# The stable homotopy category

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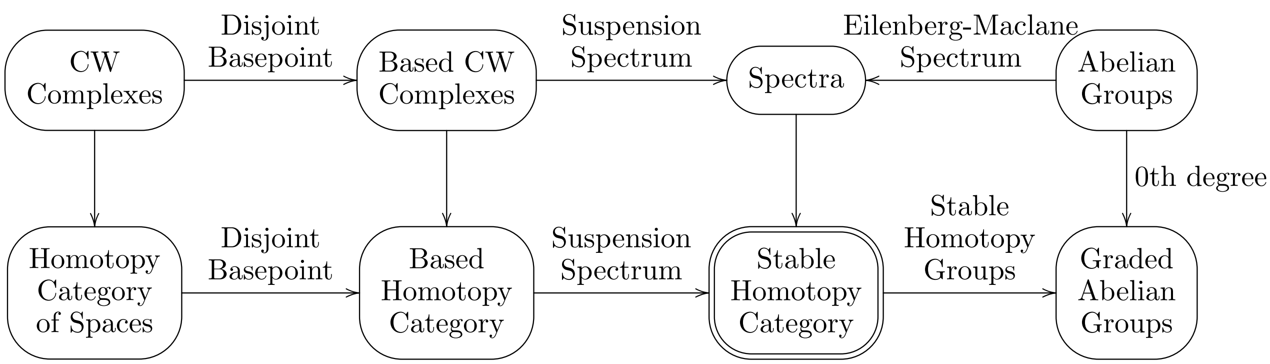
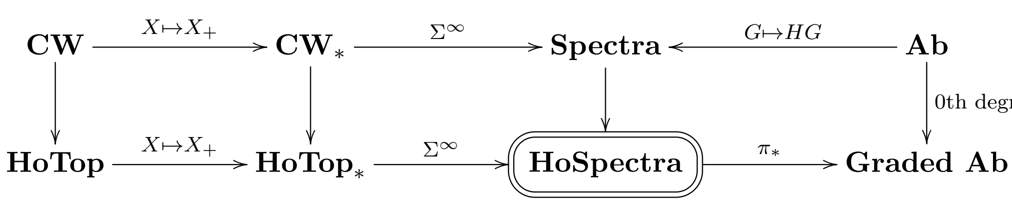
(minor tweaks since then)

These goal of these notes is to explain what a spectrum is. There are different definitions of "spectrum" in common usage today, and it is not obvious to a nonspecialist how they are equivalent. Therefore we will begin with a description of the *properties* that we want spectra to have, before actually *defining* them. We assume familiarity with homology, cohomology, and homotopy groups, along with categories, functors, and natural transformations.

To start, spectra should form a category, with functors coming in and going out to other categories that we care about. We can capture this in a commuting diagram of functors:




or in shorthand,




As we have already mentioned, there are different definitions of the category labelled **Spectra**, most of which are not equivalent. Each of them in turn gives a definition of the stable homotopy category **HoSpectra**, but here they are almost always equivalent. So it is sometimes easier to write proofs using only abstract properties of the stable homotopy category **HoSpectra**, avoiding the "implementation details" found in **Spectra**.

These notes are an ongoing project, aimed at an elementary grad-student level. I greatly appreciate feedback. I'd like to thank Aaron Mazel-Gee for providing a lot, and Ilya Grigoriev for insisting that I put bubbles on the first page.

## Contents



# 1 Desired Properties of HoSpectra

## 1.1 Topological Spaces

We care about the category of compactly generated weak Hausdorff (CGWH) topological spaces, which we'll denote **Top**. Let **CW** denote the full subcategory of spaces that are homeomorphic to CW complexes. When we say "full subcategory" we mean to allow all continuous maps between CW complexes (as opposed to only allowing cellular maps).

The *homotopy category of CW complexes* **HoCW** has the same objects as **CW**, but the arrows are homotopy classes of maps instead of actual maps. Obviously, we can pass from maps to homotopy classes of maps, which defines a functor

$$\mathbf{CW} \longrightarrow \mathbf{HoCW}$$

A map **CW** is a homotopy equivalence iff it becomes an isomorphism in **HoCW**.

Next, define the *homotopy category of spaces* **HoTop** to have the same objects as **Top**, but the morphisms from $X$ to $Y$ are homotopy classes of maps between CW approximations $[\Gamma(X), \Gamma(Y)]$. By Whitehead's theorem, a map in **Top** is a weak equivalence iff it becomes an isomorphism in **HoTop**. In fact, **HoTop** satisfies a universal property: it is initial among all categories under **Top** in which every weak equivalence becomes an isomorphism.

**Exercises.**

- Define a functor $\Gamma : \mathbf{Top} \longrightarrow \mathbf{CW}$ that takes every space to a weakly equivalent CW complex.
- Check that the inclusion $\mathbf{HoCW} \longrightarrow \mathbf{HoTop}$ is an equivalence of categories.

Similarly, we may consider the category of based spaces $\mathbf{Top}_*$, or the category of based CW complexes $\mathbf{CW}_*$. We may define homotopy categories $\mathbf{HoCW}_*$ and $\mathbf{HoTop}_*$ as before, only now every map and homotopy of maps has to preserve the basepoint.

**Remark.** If you know about model categories, our definition of **HoTop** is isomorphic to the homotopy category given by the Quillen model structure on **Top** (and similarly for $\mathbf{HoTop}_*$).

## 1.2 Suspension and Abelian Groups

We claim there is a category called the *stable homotopy category*, denoted **HoSpectra**, with the following properties.

- There is a functor $\Sigma^\infty : \mathbf{HoTop}_* \longrightarrow \mathbf{HoSpectra}$. So every space gives an object of the stable category, though there are many objects in **HoSpectra** that do not come from spaces.
- There is a suspension functor $\Sigma : \mathbf{HoSpectra} \longrightarrow \mathbf{HoSpectra}$. It agrees with the usual reduced suspension of based CW complexes:

$$\begin{array}{ccc} \mathbf{CW}_* & \xrightarrow{\Sigma} & \mathbf{CW}_* \\ \downarrow & & \downarrow \\ \mathbf{HoTop}_* & \xrightarrow{\Sigma\circ\Gamma} & \mathbf{HoTop}_* \\ \downarrow{\scriptstyle \Sigma^\infty} & & \downarrow{\scriptstyle \Sigma^\infty} \\ \mathbf{HoSpectra}_* & \xrightarrow{\Sigma} & \mathbf{HoSpectra}_* \end{array}$$

  Moreover, $\Sigma$ is an equivalence of categories from **HoSpectra** to itself. So every object of **HoSpectra** is isomorphic to the suspension of some other object. This certainly wasn't true for $\mathbf{Top}_*$ or $\mathbf{HoTop}_*$. As a technical point, in the homotopy category $\mathbf{HoTop}_*$ we must use $\Sigma \circ \Gamma$, where $\Gamma$ is a CW replacement functor. The reason for this is that CW complexes have *nondegenerate basepoints*. If $X$ has a degenerate basepoint, then the construction $X \mapsto \Sigma X$ may not preserve weak equivalences, so it does not give a functor on the homotopy category.

- The functor $\Sigma^\infty$ has a right adjoint $\Omega^\infty : \mathbf{HoSpectra} \longrightarrow \mathbf{HoTop}_*$. This means that for a based space $K$ and a spectrum $X$,

$$[\Sigma^\infty K, X] \cong [K, \Omega^\infty X]$$

- There is a loopspace functor $\Omega : \mathbf{HoSpectra} \longrightarrow \mathbf{HoSpectra}$ that agrees with the usual based loopspace:

$$\begin{array}{ccc}
\mathbf{Top}_* & \xleftarrow{\ \Omega\ } & \mathbf{Top}_* \\
\downarrow & & \downarrow \\
\mathbf{HoTop}_* & \xleftarrow{\ \Omega\ } & \mathbf{HoTop}_* \\
{\scriptstyle \Omega^\infty}\uparrow & & {\scriptstyle \Omega^\infty}\uparrow \\
\mathbf{HoSpectra}_* & \xleftarrow{\ \Omega\ } & \mathbf{HoSpectra}_*
\end{array}$$

In $\mathbf{HoSpectra}$, the functors $\Sigma$ and $\Omega$ are inverse equivalences, so $\Sigma \circ \Omega$ and $\Omega \circ \Sigma$ are naturally isomorphic to the identity. Every object is isomorphic to the loopspace of some other object. Again, this isn't true in $\mathbf{Top}_*$ or $\mathbf{HoTop}_*$.

- Given objects $X$ and $Y$ in $\mathbf{HoSpectra}$, the set of morphisms $[X, Y]$ can be turned into an abelian group. Intuitively, we think of $X$ as a suspension $\Sigma X'$, and we use the usual "pinching" and "flipping" constructions on $[S^n, Z] = \pi_n(Z)$ to add or negate maps $\Sigma X' \longrightarrow Y$. There is also an analogue of the Eckmann-Hilton argument to show that addition in $[\Sigma^2 X'', Y]$ is commutative, and a natural bijection $[\Sigma X', Y] \cong [X', \Omega Y]$. Composition of morphisms $[X, Y] \times [Y, Z] \longrightarrow [X, Z]$ is bilinear, so it induces a homomorphism of abelian groups $[X, Y] \otimes [Y, Z] \longrightarrow [X, Z]$.

- The category $\mathbf{HoSpectra}$ has coproducts (wedge sums) $X \vee Y$ and products $X \times Y$. There is a zero object $*$, coming from the one-point based space $*$ in $\mathbf{Top}_*$. This means that for every object $X$, there are unique maps $* \longrightarrow X \longrightarrow *$. This gives natural maps

$$\begin{cases} X \vee * \longrightarrow X \\ X \longrightarrow X \times * \\ X \vee Y \longrightarrow X \times Y \end{cases}$$

The first two rows are always isomorphisms, using the data we gave above. In $\mathbf{HoSpectra}$, the third map is also an isomorphism. This was not true for based spaces!

- The last two bullet points combine to tell us that $\mathbf{HoSpectra}$ is an *additive category*. It is not, however, an abelian category.

- Suppose that $A$ is a retract of $X$ in **HoSpectra**. By this we mean that there are arrows in **HoSpectra**
$$A \longrightarrow X \longrightarrow A$$
which compose to the identity. Then $X$ contains $A$ as a summand:
$$X \cong A \vee B$$

- We can extend the abelian group $[X, Y]$ into a graded abelian group $[X, Y]_*$, containing $[X, Y]$ as the 0th level. We simply define
$$[X, Y]_n = [\Sigma^n X, Y]$$
Notice that $n$ can be any integer, since suspension $\Sigma$ has an inverse equivalence $\Omega$. Notice also that it was a little bit arbitrary whether to put the suspension on the left or on the right. When we want to do both, we call the above convention *homological grading*. The opposite convention $[X, Y]_n = [X, \Sigma^n Y]$ is called *cohomological grading*.

These properties are analogous to the basic properties of graded abelian groups. Suspension is the operation that shifts the grading by one. Looping shifts the grading by one in the opposite direction. If $G$ and $H$ are two graded abelian groups, the set of graded homomorphisms $\{G_i \longrightarrow H_i\}_i$ between them forms an abelian group. This can be extended to a graded abelian group of "shifted homomorphisms" $\{G_i \longrightarrow H_{i+n}\}_i$. The coproduct $G \oplus H$ and the product $G \times H$ are naturally isomorphic. Finally, if $G$ contains $H$ as a retract, then $G \cong H \oplus G/H$.

Before we continue the analogy with abelian groups, let's list a few more topological properties of **HoSpectra**:

- Define the *sphere spectrum* to be $\mathbb{S} = \Sigma^\infty S^0$. Given an object $X$ in the stable homotopy category, we define its *stable homotopy groups*
$$\pi_n(X) = [\mathbb{S}, X]_n = [\Sigma^n \mathbb{S}, X]$$
Again, notice that $n$ can be a negative integer and this still makes sense. If $K$ is a based CW complex, then $\pi_n(\Sigma^\infty K)$ is naturally isomorphic to the usual stable homotopy groups
$$\begin{aligned} \pi_n^S(K) &:= \operatorname{colim}_{k\to\infty} \pi_{k+n}(\Sigma^k K) \\ &= \operatorname{colim}_{k\to\infty} \pi_n(\Omega^k \Sigma^k K) \\ &= \pi_n(\Omega^\infty \Sigma^\infty K) \end{aligned}$$
Notice that $\pi_n(\Sigma^\infty K)$ is zero for negative $n$. On the other hand, the adjunction
$$[\Sigma^\infty K, X] \cong [K, \Omega^\infty X]$$
tells us $\pi_n(X) \cong \pi_n(\Omega^\infty X)$ for $n \geq 0$. Of course, $\Omega^\infty X$ is a space, so it has no negative homotopy groups.

- The objects $X$ in **HoSpectra** whose homotopy groups $\pi_n(X)$ vanish for negative $n$ are called *connective spectra*. By the above, $\Sigma^\infty$ takes every based space to a connective spectrum.

- Whitehead's Theorem for spectra: If a map $f : X \longrightarrow Y$ in **HoSpectra** induces an isomorphism $\pi_*(X) \xrightarrow{\cong} \pi_*(Y)$, then $f$ is an isomorphism. In other words, the stable homotopy groups detect the isomorphisms in **HoSpectra**.

### 1.3 Tensor Products and Rings

Carrying the analogy with abelian groups even further, we can define a tensor product on objects of **HoSpectra**. Before describing its properties, let's recall the basic properties of the tensor product $\otimes = \otimes_{\mathbb{Z}}$ of abelian groups:

$$\otimes : \mathbf{Ab} \times \mathbf{Ab} \longrightarrow \mathbf{Ab}$$

Here $\mathbf{Ab} \times \mathbf{Ab}$ is a *product category*, whose objects are pairs of abelian groups and morphisms are pairs of morphisms. So we can think of the tensor product as an operation on abelian groups that is a functor in each slot.

Let's follow the convention that $\mathbf{Ab}(A, B)$ is the set of linear maps from $A$ to $B$, and $\mathrm{Hom}(A, B)$ is the abelian group of linear maps. So if we forget that $\mathrm{Hom}(A, B)$ is a group, we get the set $\mathbf{Ab}(A, B)$. The defining property of $\otimes$ is that linear maps $A \otimes B \longrightarrow C$ correspond naturally to bilinear maps $A \times B \longrightarrow C$. A bilinear map is the same thing as a linear map $A \longrightarrow \mathrm{Hom}(B, C)$. Therefore we get a bijection of sets

$$\mathbf{Ab}(A \otimes B, C) \longleftrightarrow \mathbf{Ab}(A, \mathrm{Hom}(B, C))$$

The tensor product is unital, associative, and commutative. This means there is a unit object $I$ and natural isomorphisms

$$l_A : I \otimes A \xrightarrow{\cong} A$$
$$r_A : A \otimes I \xrightarrow{\cong} A$$
$$a_{A,B,C} : (A \otimes B) \otimes C \xrightarrow{\cong} A \otimes (B \otimes C)$$
$$s_{A,B} : A \otimes B \xrightarrow{\cong} B \otimes A$$

The unit object is the group of integers $\mathbb{Z}$. We can start with a tensor product of a bunch of groups and start applying these isomorphisms willy-nilly to regroup the parentheses and rearrange terms:

$$(A \otimes (B \otimes C)) \otimes D \xrightarrow{\cong} (A \otimes (C \otimes B)) \otimes D \xrightarrow{\cong} ((A \otimes C) \otimes B) \otimes D \xrightarrow{\cong} \dots$$

If we ever come back to the expression we started with, then the composition of the maps we applied becomes the identity map. (This means that the isomorphisms $l, r, a, s$ are *coherent*. Heuristically, this means we can drop the parentheses around the tensor products without getting into trouble.) If we carefully rewrite the above properties of $(\mathbf{Ab}, \mathbb{Z}, \otimes, \mathrm{Hom}, l, r, a, s)$ using only notation from category theory, we get the concept of a *closed symmetric monoidal category.* If we drop everything involving Hom, then $(\mathbf{Ab}, \mathbb{Z}, \otimes, l, r, a, s)$ gives a *symmetric monoidal category.*

We've been building up to a statement about **HoSpectra**, so here it is: **HoSpectra** is a closed symmetric monoidal category. Its unit object is the sphere spectrum $\mathbb{S}$. Its tensor product is called the smash product $\wedge$, since it is based on the smash product of based spaces

$$X \wedge Y = (X \times Y)/(X \vee Y)$$

(It is more accurate to call it the *left derived smash product* $\wedge^L$, especially when we need to distinguish it from the smash product $\wedge$ in **Spectra**.) The internal hom of **HoSpectra** is denoted $F(X, Y)$, the $F$ standing for *function spectrum.* So if $X$, $Y$, and $Z$ are spectra, there are natural coherent isomorphisms in **HoSpectra**

$$\begin{gathered}
\mathbb{S} \wedge X \cong X \\
(X \wedge Y) \wedge Z \cong X \wedge (Y \wedge Z) \\
X \wedge Y \cong Y \wedge X \\
[X \wedge Y, Z] \cong [X, F(Y, Z)] \\
F(\mathbb{S}, X) \cong X \\
F(X \wedge Y, Z) \cong F(X, F(Y, Z))
\end{gathered} \tag{1}$$

**Exercises.**

- Show that the last two isomorphisms follow from the first four. (Use the Yoneda Lemma.)
- Define natural maps
$$\begin{gathered}
X \wedge F(X, Y) \longrightarrow Y \\
X \longrightarrow F(Y, X \wedge Y)
\end{gathered}$$
- Let's define suspension and looping more explicitly:
$$\begin{gathered}
\Sigma X = (\Sigma^\infty S^1) \wedge X \\
\Omega X = F(\Sigma^\infty S^1, X)
\end{gathered}$$
Using the above isomorphisms, together with the fact that $\Sigma^\infty$ is a functor, prove that suspension and looping are adjoint:
$$[\Sigma X, Y] \cong [X, \Omega Y]$$
and construct the operation on either of these two sets that turns it into an abelian group. (Can we prove that $\Sigma$ and $\Omega$ are inverses yet? Why not?)
- Prove that there are natural isomorphisms
$$\begin{gathered}
(\Sigma X) \wedge Y \cong \Sigma(X \wedge Y) \cong X \wedge (\Sigma Y) \\
\Omega F(X, Y) \cong F(\Sigma X, Y) \cong F(X, \Omega Y)
\end{gathered}$$

There are many concepts in algebra that have an analogue in the world of spectra. Here's an important example. Start with a symmetric monoidal category **C** and an object $M$ of **C**. If we can give a "multiplication" morphism $\mu : M \otimes M \longrightarrow M$ that is associative

$$\begin{array}{ccc} M \otimes M \otimes M & \xrightarrow{\mathrm{id}\otimes\mu} & M \otimes M \\ \downarrow{\scriptstyle \mu\otimes\mathrm{id}} & & \downarrow{\scriptstyle \mu} \\ M \otimes M & \xrightarrow{\quad\mu\quad} & M \end{array}$$

and a "unit" morphism $i : I \longrightarrow M$ of this multiplication

$$\begin{array}{ccccc} I \otimes M & \xrightarrow{i\otimes\mathrm{id}} & M \otimes M & \xleftarrow{\mathrm{id}\otimes i} & M \otimes I \\ & {\scriptstyle l_M}\searrow & \downarrow{\scriptstyle \mu} & \swarrow{\scriptstyle r_M} & \\ & & M & & \end{array}$$

then we say that $(M, \mu, i)$ is a *monoid* in **C**. Here's a fun fact: a monoid in **Ab** is the same thing as a ring! By analogy, we call a monoid in **HoSpectra** a *ring spectrum.* If you understand this example, you should be able to define a *commutative monoid* in **C**. If you define it correctly, a commutative monoid in **Ab** will be a commutative ring.

Here's a list of some symmetric monoidal categories, and common names for monoids in those categories:

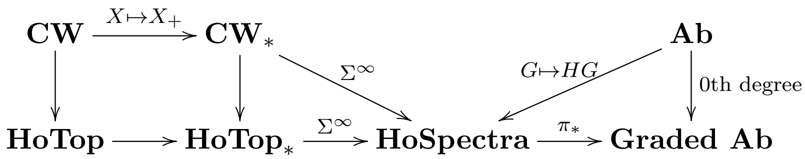
| Category | Product | Unit | Monoid | Commutative Monoid |
|---|---|---|---|---|
| **Set** | $\times$ | $\{*\}$ | Monoid | Commutative Monoid |
| **Ab** | $\otimes$ | $\mathbb{Z}$ | Ring | Commutative Ring |
| **Graded Ab**$^{(1)}$ | $\otimes$ | $\mathbb{Z}$ | Graded Ring | Commutative Graded Ring |
| **Graded Ab**$^{(2)}$ | $\otimes$ | $\mathbb{Z}$ | Graded Ring | Skew-Commutative Ring* |
| $\mathbf{Mod}_k$ | $\otimes_k$ | $k$ | $k$-Algebra | Commutative $k$-Algebra |
| $\mathbf{ChMod}_k$ | $\otimes_k$ | $k$ | Differential Graded $k$-Algebra (DGA) | Commutative Differential Graded Algebra (CDGA) |
| **CW** or **Top** | $\times$ | $\{*\}$ | Topological Monoid | Commutative Topological Monoid |
| **HoTop** | $\times$ | $\{*\}$ | Associative $H$-Space | Commutative $H$-Space |
| $\mathbf{CW}_*$ or $\mathbf{Top}_*$ | $\wedge$ | $S^0$ | Based Topological Monoid | Based Commutative Topological Monoid |
| $\mathbf{Sp}^\Sigma$ or $\mathbf{Sp}^O$ | $\wedge$ | $\mathbb{S}$ | (Strict) Ring Spectrum | (Strict) Commutative Ring Spectrum |
| **HoSpectra** | $\wedge^{(L)}$ | $\mathbb{S}$ | Ring Spectrum (up to homotopy) | Commutative Ring Spectrum (up to homotopy) |

* There are two common conventions for the symmetry isomorphism for graded abelian groups:

$$\begin{aligned} a \otimes b &\mapsto b \otimes a \\ a \otimes b &\mapsto (-1)^{|a|\cdot|b|}(b \otimes a) \end{aligned}$$

Under the first convention, a commutative monoid is a commutative ring that happens to be graded. Under the second convention, a commutative monoid is a *skew-commutative ring*. This means that even-degree elements commute with everything, and odd-degree elements introduce a $-1$ when switched past each other. Skew-commutative rings are often called *graded-commutative* or even just *commutative*, but don't confuse them with commutative graded rings like $\mathbb{Z}[x]$. Skew-commutative rings show up all over algebraic topology: the cohomology of a space $H^*(X)$ and the stable homotopy groups of spheres $\pi_*^S(S^0)$ are two examples. In these notes, we will always follow the second convention and work with skew-commutative rings.

Now we have a language that relates spectra to abelian groups. But we really want much more. Consider the diagram we gave at the beginning, with **Spectra** deleted because it doesn't always have a good smash product:




We claim that every functor in this diagram agrees with tensor products. To be more specific, if $F : C \longrightarrow D$ is any functor in the diagram, $X$ and $Y$ are objects of $C$, and $I_C$ and $I_D$ are the units of $C$ and $D$, respectively, then there are natural transformations

$$F(X) \otimes F(Y) \longrightarrow F(X \otimes Y)$$
$$I_D \longrightarrow F(I_C)$$

that commute with the unit, associativity, and symmetry isomorphisms of $C$ and $D$.

**Exercise.** Prove that a functor $F$ with these properties takes monoids to monoids.

A functor $F$ satisfying these properties is called *lax monoidal.* If the above maps are isomorphisms, then $F$ is called *strong monoidal.* In the above diagram, every functor is at least lax monoidal. So if we start with a (commutative) monoid anywhere on the diagram, and follow any route, we end up at another (commutative) monoid. For example, if $X$ is a ring spectrum, then $\pi_*(X)$ is a graded ring.

**Exercises.**

- Prove that the forgetful functor **Ab** $\longrightarrow$ **Set** is lax monoidal.
- Prove that its left adjoint, the "free abelian group on a set" construction, is strong monoidal.

Here's another example. Consider the one-point space $\{*\}$ in **CW**. This clearly forms a commutative monoid. Its image in **HoSpectra** is the sphere spectrum $\mathbb{S}$. Therefore $\mathbb{S}$ is a commutative ring spectrum! Applying $\pi_*$, we deduce that the stable homotopy groups of spheres $\pi_*^S(S^0) \cong \pi_*(\mathbb{S})$ form a skew-commutative ring. (Of course, this "proof" relies on the above claim that every functor in the diagram is lax monoidal.)

## 1.4 Exact Sequences

One of the main goals in algebraic topology is to actually calculate the invariants we define for interesting objects, and for this purpose, the exact sequence is one of our most basic tools. It turns out that **HoSpectra** has a notion of "short exact sequences" of objects, which generalize the classical cofiber sequences

$$A \longrightarrow X \longrightarrow X/A \longrightarrow \Sigma A$$

and the classical fiber sequences

$$\Omega B \longrightarrow F \longrightarrow E \longrightarrow B$$

To be more precise, we can form triples of objects $(X, Y, Z)$, and triples of maps $(f, g, h)$ of the form

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

We call $(X, Y, Z, f, g, h)$ a *triangle*. Now here is our claim. There is a collection of triangles in **HoSpectra**, called the *distinguished triangles*, that satisfy the following properties:

- For each distinguished triangle

  $$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \Sigma X$$

  and each object $W$, there are long exact sequences of abelian groups

  $$\ldots \longrightarrow [W, X]_n \longrightarrow [W, Y]_n \longrightarrow [W, Z]_n \longrightarrow [W, X]_{n-1} \longrightarrow \ldots$$

  $$\ldots \longleftarrow [X, W]_n \longleftarrow [Y, W]_n \longleftarrow [Z, W]_n \longleftarrow [X, W]_{n+1} \longleftarrow \ldots$$

  Taking $W = \mathbb{S}$, we see that the stable homotopy groups form a long exact sequence

  $$\ldots \longrightarrow \pi_n(X) \longrightarrow \pi_n(Y) \longrightarrow \pi_n(Z) \longrightarrow \pi_{n-1}(X) \longrightarrow \ldots$$

- Every map $X \xrightarrow{f} Y$ has a *(homotopy) cofiber* $C(f)$, unique up to non-unique equivalence, which fits into a distinguished triangle

  $$X \xrightarrow{f} Y \longrightarrow C(f) \longrightarrow \Sigma X$$

- Every map $X \xrightarrow{f} Y$ has a *(homotopy) fiber* $F(f)$, unique up to non-unique equivalence, which fits into a distinguished triangle

  $$F(f) \longrightarrow X \xrightarrow{f} Y \longrightarrow \Sigma F(f)$$

- There is a natural isomorphism between the cofiber and the suspension of the fiber:

  $$C(f) \cong \Sigma F(f)$$

- $\Sigma^\infty$ takes cofiber sequences in $\mathbf{Top}_*$ to distinguished triangles in **HoSpectra**.

- $\Omega^\infty$ takes distinguished triangles in **HoSpectra** to fiber sequences in $\mathbf{Top}_*$.

- If $(X, Y, Z, f, g, h)$ is distinguished and $W$ is another object, then

  $$W \wedge X \xrightarrow{f} W \wedge Y \xrightarrow{g} W \wedge Z \xrightarrow{h} \Sigma(W \wedge X)$$
  $$F(W, X) \xrightarrow{f} F(W, Y) \xrightarrow{g} F(W, Z) \xrightarrow{h} \Sigma F(W, X)$$
  $$\Sigma^{-1} F(X, W) \xrightarrow{-h} F(Z, W) \xrightarrow{g} F(Y, W) \xrightarrow{f} F(X, W)$$

  are distinguished.

These properties, along with some that we have missed, allow us to say that **HoSpectra** is a *triangulated category* [9]. The formalism of triangulated categories allows us to cleanly prove many nice statements about spectra. Here, we will simply take the above statements and apply them to homology and cohomology.

### 1.5 Homology and Cohomology

The objects of **HoSpectra** define (reduced) homology and cohomology theories on $\mathbf{CW}_*$. To see this, take a based CW complex $X$ and a spectrum $E$ in **HoSpectra**. Then the abelian groups

$$\widetilde{E}_n(X) = [\mathbb{S}, (\Sigma^\infty X) \wedge E]_n \cong \pi_n((\Sigma^\infty X) \wedge E)$$
$$\widetilde{E}^n(X) = [\Sigma^\infty X, E]_{-n} \cong \pi_{-n}(F(\Sigma^\infty X, E))$$

define an (extraordinary, reduced) homology theory and a cohomology theory.

**Exercise.** Use the statements from the last section to prove that these satisfy the axioms of a reduced (co)homology theory. Equivalently, show that

$$E_n(X) = \widetilde{E}_n(X_+)$$
$$E^n(X) = \widetilde{E}^n(X_+)$$

form an unreduced cohomology theory.

Now we can easily generalize from the homology of spaces to the homology of spectra. If $Y$ and $E$ are objects in **HoSpectra**, define the abelian groups

$$\widetilde{E}_n(Y) = [\mathbb{S}, Y \wedge E]_n \cong \pi_n(Y \wedge E)$$
$$\widetilde{E}^n(X) = [Y, E]_{-n} \cong \pi_{-n}(F(Y, E))$$

The tildes are not standard notation, but they remind us that these theories are always “reduced.”

**Exercises.**

- Construct the above two isomorphisms, using only the properties we have already given for **HoSpectra**.
- Show that $E_*(\mathrm{pt}) \cong \widetilde{E}_*(\mathbb{S}) \cong \pi_*(E) \cong \widetilde{E}^{-*}(\mathbb{S}) \cong E^{-*}(\mathrm{pt})$. If $E$ is a ring spectrum, this is the *coefficient ring* of the homology and cohomology theories associated to $E$. It is sometimes simply denoted $E_*$ or $E^*$.
- If $E$ is a ring spectrum, and $X$ is an unbased CW complex, show that $E^*(X)$ is a graded ring. If $X$ is based, then of course $\widetilde{E}^*(X)$ will be a non-unital algebra.

- If $E$ is a ring spectrum and $Y$ is a spectrum, then we might expect $\widetilde{E}^*(Y)$ to be a graded (non-unital) algebra in a meaningful way. In fact, this is false: the only natural bilinear multiplication on $\widetilde{E}^*(Y)$ is zero. Nevertheless, show that $\widetilde{E}^*(Y)$ is still a bimodule over the coefficient ring $E^*$. So in particular, $E^*(X)$ is a graded $E^*$-algebra.

- If $E$ is a commutative ring spectrum and $X$ is a space, show that $E^*(X)$ is skew-commutative.

So every object of **HoSpectra** gives an extraordinary cohomology theory, on CW-complexes or even on spectra. It turns out that the converse is true: every cohomology theory on finite CW-complexes can be extended to a cohomology theory on **HoSpectra**, and is represented by an object in **HoSpectra** which is unique up to non-canonical isomorphism. This is called *Brown Representability.* There is a similar representability theorem for homology, due to G.W. Whitehead. This allows us to pair extraordinary homology and cohomology theories together when they are represented by the same spectrum.

**Remark.** We may extend the above discussion from CW complexes to all spaces by defining the (co)homology of a space $X$ to be the (co)homology of its CW replacement $\widetilde{X}$.

Let's consider $\widetilde{H}^*(X;G)$, the theory of ordinary (singular or cellular) cohomology with coefficients in an abelian group $G$. By the above statement, there is an object in **HoSpectra** called $HG$, the *Eilenberg-Maclane spectrum* associated to $G$, and a natural isomorphism

$$\widetilde{(HG)}^n(X) \cong \widetilde{H}^n(X;G)$$

The associated homology theories also agree:

$$\widetilde{(HG)}_n(X) \cong \widetilde{H}_n(X;G)$$

The assignment $G \mapsto HG$ may be turned into a lax monoidal functor $H : \mathbf{Ab} \longrightarrow \mathbf{HoSpectra}$. Therefore if $R$ is a commutative ring, then $HR$ is a commutative ring spectrum. The multiplication on $\widetilde{(HR)}^*(X)$ is just the cup product on $\widetilde{H}^*(X;R)$! If $X$ is an unbased topological monoid, then the multiplication on $\widetilde{(HR)}_*(X_+)$ is the Pontryagin product on $\widetilde{H}_*(X_+;R) \cong H_*(X;R)$.

Similarly, there is a spectrum $KU$ for complex $K$-theory, $KO$ for real $K$-theory, $MU$ for complex cobordism, and $MO$ for real cobordism. There is a long list of interesting cohomology theories, and we won't try to exhaust it here. But we can still list infinitely many of them: any based space $X$ becomes a cohomology theory $\Sigma^\infty X \in$ **HoSpectra**, whose groups are "shifted stable maps into $X$." Classically, we had to think of cohomology theories and the spaces we took cohomology of as different objects. In **HoSpectra**, we can think of them on equal terms, and state theorems that apply to both.

## 1.6 In Summary

The stable homotopy category **HoSpectra** has:

- A functor $\Sigma^\infty$ coming in from based spaces.
- Suspension that is invertible up to natural isomorphism.
- Morphism sets $[X,Y]$ that are abelian groups, which extend to graded abelian groups $[X,Y]_*$. Composition of morphisms is graded and bilinear.
- Stable homotopy groups $\pi_*(X)$.
- A zero object $*$, and a natural isomorphism $X \vee Y \xrightarrow{\cong} X \times Y$ from coproducts to products.
- A unit object $\mathbb{S}$, a smash product $X \wedge Y$, and an internal hom $F(X,Y)$, together with some natural isomorphisms, that make it a closed symmetric monoidal category.
- Distinguished triangles that form long exact sequences of homotopy groups, and that agree with the smash product and internal hom.
- Objects which represent cohomology theories.

## 1.7 Atiyah Duality of Manifolds

Here's a geometric application of the above properties. Two based finite CW-complexes $A, B$ are *strongly $n$-dual* if there is an embedding $\Sigma^k A \hookrightarrow S^{k+n+1}$ and a homotopy equivalence $\Sigma^l B \xrightarrow{\simeq} \Sigma^l(S^{k+n+1} - \Sigma^k A)$. By Alexander duality, this gives isomorphisms

$$\widetilde{H}_q(A) \cong \widetilde{H}^{n-q}(B)$$
$$\widetilde{H}^q(A) \cong \widetilde{H}_{n-q}(B)$$

Suppose that $A$ and $B$ are strongly $n$-dual. Then we may define a map

$$\Sigma^{k+l}(A \wedge B) \longrightarrow \Sigma^{k+l} S^n$$

such that if we pull back the top-dimensional cohomology class of $S^n$ to $A \wedge B$, the slant product with this class gives isomorphisms as above

$$\widetilde{H}_q(A) \cong \widetilde{H}^{n-q}(B)$$
$$\widetilde{H}^q(A) \cong \widetilde{H}_{n-q}(B)$$

So we may form a weaker definition: $A$ and $B$ are simply *$n$-dual* if there is a map $\Sigma^{k+l}(A \wedge B) \longrightarrow \Sigma^{k+l} S^n$ which gives these isomorphisms. We no longer require $A$ to actually embed into a sphere. These two (different) notions are often both called *Spanier-Whitehead duality.*

It turns out that Spanier-Whitehead duality is much easier to state and work with in the stable homotopy category. We say that two objects $A$, $B$ of **HoSpectra** are *dual* if there are maps

$$A \wedge B \longrightarrow \mathbb{S}$$
$$\mathbb{S} \longrightarrow B \wedge A$$

such that the following two composites are the identity map:

$$A \cong A \wedge \mathbb{S} \longrightarrow A \wedge B \wedge A \longrightarrow \mathbb{S} \wedge A \cong A$$
$$B \cong \mathbb{S} \wedge B \longrightarrow B \wedge A \wedge B \longrightarrow B \wedge \mathbb{S} \cong B$$

This also implies

$$A \cong F(B, \mathbb{S})$$
$$B \cong F(A, \mathbb{S})$$

Notice the parallel with vector spaces, where $\mathrm{Hom}(V, k)$ is defined to be the dual of $V$, and the double dual of $V$ is naturally isomorphic to $V$ itself. (Exercise: Can you describe the corresponding map $k \longrightarrow V \otimes \mathrm{Hom}(V, k)$?)

Two spectra $A$ and $B$ are *$n$-dual* if $A$ and $\Sigma^{-n}B$ are dual, or equivalently $\Sigma^{-n}A$ and $B$ are dual. In this case we get a map

$$A \wedge B \longrightarrow \mathbb{S}^n = \Sigma^n \mathbb{S}$$

inducing isomorphisms

$$A \cong F(B, \Sigma^n \mathbb{S}) \quad \Leftrightarrow \quad \Sigma^{-n} A \cong F(B, \mathbb{S})$$
$$B \cong F(A, \Sigma^n \mathbb{S}) \quad \Leftrightarrow \quad \Sigma^{-n} B \cong F(A, \mathbb{S})$$

If $A$ and $B$ are $n$-dual spaces, then $\Sigma^\infty A$ and $\Sigma^\infty B$ are $n$-dual spectra. So we can import the entire theory of Spanier-Whitehead duality to the stable homotopy category, and the statements become cleaner and more categorical.

If $M$ is an $m$-manifold, then there is a smooth embedding $e : M \hookrightarrow \mathbb{R}^n$ for sufficiently large $n$. It follows that $M_+$ and $\mathbb{R}^n - M$ are (strongly) $(n-1)$-dual. But then the Thom space of the normal bundle $M^\nu$ is homotopy equivalent to the suspension of $\mathbb{R}^n - M$, so $M_+$ and $M^\nu$ are $n$-dual. This was classically called "Atiyah duality". The $n$-duality map can be described explicitly as

$$M^\nu \wedge M_+ \longrightarrow S^n$$
$$\mathbb{R}^n/(\mathbb{R}^n - \nu_\epsilon(M)) \wedge M_+ \longrightarrow \mathbb{R}^n/(\mathbb{R}^n - B_\epsilon(0))$$
$$(x, y) \mapsto x - e(y)$$

As an immediate corollary, we get isomorphisms in **HoSpectra**

$$\Sigma^\infty M^\nu \simeq F(M_+, \Sigma^n \mathbb{S}) \quad \Rightarrow \quad M^{-TM} := \Sigma^{-n}\Sigma^\infty M^\nu \simeq F(M_+, \mathbb{S})$$

coming from the Alexander map above. This is *Atiyah duality* in the stable homotopy category. From this, and the properties discussed in previous sections, we can take any cohomology theory $E$ in **HoSpectra** and get isomorphisms

$$\widetilde{E}_q(M^\nu) \cong E^{n-q}(M)$$
$$\widetilde{E}^q(M^\nu) \cong E_{n-q}(M)$$

(Recall that the tilde means the theory is reduced, and $E_q(M) := \widetilde{E}_q(M_+)$.)

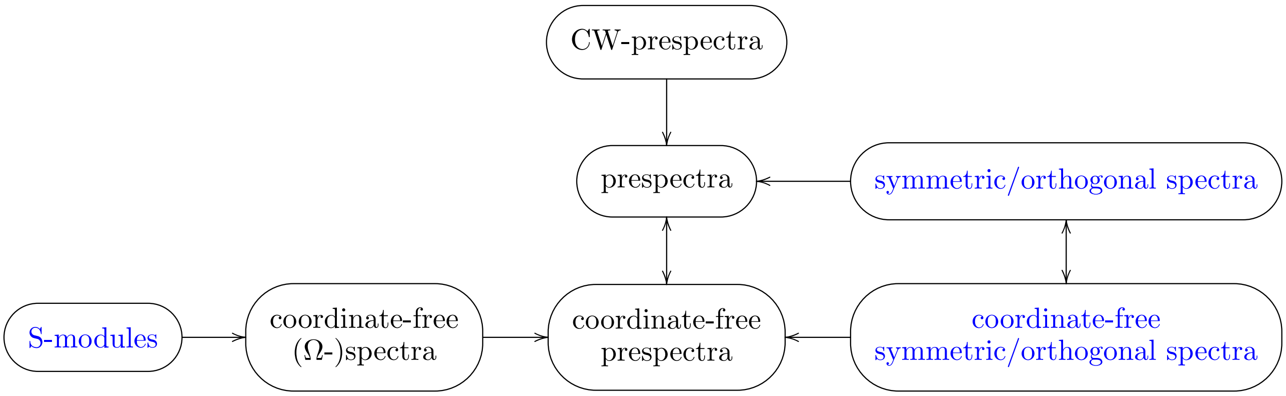
Notice that we have not assumed any kind of orientability for $M$. If $M$ is orientable in the cohomology theory $E$, then applying the Thom isomorphism to these gives Poincaré duality in $E$:

$$E_{m-q}(M) \cong E^q(M)$$

We can put a product on $M^{-TM}$ that gives the intersection product on homology; then the Atiyah duality isomorphism is an isomorphism of ring spectra [2]. As an easy consequence, Poincaré duality takes the intersection product on $E_{m-q}(M)$ to the cup product on $E^q(M)$.

# 2 Constructions of HoSpectra

Now that we've made a wish list of all the properties we desire in the category **HoSpectra**, we'll give some actual constructions of this category. These constructions all require a category called **Spectra**, which could be the category of prespectra, CW-prespectra, symmetric spectra, orthogonal spectra, coordinate-free spectra of various kinds, or S-modules. We give them here with forgetful functors:




The blue-colored categories have good smash products in **Spectra** before descending to the stable homotopy category **HoSpectra**.

## 2.1 The Adams Category

A *prespectrum* $E$ is a sequence of based spaces $E_0$, $E_1$, $E_2$, ... along with *structure maps* $\Sigma E_n \longrightarrow E_{n+1}$. We call $E_n$ the $n$th *level* of $E$. A map of prespectra $f : X \longrightarrow Y$ is a sequence of maps $f_n : X_n \longrightarrow Y_n$ that commute with the structure maps:

$$\begin{array}{ccc} \Sigma X_n & \longrightarrow & X_{n+1} \\ \downarrow{\scriptstyle \Sigma f_n} & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma Y_n & \longrightarrow & Y_{n+1} \end{array}$$

A *CW-prespectrum* is a prespectrum $E$ with the following properties: Each level $E_n$ is a CW-complex. One of the 0-cells is chosen to be the basepoint. Therefore, the reduced suspension of each cell $D^m$ is $D^{m+1}$, glued to the suspensions of the lower-dimensional cells. Therefore we can view $\Sigma E_n$ as a CW-complex with one $(m+1)$-cell for every $m$-cell of $E_n$ other than the basepoint. Using this cell structure on $\Sigma E_n$, we require that the structure map $\Sigma E_n \hookrightarrow E_{n+1}$ be the inclusion of a subcomplex.

Now every $k$-cell of $E_n$ becomes a $(k+1)$-cell of $E_{n+1}$, a $(k+2)$-cell in $E_{n+2}$, etc. We call this a *stable* $(k-n)$*-cell*. It's clear that we can have stable $m$-cells for all integer values of $m$, so a CW-prespectrum is like a CW-complex in which we have somehow allowed negative-dimensional cells. One may use these cells to define a cellular chain complex graded by $\mathbb{Z}$ instead of $\mathbb{N}$.

Our first definition of **HoSpectra** is a category **Ad** whose objects are CW-prespectra, and whose morphisms are "eventually-defined maps up to eventually-defined homotopy". More precisely, each map $f : X \longrightarrow Y$ is a map on each stable $m$-cell of $X$ that is defined on the $(m-n)$-cell in $E_n$ for all sufficiently large values of $n$. Of course, the maps on different cells have to agree with the attaching maps of those cells. So if $X$ has finitely many stable cells, then the map is eventually defined on all of $X_n$. In general, though, a map need not ever be defined on all of $X_n$ for sufficiently large $n$. To define a homotopy between such maps, we give the reduced cylinder $X_n \wedge I_+$ a CW-complex structure coming from the usual one on $I$, and we require that our homotopy be a map of spectra $\{X_n \wedge I_+\} \longrightarrow \{Y_n\}$ that is also eventually defined.

Define a functor $\Sigma^\infty : \mathbf{HoTop}_* \longrightarrow \mathbf{Ad}$ by taking each based CW-complex $X$ to the prespectrum whose $n$th level is $\Sigma^n X$, and whose structure maps are the identity.

In a previous version of these notes, we called this the Boardman category, but this construction was actually given by Adams in his classic notes [1]. Historically, this was one of the first constructions of the stable homotopy category, and it is perhaps the easiest one to understand.

**Exercises.**

- Describe the zero object $* := \Sigma^\infty(\{\text{pt}\})$ and the sphere spectrum $\mathbb{S} := \Sigma^\infty S^0$.
- Define the suspension functor $\Sigma : \mathbf{Ad} \longrightarrow \mathbf{Ad}$ so that it agrees with suspension in $\mathbf{HoTop}_*$.
- Define the shift functor sh : $\mathbf{Ad} \longrightarrow \mathbf{Ad}$ by $(\text{sh}E)_n = E_{n+1}$, with the obvious structure maps. Show that sh is a functor, and has an inverse up to natural isomorphism.
- Give a natural isomorphism between $\Sigma$ and sh. (Harder than it looks. Be careful with the structure maps.)
- Describe the abelian group structure of $[X, Y]$ and $[X, Y]_n$. In particular, this gives us the stable homotopy groups $\pi_n(X) := [\mathbb{S}, X]_n$. Prove that $\pi_n(X) = \operatorname*{colim}_{k\longrightarrow\infty} \pi_{n+k}(X_k)$.
- **Ad** has all of the properties that we claimed for **HoSpectra** in section 1.2. Prove as many of these as you can. Don't try to prove them all, since our next construction of **HoSpectra** will be much easier to work with.
- Let $\mathbf{Ad}'$ be defined as above, but we require that the attaching map of every cell in $E_n$ to be a based map $S^{m-1} \longrightarrow X$, instead of an unbased map. (This is the definition given in a book by Switzer.) Give an equivalence of categories between **Ad** and $\mathbf{Ad}'$.

## 2.2 All Prespectra

Let **Prespectra** denote the category of all prespectra (not just the CW ones) with maps that are defined on every level (not just eventually defined). So an object in **Prespectra** is a sequence of spaces $\{E_n\}_{n=1}^\infty$, together with maps $\Sigma E_n \longrightarrow E_{n+1}$. Notice that these maps always have adjoints

$E_n \longrightarrow \Omega E_{n+1}$. We say that $E$ is a *(weak) $\Omega$-spectrum* if these adjoints are all weak homotopy equivalences. We still define the stable homotopy groups of $X$ to be $\pi_k(X) = \text{colim}_k \pi_{n+k}(X_k)$. It is easy to see that if $X$ is a weak $\Omega$-spectrum, then

$$\pi_k(X) \cong \begin{cases} \pi_k(X_0) & k \geq 0 \\ \pi_0(X_k) & k \leq 0 \end{cases}$$

**Exercises.**

- Let $X$ be any prespectrum. Construct a CW-prespectrum $\widetilde{X}$ and a map of prespectra $\widetilde{X} \longrightarrow X$ that is a weak homotopy equivalence on each level.
- Construct a (weak) $\Omega$-spectrum $\hat{X}$, and a map $X \longrightarrow \hat{X}$ that induces isomorphisms on the stable homotopy groups $\pi_*(X) \xrightarrow{\cong} \pi_*(\hat{X})$.

A homotopy of maps $f$ and $g$ of prespectra is a choice of homotopy at level $n$ between $f_n$ and $g_n$ which commutes with the structure maps in the obvious way. We can define **HoPrespectra** to have the same objects as **Prespectra**, but morphisms from $X$ to $Y$ are $[\widetilde{X}, \hat{Y}]$. Note that each map or homotopy is defined on *every* level, not just eventually defined.

Intuitively, this will agree with Adams' category because an eventually-defined map $\widetilde{X}_n \longrightarrow \hat{Y}_n$ can always be looped to give $\widetilde{X}_0 \longrightarrow \Omega^n \widetilde{X}_n \longrightarrow \Omega^n \hat{Y}_n \simeq \hat{Y}_0$. Now we have a good concrete description of the maps in the homotopy category, but unfortunately, in this description it is difficult to describe how we compose the maps.

One way to circumvent this is to redefine the maps as zig-zags $X \longleftarrow A \longrightarrow B \longleftarrow \ldots \longrightarrow Y$, where the backwards maps are required to be $\pi_*$-isomorphisms. This description is very useful in practice, though it must be supplemented by a theory that guarantees that the set of all zig-zags (up to an appropriate equivalence relation) forms a set and not a proper class. For this purpose, the theory of model categories does quite nicely.

There is a model structure on **Prespectra** with the following description, found in [7]. A cofibration is a retract of a relative stable cell complex. Here the "stable cells" are defined exactly as in the previous section. However, we allow ourselves to attach lower-dimensional cells to higher-dimensional ones. A weak equivalence is a map inducing isomorphisms on the stable homotopy groups. A fibration is a map $E \longrightarrow B$ such that every level $E_n \longrightarrow B_n$ is a Serre fibration, and in the square

$$\begin{array}{ccc} E_n & \longrightarrow & \Omega E_{n+1} \\ \downarrow & & \downarrow \\ B_n & \longrightarrow & \Omega B_{n+1} \end{array}$$

the natural map $E_n \longrightarrow B_n \times_{\Omega B_{n+1}} \Omega E_{n+1}$ is a weak homotopy equivalence. Now we can pass to **HoPrespectra** by taking the set of maps from $X$ to $Y$ to be

$$[QRX, QRY] \cong [QX, RY]$$

where $Q$ denotes replacement by a weakly equivalent cofibrant prespectrum, and $R$ denotes replacement by a weakly equivalent fibrant spectrum. This category is isomorphic to the one defined above.

**HoPrespectra** is slightly less concrete than **Ad**, but in practice it is very easy to work with. For example, if $X$ is a prespectrum and $A$ is a based space, one may construct a "mapping spectrum" of $A$ into $X$ by taking at level $n$ the space

$$\mathrm{Map}_*(A, X_n)$$

Taking $A = S^1$ gives the loopspace operation $\Omega$ on prespectra. The reader may verify that it is much easier to work out the properties we gave in section 1.2 using this construction.

**Exercises.**

- Define an equivalence of categories **HoPrespectra** $\longrightarrow$ **Ad**.
- Define

  $$\mathbf{HoTop}_* \xrightarrow{\Sigma^\infty} \mathbf{HoPrespectra}$$

  by taking a space $X$ to the prespectrum whose $n$th level is $\Sigma^n QX$. Here $QX$ is any cofibrant approximation of $X$ (a CW complex will do). Define

  $$\mathbf{HoTop}_* \xleftarrow{\Omega^\infty} \mathbf{HoPrespectra}$$

  by taking a spectrum $Y$ to the 0th level of $RY$, where $RY$ is any fibrant approximation of $Y$. Verify that these commute with $\Sigma$ and $\Omega$ for spaces, as outlined in section 1.2.

- Define the wedge sum $X \vee Y$ and product $X \times Y$ of prespectra. They must satisfy the usual universal properties.
- If $X \xrightarrow{f} Y$ is a map of prespectra, the *fiber* of $f$ is a prespectrum $F(f)$ which at level $n$ is the homotopy fiber

  $$X_n \times_{Y_n} \mathrm{Map}_*(I, Y_n)$$

  Here the unit interval $I$ has 0 as its basepoint. Prove that there is a long exact sequence of stable homotopy groups

  $$\ldots \longrightarrow \pi_{n+1}(Y) \longrightarrow \pi_n(F(f)) \longrightarrow \pi_n(X) \longrightarrow \pi_n(Y) \longrightarrow \ldots$$

- If $X \xrightarrow{f} Y$ is a map of prespectra, the *cofiber* of $f$ is a prespectrum $C(f)$ which at level $n$ is the reduced mapping cone

  $$(X_n \wedge I) \cup_{X_n} Y_n$$

  Prove that there is a long exact sequence of stable homotopy groups

  $$\ldots \longrightarrow \pi_n(X) \longrightarrow \pi_n(Y) \longrightarrow \pi_n(C(f)) \longrightarrow \pi_{n-1}(X) \longrightarrow \ldots$$

- Suppose we have a triple of spectra $X \xrightarrow{f} Y \xrightarrow{g} Z$ equipped with a homotopy $h$ from $g \circ f$ to the identity. Construct two natural maps

$$C(f) \longrightarrow Z, \quad X \longrightarrow F(g)$$

and show that one is a $\pi_*$-isomorphism iff the other is as well.

- Prove that the natural inclusion $X \vee Y \longrightarrow X \times Y$ is an equivalence of prespectra ($\pi_*$-isomorphism).

- If $X$ and $Y$ are prespectra and $X$ is a retract of $Y$, prove that the cofiber of $X \longrightarrow Y$ is equivalent to the fiber of $Y \longrightarrow X$. Denoting this prespectrum by $Z$, prove that

$$Y \simeq X \vee Z$$

where $\simeq$ denotes isomorphism in the stable homotopy category.

- If $X$ is a based CW complex, prove that there is an isomorphism in **HoPrespectra**

$$\Sigma^\infty(X \vee S^0) \simeq \Sigma^\infty(X_+)$$

Use this to prove

$$\Sigma^\infty(X \times X) \simeq \Sigma^\infty(X \wedge X) \vee \Sigma^\infty X \vee \Sigma^\infty X$$

- (Binomial Theorem.)

$$\Sigma^\infty(X^n) \simeq \bigvee_{i=1}^{n} \binom{n}{i} \Sigma^\infty(X^{\wedge i})$$

Moving on, we want to get a definition of the smash product that satisfies all the properties we gave in section 1.3. We can accomplish this in either **Ad** or **HoPrespectra**. The smash product of $X$ and $Y$ is defined by taking an arbitrary sequence $p(n) \longrightarrow \infty$ such that $(n - p(n)) \longrightarrow \infty$, and setting $(X \wedge Y)_n = X_{p(n)} \wedge Y_{n-p(n)}$. If we fix one such sequence, this defines a smash product that has all of the properties that we claimed in section 1. Unfortunately, this relies on a non-canonical choice of sequence, and we hate choices because they make things hard in practice. Fortunately, there exist some tricks for making it canonical. Here is one such trick:

## 2.3 Symmetric and Orthogonal Spectra

A *symmetric spectrum* $E$ is a sequence of based spaces $E_0$, $E_1$, $E_2$, ..., structure maps $\Sigma E_n \longrightarrow E_{n+1}$, and a $\Sigma_n$-action on $E_n$ for all $n \geq 0$, such that the composite

$$S^p \wedge E_q \longrightarrow S^{p-1} \wedge E_{1+q} \longrightarrow \cdots \longrightarrow S^1 \wedge E_{(p-1)+q} \longrightarrow E_{p+q}$$

is $(\Sigma_p \times \Sigma_q)$-equivariant. A map $f : X \longrightarrow Y$ between symmetric spectra is a sequence of maps $f_n : X_n \longrightarrow Y_n$ that agree with suspension, such that $f_n$ is $\Sigma_n$-equivariant. This defines a category called $\mathbf{Sp}^{\Sigma}$. To define an *orthogonal spectrum*, we take the above definition and replace $\Sigma_n$ with $O(n)$ everywhere; this gives a category $\mathbf{Sp}^{O}$. (The $O(n)$ actions must be continuous.)

Using the inclusion of groups $\Sigma_n \hookrightarrow O(n)$, we see that an orthogonal spectrum defines a symmetric spectrum. We could forget the actions entirely and get a prespectrum; this operation has a left adjoint that takes prespectra to symmetric (or orthogonal) spectra. The derived forms of these two functors give an equivalence between **HoPrespectra** (defined in the last section) and $\mathbf{HoSp}^{\Sigma}$ or $\mathbf{HoSp}^{O}$ (defined later in this section).

Unlike prespectra, symmetric (or orthogonal) spectra form a closed symmetric monoidal category. If we let **Spectra** denote either symmetric or orthogonal spectra, then we have the diagram

$$\begin{array}{ccccccc}
\mathbf{CW} & \xrightarrow{X \mapsto X_+} & \mathbf{CW}_* & \xrightarrow{\Sigma^\infty} & \mathbf{Spectra} & \xleftarrow{G \mapsto HG} & \mathbf{Ab} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \text{0th degree} \\
\mathbf{HoTop} & \xrightarrow{X \mapsto X_+} & \mathbf{HoTop}_* & \xrightarrow{\Sigma^\infty} & \mathbf{HoSpectra} & \xrightarrow{\pi_*} & \mathbf{Graded\ Ab}
\end{array}$$

and every functor is at least lax monoidal. So we can define monoids in **Spectra**, which then become monoids in **HoSpectra**. These two notions are not the same. A monoid in **Spectra** is a *symmetric/orthogonal ring spectrum*, whereas a monoid in **HoSpectra** is just a ring spectrum "up to homotopy." We've neglected to actually define **HoSpectra**, and we'll continue to neglect this while we discuss the closed symmetric monoidal structure.

Let's describe this structure more explicitly. The unit object is the sphere spectrum $\mathbb{S}_n = S^n$. The mapping space is

$$F(X,Y)_n \subset \prod_i F(X_i, Y_{i+n})$$

the subspace of all collections of $\Sigma_i$-equivariant maps $\{X_i \longrightarrow Y_{i+n}\}_i$ that commute with suspension. Notice that $F(X,Y)_0$ is just the based space of all maps of symmetric spectra. The smash product is

$$(X \wedge Y)_n = \bigvee_{p+q=n} \Sigma_{p+q_+} \wedge_{\Sigma_p \times \Sigma_q} (X_p \wedge Y_q)/\sim$$

The quotient relation identifies the images of the two maps

$$\begin{array}{ccccc}
\Sigma_{(p+q+r)_+} \wedge_{\Sigma_q \times \Sigma_{p+r}} (X_q \wedge Y_{p+r}) & \longleftarrow & \Sigma_{(p+q+r)_+} \wedge (S^p \wedge X_q \wedge Y_r) & \longrightarrow & \Sigma_{(p+q+r)_+} \wedge_{\Sigma_{p+q} \times \Sigma_r} (X_{p+q} \wedge Y_r) \\
 & & & & \\
(\sigma \circ \tau_{q,p}, x, sy) & \longleftarrow & (\sigma, s, x, y) & \longrightarrow & (\sigma, sx, y)
\end{array}$$

Here $(s,x) \mapsto sx$ is shorthand for the structure map

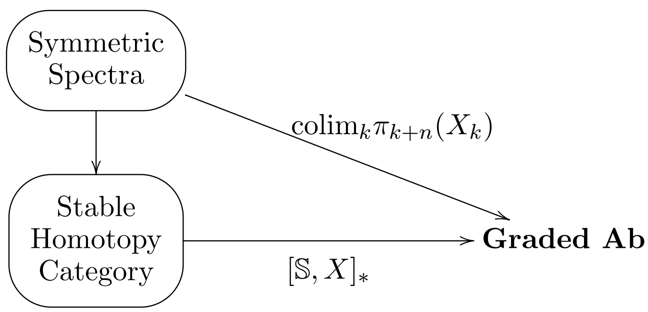
$$S^p \wedge X_q \cong \Sigma^p X_q \longrightarrow X_{p+q}$$

and $\tau_{q,p}$ is a permutation in $\Sigma_{p+q+r}$ moves the first block of $q$ elements past the second block of $p$ elements and leaves the last block of $r$ elements alone.

Let's describe this more heuristically. $S^p$ has $p$ sphere coordinates, $X_q$ has $q$ sphere coordinates, and $Y_r$ has $r$ sphere coordinates. They are naturally arranged with the $p$ coordinates first, then the $q$ coordinates, then the $r$ coordinates. The permutation $\sigma$ takes this natural arrangement and gives us the arrangement we desire. Now if we smash $S^p$ into $X_q$, we get a space $X_{p+q}$ with $(p+q)$ sphere coordinates, still lined in order with the $p$ coordinates first and the $q$ coordinates second. So in $X_{p+q} \wedge Y_r$, the $p$ coordinates come first, then the $q$ coordinates, then the $r$ coordinates. Applying $\sigma$, we again get the desired arrangement of sphere coordinates.

However, if we smash $S^p$ into $Y_r$, we get $X_q \wedge Y_{p+r}$. The $q$ coordinates come first, then the $p$ coordinates, then the $r$ coordinates. Applying $\sigma$, we get the wrong arrangement. We fix the problem by applying $\sigma \circ \tau_{q,p}$ instead. The $\tau_{q,p}$ pulls the $p$ coordinates back to the beginning where they belong. Therefore $\sigma \circ \tau_{q,p}$ gives us the correct arrangement of sphere coordinates. We remember to include $\tau_{q,p}$ by feeling a pang of guilt whenever we try to move $S^p$ past $X_q$. The permutation $\tau_{q,p}$ alleviates that guilt.

To recap, symmetric (or orthogonal) spectra form a closed symmetric monoidal category. We can define a symmetric ring spectrum to be a monoid object in this category; this always descends to a monoid object in the homotopy category. Unfortunately, symmetric spectra sometimes have the "wrong" homotopy groups. If we try to define $\pi_n$ of a symmetric spectrum $X$ as $\text{colim}_{k\to\infty}\pi_{k+n}(X_k)$, then we get a diagram




This diagram does NOT commute, so the naïve homotopy groups $\pi_n(X) = \text{colim}_{k\to\infty}\pi_{k+n}(X_k)$ are not equal to our original definition $\pi_n(X) = [\mathbb{S}, X]_n$. Moreover, the naïve homotopy groups do not define a monoidal functor into **Graded Ab**. Therefore, the "correct" definition of homotopy groups is $\pi_*(X) = [\mathbb{S}, X]_*$. Fortunately, our two definitions coincide for the class of *semistable* symmetric spectra, as defined in [10].

Every orthogonal spectrum gives a semistable symmetric spectrum, so the naïve definition of $\pi_*$ gives the right answer when $X$ is an orthogonal spectrum. So orthogonal spectra enjoy the convenience of having a good smash product and internal hom, and the naïve definition of their homotopy groups is the correct one. The smash product of orthogonal spectra is defined as above, but with $O(n)$ everywhere instead of $\Sigma_n$. (Then why use symmetric spectra at all? Well, symmetric spectra are a much more reasonable choice when one wants to use *simplicial sets* instead of spaces.)

It's sometimes necessary to write an explicit model for the homotopy groups of the smash product of orthogonal spectra $X \wedge Y$. Using the above definition directly, this looks like a nightmare.

Fortunately, (using [7]) these homotopy groups are equal to the ones we obtain from the "handicrafted smash product" of $X$ and $Y$ as prespectra. So they are given by the colimit of the following commuting grid of abelian groups:

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \text{colim} = \pi_k(X \wedge Y) \\
\uparrow + & & \uparrow - & & \uparrow + & & \uparrow - & & \\
\pi_{2+k}(X_0 \wedge Y_2) & \xrightarrow{+} & \pi_{3+k}(X_1 \wedge Y_2) & \xrightarrow{+} & \pi_{4+k}(X_2 \wedge Y_2) & \xrightarrow{+} & \pi_{5+k}(X_3 \wedge Y_2) & \xrightarrow{+} & \dots \\
\uparrow + & & \uparrow - & & \uparrow + & & \uparrow - & & \\
\pi_{1+k}(X_0 \wedge Y_1) & \xrightarrow{+} & \pi_{2+k}(X_1 \wedge Y_1) & \xrightarrow{+} & \pi_{3+k}(X_2 \wedge Y_1) & \xrightarrow{+} & \pi_{4+k}(X_3 \wedge Y_1) & \xrightarrow{+} & \dots \\
\uparrow + & & \uparrow - & & \uparrow + & & \uparrow - & & \\
\pi_k(X_0 \wedge Y_0) & \xrightarrow{+} & \pi_{1+k}(X_1 \wedge Y_0) & \xrightarrow{+} & \pi_{2+k}(X_2 \wedge Y_0) & \xrightarrow{+} & \pi_{3+k}(X_3 \wedge Y_0) & \xrightarrow{+} & \dots
\end{array}$$

Here $(+)$ means that we use the usual suspension homomorphism, and $(-)$ means that we negate it. The signs are explained by the fact that a new sphere coordinate must be switched past $X_p$ before it can be smashed into $Y_q$. Note that the colimit can be computed in at least three different ways: we can compute the colimit of each column and then take the colimit of the results, or we could do the same thing with rows, or we could take a path from the bottom-left corner out to infinity that eventually reaches each row and column, and take the colimit along that path.

How do we go to **HoSpectra**? We need a notion of an unbased cell; we get it by creating a "free" symmetric/orthogonal spectrum out of the map of based spaces $S^{n-1}_+ \hookrightarrow D^n_+$. To be more precise, there is a forgetful functor from symmetric/orthogonal spectra to spaces by taking the space level $k$; we take the left adjoint $F_k$ of this functor to the basic cell $S^{n-1}_+ \hookrightarrow D^n_+$ to get our unbased cell of spectra. Here $S^{n-1}_+$ is a sphere with a disjoint basepoint, NOT the upper hemisphere of $S^{n-1}$. This is precisely analogous to what we did for prespectra, only that our free cell spectra now come with extra $\Sigma_n$s or $O(n)$s built in because they need that extra structure.

There is a model structure in which the cofibrations are the retracts of the relative cell complexes, the fibrations are the levelwise fibrations $E \longrightarrow B$ giving homotopy pullbacks

$$\begin{array}{ccc}
E_n & \longrightarrow & \Omega E_{n+1} \\
\downarrow & & \downarrow \\
B_n & \longrightarrow & \Omega B_{n+1}
\end{array}$$

and the weak equivalences $X \longrightarrow Y$ are the maps that induce isomorphisms $[Y, E] \xrightarrow{\cong} [X, E]$ for every (weak) $\Omega$-spectrum $E$. (These will coincide with the idea that $X \longrightarrow Y$ gives isomorphisms on the "correct" homotopy groups, but to define the "correct" groups we need to actually construct **HoSpectra** first.) This is the model structure given in [7]; another one is given in [10].

The derived smash product $\wedge^{\mathbb{L}}$ on **HoSpectra** is obtained from the smash product $\wedge$ on **Spectra** in the following way:

$$X \wedge^{\mathbb{L}} Y := QX \wedge QY$$

where $Q$ denotes cofibrant replacement. This process of "deriving" the smash product is critically important if we want to end up with a construction that preserves equivalences of spectra. Of course, the sphere spectrum $\mathbb{S}$ is cofibrant, so it is the unit object of the derived smash product as well. The natural map $X \wedge^{\mathbb{L}} Y \longrightarrow X \wedge Y$ is an equivalence when $X$ and $Y$ are cofibrant spectra. It is also an equivalence if either $X$ or $Y$ is "flat," as defined in [10]. Adjoint to this derived smash product is a derived internal hom:

$$\mathbb{R}F(X,Y) := F(QX, RY)$$

This is equivalent to $F(X,Y)$ when $X$ is cofibrant and $Y$ is fibrant. The derived smash product and derived internal hom give the closed symmetric monoidal structure on $\mathbf{HoSp}^{\Sigma}$ and $\mathbf{HoSp}^{O}$. This is true for quite general reasons: see chapter 4 of [4] for an explanation.

### 2.4 Coordinate-Free Spectra

We may also present symmetric and orthogonal spectra in a "coordinate-free" way. Throughout this section, one should keep in mind that these coordinate-free constructions are equivalent to the simpler definitions we gave above. The reason for introducing them at all is that they make some constructions more natural, and they lead to a strong theory in the equivariant setting.

If $A$ is a finite set, let $\mathbb{R}^A$ denote the space of all functions $A \longrightarrow \mathbb{R}$, and $S^A$ its the one-point compactification. A *coordinate-free symmetric spectrum* is an assignment of a space $X(A)$ to each finite set $A$ (in some appropriate universe), and a map $S^{B-i(A)} \wedge X(A) \xrightarrow{\xi_i} X(B)$ to each inclusion $i : A \hookrightarrow B$. The identity map $A \hookrightarrow A$ must induce the identity $S^0 \wedge X(A) \longrightarrow X(A)$, and for each composition $A \overset{i}{\hookrightarrow} B \overset{j}{\hookrightarrow} C$ the evident diagram commutes:

$$\begin{array}{ccc}
S^{C-j(B)} \wedge S^{B-i(A)} \wedge X(A) & \xrightarrow{\xi_i} & S^{C-j(B)} \wedge X(B) \\
\downarrow \cong & & \downarrow \xi_j \\
S^{C-j(i(A))} \wedge X(A) & \xrightarrow{\xi_{j\circ i}} & X(C)
\end{array}$$

**Exercise.** Let $\mathbf{n}$ be the finite set $\{1, \ldots, n\}$. If $X$ is a coordinate-free symmetric spectrum, construct an ordinary symmetric spectrum whose levels are $\{X(\mathbf{n})\}_{n=0}^{\infty}$.

Since every finite set is isomorphic to some $\mathbf{n}$, it's also possible to go backwards and turn any symmetric spectrum into a coordinate-free one. So the theory of coordinate-free symmetric spectra is essentially the same as the the theory of symmetric spectra.

If $V$ is an inner product space, let $S^V$ denote its one-point compactification. A *coordinate-free orthogonal spectrum* is an assignment of a space $X(V)$ to each finite-dimensional inner product space $V$, and a map $S^{W-i(V)} \wedge X(V) \xrightarrow{\xi_i} X(W)$ to each linear isometric inclusion $i : V \hookrightarrow W$. Here $W - i(V)$ is the orthogonal complement of $i(V) \subset W$. The maps must depend continuously on $i$. To state this precisely, let $O(V,W)$ be the space of linear isometries $V \hookrightarrow W$, and let $O(V,W)^{W-V}$

be the Thom space of the canonical bundle over the Grassmannian $O(V, W)$, whose fiber over $i$ is $W - i(V)$. Then we require that the following map be continuous:

$$O(V, W)^{W-V} \wedge X(V) \longrightarrow X(W)$$

The identity map $V \hookrightarrow V$ must induce the identity $S^0 \wedge X(V) \longrightarrow X(V)$, and for each composition $V \hookrightarrow V' \hookrightarrow V''$ the evident diagram commutes. As above, if $X$ is a coordinate-free orthogonal spectrum, then the sequence of spaces $X(\mathbb{R}^n)$ forms an orthogonal spectrum that captures all the information in $X$ up to isomorphism. We can also avoid set-theoretic difficulties by declaring that our "universe" is just some infinite-dimensional real inner product space $U \cong \mathbb{R}^\infty$, and that we only consider the finite-dimensional subspaces $V \subset U$. Note however that we work with all linear isometric injective maps $V \hookrightarrow W$, not just the inclusions of subspaces $V \subset W \subset U$.

For completeness, we will briefly discuss coordinate-free ($\Omega$-)spectra. It is very important not to confuse this theory with coordinate-free symmetric/orthogonal spectra, even though they share some similar notation. As above, we fix a universe $U \cong \mathbb{R}^\infty$ with an inner product, and for each finite-dimensional $V \subset U$ we let $S^V$ be its one-point compactification. If $K$ is any based space, let $\Omega^V K = F(S^V, K)$ be space of based maps in the (CGWH) compact-open topology. A *coordinate-free prespectrum* $X$ associates to every finite-dimensional subspace $V \subset U$ a based space $X(V)$, and to every inclusion $V \subset W$ of subspaces a continuous map $S^{W-V} \wedge X(V) \longrightarrow X(W)$. Equivalently, there is a continuous map $X(V) \longrightarrow \Omega^{W-V} X(W)$. We have identity and composition axioms: the inclusion $V \subset V$ must induce the identity map $X(V) \longrightarrow X(V)$, and a triple of inclusions $V \subset V' \subset V''$ yield three maps that must agree. A *coordinate-free spectrum* $X$ is a prespectrum for which the maps $X(V) \longrightarrow \Omega^{W-V} X(W)$ are homeomorphisms. These are discussed in classic notes by Lewis, May and Steinberger [5].

**Remark.** Note that coordinate-free ($\Omega$-)spectra only have maps for *inclusions* of spaces $V \subset W$, whereas coordinate-free orthogonal spectra have maps for *every injective map* $V \hookrightarrow W$ that preserves the inner product. It is not difficult to see that the only spectrum satisfying both definitions is the zero object, $X(V) = *$ for all $V$. (To do this, consider the $O(V)$-equivariant map

$$X(V) \xrightarrow{\cong} \Omega^{W-V} X(W)$$

when $\dim(W - V) = 2$.)

Coordinate-free $\Omega$-spectra do not form a closed symmetric monoidal category. One may construct a richer category of $\mathbb{L}$-*spectra* and then pass to a subcategory of *S-modules*, which does form a closed symmetric monoidal category. This is done in work of Elmendorf, Kriz, Mandell, and May [3]. The homotopy category of S-modules is then equivalent to **Ad**, **HoPrespectra**, $\mathbf{HoSp}^\Sigma$, and $\mathbf{HoSp}^O$ as defined in previous sections. S-modules in the sense of EKMM are not as elementary to construct as orthogonal spectra, though all the objects are already fibrant, which makes some applications cleaner.

# 3 Equivariant and Parametrized Spectra

## 3.1 Equivariant Spectra

Equivariant spectra seem to be one of the most intimidating objects in homotopy theory; here we'll try our best to bring them down to earth. Let $G$ be a compact Lie group. For simplicity, let's assume that it is 0-dimensional, i.e. a finite discrete group. Then an *orthogonal $G$-spectrum* is an orthogonal spectrum $X$ together with a continuous $G$-action on every level $X_n$. The $G$-action must commute with the $O(n)$ action and the structure maps. The most obvious model structure to put on these objects is the *projective model structure*: the weak equivalences and fibrations are obtained by forgetting the $G$-action. The cofibrations are retracts of cell complexes built from the "free $G$-cells"

$$F_k(G \times S^{n-1}) \longrightarrow F_k(G \times D^n)$$

where $F_k$ is the left adjoint to the forgetful functor $X \rightsquigarrow X_k$ from orthogonal spectra to unbased spaces.

To distinguish from later notions, let us call these weak equivalences *naïve $G$-equivalences.* To reiterate, a naïve $G$-equivalence of orthogonal $G$-spectra is a map of orthogonal spectra $X \longrightarrow Y$ which preserves the $G$ action, and which is a stable equivalence when we forget the $G$ action.

You'll sometimes hear that this is the "wrong" model of $G$-spectra but it's perfectly fine for some applications. Let's give some examples. First, suspension spectrum and 0th space

$$\Sigma^\infty : \mathbf{Spaces}_* \leftrightarrow G\mathbf{Sp}^O : (-)_0$$

define a Quillen adjunction with $\Sigma^\infty$ the left adjoint. Second, for a $G$-spectrum $X$ we can define an orbit spectrum $X_G$ by taking orbits of each level:

$$(X_G)_n := (X_n)_G$$

This gives a well-defined functor to ordinary orthogonal spectra, which is left adjoint to the functor which endows an orthogonal spectrum $Y$ the constant $G$-action:

$$(-)_G : G\mathbf{Sp}^O \leftrightarrow \mathbf{Sp}^O : \text{constant } G\text{-action}$$

This is also a Quillen adjunction! The left derived functor of orbits is *homotopy orbits* $X_{hG}$, and is related by a zig-zag of weak equivalences to the functor $X \wedge_G EG_+$. Here $EG$ is any contractible CW-complex on which $G$ acts freely by permuting cells.

As left adjoints, $\Sigma^\infty_+ -$ and $(-)_{hG}$ commute with each other; therefore we get things like

$$(\Sigma^\infty_+ *)_{hG} \cong \Sigma^\infty_+(*_{hG}) \cong \Sigma^\infty_+ BG$$

Finally we will define the homotopy fixed points of $X$. We could of course take fixed points by taking fixed points on each level of the spectrum. This defines the right adjoint to the "constant $G$-action" functor. Unfortunately, it is not a right Quillen functor with the chosen model structure.

It might be possible to rectify this by changing the model structure, but there is a quicker fix. Recall that we can think of $G$-fixed points of a space as equivariant maps in from the one-point space:

$$X^G \cong F^G(*, X)$$

We derive this construction by replacing $*$ with $EG$, and replacing $X$ with a fibrant $G$-spectrum $RX$ in the above model structure. Since the fibrations are obtained by forgetting the $G$-action, all we need is that $RX$ is nonequivariantly a weak $\Omega$-spectrum, and $X \longrightarrow RX$ is any naïve $G$-equivalence. Then we define *homotopy fixed points* of $X$ by

$$X^{hG} := F^G(EG, RX)$$

that is, a spectrum which at level $n$ is the space of $G$-equivariant maps from $EG$ to $(RX)_n$.

To recap, we can define homotopy orbits and homotopy fixed points, and these constructions both turn naïve $G$-equivalences into ordinary stable equivalences. This is all you need for lots of applications.

That's it for the projective model structure. What more could we want? Well for starters, since every spectrum is equivalent to one with a free $G$-action (by cofibrant replacement), there isn't really any theory of fixed points. This is exactly the same issue that comes up when you look at $G$-spaces: you can make your equivalences naïve, but then everything is equivalent to a free $G$-CW complex so there's no theory of fixed points. To correct this, you refine your notion of weak equivalence:

**Definition 3.1.** A map $X \longrightarrow Y$ of $G$-spaces is a *$G$-equivalence* if for each subgroup $H \leq G$ the induced map of $H$-fixed points $X^H \longrightarrow Y^H$ is a weak equivalence.

To obtain a model structure on $G$-spaces with these weak equivalences, we use a variant of the projective model structure. Instead of considering just the space $X$ with the $G$ action, we consider the spaces $X^H$ for varying $H$ as levels of a diagram. The indexing category for this diagram has one object labelled $G/H$ for each subgroup $H \leq G$. The labelling is suggestive, since we define the maps from $G/H$ to $G/K$ to be maps $G/H \longrightarrow G/K$ commuting with the left $G$-action on both spaces. If $X$ is a based space with a $G$-action, then create a diagram over this bigger indexing category by assigning $X^H$ to the object $G/H$. Each map $G/H \longrightarrow G/K$ induces a map $X^K \longrightarrow X^H$, by seeing where the identity coset goes, taking any element in the resulting coset, and having that element act on $X^K \subset X$, landing inside $X^H \subset X$. More to the point, we may think of $X^H$ as $F^G(G/H, X)$ and pre-compose with the given map $G/H \longrightarrow G/K$. Now, as above, we get a new projective model structure in which the weak equivalences are the maps $X \longrightarrow Y$ which give weak equivalences $X^H \longrightarrow Y^H$ (i.e. the $G$-equivalences defined above), and similarly for the fibrations. The cofibrations are then generated by

$$(G/H \times S^{n-1})_+ \longrightarrow (G/H \times D^n)_+$$

for all subgroups $H$, not just the trivial subgroup.

This is great - a model structure that keeps track of fixed point data! We can apply this method almost verbatim to orthogonal spectra: we define a *trivial universe $G$-equivalence* to be a map of orthogonal spectra $X \longrightarrow Y$ which gives a stable equivalence of fixed point spectra $X^H \longrightarrow Y^H$ for each subgroup $H$. As for spaces, there is then a model structure with these weak equivalences, where the fibrations are also determined on $X^H \longrightarrow Y^H$, and where the cofibrations are generated by

$$F_k(G/H \times S^{n-1}) \longrightarrow F_k(G/H \times D^n)$$

for all $n, k \geq 0$ and $H \leq G$.

Unfortunately, as soon as we do this our equivariant stable category becomes much less useful. In order to support Poincare duality and Atiyah duality, it is necessary to make the operation $\wedge S^V$ invertible in the homotopy category. (Here $S^V$ is the one-point compactification of a finite-dimensional $G$-representation $V$.) This was actually true for naïve orthogonal spectra, but it is not true for trivial-universe $G$-spectra; we lost this useful property in the process of trying to track fixed-point data in our homotopy category. We have to somehow make the representations more involved in the notion of "equivalence of spectra" - this leads to the following more sophisticated definition.

Let $U \cong \mathbb{R}^\infty$ be an infinite-dimensional real inner product space, equipped with a $G$-action. We require that it contains infinitely many copies of each irreducible $G$-representation. Call such a $U$ a *complete $G$-universe.* Then a *coordinate-free orthogonal $G$-spectrum* is an assignment of a space $X(V)$ to each finite-dimensional inner product space $V$, and a map $S^{W-i(V)} \wedge X(V) \xrightarrow{\xi_i} X(W)$ to each linear isometric inclusion $i : V \hookrightarrow W$, satisfying the same conditions as in the previous section. In addition, there is an equivariance condition, though perhaps not the one you might expect.

As before, let $O(V, W)$ denote the space of all linear isometric inclusions $V \hookrightarrow W$, which are not necessarily equivariant. Since $V$ and $W$ have left $G$ actions, the Grassmannian $O(V, W)$ inherits a left $G$-action by conjugation. In particular, the space of equivariant maps is obtained by restricting to the fixed points $O(V, W)^G$. Next, the canonical bundle over $O(V, W)$ inherits a $G$ action by restricting from $O(V, W) \times W$. This gives the Thom space $O(V, W)^{W-V}$ a basepoint-preserving $G$ action. Finally, we give the smash product

$$O(V, W)^{W-V} \wedge X(V)$$

the diagonal $G$-action and we require that the map

$$O(V, W)^{W-V} \wedge X(V) \longrightarrow X(W)$$

be $G$-equivariant.

This takes a little time to unwind, but the result is a definition that is completely equivalent to the one above. Namely, the spaces $X(\mathbb{R}^n)$ indexed by trivial representations give an orthogonal $G$-spectrum as above, and up to isomorphism the rest of the coordinate-free spectrum may be reconstructed from this.

However, the coordinate-free perspective leads to a different, stronger notion of $G$-equivalence of spectra. We say that a coordinate-free orthogonal $G$-spectrum $X$ is *fibrant* if for each inclusion $V \subset W$ of subspaces of $U$, the structure map

$$X(V) \longrightarrow \Omega^{W-V} X(W)$$

induces for every closed subgroup $H \leq G$ a weak equivalence of fixed point spaces

$$X(V)^H \longrightarrow (\Omega^{W-V} X(W))^H$$

(Notice that the structure map is equivariant because the inclusion $V \hookrightarrow W$ is equivariant by definition.) If $X$ is fibrant and $H \leq G$ is a closed subgroup, define the *(genuine) fixed point spectrum* $X^H$ to be the orthogonal spectrum which at level $n$ is simply the fixed points

$$X(\mathbb{R}^n)^H$$

Now we say that a map $X \longrightarrow Y$ of coordinate-free orthogonal $G$-spectra is a *(genuine) $G$-equivalence* if it induces equivalences or orthogonal spectra

$$X^H \xrightarrow{\sim} Y^H$$

for each closed subgroup $H$.

This definition is in line with the kind of homotopy theory of $G$-spaces where we want to keep track of fixed point spaces $X^H$, but suspending by a representation is now invertible, so we can do Atiyah duality and Poincaré duality for $G$-manifolds. The homology/cohomology theories classified by these spectra have not just an integer grading but an $RO(G)$-grading, where $RO(G)$ is the usual ring on the group completion of the set of isomorphism classes of finite-dimensional orthogonal $G$-representations. For an elegant but detailed treatment of this theory see [6].

## 3.2 Parametrized Spectra

Loosely, a *parametrized spectrum* or *fibered spectrum* is an object over some (CGWH) topological space $B$ such that the "fiber" over each point $b \in B$ is a spectrum, using one of the definitions we gave in the previous section. If $B$ is a point, a fibered spectrum over $B$ should just be a spectrum. These objects are very useful tools for collecting together unstable information (in the base $B$) with stable information (in the fibers). As a basic application, one can prove a version of twisted Poincaré duality that is much more powerful and general than the usual one using ordinary (co)homology with twisted coefficients.

There are at least three approaches to parametrized spectra: the May-Sigurdsson approach uses coordinate-free orthogonal spectra, but there is another approach using S-modules, and other approaches using $\infty$-categories and/or homotopy sheaves. We will follow May and Sigurdsson and describe the model category of parametrized spectra found in [8].

Let $B$ be an unbased (CGWH) topological space. An ex-space over $B$ is a topological space $X$ (which for technical reasons must be a $k$-space but need not be weak Hausdorff) together with maps $B \longrightarrow X \longrightarrow B$ that compose to the identity. The category of such spaces is denoted $\mathcal{K}_B$. This category has products $X \times_B Y$, quotients $X/_B Y$, wedge sums $X \vee_B Y$, and smash products $X \wedge_B Y$, and each of these constructions does the obvious thing on each fiber. It also has mapping spaces $F_B(X, Y)$, which on each fiber is the mapping space of fibers $F(X_b, Y_b)$, but its construction is a bit subtle.

Let's define the $qf$-model structure on $\mathcal{K}_B$, which is nicely behaved and yields a parametrized version of $\mathbf{HoTop}_*$. This structure is *compactly generated*, which means that there is a collection of cells $I$ and trivial cells $J$ that are compact in some sense, such that the cofibrations are the retracts of relative $I$-cell complexes, the acyclic fibrations have the RLP with respect to maps in $I$, the acyclic cofibrations are the retracts of the relative $J$-cell complexes, and the fibrations have the RLP with respect to maps in $J$. It's also *well grounded*, which means that there is a forgetful functor to spaces (total space), and the model structure on spaces interacts in the correct way with the $qf$-model structure on $\mathcal{K}_B$. As a consequence, arguments like the Puppe cofibration sequence go through. (Other model structures run into problems with this.)

To define the $qf$-model structure, we first define an "$f$-model structure" as in ([8],p.80-84). The classes of $f$-cofibrations, fibrations, and weak equivalences are defined using the fiberwise versions of the homotopy extension property (HEP), the homotopy lifting property (HLP), and homotopy equivalence. There are $\bar{f}$-cofibrations, but these end up being $f$-cofibrations that are also closed inclusions. These give an $f$-model structure on spaces over $B$ and ex-spaces over $B$.

Now we can give the $qf$-model structure on ex-spaces over $B$. An $f$-disc is a disc $D^n \longrightarrow B$ such that $S^{n-1} \hookrightarrow D^n$ is an $f$-cofibration; this is morally the same as saying that the map $D^n \longrightarrow B$ is constant on some collar neighborhood of the boundary of $D^n$. A relative $f$-disc is a diagram of $f$-cofibrations over $B$

$$\text{upper hemisphere} \longrightarrow S^n \longrightarrow D^{n+1}$$

Equivalently, $D^{n+1}$ and its lower hemisphere are both $f$-discs. Then $I$ is the collection of $f$-discs $S^{n-1} \longrightarrow D^n$ (with a disjoint section attached) and $J$ is the collection of relative $f$-discs upper hemisphere $\longrightarrow D^{n+1}$ (with a disjoint section attached). These collections generate the cofibrations and acyclic cofibrations of the $qf$-model structure. The $qf$-equivalences are the weak homotopy equivalences on total spaces. This is enough to determine the $qf$ model structure: the cofibrations are the retracts of the $f$-disc complexes, whereas the $qf$-fibrations have the usual lifting property with respect to every relative $f$-disc. Every $qf$-cofibrant object is $f$-cofibrant, $\bar{f}$-cofibrant, and $q$-cofibrant. Every $f$-fibrant object is $qf$-fibrant. Every $qf$-fibrant object is a quasifibration, i.e. for every point $b \in B$ there is a long exact sequence of homotopy groups. So for the fibrant objects, the homotopy groups of each fiber capture the homotopy type.

Now we'll move from spaces $\mathcal{K}_B$ to spectra $\mathcal{S}_B$. Fix a universe $U$. For each finite-dimensional subspace $V$, let $S^V_B$ be the fiberwise one-point compactification of the trivial bundle $B \times V \longrightarrow B$. Now a *parametrized coordinate-free orthogonal spectrum* is an assignment of an ex-space $X(V)$ to each finite-dimensional inner product space $V$, and a map of ex-spaces $S_B^{W-i(V)} \wedge_B X(V) \xrightarrow{\xi_i} X(W)$ to each linear isometric inclusion $i : V \hookrightarrow W$. As before, the maps must depend continuously on $i$:

$$(O(V,W)^{W-V} \times B) \wedge_B X(V) \longrightarrow X(W)$$

As before, these maps also respect the identity and composition.

To construct the homotopy category $\mathbf{Ho}\mathcal{S}_B$, we restrict attention to orthogonal spectra whose levels $X(V)$ are well-grounded ($\bar{f}$-cofibrant and CGWH). Construct shift desuspensions $F_V : \mathcal{K}_B \longrightarrow \mathcal{S}_B$ just as in the nonparametrized case:

$$F_V(A)(W) = (O(V,W)^{W-V} \times B) \wedge_B A$$

Then the level model structure has as its weak equivalences the levelwise weak homotopy equivalences of total spaces over $B$. The cofibrations and acyclic cofibrations generated by the shift desuspensions of the $f$-discs and the relative $f$-discs, respectively. The stable model structure has weak equivalences the maps that induce isomorphisms on the stable homotopy groups of each fiber. The cofibrations are the same as in the level case; this is enough to determine the fibrations. The fibrant objects are levelwise $qf$-fibrant and are $\Omega$-spectra in the sense that the maps from one space to fiberwise loops of the next is a weak homotopy equivalence on the total space.

Notice that if $B = *$ then we get the category of coordinate-free orthogonal spectra from a previous section, with the same stable model structure, yielding the same homotopy category.

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