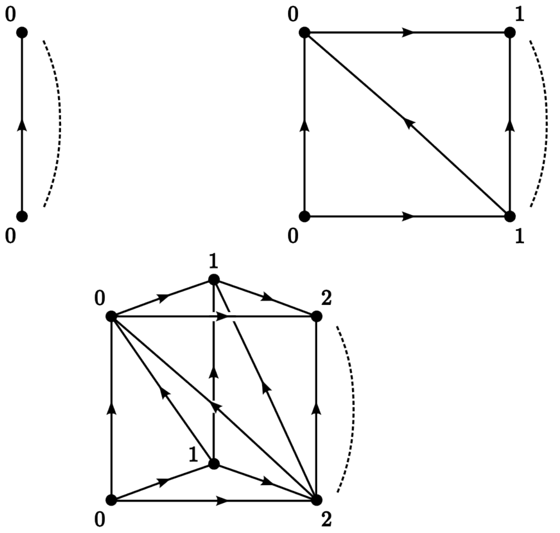
The circle action.



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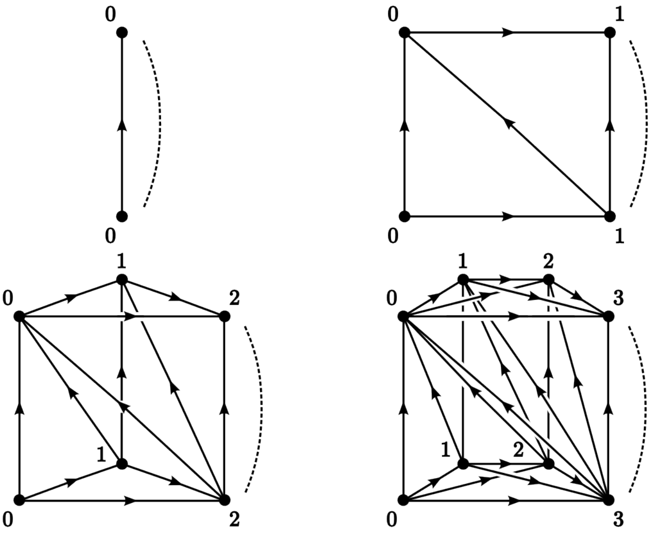
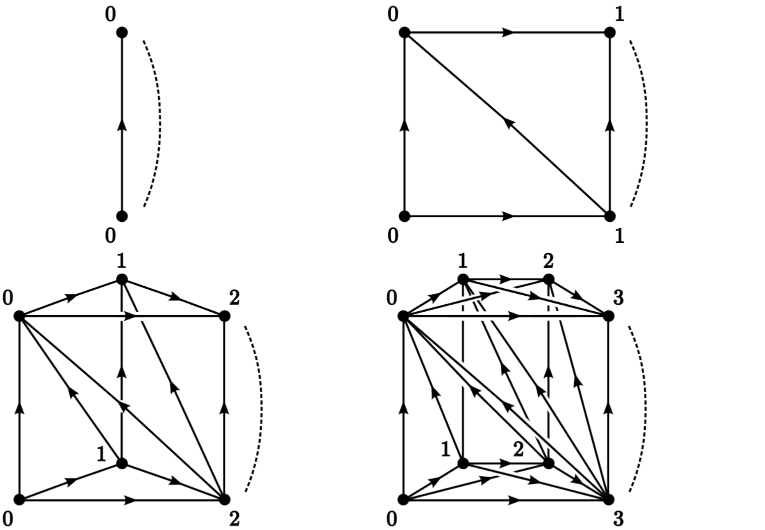


Glue the top to the bottom: $\Lambda^3 \cong \Delta^3 \times S^1$



Glue the top to the bottom: $\Lambda^3 \cong \Delta^3 \times S^1$ and so on. □

Fixed points.

$C_n \leq S^1$ cyclic subgroup – what are its fixed points?

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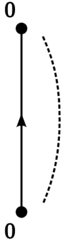
Simplicial level 0: get one copy of Λ^0 for each $g \in G$



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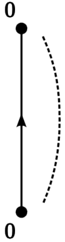
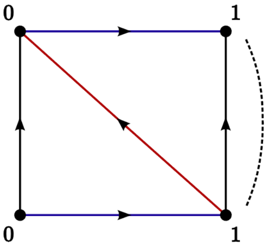


Degenerate if $g = 1$, nondegenerate otherwise.

Simplicial level 1: we get a $\Lambda^1 = \Delta^1 \times S^1$ for each pair (g_1, g_2).



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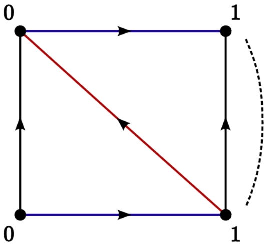

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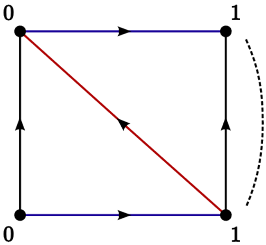
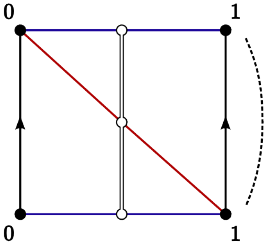


The bottom triangle for (g_1, g_2) is glued to top triangle for (g_2, g_1) and vice-versa.
Are any blue points fixed by some nontrivial element of S^1?

Answer: only the midpoint, and only if $g_1 = g_2$:



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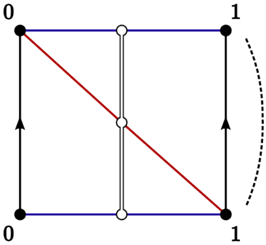

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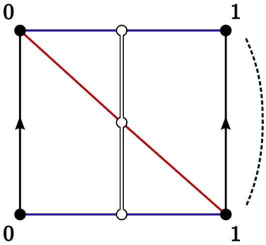

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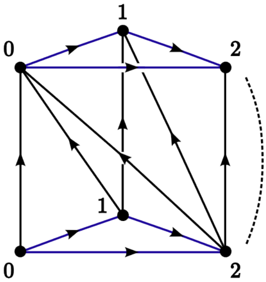
Simplicial level 2: play the same game. One prism for each triple (g_1, g_2, g_3)



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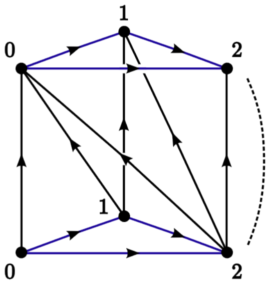

glued by rotating the triple (g_1, g_2, g_3) and rotating the three 3-simplices in the figure.

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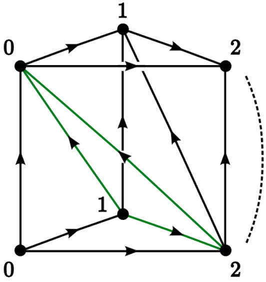

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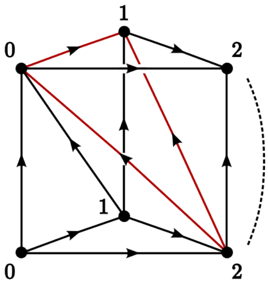

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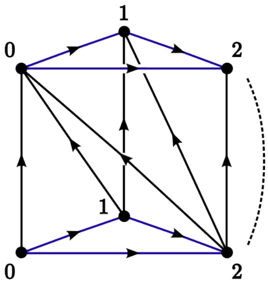

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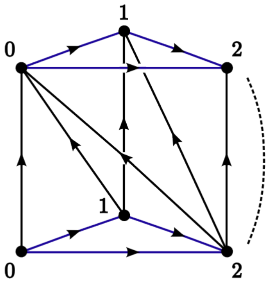
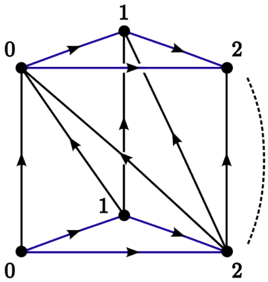


Which points in the blue simplex are fixed?
Triple must be (g_1, g_1, g_1), point must be fixed under rotation of vertices of Δ^2



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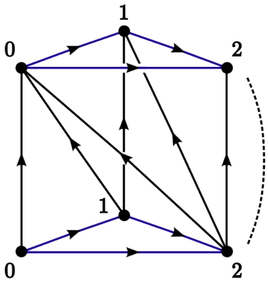

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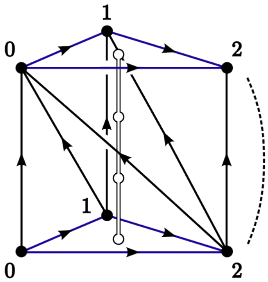
We get C_3-fixed points:



Get another $G \times \Lambda^0$ in the C_3-fixed points.

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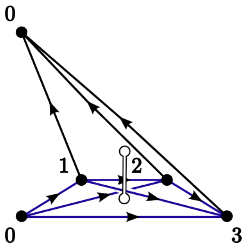
First chance to get mapped to yourself, by C_4:



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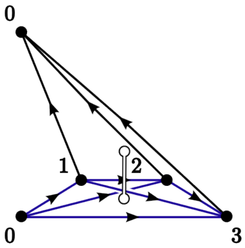

We get a $G \times \Lambda^0$ in the C_4-fixed points.

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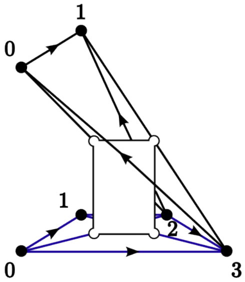

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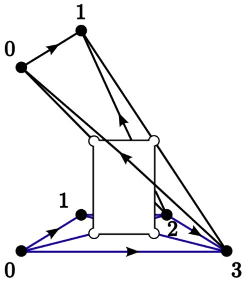
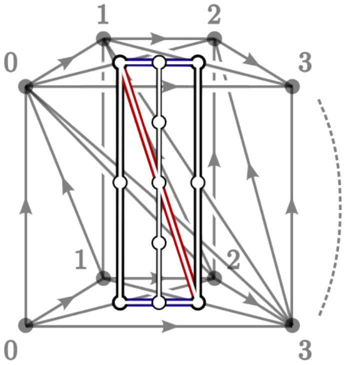


More fixed points! C_2 acts on Δ^3 by rotating the coordinates twice:

$$(t_0, t_1, t_2, t_3) \mapsto (t_2, t_3, t_0, t_1)$$

The fixed points form a line Δ^1.

So, get a copy of $G^2 \times \Lambda^1$ in the C_2-fixed points.



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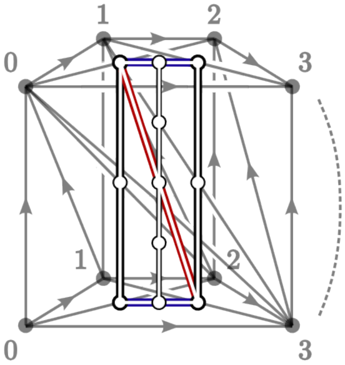

Can easily formalize now: if $r \mid n$, the piece $G^n \times \Lambda^{n-1}$ has C_r-fixed points $G^{n/r} \times \Lambda^{n/r-1}$.

Collect it all together:

simp. level	S^1	C_1	C_2	C_3	C_4
0	$\{1\} \times \Delta^0$	$G \times \Lambda^0$			
1		$G^2 \times \Lambda^1$	$G \times \Lambda^0$		
2		$G^3 \times \Lambda^2$		$G \times \Lambda^0$	
3		$G^4 \times \Lambda^3$	$G^2 \times \Lambda^1$		$G \times \Lambda^0$
4		$G^5 \times \Lambda^4$			
5		$G^6 \times \Lambda^5$	$G^3 \times \Lambda^2$	$G^2 \times \Lambda^1$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Notice anything?

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Notice anything?

$$(B^{\mathrm{cyc}}G)^{C_n} \cong (B^{\mathrm{cyc}}G)^{C_1} = B^{\mathrm{cyc}}G$$

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Notice anything?

$$(B^{\mathrm{cyc}} G)^{C_n} \cong (B^{\mathrm{cyc}} G)^{C_1} = B^{\mathrm{cyc}} G$$

An S^1-space with this property is *cyclotomic*.

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In fact

Proposition

$B^{\mathrm{cyc}} G \simeq L(BG)$

Switch to based spaces and smash product.

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$B^{cyc}(S^0 \vee X) \cong$
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Very similar! (Koszul duality)

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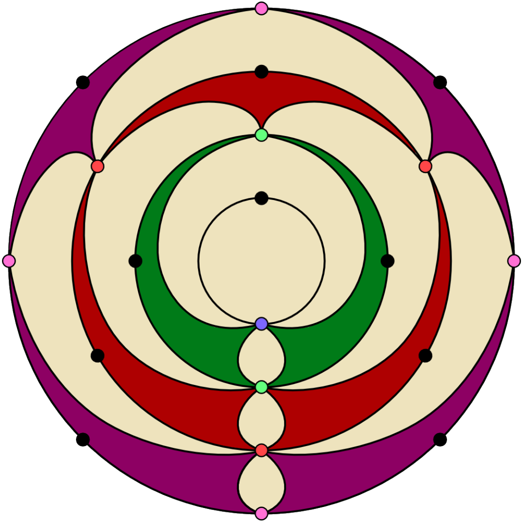
Earlier model (Bökstedt): extra coherence machinery
Applications: $THH(DX)$ and its dual, mapping spectra between cyclotomic spectra, bivariant algebraic K-theory.

Takeaway: *THH* is cool!



The face maps of the cyclic bar construction, superimposed on the objects of $\mathbf{\Lambda}$.