Crystalline invariants of integer and fractional Chern insulators

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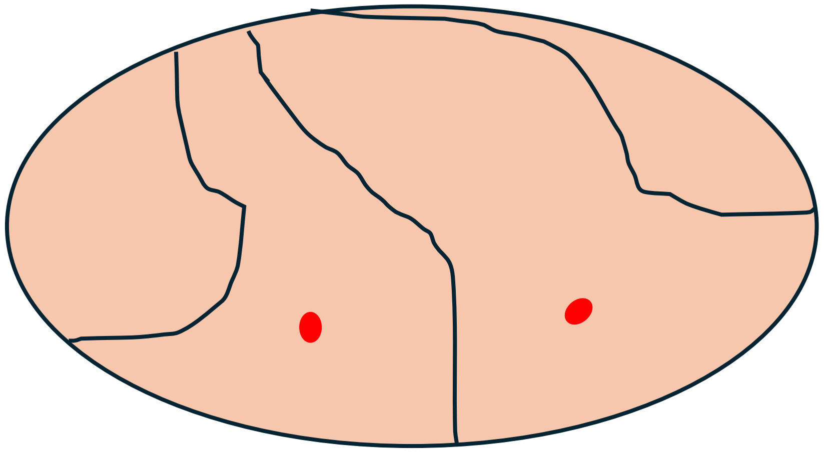
- The classification of topological phases is enriched by symmetry (states acquire quantized symmetry-protected topological invariants)

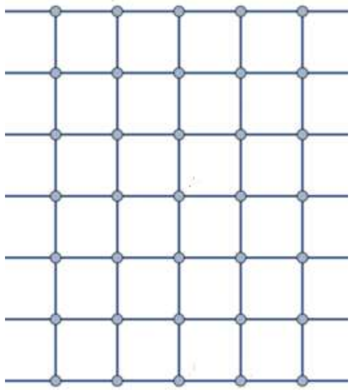
- How many different phases of matter exist for a given symmetry?

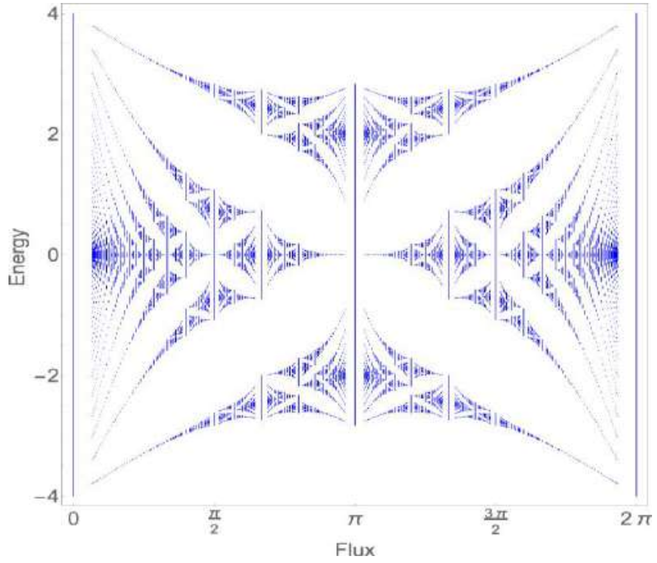


- How do we determine the phase of a particular numerical model or experimental system (i.e. how to actually measure all the topological invariants)?

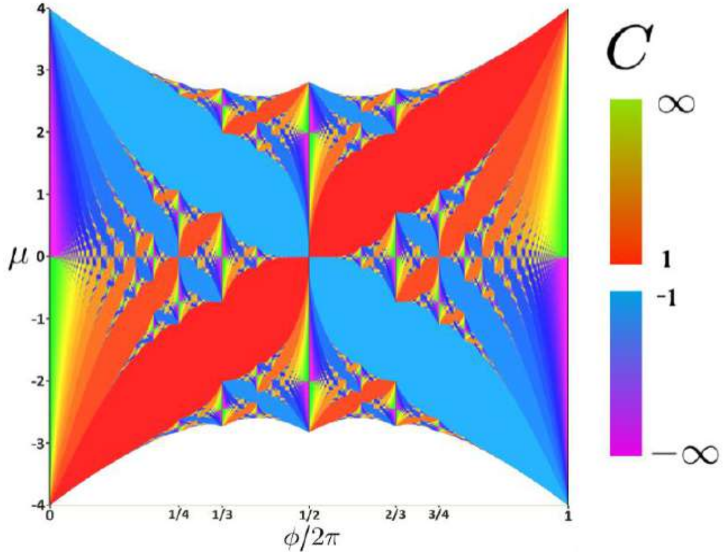
- We are interested in crystalline symmetries- full answer still not known!

$$H = -\mu \sum_i c_i^\dagger c_i + \sum_{<ij>} e^{-iA_{ij}} c_i^\dagger c_j + \text{h.c.}$$



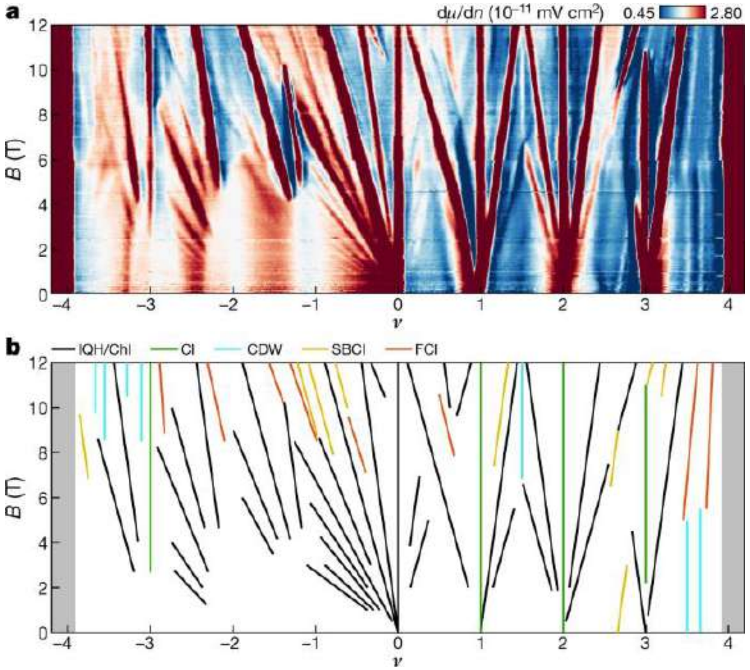


Hofstadter (1976); Harper (1955)



TKNN (1982); Osadchy, Avron (2001)

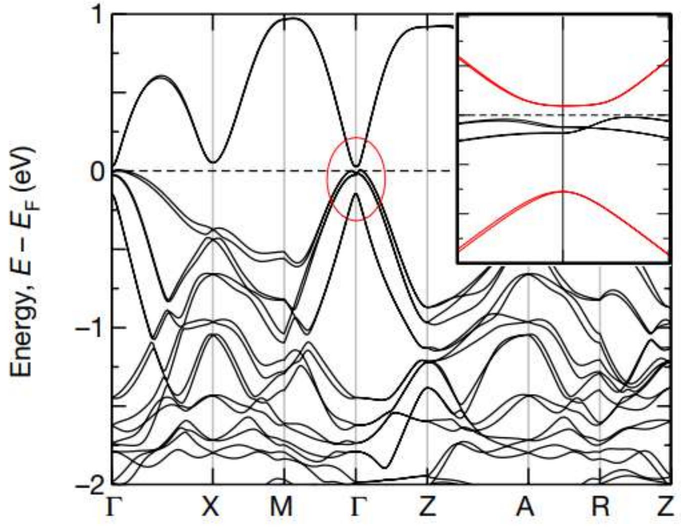
?



Xie et al, Nature 2021

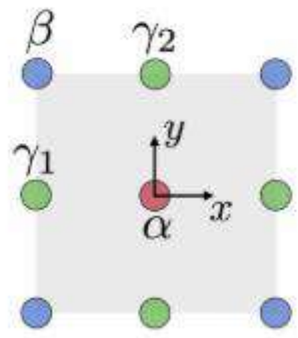
- In free fermion crystalline systems, the C&C problems have been answered to a large extent (classify representations of bands in momentum space)

Shiozaki-Sato, K-theory, 2014
Kruthoff et al, Band structure combinatorics, 2017
Watanabe, Po, Vishwanath, Symmetry indicators, 2017
Bradlyn, Cano, Bernevig et al, Topological quantum chemistry, 2020
Shiozaki-Ono, AHSS, 2023
...



- We are interested in interacting systems where band theory is no longer valid

Our focus: Chern insulators and fractional Chern insulators on square lattice



$$\mathrm{U}(1) \times_\phi G_{\mathrm{space}}, \quad G_{\mathrm{space}} = \mathrm{p4} = \mathbb{Z}^2 \rtimes \mathbb{Z}_4$$

But more generally:

- Topological crystalline insulator (+ interacting versions)
- 2d SPT/invertible states with above symmetry
- Quantum spin liquids
- Can generalize to the space groups p1, p2, p3, p6

- Two broad approaches for classification:

1. Construct exactly solvable ground state wave functions in real space

Huang, Song, Huang, Hermele, PRB 2017
Song, Fang, Qi, Nature 2020
Zhang, Qi, Gu, PRR 2022
...

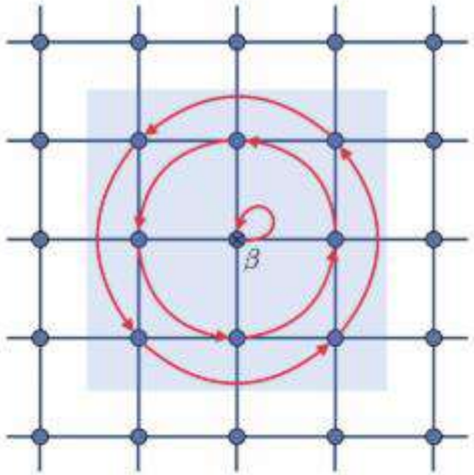
2. Topological quantum field theory (TQFT)

$$\mathcal{L} = \frac{C}{4\pi} A \wedge dA + \ldots$$

- These approaches have addressed the classification problem for our symmetry choice

(today: what do the predicted invariants mean and how do we measure them?)

Take home message



There are many new crystalline invariants in both ordinary and fractional Chern insulators

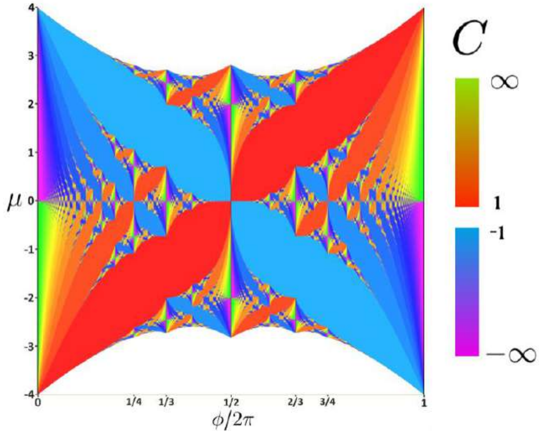
Can be fully characterized by partial rotations (+TQFT/CFT/real space ideas)

$$\langle\Psi|\, \hat{C}_{4,\text{o}}|_D \,|\Psi\rangle = e^{-\gamma|\partial D| + i\frac{\pi}{2}K_\text{o}}$$

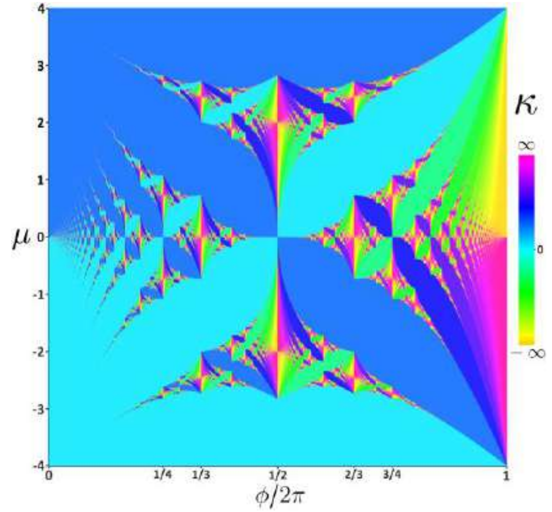
Leading term with universal part
+ subleading terms

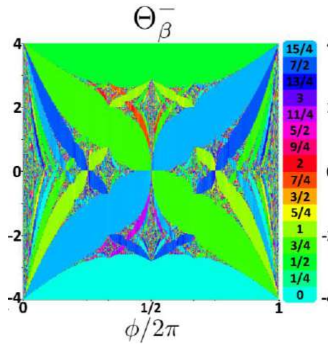
Nontrivial invariants should exist in current experimental setups, e.g. zero-field FCI at fractional filling

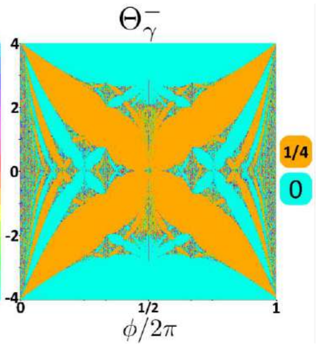
Results for Chern insulators (Hofstadter model)

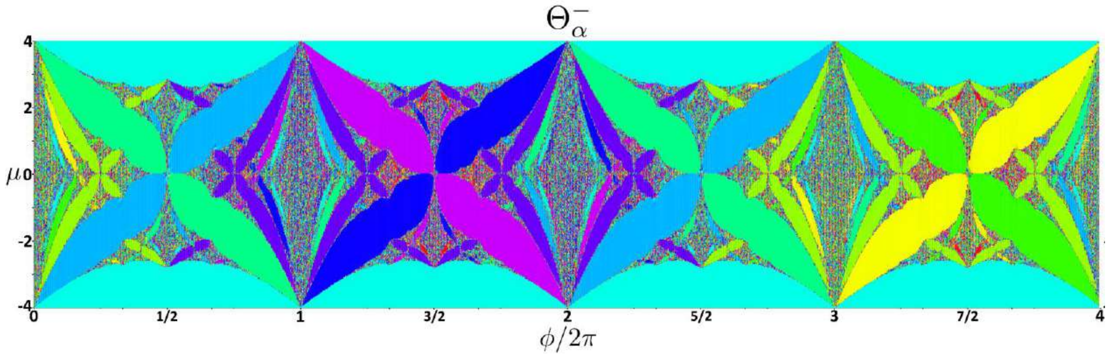


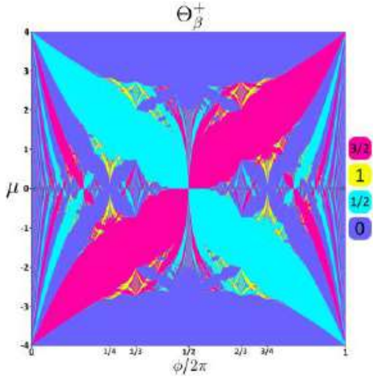
$$\mathbb{Z}^3 \times \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$$





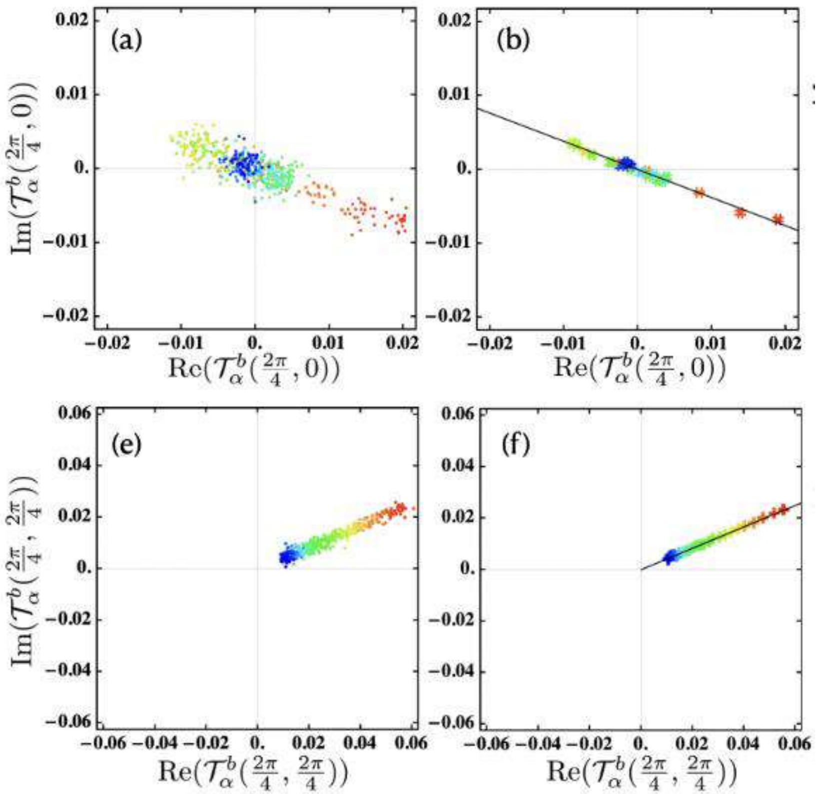






Results for fractional Chern insulators

- Hofstadter ground states → projected parton wave functions for ½ Laughlin topological order
- Predict crystalline invariants + partial rotation results using TQFT/CFT
- Verify partial rotation prediction numerically (Monte Carlo)
- We can extract the full set of crystalline invariants in good agreement with theory



- Part 1: Complete characterization of Chern insulators through partial rotations
  - Real space/TQFT classification
  - Partial rotation prediction from CFT
  - Numerics (Hofstadter model)

- Part 2: Fractional Chern insulator (1/2 Laughlin state)
  - Formal theory (Chern-Simons theory + real-space)
  - Parton construction of ½ Laughlin state
  - Partial rotation prediction
  - Numerical details

Part 1

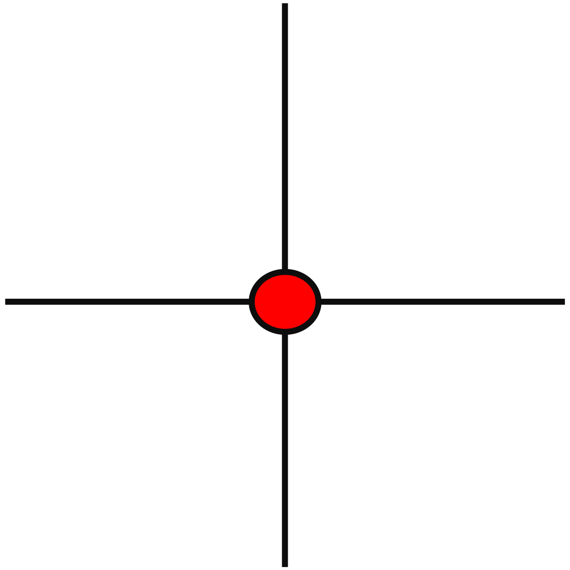
Zhang, NM, Kobayashi, Barkeshli, PRL (2023)

Yuxuan Zhang

Ryohei Kobayashi

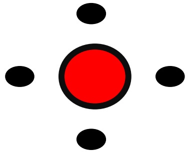
Maissam Barkeshli

- Consider (2+1)D invertible states (unique ground state on torus, no anyons), $G = \mathrm{U}(1)^f \times C_4$
- Single rotation center: assume orbitals can be exponentially localized at center

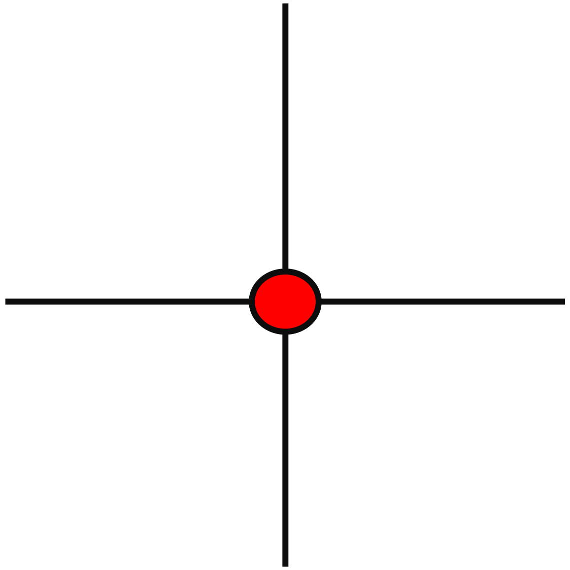


(n_o, m_o)

$n_o \in \mathbb{Z}, \quad m_o \in \mathbb{Z}_4$



$\delta n_o = 4; \delta m_o = 2 \mod 4$

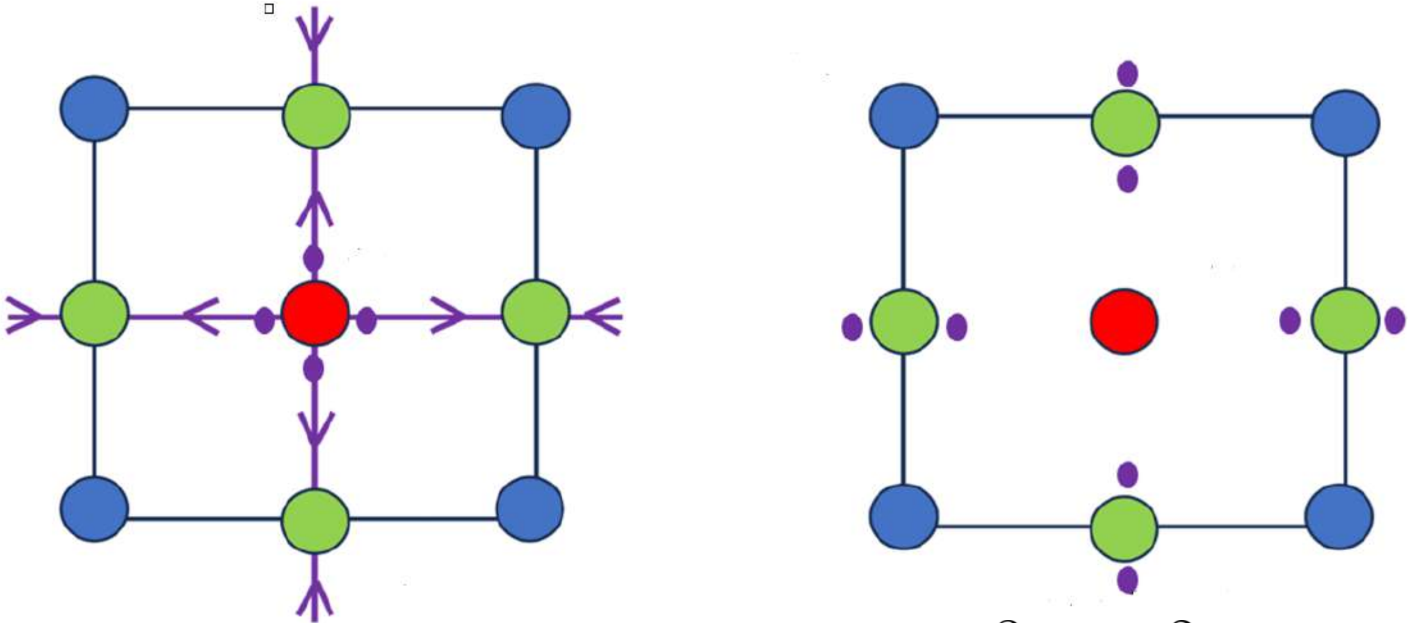


$(n_o + 4, m_o + 2)$

$$(n_o + 2m_o \mod 8, m_o \mod 2) \in \mathbb{Z}_8 \times \mathbb{Z}_2$$

Multiple rotation centers: need to assume that all orbitals can be exponentially localized near high-symmetry points

Zhang, Qi, Gu, PRR 2022



$$(n_\alpha, n_\beta, n_\gamma | m_\alpha, m_\beta, m_\gamma) \in \mathbb{Z}^3 \times \mathbb{Z}_4^2 \times \mathbb{Z}_2$$

$$\simeq (n_\alpha - 4, n_\beta, n_\gamma + 2 | m_\alpha + 2, m_\beta, m_\gamma + 1) \quad (\alpha \to \gamma)$$
$$\simeq (n_\alpha, n_\beta - 4, n_\gamma + 2 | m_\alpha, m_\beta + 2, m_\gamma + 1) \quad (\beta \to \gamma)$$

$$(n_\alpha + n_\beta + 2n_\gamma, n_\alpha + 2m_\alpha, m_\alpha, n_\beta, m_\alpha + m_\beta + 2m_\gamma) \in \mathbb{Z} \times \mathbb{Z}_8 \times \mathbb{Z}_2 \times \mathbb{Z}_4^2$$

$$\mathbb{Z}^3 \times \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$$

This construction fully captures the rotation invariants but misses two integer invariants

1. Chern number C (electrical Hall conductance)

2. Chiral central charge c_- ($= C$ for free fermions but not necessarily so in general)
Sets thermal Hall conductance

3. For general flux, the filling relation is modified: $\nu = C\frac{\phi}{2\pi} + \kappa, \quad \kappa \in \mathbb{Z}$

Limitation: Construction applies to ideal wave functions; does not suggest order parameters/experimental measures

To resolve this, appeal to TQFT!

Manjunath, Calvera, Barkeshli, PRB 2024

TQFT approach (internal symmetries):

e.g. $U(1)^f$ charge conservation: define background gauge field A

Flux of $A \sim \mathrm{d}A$ (magnetic flux)

$$\mathcal{L} = \frac{C}{4\pi} A \wedge dA$$

In general: define a background G gauge field on a triangulated manifold

Find all possible 'topological' effective actions
(partition function on closed manifold is invariant under retriangulations)

Dijkgraaf, Witten (1990)
Chen, Liu, Gu, Wen, PRB (2013)

Bosonic SPTs -> Group cohomology classes
Invertible fermionic states: more complicated
(Cobordism/ G-crossed BTC theory)

Barkeshli, Bonderson, Cheng, Wang, PRB (2019)
Kapustin, Thorngren, Turzillo, Wang (2015)
Barkeshli, Chen, Hsin, NM, PRB (2022)
Aasen, Bonderson, Knapp (2021)

Extension to crystalline symmetries (Crystalline equivalence principle):

Classification/response for spatial symmetry G

$\updownarrow$

Classification/response for effective internal symmetry G^{eff}

Thorngren, Else, PRX (2018)
Debray (2021)
NM, Calvera, Barkeshli, PRB (2022)
...

For bosons $G^{eff} \cong G$

For fermions: symmetry G_f has a subgroup of fermion parity, $\mathbb{Z}_2^f$; define $G_b = G_f/\mathbb{Z}_2^f$

Operators which act trivially on bosons can act in two ways on fermions (group extension)

$$\hat{C}_4^4 = +1 \text{ or } (-1)^F$$

$G_b^{eff} \cong G_b$ but $\hat{C}_4^4 = +1 \rightarrow \mathbf{g}^4 = (-1)^F$ and vice versa

This is a conjecture, no full proof but many explicit checks (match to real-space constructions)

General effective action:

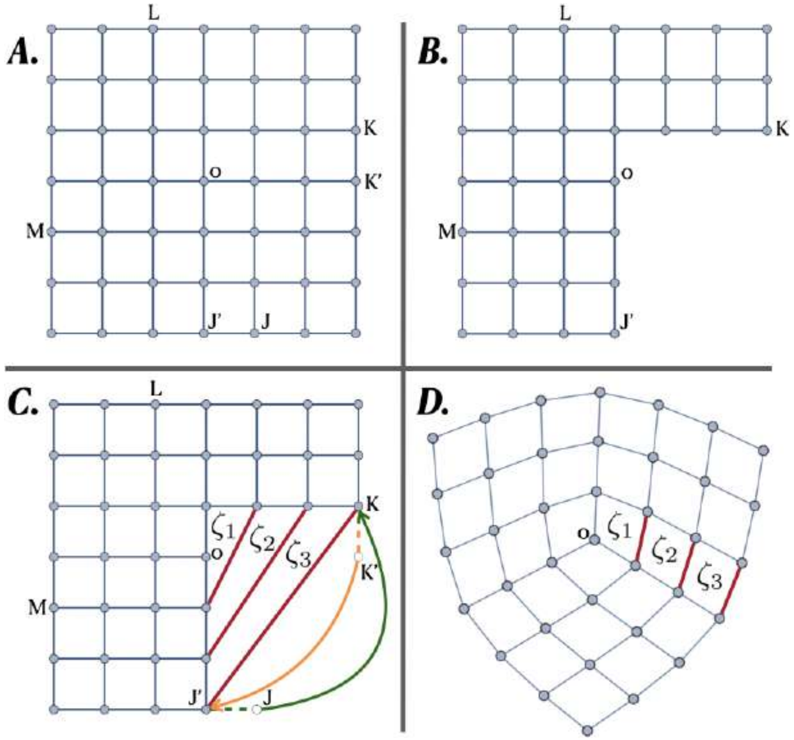
$$\mathcal{L} = \frac{C}{4\pi} A \wedge dA + \frac{\mathcal{S}_o}{2\pi} A \wedge d\omega + \frac{\ell_o - c_-/12}{4\pi} \omega \wedge d\omega$$

- If we interpret ω as a rotation gauge field, $d\omega = \pi/2$ represents a $\pi/2$ disclination

- Relation to real-space data:

$$\mathcal{S}_o = n_o, l_o = 2m_o$$

- The second term describes quantized charge bound to disclinations

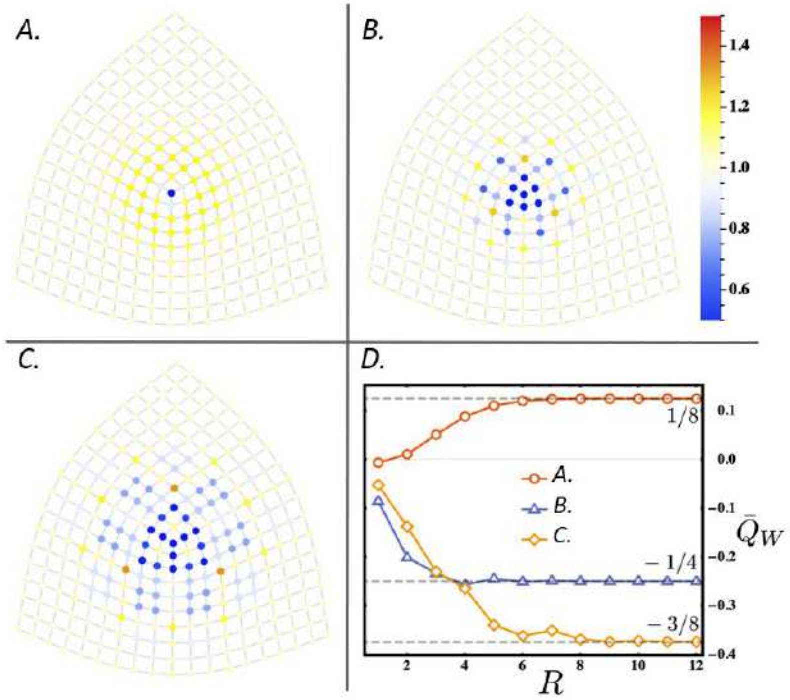


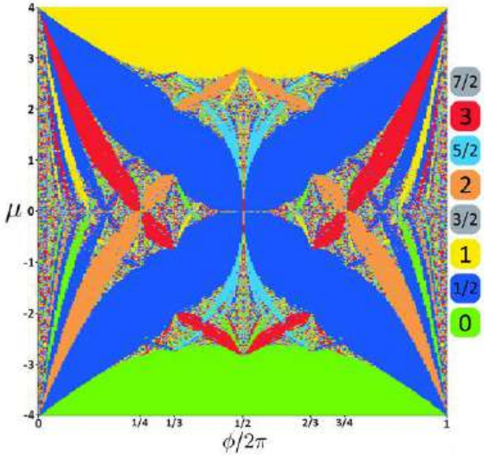
Brief digression: Quantized charge at lattice defects

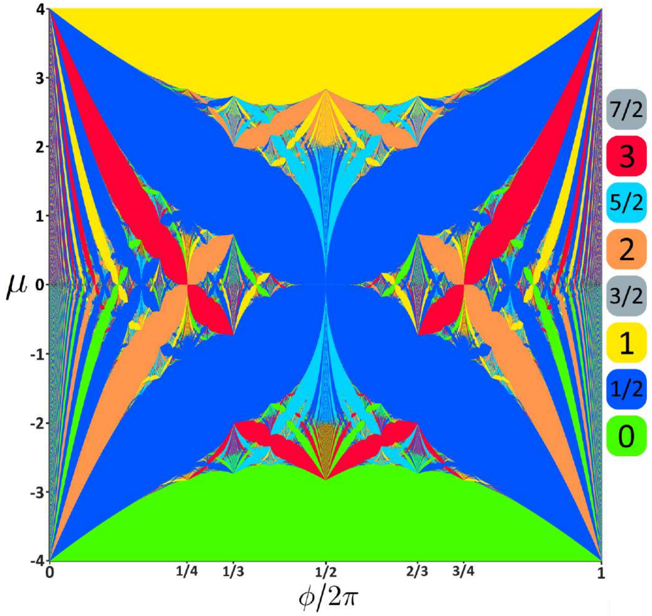
$$Q_W = \int_W \frac{\delta \mathcal{L}}{\delta A_0} = C\left(\frac{\phi}{2\pi} N_{W,\mathrm{o}} + \frac{\delta \Phi_{W,\mathrm{o}}}{2\pi}\right) + \mathscr{S}_\mathrm{o} \frac{\Omega_W}{2\pi} + \kappa N_{W,\mathrm{o}}$$

$$= \cancel{C \frac{\delta \Phi_W}{2\pi}} + \mathscr{S}_\mathrm{o} \frac{\Omega_W}{2\pi} + \nu N_{W,\mathrm{o}}$$

$$= \frac{\mathscr{S}_\mathrm{o}}{4} + \nu N_{W,\mathrm{o}}$$



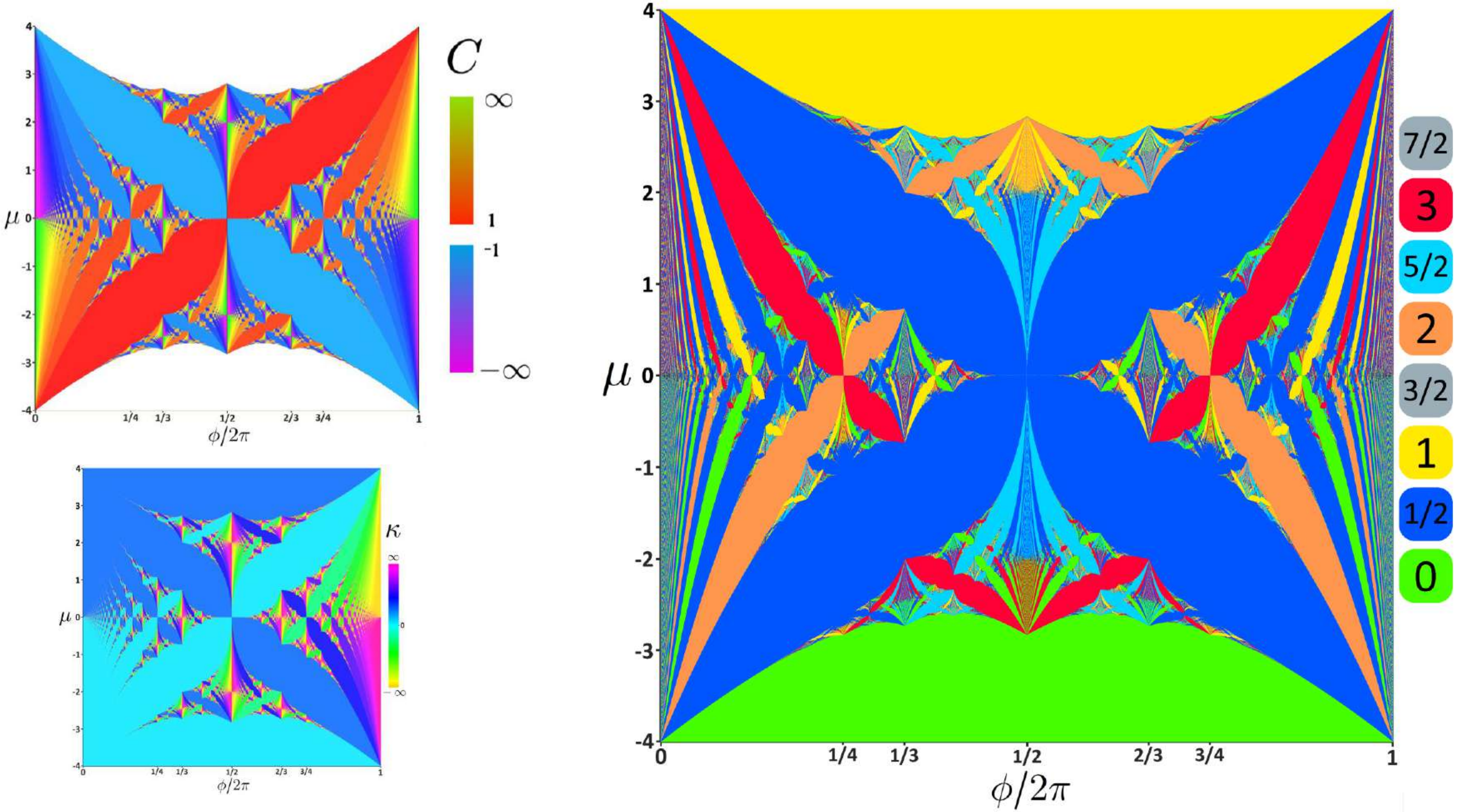




Consistency check: Landau level limit

Zhang, NM, Nambiar, Barkeshli, PRL (2022)

States with fixed C, κ can have different invariants



$$\mathbb{Z}^3 \times \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$$

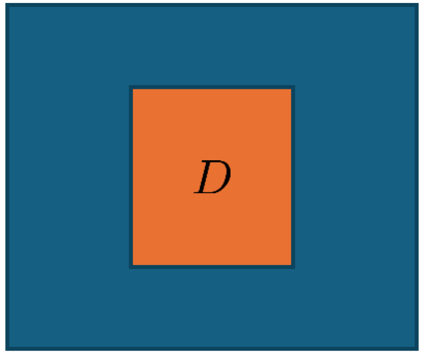
- Measuring shift at different points gives at most $\mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$

(we have missed the l_o invariant)

$$\mathcal{L} = \frac{C}{4\pi} A \wedge dA + \frac{\mathscr{S}_o}{2\pi} A \wedge d\omega + \frac{\ell_o - c_-/12}{4\pi} \omega \wedge d\omega$$

- For complete characterization, need a different approach!

- Real-space construction suggests that we should measure a ‘localized’ angular momentum; can we do a localized, ‘partial’ rotation?



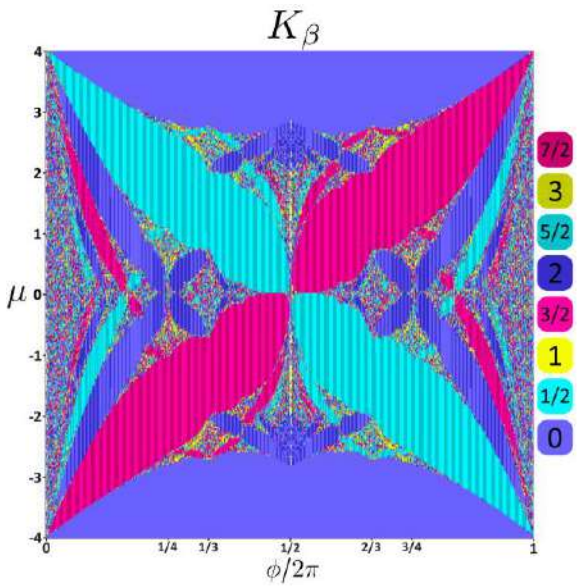
Shiozaki, Shapourian, Ryu, PRB (2017)
Zhang, NM, Kobayashi, Barkeshli (2023)

$$\tilde{C}^{+}_{M_o}, \quad (\tilde{C}^{+}_{M_o})^{M_o} = +1$$

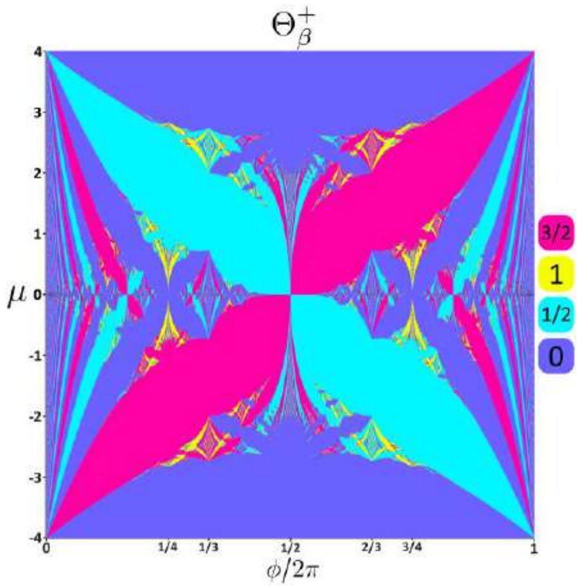
$$\langle\Psi| \tilde{C}_{M_{\alpha(\beta)}}|_D |\Psi\rangle = e^{-\gamma_D + i\frac{\pi}{2} l^{+}_{D,\alpha(\beta)}}$$

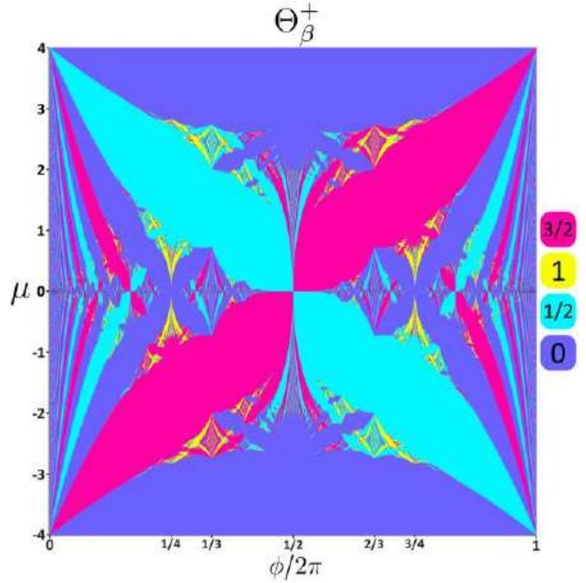
$$l^{+}_{D,\alpha(\beta)} = K^{+}_{\alpha(\beta)} \mod 4$$

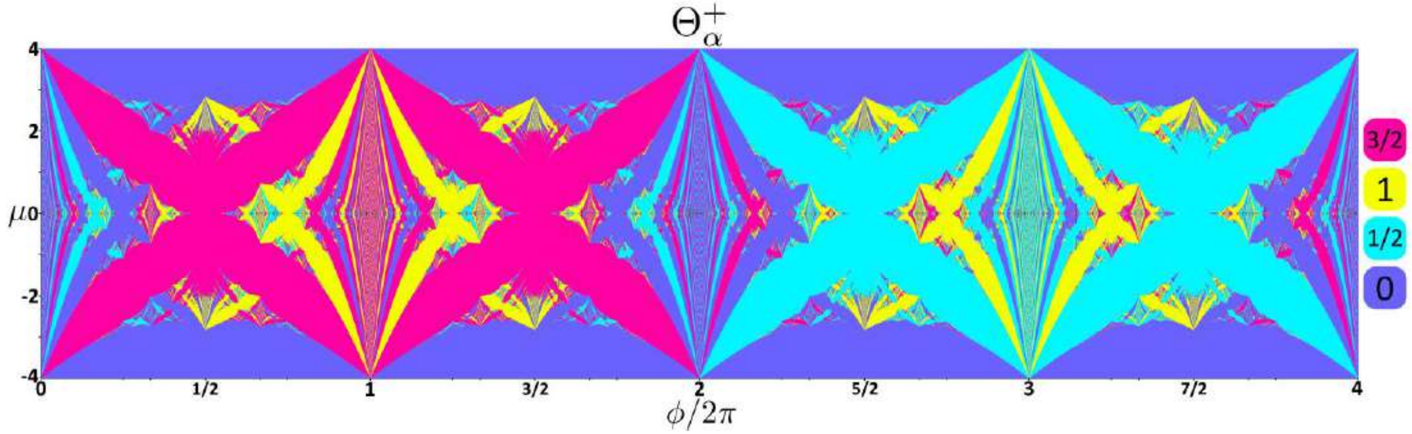
$$(K^{+}_{\gamma} = 0)$$



$$\Theta^{+}_{\alpha(\beta)} := K^{+}_{\alpha(\beta)} \mod 2$$



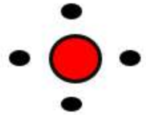




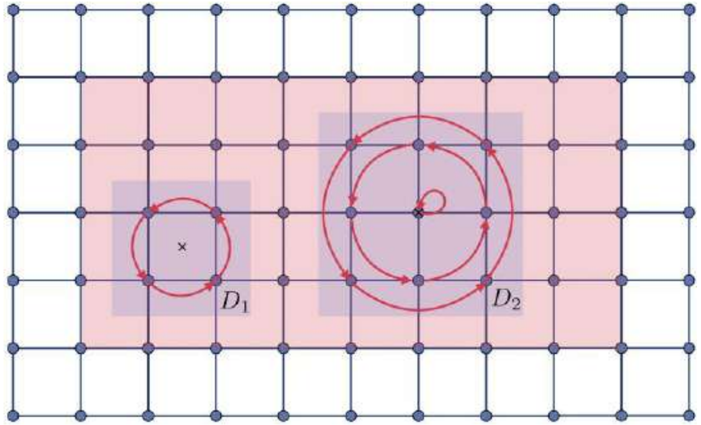
$$\Theta_{\rm o}^{+} = k_{3,{\rm o}}^{+} - \frac{C}{2} \mod 2, \quad {\rm o} = \alpha, \beta$$

After fixing C: Θ_o^+ gives a $\mathbb{Z}_2$ invariant for o= α, β

Why $\mathbb{Z}_2$ and not $\mathbb{Z}_4$?



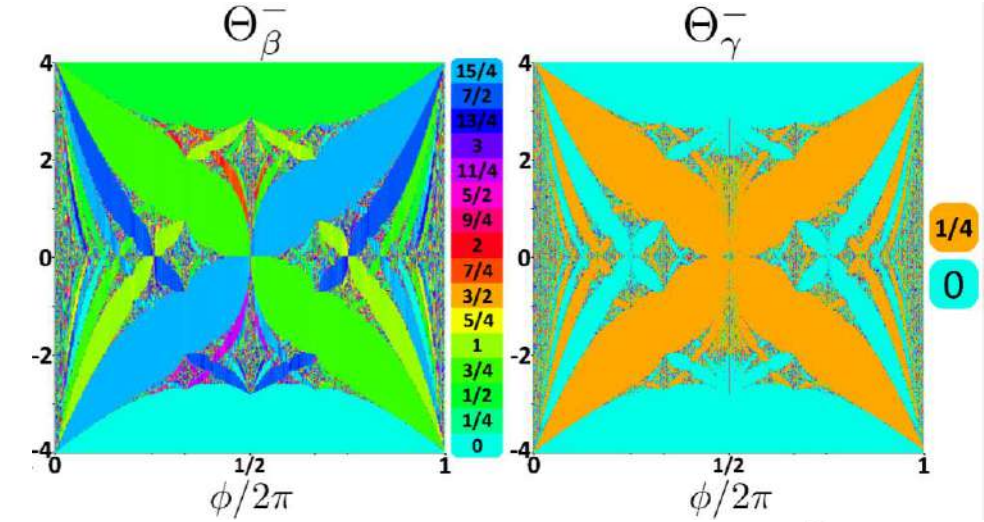
$$\delta n_{\rm o} = 4; \delta m_{\rm o} = 2 \mod 4$$



$$\tilde{C}^{-}_{M_{\mathrm{o}}} := e^{i\frac{\pi}{M_{\mathrm{o}}}\hat{N}}\tilde{C}^{+}_{M_{\mathrm{o}}}, \quad (\tilde{C}^{-}_{M_{\mathrm{o}}})^{M_{\mathrm{o}}} = (-1)^F$$

$$\langle\Psi|\,\tilde{C}^{-}_{M_{\mathrm{o}}}|_D\,|\Psi\rangle = e^{-\gamma_D + i\frac{2\pi}{M_{\mathrm{o}}}l^{-}_{D,\mathrm{o}}}$$

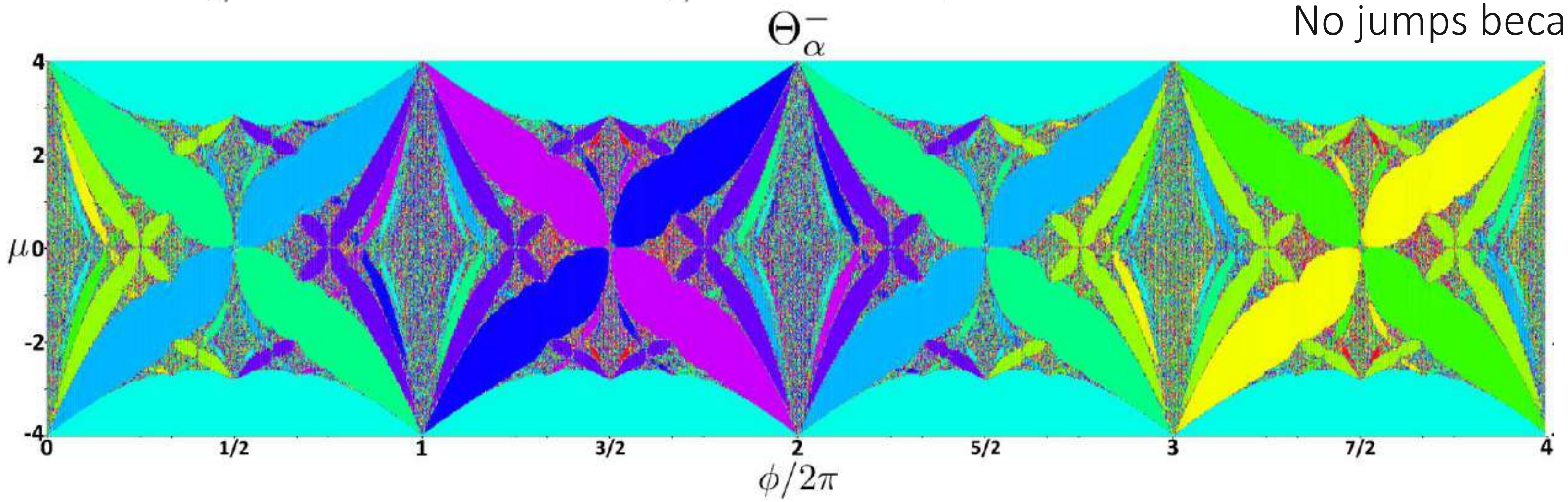
$$l^{-}_{D,\mathrm{o}} = \Theta^{-}_{\mathrm{o}} \mod M_{\mathrm{o}}$$



$$\Theta^{-}_{\mathrm{o}} = \begin{cases} k^{-}_{3,\mathrm{o}} + \frac{k_{s,\mathrm{o}}}{2} - \frac{3C}{4} \mod 4, & \mathrm{o} = \alpha, \beta \\ k^{-}_{3,\mathrm{o}} + \frac{k_{s,\mathrm{o}}}{2} - \frac{C}{4} \mod 2, & \mathrm{o} = \gamma, \end{cases}$$

After fixing C:
Θ^{-}_{o} gives a $\mathbb{Z}_8, \mathbb{Z}_8, \mathbb{Z}_4$ invariant for o$= \alpha, \beta, \gamma$

No jumps because extra orbit contributes l=0



Li, Haldane, PRL (2008)
Qi, Katsura, Ludwig, PRL (2012)
Shiozaki, Shapourian, Ryu, PRB (2017)
Zhang, NM, Kobayashi, Barkeshli (2023)

CFT calculation: $\rho_D = \rho_{CFT}$

$$\langle\Psi|\, \tilde{C}^{\pm}_{M_o}|_D\, |\Psi\rangle = \frac{\mathrm{Tr}[e^{iQ_{M_o}\frac{\pi}{M_o}} e^{i\tilde{P}\frac{L}{M_o}} e^{-\frac{\xi}{v}H}]}{\mathrm{Tr}[e^{-\frac{\xi}{v}H}]}$$

$$= e^{-\frac{2\pi i}{24 M_o}c_-} \frac{\sum_{a=1,\psi}\chi_a(\frac{i\xi}{L} - \frac{1}{M_o}; [\mathrm{AP},0],[\mathrm{AP},1])}{\sum_{a=1,\psi}\chi_a(\frac{i\xi}{L}; [\mathrm{AP},0],[\mathrm{AP},0])}$$

$$\tilde{P} := \frac{1}{v}(H - E_0) = \frac{2\pi}{L}\left[L_0 - \frac{c_-}{24} - \langle L_0 - \frac{c_-}{24}\rangle\right]$$

$$\chi_a(\tau;[\mathrm{AP},j],[\mathrm{AP},j']) = \mathrm{Tr}_{a,[\mathrm{AP},j]}[e^{iQ_M\frac{j'\pi}{M}} e^{2\pi i\tau(L_0 - \frac{c_-}{24})}]$$

$$\sum_{n\geq 0} e^{2\pi i\tau(h_a + n - \frac{c_-}{24})} \times \ldots$$

$$\langle\Psi|\, \tilde{C}^{+}_{M_o}|_D\, |\Psi\rangle_{\mathrm{CFT}} \propto e^{-\frac{2\pi i}{24}(M_o - \frac{1}{M_o})c_-}\mathcal{I}^{+}_{M_o} = e^{\frac{2\pi i}{M_o}\Theta_o^+}$$

$$\langle\Psi|\, \tilde{C}^{-}_{M_o}|_D\, |\Psi\rangle_{\mathrm{CFT}} \propto e^{-\frac{2\pi i}{24}(M_o + \frac{2}{M_o})c_-}\mathcal{I}^{-}_{M_o} = e^{\frac{2\pi i}{M_o}\Theta_o^-}$$

$$\chi_a\left(\frac{i\xi}{L} - \frac{1}{M}; [\mathrm{AP},0],[\mathrm{AP},1]\right)$$

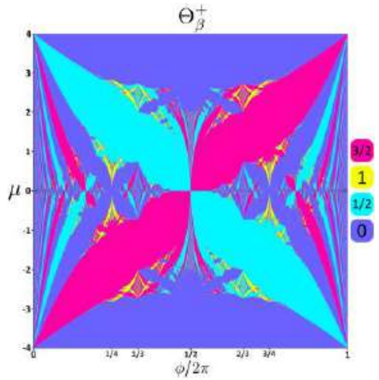
$$= \sum_{b\in\mathcal{C}_1}(ST^M)_{ab}\chi_b\left(\frac{-iM\frac{\xi}{L}}{\frac{i\xi}{L}+\frac{1}{M}}; [\mathrm{AP},1],[\mathrm{AP},0]\right)$$

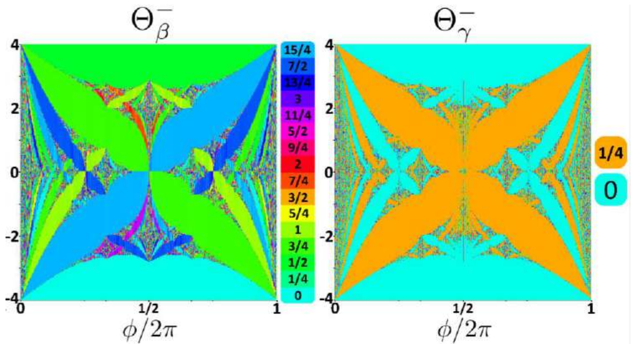
$$\chi_a\left(\frac{i\xi}{L}; [\mathrm{AP},0],[\mathrm{AP},0]\right) = \sum_{b\in\mathcal{C}_0} S_{ab}\chi_b\left(\frac{iL}{\xi}; [\mathrm{AP},0],[\mathrm{AP},0]\right)$$

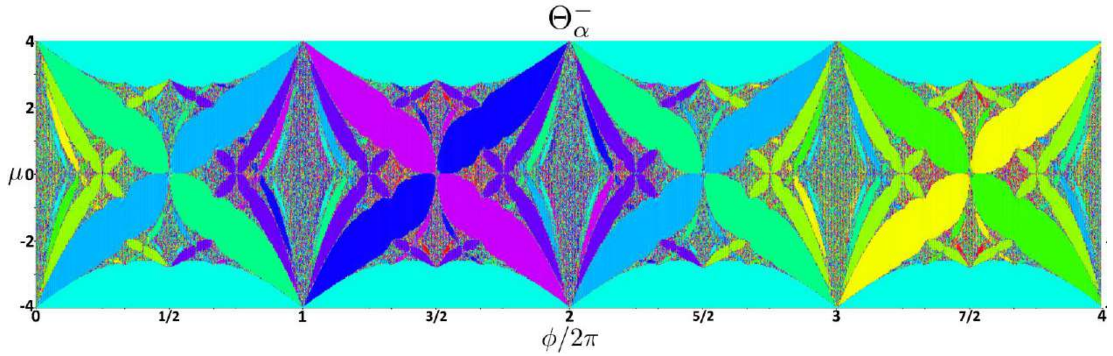
S,T matrices of CFT are directly related to the field theory!

$$\mathcal{I}^{\pm}_{M_o} := e^{\frac{2\pi i}{M_o}\frac{\ell^{\pm}_{s,o}}{2}} \qquad \mathcal{L} = \frac{C}{4\pi}A\wedge dA + \frac{\mathscr{S}_o}{2\pi}A\wedge d\omega + \frac{\ell_o - c_-/12}{4\pi}\omega\wedge d\omega$$

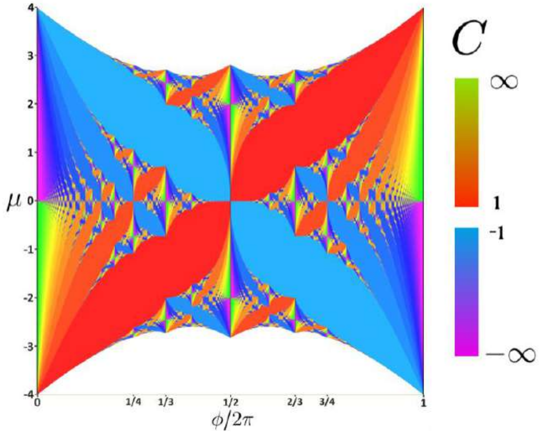
We know the quantization of field theory coefficients; can prove the empirical formulas for $\Theta_o^{\pm}$



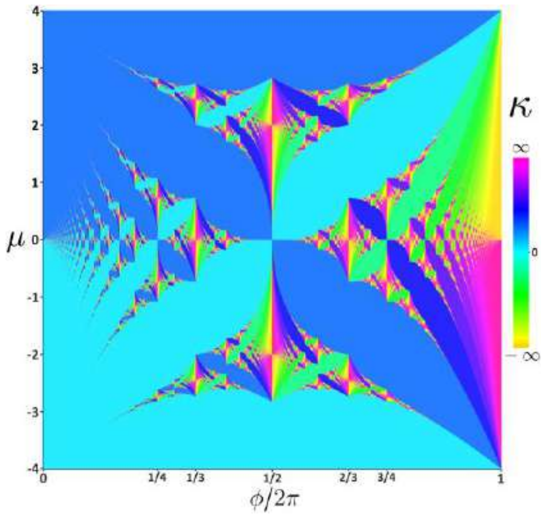




$$\mathbb{Z}^3 \times \mathbb{Z}_8 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \times \mathbb{Z}_2$$



From a single wave function
No defects necessary



Part 2

Zhang, NM, Kobayashi, Barkeshli, 2405.17431 (2024)

Ryohei Kobayashi

Yuxuan Zhang

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If a quantized invariant exists for Chern insulators, a 'fractional' version should exist for FCI!
Is this true for the partial rotation invariant?

We work with the simplest topological order: $U(1)_2$ (anyons: 1, a = semion; a x a = 1)

Ordinary CI case:

$$\mathcal{L} = \frac{C}{4\pi} A \wedge dA + \frac{\mathscr{S}_o}{2\pi} A \wedge d\omega + \frac{\ell_o - c_-/12}{4\pi} \omega \wedge d\omega$$

FCI case:

$$\mathcal{L} = -\frac{2}{4\pi} ada + \frac{v}{2\pi} Ada + \frac{s}{2\pi} \omega da + \frac{k_1}{2\pi} AdA + \frac{k_2}{4\pi} \omega dA + \frac{k_3}{2\pi} \omega d\omega$$

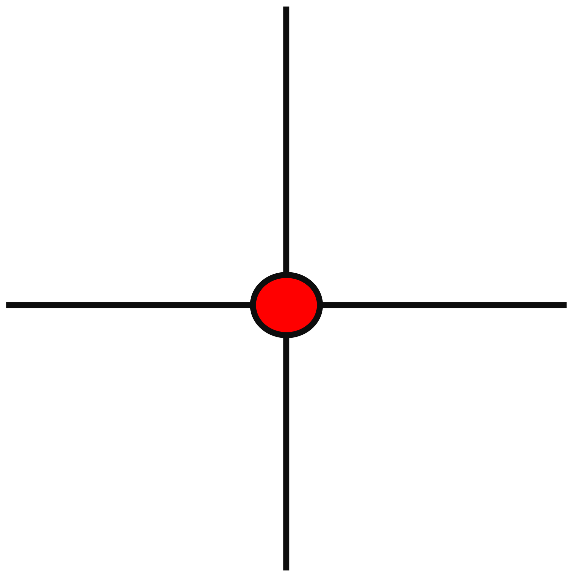
Fractional charge/angular momentum of semion:

$$Q_a = v/2 \mod 1; v \in \mathbb{Z}_2$$
$$L_a = s/2 \mod 1; s \in \mathbb{Z}_2$$

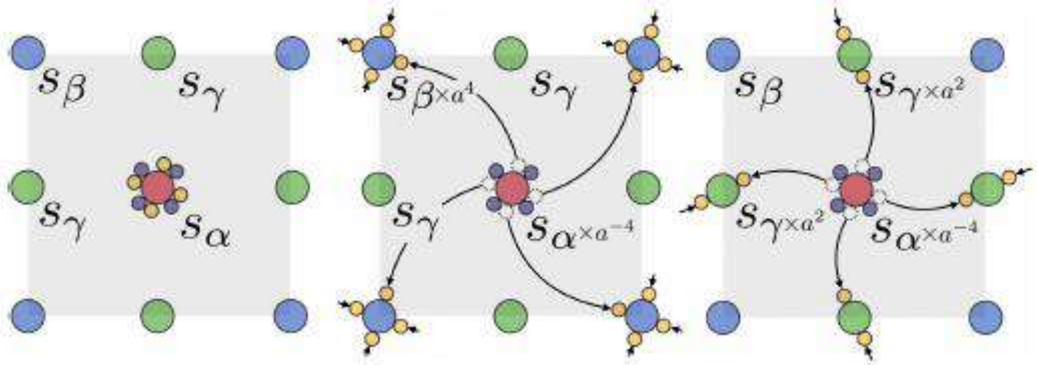
Real space construction:

1. For symmetry fractionalization: start with a state having some fixed ν but trivial crystalline sym frac
Then decorate high-symmetry points with anyons to get a new state

2. For the SPT indices, decorate integer charges at high symmetry points as before



$$s_o \in \mathbb{Z}_2; s_o \to s_o \times a^{-4}$$



$$(s_\alpha, s_\beta, s_\gamma) \in \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$$

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Parton wave functions for ½ state:

1. Decompose boson into fermionic partons, $b = f_1 f_2$
2. Each parton flavor is in a CI ground state, e.g. from Hofstadter model
3. Projection:

$$\Psi_b(\{\vec{r}_i\}) = \psi_1(\{\vec{r}_i\})\psi_2(\{\vec{r}_i\})$$

Determining the invariants of the parton ground state:

$$\mathcal{L} = \frac{C}{4\pi} A \wedge dA + \frac{\mathcal{S}_o}{2\pi} A \wedge d\omega + \frac{\ell_o - c_-/12}{4\pi} \omega \wedge d\omega$$

→

$$\mathcal{L} = -\frac{2}{4\pi} ada + \frac{v}{2\pi} Ada + \frac{s}{2\pi} \omega da + \frac{k_1}{2\pi} AdA + \frac{k_2}{4\pi} \omega dA + \frac{k_3}{2\pi} \omega d\omega$$

$$v = 1, \quad s = \mathcal{S}_1 + \mathcal{S}_2, \quad k_1 = k_2 = 0$$
$$k_3 = \ell_{s,1} + \ell_{s,2} - \mathcal{S}_1^2 - \mathcal{S}_2^2.$$

Derivation:

1. Each parton state has its own K-matrix theory $a^I_{(1)}, a^I_{(2)}$
2. Parton decomposition has U(1) redundancy with gauge field α
3. Integrate out α + other dynamical gauge fields

$$K = 1 \oplus \sigma_x; \quad v_f = (1, 0, 0); s_f = (\mathcal{S}, l_s - \mathcal{S}^2, 1)$$

$$\mathcal{L} = \frac{1}{4\pi} K_{IJ} a^I_{(i)} da^J_{(i)} - \frac{s^I_{f,i}}{2\pi} \omega da^I_{(i)} - \frac{v^I_{f,1}}{2\pi} Ada^{(I)}_{(1)} + \frac{1}{2\pi} \alpha d(v^I_{f,1} a^I_{(1)} - v^I_{f,2} a^I_{(2)}).$$

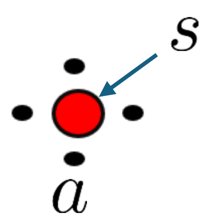
Predictions for partial rotations about α, β from CFT (to leading order): $\hat{C}_{4,l} \equiv e^{i\frac{2\pi l}{4}\hat{N}}\hat{C}_4$

$$\mathcal{T}^b\left(\frac{2\pi}{4};\frac{2\pi l}{4}\right) := \langle\Psi|\hat{C}_{4,l}|_D|\Psi\rangle \approx e^{-\gamma|\partial D|}e^{i\frac{2\pi}{4}K_l}$$

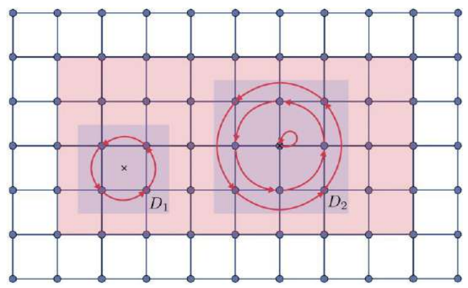
$$K_l = -\frac{3}{4} + K_l^{\text{frac}} + K_l^{\text{SPT}} + A_l,$$

$$K_l^{\text{frac}} = \frac{(s+lv)^2}{4}; \quad K_l^{\text{SPT}} = l^2k_1 + lk_2 + k_3 \mod 4; \quad A_l = \frac{\delta(lv+s+1)}{4}$$

$$\Theta_l := \begin{cases} K_l \mod 2, & \text{if } s+vl=1 \mod 2 \\ K_l \mod 4, & \text{if } s+vl=0 \mod 2 \end{cases}$$



$$e^{\frac{i\pi}{4}sa\times 4} = (-1)^{sa}$$



Note:

1. Quantization (mod 2 v/s mod 4) already gives s, v
2. Formulas greatly generalize (bosonic + fermionic TO, other rotation symmetries)

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$L_x \times L_y = 60 \times 60$ torus

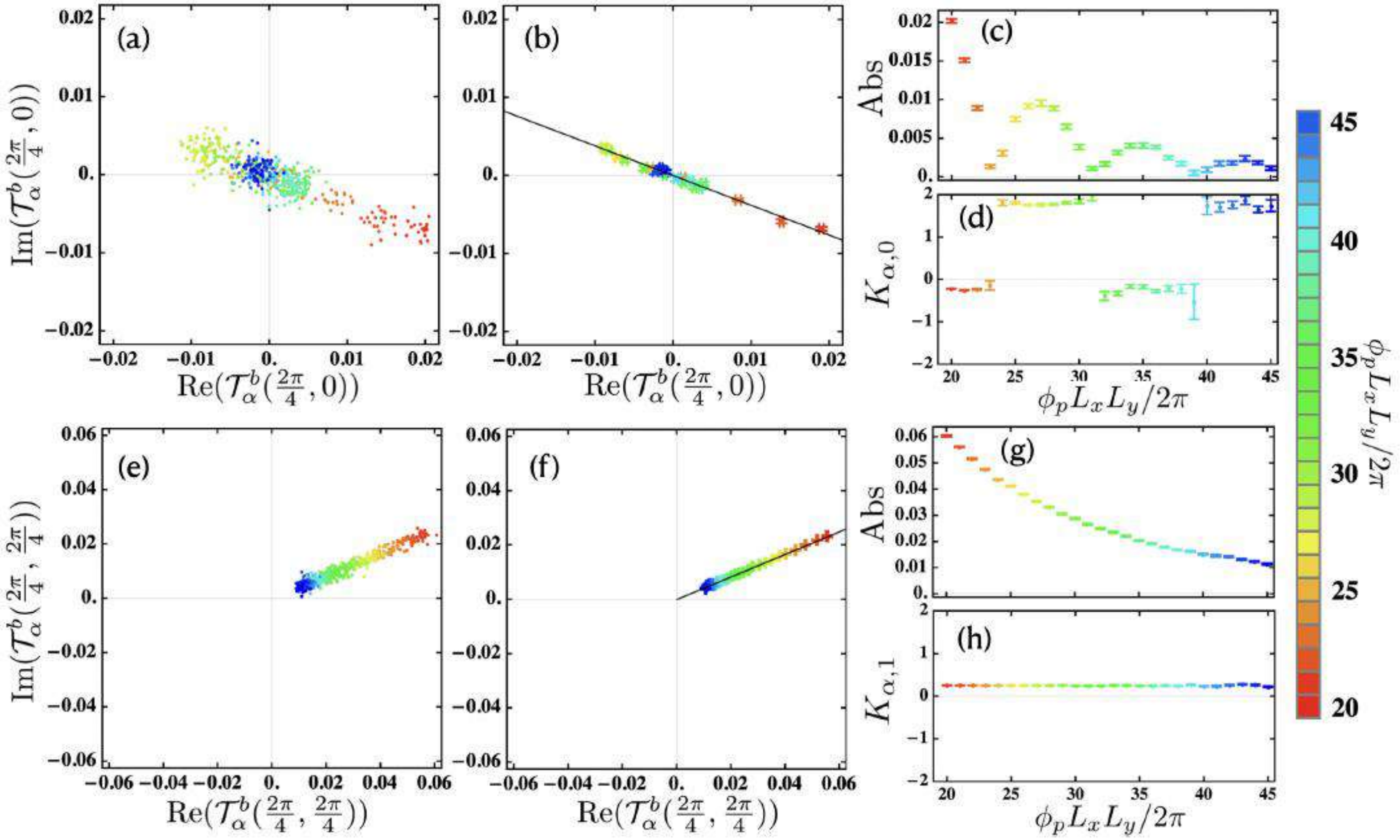
disk D of size 30×30

$o = \alpha$ (plaquette center)

$\{C_1, \kappa_1, \mathcal{S}_{\alpha,1}, \ell_{s,\alpha,1}\} = \{C_2, \kappa_2, \mathcal{S}_{\alpha,2}, \ell_{s,\alpha,2}\}$
$= \{1, 0, 1/2, 1/4\}$.

a-d) l=0
5×10^6 Metropolis steps = 1 point
Color = flux value
b) Average over 20 batches
π phase ambiguity for different flux
implies $s = 1$ (correct)

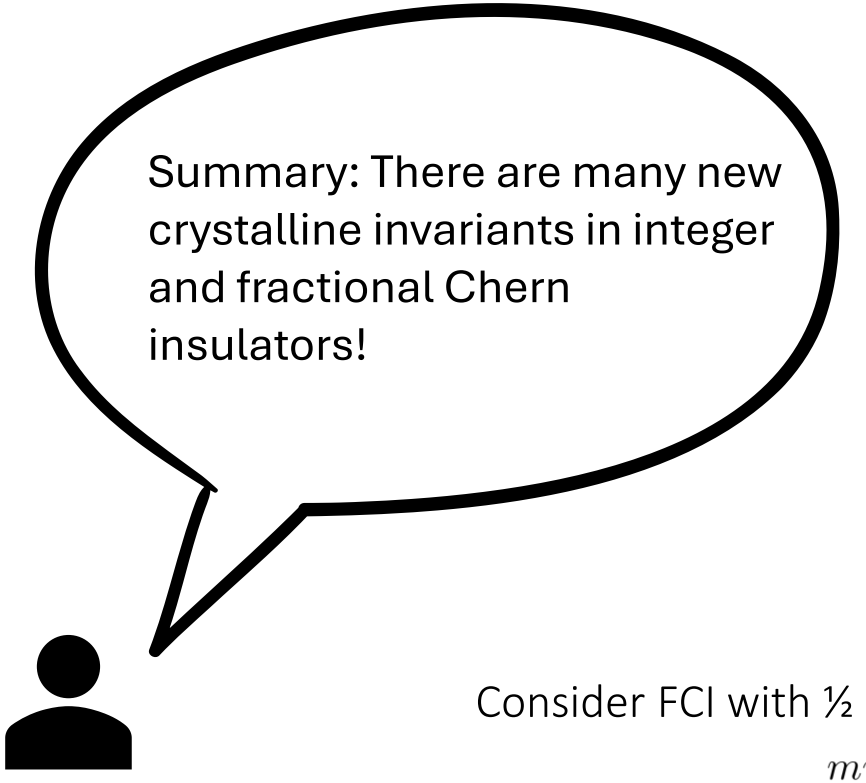
e-h) l=1
No phase ambiguity, implies $s + v = 0$
(correct)



Square Lattice $U(1)_{\pm 2}$

	$\lvert\psi^2_{C,\kappa}\rangle$				$\Theta_{o,l}$			
o	C	κ	$\mathcal{S}_o$	$\ell_{s,o}$	$l=0$	$l=1$	$l=2$	$l=3$
β	1	0	1/2	1/4	-0.247	0.253	-0.217	-0.746
β	-1	1	1/2	15/4	0.236	0.750	0.227	-0.251
α	1	0	1/2	1/4	-0.232	0.251	-0.227	-0.750
α	-1	1	3/2	7/4	0.208	-0.251	0.228	0.748
α	1	-1	7/2	1/4	-0.220	-0.749	-0.233	0.253
α	-1	2	5/2	7/4	0.245	0.747	0.218	-0.251

2405.17431



Consider FCI with ½ Laughlin TO at zero magnetic field:

$$\nu = \frac{mv}{2} \mod 1; \quad m = s_\alpha \times s_\beta \times s_\gamma^2$$
$$(= Q_\alpha + Q_\beta + 2Q_\gamma)$$

Fractional filling implies both m, v are nontrivial; implies one of the s_o are nontrivial! Similar argument for other fractional fillings

Outlook: Can we measure these in an experiment?