String Topology and the Based Loop Space

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- $LM = \mathrm{Map}(S^1, M)$

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Make $H_{*+d}(LM)$ a Batalin-Vilkovisky (BV) algebra:

- $\circ$ and Δ combine to produce a degree-1 Lie bracket on $H_{*+d}(LM)$, called the *loop bracket*.

Also an algebra over H_* of the framed little discs operad. (Getzler)

Hochschild Homology and Cohomology

The Hochschild homology and cohomology of an algebra A exhibit similar operations:

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Goal: relate these structures to string topology of M for certain DG algebras associated to M:

- C^*M, cochains of M
- $C_*\Omega M$, chains on the based loop space ΩM

Results

Theorem (M.)

Let M be a connected, k-oriented Poincaré duality space of formal dimension d. Then Poincaré duality induces an isomorphism

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Uses "derived" Poincaré duality (Klein, Dwyer-Greenlees-Iyengar):

- Generalize co/homology with local coefficients E to allow $C_*\Omega M$-module coefficients
- Cap product with $[M]$ still induces an isomorphism

$$H^*(M; E) \to H_{*+d}(M; E).$$

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When M is a manifold, the composite of D with the Goodwillie isomorphism $HH_*(C_*\Omega M) \to H_*(LM)$ takes this BV structure to that of string topology.

Resolves an outstanding conjecture about string topology and Hochschild cohomology.

Previous Results

Pre-String Topology

- $HH_*(C_*\Omega X) \cong H_*LX$, taking B to Δ (Goodwillie)
- $HH_*(C^*X) \cong H^*LX$, taking B to Δ, for X 1-conn (Jones)

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String Topology and C^*M

- Thom spectrum LM^{-TM} an algebra over the cactus operad (equivalent to the framed little discs operad) (Cohen-Jones)
- Cosimplicial model for LM^{-TM} shows $HH^*(C^*M) \cong H_{*+d}(LM)$ as rings, M 1-conn
- When char $k = 0$, $HH^*(C^*M)$ a BV algebra, isom to $H_{*+d}(LM)$, still need M 1-conn (Félix-Thomas)

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Koszul Duality

- C a 1-conn finite-type coalgebra, $HH^*(C^\vee) \cong HH^*(\mathrm{Cobar}(C))$, preserving the cup and bracket (Félix-Menichi-Thomas)
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Group Rings

G a discrete group, M an aspherical $K(G, 1)$ manifold.

- $H_{*+d}(G, kG^{\mathrm{conj}})$ is a ring, isomorphic to $H_{*+d}(LM)$ (Abbaspour-Cohen-Gruher)
- $HH^*(kG)$ a BV algebra, isomorphic to $H_{*+d}(LM)$ (Vaintrob)

In this case, $\Omega M \simeq G$ so our result generalizes these ones

Homological Algebra of $C_*\Omega M$

Models for Homological Algebra

Replace ΩM with an equivalent top group so $C_*\Omega M$ a DGA

- $C_*\Omega M$ a cofibrant chain complex, so category of modules has cofibrantly generated model structure
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Rothenberg-Steenrod Constructions

Connect these bar constructions over $C_*\Omega M$ to topological settings

- $C_*M \simeq B(k, C_*\Omega M, k)$
- $C_*(F \times_G EG) \simeq B(C_*F, C_*G, k)$ for G a top group

Derived Poincaré Duality

Co/homology with local coefficients: for E a $k[\pi_1 M]$-module,

$$H_*(M; E) \cong \mathrm{Tor}_*^{C_*\Omega M}(E, k), \quad H^*(M; E) \cong \mathrm{Ext}^*_{C_*\Omega M}(k, E)$$

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- View $[M] \in H_d M$ as a class in $\operatorname{Tor}_d^{C_*\Omega M}(k,k)$. Then

$$\mathrm{ev}_{[M]} : R\operatorname{Hom}_{C_*\Omega M}(k,E) \to E \otimes^L_{C_*\Omega M} k[d]$$

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- Algebraic Postnikov tower, compactness of k as a $C_*\Omega M$-module show a weak equivalence for all $C_*\Omega M$-modules E.

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- Need to insert SH-linear maps, though

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- Smash with Hk, pass to equivalent derived category $\mathrm{Ho}\, k$-Mod to recover chain-level equivalences

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- BV Lie bracket also agrees with Hochschild Lie bracket

D and Goodwillie isom take $\cup$ to loop product and $-D^{-1}BD$ to Δ

Future Directions

- Develop similar models for loop coproduct, string topology operations from fat graphs (Godin)
- Explore similar models using $C_*\Omega^n M$ for higher string topology on $H_*(\mathrm{Map}(S^n, M))$, relate to Hu's work on $HH^*_{(n)}(C^*M)$
- Connect this description of string topology to topological field theories via Cobordism Hypothesis

`http://math.stanford.edu/~emalm/`