What is Category Theory?

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What is category theory? As everyone knows, it is the theory of general abstract nonsense. Unfortunately, most people believe that this statement was meant to mock category theory. In fact, it was uttered by Norman Steenrod, who took category theory very seriously early on, since it allowed him to solve quickly one of the problems he had been struggling with for some time, namely to find a proper axiomatic treatment of homology theory.

Steenrod's statement illustrates perfectly the difficulty one faces in answering the question 'what is category theory?'. It would be nice if one could answer that question in the same way that one can answer the question 'what is number theory?' or 'what is topology?'. In the latter cases, the answers come immediately: number theory is the study of properties of natural numbers; topology is the study of invariant properties of spaces under continuous transformations or deformations. Some reasonably concrete intuitions underlie these fields: everybody knows what the natural numbers are (although knowing what the *relevant* properties are, is another matter) and everyone has some informal idea of what a space is. But the case of category theory is different, as are all cases of *algebraic* structures, e.g. monoids, groups, rings, fields, algebras, etc. Algebra has not attracted the attention of philosophers of mathematics lately, at least not as much as numbers, sets or even the concept of space. This is in a way a shame since algebra has certainly become one of the key components of twentieth century mathematics. A decent philosophy of algebra is still awaiting a proper development and I believe that it would be in this context that the foregoing question about category theory could find a reasonably satisfactory answer.

One of the problems is that the objects dealt with have an ambiguous status. The easy answer "category theory is the theory of categories" does not help much. For it does not say what category theory *is*, what categories *are for* and why they would be of any interest to mathematicians, logicians, computer scientists, philosophers, cognitive scientists, mathematical physicists and even theoretical biologists. An analogy with group theory or, better, the theory of Lie groups, is in order and should allow us to see more clearly the nature of the problem we are facing. It is extraordinarily easy to define groups but it is another matter to *understand* what groups are about or for that matter what *Lie* groups are about. What one has to grasp is their roles *within* mathematics and related disciplines. Groups and Lie groups are structures at the core of mathematics *itself* (with the bonus of occupying a central role in the applications of mathematics to other sciences, e.g. physics, chemistry and biology). Notice that I am talking of functions or roles in the plural, a typical situation for algebraic structures. I believe that the same is true of categories. Furthermore, when categories were introduced, only certain roles were foreseen by mathematicians at the time. In fact, categories were introduced with certain specific functions in mind. The concept had a certain *form*, given by the axioms of the original theory, precisely to capture these roles. Some creative mathematicians then saw that this form, perhaps slightly modified, could serve other original functions and these led to the modification and introduction of new forms associated with the theory. This interplay is still going on as I am writing this essay.

My claim, thus, is that to understand what category theory is, and I believe that this claim could be made and should be made for *any* algebraic structure, one has to understand how a specific algebraic form is introduced for a specific *usage* in a given context and how this usage leads, via analogies, abstractions and generalizations, to the introduction of new contexts, new usages and new forms, the latter having sometimes an impact on our understanding of the original form. Thus, I believe that it is not possible to understand what category theory is without understanding what it allows us to *do*, and the latter, in turn, cannot be understood without understanding the *form* used, i.e. the definition of a category, in that specific context. It is important to understand that, as one of the elements changes, the others are forced to change accordingly. These changes might sometimes take longer than might be expected and in certain circumstances they depend on social or cognitive constraints. Thus, as the context changes, the form and the usage might change accordingly, although from one context to the next, it is easy to see how the various forms are connected (and usually this connection becomes a precise mathematical statement or theory). In other cases, it is the usage that changes, bringing with it a change in the context and the form.

I will concentrate here on the first twenty-five years of the history of category theory, for I believe that it is during these years that the most important shifts occurred. I claim that during the first fifteen years approximately, categories, functors and natural transformations, together with the language associated with these notions, had *heuristic* roles. These roles were surely important and deserve to be analyzed, understood and clarified, even from a philosophical point of view, but they do not cover what I take to be the most important roles of category *theory*. For it is precisely what happened historically: category *theory* really started only after the discovery and use of Abelian categories, on the one hand and adjoint functors, on the other hand. These discoveries were made approximately at the same time, namely in 1955 and 1956. It took another five to six years before the community could measure the extent of the changes these concepts brought about. It is approximately between 1963 and 1970 that category theory, as a genuine theory, arose and acquired a status that is still not clear to many mathematicians, logicians and philosophers. I will thus try to to make these changes as clear as possible by looking carefully at the work done by one of the main proponents of these changes, namely Bill Lawvere. Although progress since then has been steady and wide, the main *philosophical* features of the theory are, I believe, still the same. I should point out, however, that we might be on the verge of another important conceptual shift, this time coming from work done on what are called weak higher-dimensional categories. Be that as it may, it is clear that even a new shift will only confirm the main point, namely that category theory is indeed general abstract nonsense and that it is precisely because of that that it is so important both mathematically and philosophically.

1 Introducing categories: the context, the form and the usage

The context

Category theory made its official public appearance in 1945 in the paper entitled « General Theory of Natural Equivalences » written Samuel Eilenberg, a topologist then at the University of Michigan, and Saunders Mac Lane, an algebraist at Harvard. This paper, qualified by Mac Lane as being "off beat" and "far out" (Mac Lane 2002, 130) for that period, was written foremost with the goal of providing a totally general framework for a concept that was essential in their work but that also seemed to deserve a general and autonomous treatment: it is the concept of natural transformation. As Mac Lane has emphasized many times, functors were created to define natural transformations and categories to define functors.

Natural transformations showed up in Eilenberg and Mac Lane very first collaboration in 1942. In the late nineteen thirties, algebraic topology was slowly but surely taking shape: although cohomology and homotopy groups as well as some homology theories had been defined a few years earlier, cohomology operations and crucial developments connecting these various notions were still to be made.

Eilenberg was interested in understanding and computing various homology, cohomology and homotopy groups and Mac Lane was interested in understanding and computing group extensions. At first, these two topics seem to be unrelated. Group extensions belong to class field theory, a part of algebraic number theory. Homology, cohomology and homotopy groups are constructed from topological

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spaces and are aimed at translating topological information into algebraic, i.e. computable, data. Their function is to lead to the classification of topological spaces under continuous deformations.

While Mac Lane was visiting the University of Michigan to give a series of lectures on group extensions, Eilenberg observed the surprising fact that Mac Lane's calculations on a specific case of a group extension yielded exactly the same result as Steenrod's calculation of the homology of the solenoid, an important test case in topology. Eilenberg and Mac Lane's long and extraordinarily fruitful collaboration – 14 years and about 26 papers – resulted from their attempt at getting at the bottom of this unexpected coincidence. As Mac Lane has recounted many times – see for instance Mac Lane 1976, 1989, 1996, 2002 –, the basic fact explaining this coincidence was what is now called the "universal coefficient theorem", that is the existence of a very useful short exact sequence connecting the integral homology groups $H_n(X)$ or $H_n(X,\mathbb{Z})$ of a space X with the cohomology groups $H^n(X,G)$ of the same space with coefficients in an abelian group G. The short exact sequence is now written as follows:

$$0 \to \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X),G) \to H^n(X,G) \to \mathrm{Hom}(H_n(X),G) \to 0 \cdot$$

The fact that the group extension $\mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X),G)$, whose precise definition is not necessary, shows up in the sequence, as the kernel of the homomorphism $H^n(X,G) \to \mathrm{Hom}(H_n(X),G)$, explains its topological use. But Eilenberg and Mac Lane had to investigate the behavior of the universal coefficient theorem with respect to continuous maps $f : X \to Y$. This means that one has to consider diagrams like the following:

$$\begin{array}{ccccccccc}
0 & \to & \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(X,G)) & \to & H^n(X,G) & \to & \mathrm{Hom}(H_n(X),G) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \mathrm{Ext}^1_{\mathbb{Z}}(H_{n-1}(Y,G)) & \to & H^n(Y,G) & \to & \mathrm{Hom}(H_n(Y),G) & \to & 0
\end{array}$$

and prove that the vertical homomorphisms make the various squares commute for any continuous mapping $f : X \to Y$. Thus, there are two issues here: 1. How do the vertical arrows behave when one considers a mapping $f : X \to Y$, i.e. do they always exist? and 2. How do the horizontal mappings behave with respect to these vertical mappings, i.e. if they exist, do the squares commute? The first question leads to the notion of a functor and the second to the notion of natural transformation. Notice that it is the latter here that does the work Eilenberg and Mac Lane were interested in, but it had to do it with respect to the vertical mappings for it to be of any value. Thus, one has to define what a *functor* is first. This is what Eilenberg and Mac Lane did in 1942 in a short note for functors between groups and in which categories are *not* defined. In the same paper, they restrict themselves to natural *isomorphisms* between functors. They were however aware that these concepts were entirely general and could be applied to any mathematical concept for which a notion of homomorphism was well defined. But for the general definition to be given, an axiomatic approach was required and this is where categories had to come in. Functors were thought of as pairs of functions and these functions had to have a domain and a codomain with completely general properties adequate for the general definition of functor to be written: the concept of category was tailored for that purpose. They announced in 1942 that they would present this general axiomatic approach in a subsequent paper which appeared in 1945 under the name “General Theory of Natural Equivalences” and which, as reported by Mac Lane 2002, Eilenberg believed would be the only paper needed for category theory.

The form

The 1945 paper is interesting and revealing for what it contains and what it does *not* contain. Let us start with Eilenberg and Mac Lane’s definitions of category, functor and natural transformation, as they are given in the original paper (but with a different notation).

A *category* $\mathcal{C}$ is an aggregate of abstract elements X, called the *objects* of the category, and abstract elements f, called *mappings* of the category. Certain pairs of mappings f,g of $\mathcal{C}$ determine uniquely a product mapping gf, satisfying the axioms C_1, C_2, C_3 below. Corresponding to each object X of $\mathcal{C}$, there is a unique mapping, denoted by 1_X satisfying the axioms C_4 and C_5. The axioms are:

C_1. The triple product $h(gf)$ is defined if and only if $(hg)f$ is defined. When either is defined, the associative law

$$h(gf) = (hg)f$$

holds. This triple product will be written as hgf.

C_2. The triple product hgf is defined whenever both products hg and gf are defined.

A mapping 1 of $\mathcal{C}$ will be called an *identity* of $\mathcal{C}$ if and only if the existence of any product $1f$ and $g1$ implies that $1f = f$ and $g1 = g$.

C_3. For each mapping f of $\mathcal{C}$ there is at least one identity 1_r such that $f1_r$ is defined, and at least one identity 1_l such that $1_l g$ is defined.

C_4. The mapping 1_X corresponding to each object X is an identity.

C_5. For each identity 1 of $\mathcal{C}$ there is a unique object X of $\mathcal{C}$ such that $1_X = 1$.

The last two axioms “assert that the rule $X \to 1_X$ provides a one-to-one correspondence between the set of all objects of the category and the set of all its identities. It is thus clear that the objects play a secondary role, and could be entirely omitted from the definition of a category. However, the manipulation of the applications would be slightly less convenient were this done.” (Eilenberg & Mac Lane, 1945, 238) Thus, from a theoretical point of view, a category is determined by its mappings, but from a practical point of view, it is convenient to

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distinguish the objects from the mappings. Eilenberg and Mac Lane then state as a lemma that each mapping has a unique domain (source) and a unique codomain (target or range) and write $f : X \to Y$.

Eilenberg and Mac Lane defined a category in a purely *algebraic* fashion, as an abstract structure satisfying certain identities. In particular, it is immediate that if a category $\mathcal{C}$ has *a unique object*, then it is the same as a monoid, another algebraic structure. Thus, the concept of monoid is subsumed under the concept of category.

Eilenberg and Mac Lane then proceed to define *equivalences* in a category, nowadays called *isomorphisms*, thus: a mapping f is an *isomorphism* if it has an inverse, i.e. if there is a mapping g such that gf and fg are defined and are identities. Two objects X_1 and X_2 are said to be *isomorphic* if there is an isomorphism between them.

Eilenberg and Mac Lane gave four basic examples of categories: the category $\mathfrak{S}$ of sets with functions between them, the category $\mathfrak{X}$ of topological spaces with continuous functions, the category $\mathfrak{G}$ of topological groups with continuous homomorphisms and the category $\mathfrak{B}$ of Banach spaces with linear transformations with norm at most 1. This is a surprisingly short list of examples. They give more examples by defining the notion of a subcategory in the obvious fashion. Thus, they point out that given a category $\mathcal{C}$, the subcategory composed of the same objects as $\mathcal{C}$ but with isomorphisms as mappings is a category, nowadays called a groupoid. However, they do not make an explicit connection to the latter concept. The category of finite sets is also mentioned as well as other subcategories of the category of sets, e.g. for a fixed cardinal k, there is a category of all sets of power less than k together with all the mappings. Another way to form subcategories is by choosing different mappings. Thus, by restricting the mappings between sets to be onto or injective, one obtains different subcategories of sets. Similarly, if one restricts the continuous maps to open maps between topological spaces, then one obtains a different subcategory of topological spaces. In §11 of their paper, Eilenberg and Mac Lane observe that any group G can be thought of as a category: it has only one object and its mappings are the elements of the group. They also point out in §20 that any preorder P can be viewed as a category. But these two last examples are not given immediately after the definition of a category, although they show that the concepts of group and preorder are also subsumed under the concept of category and that morphisms need *not* be structure-preserving functions. Whether there is any conceptual gain by considering a group G or a preorder P *as* a category is a different matter. While it clearly indicates that the concept of category is extremely general and that it unifies various fundamental mathematical notions — a feature that might certainly attract the attention of a philosopher — it does not by itself show that the concept is fruitful or lead to any new interesting mathematics which would be a generalization of group theory or the theory of ordered sets.

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Eilenberg and Mac Lane defined functors in two arguments, noting that the generalization to n arguments is immediate. We will restrict ourselves to functors in one argument, for they are simpler and just as important. Here is their definition.

A *functor* F between categories $\mathcal{C}$ and $\mathcal{D}$ is a pair of functions, an *object-function* which associates to each object X of $\mathcal{C}$ and object $Z = F(X)$ in $\mathcal{D}$ and a *mapping function* which associates to each mapping f of $\mathcal{C}$ a mapping $h = F(f)$, such that

$$1.\ F(1_X) = 1_{F(X)};$$

$$2.\ F(gf) = F(g)F(f).$$

Such a functor is said to be *covariant*. Whenever a functor satisfies the equality

$$2^{\text{o}}.\ F(gf) = F(f)F(g).$$

instead of 2, it is said to be *contravariant*.

Examples of functors given by Eilenberg and Mac Lane include: the two power set functors $\wp^+ : \mathfrak{S} \to \mathfrak{S}$ and $\wp^- : \mathfrak{S} \to \mathfrak{S}$ which differ only on maps, the first one being covariant and the second contravariant, the Cartesian product $X \times Y$ of two topological spaces, the direct product $G \times H$ of two groups, the Cartesian product of two Banach spaces, given a space Y and a locally compact Hausdorff space X, the function space functor Y^X, similarly given a topological abelian group H and a locally compact regular topological group G, there is a functor $\mathrm{Hom}(G,H)$, contravariant in the first variable and covariant in the second, yielding a topological abelian group, similarly for Banach spaces and finally the tensor product $G \otimes H$ of two abelian groups.

Functors with the same domain category and the same codomain category can be connected to one another systematically or "naturally". This is precisely what the notion of natural transformation captures. Here is Eilenberg and Mac Lane's definition, restricted to functors $F,G : \mathcal{C} \to \mathcal{D}$ in one argument.

A *natural transformation* $\tau : F \to G$ between functors $F,G : \mathcal{C} \to \mathcal{D}$ is a function which associates to each object X of $\mathcal{C}$ a mapping $\tau_X : F(X) \to G(X)$ of $\mathcal{D}$ such that for any mapping $f : X \to Y$, the following diagram commutes

$$\begin{array}{ccc} F(X) & \xrightarrow{\tau_X} & G(X) \\ {\scriptstyle F(f)}\downarrow & & \downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow[\tau_Y]{} & G(Y) \end{array}$$

that is, $G(f)\tau_X = \tau_Y F(f)$.

If each τ_X is an isomorphism, then τ is said to be a *natural isomorphism* (Eilenberg and Mac Lane said *natural equivalence*, whence the title of their paper).

Given functors and natural transformations, it is possible to define categories of functors: its objects are functors $F : \mathcal{C} \to \mathcal{D}$ and its mappings are natural transformations $\tau : F \to G$. Eilenberg and Mac Lane find categories of functors "useful chiefly in simplifying the statements and proofs of various facts about functors" (Eilenberg & Mac Lane, 1945, 250) and not in themselves. Functor

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categories will become a major tool in subsequent developments and will force a change of perspective.

Eilenberg and Mac Lane give numerous examples of natural *isomorphisms*. We won't give all the examples here. Suffice it to mention two noteworthy cases:

$$\mathrm{Hom}(X \times Y, Z) \simeq \mathrm{Hom}(X, \mathrm{Hom}(Y, Z))$$

where Z is any topological space and X and Y are locally compact Hausdorff spaces and

$$\mathrm{Hom}(G \otimes H, K) \simeq \mathrm{Hom}(G, \mathrm{Hom}(H, K))$$

where G, H and K are groups. Eilenberg and Mac Lane knew this example since their 1942 paper. With hindsight, one could wonder, as Mac Lane himself did many times after, why they did not hit upon the notion of adjoint functors from these cases. But, when one looks carefully at the other examples, including those used in their previous collaboration, and there are many, it is not surprising at all. The focus of attention here is on the notion of natural isomorphism and how *it* unifies various mathematical situations and results, *not* the notion of functor and their properties nor the notion of categories with their properties.

Eilenberg and Mac Lane defined two other important notions in their original paper: the dual $\mathcal{C}^o$ of a category $\mathcal{C}$ in §13 and limits and colimits for directed sets in §21 and §22. As we will see, dual categories do play an important conceptual role in category theory and categorical logic. Given a category $\mathcal{C}$, the dual category $\mathcal{C}^o$ has as its objects those of $\mathcal{C}$; the mappings f^o of $\mathcal{C}^o$ are in one-to-one correspondence $f \rightleftarrows f^o$ with the mappings of $\mathcal{C}$. If $f : X \to Y$ is in $\mathcal{C}$, then $f^o : Y \to X$ is in $\mathcal{C}^o$. The composition law is defined by the equation $f^o g^o = (gf)^o$, whenever gf is defined in $\mathcal{C}$. Notice that mappings in $\mathcal{C}^o$ are not mappings in the set theoretical sense, i.e. they are not functions. Thus $\mathcal{C}^o$ is not a category of structured sets with structure preserving mappings.

Before we move on, let us now quickly underline what one does *not* find in Eilenberg and Mac Lane's paper. First, although the notion of a subcategory is clearly defined in the paper, properties of the inclusion functor, or for that matter, properties of functors in general, are not identified, e.g. the properties of being faithful, full, essentially surjective and reflexive are nowhere to be found.

Although Eilenberg and Mac Lane did define the notion of isomorphism of categories, they did *not* define the notion of *equivalence* of categories. The distinction between the two concepts might seem to be formally subtle, but it is crucial in certain applications of category theory that Eilenberg and Mac Lane could not foresee. The notion of isomorphism between categories is just the same as the notion of isomorphism between objects in a category: two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *isomorphic* if there is an isomorphism between them, that is if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that $GF = 1_{\mathcal{C}}$ and $FG = 1_{\mathcal{D}}$,where

1_C and 1_D denote the obvious identity functors. Two categories $\mathcal{C}$ and $\mathcal{D}$ are said to be *equivalent* if there is an equivalence between them, that is if there are functors $F:\mathcal{C}\to\mathcal{D}$ and $G:\mathcal{D}\to\mathcal{C}$ and natural isomorphisms $\tau: GF \tilde{\to} 1_{\mathcal{C}}$ and $\rho: FG \tilde{\to} 1_{\mathcal{D}}$. Thus, in the case of an equivalence, composing the functors F and G does not yield the identity functors, but there are *systematic translations*, namely natural isomorphisms, of the compositions to the identity functors. From the point of view of category theory, the notion of equivalence of categories is fundamental.

Although Eilenberg and Mac Lane introduced functor categories, they did not mention the possibility of considering a category of categories nor did they notice that natural transformations compose in two different ways. Of course, they did not need these concepts and therefore did not have to consider them at all.

The function

One has to keep in mind that categories are truly secondary for Eilenberg and Mac Lane. They do not constitute the focus of the paper. Eilenberg and Mac Lane explicitly recognize this fact in §6 where they discuss foundational issues related to categories, e.g. the category of all sets is not a set, thus not a legitimate entity from the standard set theoretical point of view.

> It should be observed first that the whole concept of a category is essentially an auxiliary one; our basic concepts are essentially those of a *functor* and of a natural transformation (…). The idea of a category is required only by the precept that every function should have a definite class as domain and a definite class as range, for the categories are provided as the domains and ranges of functors. Thus one could drop the category concept altogether and adopt an even more intuitive standpoint, in which a functor such as "Hom" is not defined over the category of "all" groups, but for each particular pair of groups which may be given. The standpoint would suffice for the applications, inasmuch as none of our developments will involve elaborate constructions on the categories themselves. (Eilenberg & Mac Lane, 1945, 247)

Of course, Eilenberg and Mac Lane could not foresee that elaborate constructions on categories themselves would become essential. It took approximately ten to fifteen years to see this clearly. Thus, it can be said that the major function of the concept of category in Eilenberg and Mac Lane's original paper is to provide a conceptual clarification: the concept of category is required to state and understand the concepts of natural transformation and natural isomorphism in full generality.

However, it seems that Eilenberg and Mac Lane envisioned a more ambitious role. Indeed, in the introduction of their paper, Eilenberg and Mac Lane hint at a different, more foundational role, that category theory could play in mathematics. They claim that category theory can be seen as a generalization of Klein's Erlangen Program. Here is how they stated it.

> The theory also emphasizes that, whenever new abstract objects are constructed in a specified way out of given ones, it is advisable to regard the construction of the corresponding induced mappings on these new objects as an integral part of their definition. The pursuit of this program entails a simultaneous consideration of objects and their mappings (in our terminology, this means the consideration not of individual objects but of categories). This emphasis on the specification of the type of mappings employed gives more insight into the degree of invariance of the various concepts involved. For instance, we show in Chapter III, §16, that the concept of the commutator subgroup of a group is in a sense a more invariant one than that of the center, which in its turn is more invariant than the concept of the automorphism group of a group, even though in the classical sense all three concepts are invariant.
>
> The invariant character of a mathematical discipline can be formulated in these terms. Thus, in group theory all the basic constructions can be regarded as the definitions of co- or contravariant manner under induced homomorphisms. More precisely, group theory studies functors defined on well specified categories of groups, with values in another such category.
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> This may be regarded as a continuation of the Klein Erlanger Programm, in the sense that a geometrical space with its group of transformations is generalized to a category with its algebra of mappings. (Eilenberg & Mac Lane, 1945, 236-237)

The first part is unproblematic: mappings are just as important as objects. That much was clear from the development of algebraic topology. They are now suggesting that it is a general phenomenon that should be observed in any mathematical field. As such, it can be taken as a heuristic principle. What is rather obscure is the claim that category theory is a generalization of Klein's program. What Eilenberg and Mac Lane had in mind is revealed in Chapter III of their paper, a chapter entitled *Functors and Groups*, more precisely in §16 where they present examples of subfunctors. The latter notion relies on the set theoretical notion of subset as follows.

Let $T_1, T_2 : \mathcal{C} \to \mathcal{D}$ be two parallel functors. The functor T_1 is said to be a *subfunctor* of T_2 provided $T_1(X) \subset T_2(X)$ for all objects $X \in \mathrm{C}$ and $T_1(f) \subset T_2(f)$ for all $f : X \to Y$ in $\mathcal{C}$. Eilenberg and Mac Lane then observe: "many characteristics subgroups of a group may be written as subfunctors of the identity functor." (Eilenberg & Mac Lane, 1945, 263-264) By letting $\mathcal{C}$ and $\mathcal{D}$ be the category $\mathfrak{G}$ of groups, one can consider the subfunctors T of the identity functors $I : \mathfrak{G} \to \mathfrak{G}$. If each $T(G)$ is a normal subgroup of G, then one can form the quotient group I/T. A specific case of this construction includes the commutator subgroup $C(G)$, a normal subgroup of G and the quotient group I/C is the abelianization of G. The center $Z(G)$ of the group G does not determine a subfunctor of the identity functor, simply because it is not a functorial construction on the category of groups. However, if one considers instead the category of groups with *surjective* homomorphisms between them, then it is a functor and the quotient I/Z can be formed. Finally, the automorphism group $A(G)$ of a group G is a functor only if one restricts its domain of application to the groupoid of the category $\mathfrak{G}$. Thus, in this way, "various types of subgroups of G may be classified in terms of the degree of invariance of the "subfunctors" of the identity which they generate," (Eilenberg & Mac Lane, 1945, 264) As Eilenberg and Mac Lane remark, this classification is intrinsically categorical, since one looks at functors on the whole category $\mathfrak{G}$ and not at an individual group and its subgroups.

However, it is hard to see how this approach can be generalized to an arbitrary or abstract category since it relies on the notion of subset. Of course, a purely categorical version of that notion can and was eventually given, as far as we know for the first time by Grothendieck in 1957, but Eilenberg and Mac Lane's original plan turned out to be a dead end. The notion of subfunctor was to be replaced by the more general notion of subobject and a *different* notion (and its equivalent formulations) was to play a central role in the characterization of mathematical constructions and their invariant character.

Although Eilenberg and Mac Lane's suggestion of a connection between category theory and Klein's program remained without direct progeny in the literature, we *do* believe that the claim can still be made, although in slightly different way[1]. The main points that need to be underlined at this stage are the following[2]:

1. In the same way that Klein's approach to elementary geometry allows an intrinsic and natural classification of various geometrical systems, e.g. Euclidean geometry, affine geometry, projective geometry, line geometry, etc., category theory leads to an intrinsic and natural classification of various mathematical structures; thus in the same way that Klein's approach provides a *unified* treatment of geometry, category theory provides a *unified* framework for mathematical domains;
2. In the same way that a geometry is characterized by its invariant properties under a group of transformations, a mathematical domain can be characterized by its invariant properties, when the latter are understood in the proper fashion;
3. Whereas Klein's program showed clearly that the analysis of the invariant properties of a structure is revealing, category theory extends this analysis by including various *covariant* properties between mathematical concepts.

[1] I asked Mac Lane in 1993 whether he still believed that one could make a connection between category theory and Klein's Erlangen Program and he immediately replied in the affirmative.

[2] The analogy is developed in detail in my forthcoming book *From a Geometrical Point of View : the categorical perspective on mathematics and its foundations*.

> The whole of algebraic topology can be thought in these terms. In other words, invariance is a special case of covariance.

Notice that in each of these three cases, categories acquire a new function. Whereas the concept was introduced as a mean to an end and first and foremost so that one could understand in its full generality the concept of natural transformation, in the end, Eilenberg and Mac Lane hinted at some other roles it could play in the realm of mathematical concepts. They certainly did not foresee that it would quickly come to occupy center stage in various areas.

2 Finding new usages with the same forms in known and new contexts

It quickly appeared, and Eilenberg and Mac Lane suggested some of these elements explicitly themselves, that categories could be used usefully in the context of algebraic topology, more specifically by providing the framework needed to clarify: 1. What homology and cohomology theories are; 2. How various homology theories are related to one another, in particular whether two homology theories are in fact the same; 3. How homology and cohomology theories are related in general; 4. What homotopy theory is and how it is related to homology and cohomology theories. Bits and pieces of these problems were already known, but it seemed clear that within the framework of category theory, these questions could be *stated* precisely and could receive *precise mathematical* answers. In the context of algebraic topology, new usages of the concepts of categories, functors and natural transformations were possible and these usages could be transferred to define and develop a new context, namely homological algebra.

We will concentrate on two monographs and one paper which crystallized these two movements: Eilenberg and Steenrod's book *Foundations of Algebraic Topology*, published in 1952, Cartan and Eilenberg's book *Homological Algebra*, published in 1956 and Grothendieck's *Tohoku*'s paper *Sur quelques points d'algèbre homologique*, published in 1957. Our presentation and remarks will be rather brief, even though each one of these works deserves a careful analysis.

Eilenberg and Steenrod presented axioms for homology theories as early as 1945. These notes, which circulated widely but which did not contain any proof, were turned into their extremely influential book published in 1952. Eilenberg and Steenrod used essentially the concepts of categories, functors and natural transformations in their work. Their definitions are taken directly from Eilenberg and Mac Lane. Thus, the *form* of the concepts of category, functor and natural transformation remains the same.

The language of categories allows them to sharply separate the topological aspects from the algebraic aspects of the problems of algebraic topology. A *homology theory* is analyzed as a *functor* from a category of topological spaces to a category of algebraic structures, for instance the category of Abelian groups, satisfying certain conditions stipulated by a list of axioms. A *cohomology theory* is also seen as a *functor* between the same categories but with the arrows in the axioms reversed. Thus, the transcription of algebraic topology within the context of categories modifies the *form* of some of its fundamental concepts. Eilenberg and Steenrod were well aware that "homology theory and cohomology theory are dual to one another". (Eilenberg & Steenrod, 1952, xiii) However, they immediately add: "the duality between the two theories has only a semiformal status." (Eilenberg & Steenrod, 1952, xiii) Turning this semiformal duality into a fully formal duality will bring with it a change in the form and the functions of category theory itself.

Furthermore, the use of diagrams and diagram chasing method are also crucial. Eilenberg and Steenrod underlined the importance of this original feature of their approach explicitly in the preface of their book.

> Successful axiomatizations in the past have led invariably to new techniques of proof and a corresponding new language. The present system is no exception. The reader will observe the presence of numerous diagrams in the text. Each diagram is a network or linear graph in which each vertex represents a group, and each oriented edge represents a homomorphism connecting the groups at its two ends. […]
>
> The diagrams incorporate a large amount of information. Their use provides extensive savings in space and in mental effort. In the case of many theorems, the setting up of the correct diagram is the major part of the proof. (Eilenberg & Steenrod, 1952, xi)

Eilenberg and Steenrod emphasized the *usefulness* of diagrams as a notational method and in organizing ideas involved in proofs. Students who were about to learn algebraic topology from their book would not only learn the basic concepts of category theory, they would also learn the language of diagrams and diagram chasing. For many mathematicians, these two aspects of the book constituted a fundamental change in mathematical practice.

> Eilenberg and Steenrod's book effected a revolution in mathematical notation. Perhaps not since Descartes' La géométrie has a book influenced how we write Mathematics. One knew they were looking at mathematics before 1600 because of the geometric diagrams with vertices and sides labeled by alphabetic letters. La géométrie in 1637 gave us nearly modern forms of equations, especially the notation of the exponent, i.e. a^3. The diagrams of Eilenberg-Steenrod not only made algebraic topology intelligible, but eventually swept out to other parts of mathematics, providing an efficient way to express complex, functorial relationships and giving us powerful methods of proofs by means of diagram chasing. (Becker & Gottlieb, 1999, 733)

Although the reference to Descartes' *La géométrie* might seem surprising, Eilenberg and Steenrod opened the door to it themselves. Indeed, again in the preface of their book, they draw a parallel between homology theory and analytic

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geometry: whereas in the latter case, problems of geometry are solved by being translated into algebraic problems, in homology theory, problems of topology are solved by being translated into algebraic problems. In the same way that Descartes showed how the translation between geometry and algebra had to be effected, Eilenberg and Steenrod's book is a precise and rigorous codification of the translation between topology and algebra. But in the case of Eilenberg and Steenrod, there seems to be something more. The language of categories and the use of diagrams made algebraic topology "intelligible". This is an extraordinarily strong claim! The algebraic topologist J.P. May makes a similar claim, talking about category theory in general.

> A great deal of modern mathematics, by no means just algebraic topology, would quite literally be unthinkable without the language of categories, functors, and natural transformations introduced by Eilenberg and Mac Lane in their 1945 paper. It was perhaps inevitable that some such language would have appeared eventually. It was certainly not inevitable that such an early systematization would have proven so remarkably durable and appropriate; it is hard to imagine that this language will ever be supplanted. (May, 1999, 666)

To say that a field would "quite literally be unthinkable" is, once again, an extraordinarily strong claim to make and so is the claim that "it is hard to imagine that this language will ever be supplanted". When Eilenberg and Steenrod published their book on the foundations of algebraic topology, surely mathematicians of this era understood what they were doing or, at the very least, understood algebraic topology in *some sense*. What does category theory *do* to make these fields intelligible and thinkable? We should immediately notice that the use of diagrams and the method of diagram chasing are not *essentially* linked to categories as such. In their book, Eilenberg and Steenrod use extensively diagrams and the method of diagram chasing *before* they introduce categories, functors, etc. Indeed, diagrams and the method of diagram chasing are heavily used in the first three chapters of their book and categories and functors are introduced in the fourth chapter only. But, as Eilenberg and Steenrod pointed out, the point of view of categories dominated the development of the entire book. (Eilenberg & Steenrod, 1952, xii) Once diagrams and the method of diagram chasing are used, it is clearly *natural* to think of them as being *in* categories and to look for functors and natural transformations. Thus, as for Eilenberg and Mac Lane, categories in Eilenberg and Steenrod come *in some sense* after the fact, as a useful mean to codify a mathematical situation. I submit that Eilenberg, Mac Lane, Steenrod and their contemporaries who started to use categories in the late forties and early fifties did not think of it as a revolution but merely as a useful language or framework. Once it had been introduced, it seemed trivial and thus, extremely handy. Furthermore, it is true that algebraic topology was suddenly standing on clear and solid foundations. It was finally possible to say clearly what homology theory was about, e.g. it is about certain *functors*, and it was possible to compare systematically various homology theories with the help of natural transformations. For instance, one of the key results of Eilenberg and Steenrod is the proof that over the category of triangulable spaces, simplicial homology and singular homology are isomorphic. This is now a precise and provable mathematical claim. The categorical framework makes it indeed easy to organize all the data involved in that theorem and its proof.

Notice the shift: whereas for Eilenberg and Mac Lane, categories and functors are defined and used, they are prosthetics to the notion of natural transformation. In the context of algebraic topology, certain important mathematical constructions, namely homology theories, become *functors*. This is an important ontological shift that brings with it a certain ontological weight to categories themselves. Although that weight makes them appear on the scene of mathematical objects, categories are still not used as such, i.e. there are still no elaborate constructions on categories that completely justify their presence. In other words, there is still no category *theory*.

I believe that the following quote captures perfectly the way people might have envisioned categories in the fifties. Although the authors are here talking about a specific principle and a concept that was not available in the forties and most of the fifties, it is easy to generalize the claim to categories, functors, natural transformations, diagrams and diagram chasing.

> Instead of being a collection of theorems, Eckmann-Hilton duality is a principle for discovering interesting concepts, theorems, and questions. It is based on the dual category, that is, on the duality between the target and source of a morphism; and also on the duality between functors and their adjoints.
>
> *In fact it is a method wherein interesting definitions or theorems are given a description in terms of a diagram of maps, or in terms of functors.* Then there is a dual way to express the diagram, or perhaps several different dual ways. These lead to new definitions or conjectures. Some, not all, of these definitions turn out to be very fruitful and some of the conjectures turn out to be important theorems. (Becker & Gottlieb, 1999, 726) (Italics ours)

Once Eilenberg and Mac Lane had introduced the basic concepts of categories, functors and natural transformations, and Eilenberg and Steenrod had shown how to translate known problems of algebraic topology into problems about functors, natural transformations, diagrams and diagram chasing, it was possible to try to do the same with similar or analogous mathematical problems and fields. A new heuristic becomes available and this heuristic seems, at first sight, fruitful and insightful: it provides a new way of looking at mathematical problems and situations and it clearly leads to interesting and fruitful definitions, theorems, insights and even, the creation of a new mathematical discipline.

Homological algebra emerged in the nineteen forties when mathematicians observed that they could used homological and cohomological methods to study *algebraic* systems. Thus tools that were developed to classify and understand topological spaces and their morphisms were transferable to various algebraic systems and their morphisms, e.g. Lie groups. In the early fifties, Cartan and Eilenberg wrote a book that was about to have the same import on the field the same impact on homological algebra, even the name comes from that book, that Eilenberg and Steenrod's book had on algebraic topology.

> Cartan and Eilenberg began collaborating during the 1950/51 Séminaire Cartan, rewriting the foundations of all the ad hoc algebraic homology and cohomology theories that had arisen in the previous decade. Coining the term *Homological Algebra* for this newly unified subject, and using it for the title of the 1956 textbook, they revolutionized the subject. (Weibel, 1999, 812)

Nothing less than a third revolution in ten years for Eilenberg… The term 'revolution' in this case is not, however, entirely convincing, since there was no discipline to go back to or to overthrow. It is not so much that Cartan and Eilenberg transformed radically the field of homological algebra, they more or less launched it. From the point of view of the history of category theory, what is striking in that book is that categories, functors and natural transformations are not even defined! It is assumed right from the start that homology and cohomology theories are functors. Thus, once again, the form of the concept does not change and categories are not used as such, i.e. there is no construction on categories. However, two elements in their work will contribute directly to what deserves to be called the birth of category *theory*.

We have already indicated that Eilenberg and Steenrod had observed that the duality between homology and cohomology was semiformal. Was it possible to make it purely formal? Using categories, Mac Lane tried to make it so in 1948 and, more successfully, in 1950. In the latter important paper, not only does Mac Lane come very close to the notion of Abelian category, but he also uses the categorical language to *define* mathematical concepts, e.g. products, coproducts, etc. It is the first place where universal mappings are defined by categorical tools. Unfortunately, Mac Lane's paper did not reach its audience and did not have an impact on the categorical audience. Mac Lane's paper was *theoretically* interesting, but did not yield any new and striking mathematical results.

It was clear to Cartan and Eilenberg that homological algebra could be developed in a more general setting, that is that one could characterize by categorical means the *type* of categories required to define and develop the homology and cohomology theories. This was done by Buschbaum in his Ph.D. thesis in 1955 and published as an appendix in Cartan and Eilenberg's book. A second motivation came from algebraic geometry. Both Cartan and Eilenberg knew that cohomology theory for sheaves of Abelian groups on a topological space ought to fall under their general framework, but they simply could not solve a technical problem involved in the construction of the theory. The problem had to do with the construction of injective resolutions for sheaves and it seemed to be insurmountable. Grothendieck solved the problem not by giving an explicit construction, but rather by finding the appropriate categorical framework for homological algebra in general and providing a categorical property from which

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the construction required could be performed. Grothendieck's work was therefore astonishing not only because it showed how to subsume the cohomology theory of sheaves within homological algebra, a remarkable result in itself, but also in the way he used categories to obtain this result. Not only categories could be used to organize systematically given fields like algebraic topology or homological algebra, it could be genuinely used to prove important mathematical results. Thus, in Grothendieck's hand, categories take a new form, find new roles and give rise to a new context.

Both Grothendieck and Buschbaum presented a definition of categories that is formally different from the one found in Èilenberg and Mac Lane. Whereas Eilenberg and Mac Lane's original definition was entirely abstract, Buschbaum and Grothendieck both chose to define categories as sets. The reason for this seems to be that they wanted to add more structure to a category and the language of Hom-sets provided an easy and simple solution to add this structure directly and at the right place. Here is, in a slightly modified presentation, Grothendieck's definition (our translation).

A *category* $\mathcal{C}$ is given by:

1. A non-empty class of objects;
2. For any objects X and Y of $\mathcal{C}$, there is a *set* $\mathrm{Hom}(X,Y)$, the set of *morphisms* from X to Y;
3. For any objects X, Y and Z, there is an operation, called *composition*, $\mathrm{Hom}(X,Y)\times\mathrm{Hom}(Y,Z)\to\mathrm{Hom}(X,Z)$ $(f,g)\mapsto g\circ f$;

These data satisfy the following axioms:

1. Composition is associative: $h\circ(g\circ f)=(h\circ g)\circ f$;
2. For each $\mathrm{Hom}(X,X)$, there is a morphism $1_X : X\to X$ such that $f\circ 1_X = f$ and $1_Y\circ f = f$ for any $f : X\to Y$.

Grothendieck then immediately defines the dual category $\mathcal{C}^{op}$ of a category $\mathcal{C}$: the objects of $\mathcal{C}^{op}$ are the same as those of $\mathcal{C}$ and the set $\mathrm{Hom}(X,Y)^{op}$ is $\mathrm{Hom}(Y,X)$.

These definitions are certainly not profoundly different from those given by Eilenberg and Mac Lane. But the difference is not accidental either. The choice of Hom-sets to define categories allows a simple and direct definition of a new notion, the starting point of the whole work, the notion of an *additive category*. A category $\mathcal{C}$ is said to be *additive* by Grothendieck when it satisfies the following three conditions:

1. For every couple (X,Y) of objects of $\mathcal{C}$, the set $\mathrm{Hom}(X,Y)$ is an Abelian group such that composition of morphisms is bilinear;
2. $\mathcal{C}$ has binary products;
3. $\mathcal{C}$ has a zero object.

Grothendieck then observes that the dual $\mathcal{C}^{op}$ of an additive category $\mathcal{C}$ is also additive, an important property given that one of the targets is to account for the duality mentioned above between homology and cohomology theories.

A category can now have a *structure*: first its Hom-sets have an Abelian group structure, second, certain constructions are possible, namely products, and third, it has a specific object. That structure, in turn, supports various definitions, constructions, theorems, etc. Notice that the definition rests upon concepts that are defined using the language of categories: products, coproducts, zero objects, etc. These notions are all defined explicitly by Grothendieck in the first section of his paper. Grothendieck also introduced the concepts of monomorphism, epimorphism, subobject and quotient object. Again, these are used to define a *structure* in a category. Although this structure is put in place so that certain functors with specific properties can be defined, by the same token, categories start playing a real role in mathematics. Categories no longer merely encode data about objects and their morphisms so that a functor can have a domain and a codomain, their structure is relevant to a problem at hand.

Grothendieck also introduces the notion of *equivalence* of categories and states clearly that it is different from the notion of isomorphism of categories. But the real surprise at this point is that, in fact, Grothendieck defines an *adjunction* between functors and immediately restricts himself to the specific case of an equivalence of categories. Thus, Grothendieck just missed the fundamental concept of adjoint functors. A careful analysis of this interesting conceptual slip would lead us too far from our main objective. (But see Krömer 2004.)

Grothendieck defines the notion of an *Abelian category* as an additive category $\mathcal{C}$ satisfying two additional properties:

AB 1) Every morphism has a kernel and a cokernel;

AB 2) Let u be a morphism of $\mathcal{C}$. Then the canonical morphism $\bar{u}: \text{Coim } u \to \text{Im } u$ is an isomorphism.

The details of this definition are not essential here. Although it might not be obvious from the definition itself, again this definition is self-dual: the dual of an Abelian category is Abelian. Four more axioms and their dual, named AB 3, AB 4, AB 5 and AB 6, are then given and used. We will present AB 3 and AB5.

AB 3) any family $(X_i)_{i\in I}$ of objects of $\mathcal{C}$ has a direct sum $\bigoplus_{i\in I} X_i$;

AB 5) the axiom AB 3) is satisfied and if $(X_i)_{i\in I}$ is an increasing filtered family of subobjects of X and Y any subobject of X, then

$$(\sum_i X_i) \cap Y = \sum_i (X_i \cap Y).$$

The actual content of these axioms need not be analyzed in details here. They stipulate, in the first case, the existence of a specific categorical construction characterized by a universal morphism and what is called in the second case an exactness condition. They both introduce in an essential fashion infinite operations in the constructions. In the mind of Grothendieck, they serve essentially one purpose: "the foregoing axioms will be useful for the study of injective and projective limits which we will need to give flexible existence conditions for the "injective" and "projective" objects." (Grothendieck, 1957, 129-130, our translation.) As we have mentioned, it is by stipulating these properties, properties that can be found in certain Abelian categories, that Grothendieck circumvents the difficulties blocking the application of homological algebra to a cohomology of sheaves. Furthermore, Grothendieck immediately generalizes the presentation by considering diagram categories, a special case of functor categories. The constructions of direct and inverse limits are seen as special cases of universal morphisms in diagram categories, when diagrams are restricted to preorders or filtered sets. Thus, Grothendieck does not give the general construction of limits and colimits in categories, presumably because he does not need them and does not see the necessity of exploring the purely theoretical development of these concepts. Finally, Grothendieck shows in theorem 1.10.1 that any Abelian category satisfying AB 5) and having a generator, another categorical property, has enough injectives. After proving this last fundamental result, Grothendieck remarks that "in many cases, the existence of a monomorphism from X to an injective object can be seen directly in a more simple manner. Theorem 1.10.1 has the advantage of being applicable to different cases. Furthermore, the conditions of the theorem are stable when moving to certain diagram categories where the existence of enough injectives is not always clear to the naked eye." (Grothendieck, 1957, 137, our translation.) This is certainly an understatement, for it is precisely what Grothendieck uses to solve the problem of injective resolutions.

Another striking feature of Grothendieck's paper, in contrast with Eilenberg and Mac Lane, is the presence and importance of functor categories. They are central to the whole project, categories of presheaves are defined as functor categories and so are many others, and there is no doubt that the notion of equivalence of categories is justified by their presence: there are numerous examples of functor categories that are equivalent but not isomorphic. But Eilenberg and Mac Lane, as we have indicated, had no use of functor categories. Whereas Eilenberg and Mac Lane, Eilenberg and Steenrod and Cartan and Eilenberg presented certain constructions and concepts as *specific* functors, Grothendieck considers the functor category *itself* and its properties. For instance, if the category $\mathcal{C}$ is additive, then the functor category $\mathcal{C}^{\mathcal{D}}$ is also additive: thus everything that is proved from the axioms of the notion of additive category is true of every functor category $\mathcal{C}^{\mathcal{D}}$ whenever $\mathcal{C}$ is additive. In a nutshell, by the time Grothendieck published his paper in 1957, constructions on categories had become a fundamental part of the landscape.

With the appearance of the notions of additive and Abelian categories, a significant change had occurred in the mathematical landscape, a change that we believe to be more important than the introduction of diagrams and the method of diagram chasing, even though it could not have been without the introduction of the latter. Grothendieck had shown that a large part of homological algebra could be developed in the context of Abelian categories, that is in the context of a *type* of category. There was, for the first time, a clear example of a *type* of category, not defined by its objects and its morphisms like the category of sets or the category of topological spaces or the category of Abelian groups, but by the *categorical properties* it satisfied, which unified *by the categorical properties they satisfied* various parts of mathematics and was used to define, construct and prove important theorems. The search for other *types* of categories, of what will be called *abstract categories*, started to make sense. In other words, the development of category *theory* could be conceived as a legitimate and reasonable enterprise, although its mathematical benefits might seem remote at this stage. Grothendieck and his school continued to play an important role in the development of categorical concepts and their use in the context of algebraic geometry in the sixties, an endeavor which culminated in the proof of the Weil conjectures in the early seventies by Pierre Deligne, one of Grothendieck's students.

But, and this has to be emphasized, Grothendieck's paper and the use of categories in algebraic geometry, does not constitute *as such* a piece of category *theory*. It uses category theory in an essential manner but it does not present additive, Abelian categories, etc. as being intrinsically valuable. I claim that the second fundamental paper on category theory and which really paved the way to the development of category *theory* is Daniel Kan's paper *Adjoint Functors* published in 1958. Kan wrote the paper on adjoint functors because he wanted mathematicians of his time to be able to read the companion paper published immediately after, namely the paper entitled *Functors involving c.s.s. Complexes*. It is undeniable that the paper on adjoint functors is truly *theoretical*: it presents the notion of adjoint functors, unit and counit of adjunctions and systematically examines how the notion of adjointness is related to the other important notions of category theory, for instance limits and colimits. The paper also introduces what are now called 'Kan extensions'.

Although Kan assumes explicitly the notion of a category as it is given in Eilenberg and Mac Lane, he gives the definition of adjoint functors in terms of hom-sets as follows: let $\mathcal{C}\underset{G}{\overset{F}{\rightleftarrows}}\mathcal{D}$ be two contravariant functors and let $\varphi:\mathrm{Hom}(F(X),Y)\xrightarrow{\sim}\mathrm{Hom}(X,G(Y))$ be a natural isomorphism. Then F is called the *left adjoint of* G *under* φ and G *the right adjoint of* F *under* φ and the situation is denoted by $F \dashv G$. In the same paper, Kan gives, for the first time totally general definitions of limits and colimits and proves that their existence in a category is equivalent to the existence of certain adjoints to elementary functors. The paper ends with the notion of what are now called Kan extensions and results needed for the applications Kan has in mind in combinatorial homotopy theory.

In contrast with Grothendieck's 1957 paper, Kan does not obtain striking new results, he does not solve an outstanding problem in homotopy theory or in homological algebra or in homology theory. In his paper *Functors involving c.s.s. Complexes*, he presents a unified framework for various notions of complexes, singular, simplicial, Kan complexes, etc., and obtains new, more conceptual, proofs of known results, e.g. the Hurewicz homomorphism. It is a thoroughly conceptual paper in which it is shown that various notions, which were taken to be different, follow a general pattern, the latter being revealed by the presence of adjoint functors in various contexts. To the contemporary reader, the beauty of these papers is absolutely undeniable. Three missing elements are nonetheless worth mentioning. First, Kan does *not* define the notion of equivalence of categories as a special case of an adjoint situation nor does he consider the case of reflective subcategory, a special case that Peter Freyd was investigating in his undergraduate honor's thesis at about the same time. Second, the class of examples of adjoint functors mentioned by Kan is surprisingly limited. Kan's paper does not convey the idea that adjoint functors pervade mathematics although it is clear from his paper that it is a central concept of category theory. Third, Kan is totally oblivious to foundational issues even though all the constructions introduced involve large categories.

Adjoint functors introduce a basic, fundamental form in category theory. Adjoint functors are not, strictly speaking, inverses to one another. Strict inverses are isomorphisms of categories. But they *are* inverses in *some sense* and this is what makes them so important. It is tempting and certainly appealing to say that adjoint functors are *conceptual* inverses to one another. Notice immediately that a functor can have *both* a right *and* a left adjoint, up to (unique) isomorphisms, and thus it can have two conceptual inverses. Kan's definition does not exhibit clearly this feature of adjoint functors. An alternative definition, which does not rely on hom-sets, reveals it clearly. Two functors $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ form an adjunction, $F \dashv G$, whenever there are two natural transformations $\eta : I_X \to GF$ and $\varepsilon : FG \to I_Y$ such that the two following diagrams commute:

$$\begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & {\scriptstyle 1_G}\searrow & \downarrow{\scriptstyle G\varepsilon} \\ & & G \end{array} \qquad\qquad \begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & {\scriptstyle 1_F}\searrow & \downarrow{\scriptstyle \varepsilon F} \\ & & F \end{array}.$$

Thus, although the composites FG and GF are not *equal* to the appropriate identity functors, a systematic and natural connection to the identity functors exists. As the diagrams show, the identity is "lifted" to the functors themselves, for we do have that $1_G = G\varepsilon \circ \eta G$ and $1_F = \varepsilon F \circ F\eta$. Notice that when $\eta : I_X \to GF$ and $\varepsilon : FG \to I_Y$ are natural isomorphisms, we have an equivalence of categories. The best way to see how adjoint functors are conceptual inverses is by looking at specific examples in the literature. See Mac Lane 1998, chap. IV, Adamek et. Al. 1990, chap. V, Borceux 1994, chap. 3, Taylor 1999, chap. VII.

Before the advent of adjoint functors, all applications of category theory were *unidirectional*: functors were going in one direction, e.g. from a category of topological spaces or of algebraic structures into a category of algebraic structures. Grothendieck had pointed out that the notion of equivalence of categories was central and more important than the notion of isomorphism of categories, but, as we have already mentioned, he just missed the notion of adjoint functors and their significance. It took some time to realize how pervasive adjoint functors are and, in particular, to see that there were numerous mathematically *significant* examples at hand: the various reflective subcategories, free structures of various kinds, dualities, e.g. Stone, Pontryagin, etc., compactification results, Galois theory, the theory of covering spaces, etc. With hindsight, the latter cases are so compelling that Mac Lane himself was almost ready to claim that, had it not been of the

Second World War, Stone or someone else interested in dualities would have probably discovered adjoint functors very quickly. (See Mac Lane 1970.)

Be that as it may, the introduction of adjoint functors as the central concept of category theory is crucial in two complementary respects: 1. It changes drastically the impact and the role of category theory in mathematics; 2. It makes it clear that categories, functors and natural transformations are more than a purely heuristic framework. Indeed, whereas categories were introduced in order to have a complete and systematic conceptual framework underlying the definition of the concept of natural transformation, once the notion of adjoint functors is seen as being central, functors, categories and their properties acquire autonomy. Categorical *properties* can be stated in a completely uniform manner within the language of category theory itself. Furthermore, these properties not only describe the basic mathematical constructions used in various fields, in some cases they *are* pivotal, conceptual mathematical results. Finally, if Grothendieck had shown how to use categories and their properties to solve important mathematical problems, he did not free entirely categories from set theoretical constraints, e.g. in the definition of arbitrary limits and colimits, a step taken by Kan when he showed that the latter could be described by the existence of adjoints to given functors. (This is *not* to say that questions of size do not matter, but simply that they could in principle be treated internally so to speak, i.e. relative to a base category.) A *network* of categories and functors could be considered as one whole; its basic constitutive features, its skeleton, were revealed by adjoint functors going back and forth between categories. In other words, a category of categories is not merely a category, it is not merely a collection of categories and functors, it has an intrinsic frame, an 'organic' unity, principles of cohesion based on adjoint functors. The first mathematician who emphasized and used these facts was Bill Lawvere who, in 1963, in his Ph.D. thesis, suggested that a category of categories could be a taken as foundation for mathematics, suggested to defined sets in categories and not categories as sets, used in an essential manner adjoint functors to present and prove his results, thus putting adjoints at the center of his work and apply these tools in directions that were entirely original, in particular in the direction of logic and the foundations of mathematics. Lawvere pursued this work during the sixties and in 1969, in collaboration with Miles Tierney, they discovered the notion of elementary topos which has played a key role in categorical logic and categorical foundations of mathematics ever since. He thus introduced new forms, new contexts and new roles for all the notions available at that time.

3 Lawvere

I will not look at the details of Lawvere's thesis: the notions of algebraic category, algebraic functors, functorial semantics and algebraic structure. The notions as well as the results presented in the thesis were quickly absorbed by the community of category theorists and led to extensions, refinements and similar results for other types of categories. (For a presentation of Lawvere's work in a contemporary setting, see Pedicchio & Rovatti 2004.) Instead of looking at the chronological development of Lawvere's ideas and papers, I will concentrate on the three aspects that constitute the spine of my approach: new contexts, new forms and new functions introduced by Lawvere.

It could be argued that one of Lawvere's main objectives was to find and develop the proper context to define and develop functional analysis and more generally rational or continuum mechanics.

> What was the impetus which demanded the simplification and generalization of Grothendieck's concept of topos, if indeed the applications to logic and set theory were not decisive? Tierney had wanted sheaf theory to be axiomatized for efficient use in algebraic topology. My own motivation came from my earlier study of physics. The foundation of the continuum physics of general materials, in the spirit of Truesdell, Noll, and others, involves powerful and clear physical ideas which unfortunately have been submerged under a mathematical apparatus including not only Cauchy sequences and countably additive measures, but also ad hoc choices of charts for manifolds and of inverse limits of Sobolev Hilbert spaces, to get at the simple nuclear spaces of intensively and extensively variable quantities. But as Fichera lamented, all this apparatus gives often a very uncertain fit to the phenomena. This apparatus may well be helpful in the solution of certain problems, but can the problems themselves and the needed axioms be stated in a direct and clear manner? And might this not lead to a simpler, equally rigorous account? (Lawvere 2000, 726)

Although Lawvere is here talking about the notion of elementary topos, to which we will turn shortly, a fundamental leitmotiv is clearly showing up in this passage: the "mathematical apparatus" responsible for this "uncertain fit to the phenomena" derives from a certain *usage* of sets and their operations. It is not that sets *themselves* are the culprit, but rather a certain *method* based on a conception of sets, that is as essentially made up of faceless, abstract points. Such a belief is not entirely unusual among mathematicians as the following quote illustrates perfectly:

> The vogue for point-theoretical analysis situs seems to be due, in large part, to the predominating influence of analysis on mathematics in general. Nowadays we tend, almost automatically, to identify physical space with the space of three variables and to interpret physical continuity in the classical function theoretical manner. But the space of three real variables is not the only possible model of physical space, nor is it a satisfactory model for dealing with certain types of problems. Whenever we attack a topological problem by analytic methods it almost invariably happens that to the intrinsic difficulties of the problem, which we can hardly hope to avoid, there are added certain extraneous difficulties in no way connected with the problem itself, but apparently associated with the particular type of machinery used in dealing with it. (Alexander, 1932, quoted by James, 2001, 811.)

There is no doubt in my mind that Lawvere would whole-heartedly agree with this claim. It is also clear that the type of machinery referred to here by Alexander is (point-)set theoretical. It is to avoid part of this machinery and to replace it by algebraic machinery that Lawvere wanted to develop mathematics within a category of categories and place sets within that universe. It has to be emphasized immediately that the idea of developing mathematics within a category of categories remains even to this day a theoretical possibility, although as we will indicate recent work in higher-dimensional categories are opening up new avenues, and that Lawvere's axiomatization of the category of sets in 1964 did not generate a flurry of research and developments. However, from an historical point of view, both papers constitute a radical break, not to say a revolution, with the standards of the time. It is important to note that a category of categories, a category of sets and the notion of elementary topos all constitute at the same time new forms *and* new contexts. Lawvere suggested that these new forms be taken as foundational frameworks, thus having a role that is *not* radically new. What *is* new is to propose that a categorical framework could play this role. In Grothendieck's mind, a topos is a tool whose main purpose is to allow for the definition and applications of cohomology theories in algebraic geometry. Although this remains to some extent true for Lawvere, a topos still has a crucial role to play in algebraic geometry, elementary toposes have a function that was certainly not foreseen by Grothendieck: to provide a proper foundation for mathematics, or at least parts of mathematics, e.g. functional analysis or continuum mechanics. Here is, for instance, what Lawvere says in his paper on the category of categories as a foundation for mathematics:

> Having presented the axioms for the basic theory of the category of categories, we now ask what can be done with them. Besides the possibility of developing analysis which was previously alluded to, one can also define easily the full metacategories of ordered sets, groups, or algebraic theories and study these to a considerable extent. … . The general theories of triplable categories, of fibered categories, and of closed categories (when the latter is phrased so as not to refer to the category of sets) can all be developed quite nicely within the basic theory, as can many other things. (Lawvere 1966, 12)

Thus, the role of category theory becomes *metamathematical*. Once again, it is supposedly preferable to a purely set theoretical context because a categorical framework is assumed to reflect more faithfully the nature of mathematical knowledge.

> In the mathematical development of recent decades one sees clearly the rise of the conviction that the relevant properties of mathematical objects are those which can be stated in terms of their abstract structure rather than in terms of the elements which the objects were thought to be made of. The question thus naturally arises whether one can give a foundation for mathematics which expresses wholeheartedly this conviction

> concerning what mathematics is about, and in particular in which classes and membership in classes do not play any role. (Lawvere 1966a, 1)

To those familiar with set theory, the claim that a categorical foundation would capture more adequately the abstract nature of mathematical objects is startling, for set theory is usually seen as exhibiting clearly the abstract nature of mathematical objects as well as their structures, particularly in the context of model theory. But Lawvere has a different sense of 'abstract structure' in mind, a sense which is the direct continuation of Eilenberg and Mac Lane's claim according to which category theory is a generalization of Klein's program[3].

> We adjoin eight first-order axioms to the usual first-order theory of an abstract Eilenberg-Mac Lane category to obtain an elementary theory with the following properties: (a) There is essentially only one category which satisfies these eight axioms together with the additional (nonelementary) axiom of completeness, namely, the category S of sets and mappings. Thus our theory distinguishes S structurally from other complete categories, such as those of topological spaces, groups, rings, partially ordered sets, etc. (b) The theory provides a foundation for number theory, analysis, and much of algebra and topology even though no relation $\in$ with the traditional properties can be defined. *Thus we seem to have partially demonstrated that even in foundations, not Substance but invariant Form is the carrier of the relevant mathematical information*. (Lawvere 1964, 1506) [our emphasis]

Lawvere's goal is here, from the point of view of the foundations of mathematics, traditional: to give a first-order axiomatic description of a conceptual system up to "isomorphism". But here the notion of "isomorphism" is the notion of equivalence of categories: his goal is to add axioms to those of Eilenberg and Mac Lane in such a way that any two categories satisfying them are equivalent, in the categorical sense of that expression. This result, he claims, shows that we thus have a description of the Form of the universe of sets, the latter Form being to a large extent independent from specific details of the underlying Substance. This independence from the substance amounts to a claim of *invariance under relevant transformations*. In fact, this notion of invariance already underlies Lawvere's thesis: it is in a sense its main goal. Indeed, Lawvere claims that "essentially, algebraic theories are an *invariant notion* of which the usual formalism with the operations and equations may be regarded as "presentation"." (Lawvere, 1963a, ii, our emphasis.) For algebraic theories, e.g. group theory, Lawvere constructs a category which can be thought of as the invariant presentation of the theory and for that very reason ought to be taken as *the* theory of an algebraic type. The goal is

[3] This is explained in more details in my forthcoming book *From a Geometrical Point of View : the categorical perspective on mathematics and its foundations*.

the same for the category of sets and, to some extent, for the situation is clearly different, for the category of categories.

> Some years ago I began an introductory course on Set Theory by attempting to explain the *invariant content* of the category of sets, for which I had formulated an axiomatic description. (Lawvere, 1994, 5) [our emphasis]

In the same way that it could be argued that the group of transformations of an elementary geometry characterizes its invariant content in a very precise sense, Lawvere claims that the invariant content of a conceptual system can be characterized by categorical means.

Lawvere progressively came to realize that the role of adjoint functors was crucial in this enterprise, for they play a role similar to that of a group of transformations in the case of an elementary geometry.

> As posets often need to be deepened to categories to *accurately reflect the content of thought*, so should inverses, in the sense of group theory, often be replaced by adjoints. Adjoints retain the virtue of being uniquely determined reversal attempts, and very often exist when inverses do not. (Lawvere, 1994, 47) [our emphasis]

In the mid-sixties, Lawvere's project went through a fundamental shift whose explicit formulation appeared in 1969 in the paper entitled *Adjointness in Foundations*. Whereas the original plan was to give axioms for *the* category of categories and an invariant characterization of *the* category of sets, thus staying close to the traditional way of doing foundational research, in the mid-sixties, it became clear to Lawvere that some categories can be characterized *entirely* by adjoint functors and that these categories have a foundational status: they correspond in a precise sense to logical systems. Indeed, Lawvere had understood how propositional connectives could be described as adjoints between posets and that even quantifiers are, in fact, adjoints to an elementary operation, namely substitution. (This was officially presented in 1966b.) This is a key aspect of the whole situation that has unfortunately not been emphasized properly. Here is how Lawvere himself presented his views on the matter:

> This paper will have as one of its aims the giving of evidence for the universality of the concept of adjointness, which was first isolated and named in the conceptual sphere of category theory, but which also seems to pervade logic. Specifically, we describe in section III the notion of Cartesian closed category, which appears to be the appropriate abstract structure for making explicit the known analogy between the theory of functionality and propositional logic which is sometimes exploited in proof theory. The structure of a Cartesian closed category is entirely given by adjointness, as is the structure of a "hyperdoctrine", which includes quantification as well. Precisely analogous "quantifiers" occur

> in realms of mathematics normally considered far removed from the province of logic or proof theory. As we point out, recursion (at least on the natural numbers) is also characterized entirely by an appropriate adjoint; thus it is possible to give a theory, roughly proof theory of intuitionistic higher-order number theory, in which all important axioms (logical or mathematical) express instances of the notion of adjointness. (Lawvere, 1969, 282)

What Lawvere is not saying here is that these adjoints arise as adjoints to what ought to be considered *elementary* functors. As we have said, the goal here is not to give an axiomatization of *the* category of categories or *the* category of sets, even when the definite description is interpreted in categorical terms, but it is more general and abstract: it is to present strictly in terms of adjoints certain *types* of categories that are foundationally significant. It in this light that the well-known following statement about foundations found in the 1969 paper has to be understood:

> Foundations will mean here the study of what is universal in mathematics. Thus Foundations in this sense cannot be identified with any “starting-point” or “justification” for mathematics, though partial results in these directions may be among its fruits. But among the other fruits of Foundations so defined would presumably be guide-lines for passing from one branch of mathematics to another and for gauging to some extent which directions of research are likely to be relevant. (Lawvere, 1969, 281)

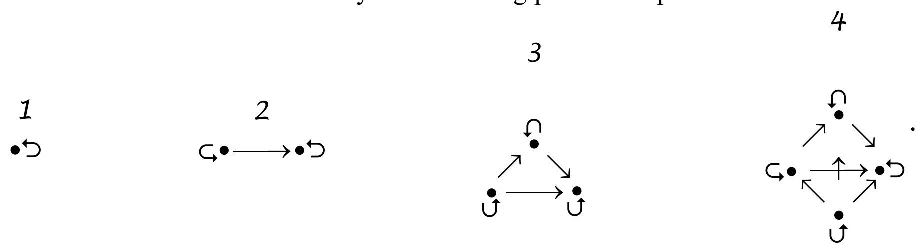
Adjoints are everywhere. As we have already indicated, important mathematical results and constructions can be described in terms of adjoint functors: compactness results, free structures of a certain kind and their variants, e.g. universal enveloping algebra or abelianization of a group, Galois theory, the adjointness between loop-space construction and the suspension construction in homotopy theory, etc. But these specific adjunctions, which illustrate the diversity and the extent of the phenomenon, are not what Lawvere was after. Lawvere had identified what deserve to be called *structural adjunctions* that happen, and this is the truly remarkable fact, to be equivalent in a very specific sense to logical concepts and systems. The categories thus described have a different status than the categories previously considered.

> More recently, the search for universals has also taken a conceptual turn in the form of Category Theory, which began with viewing as a new mathematical object the totality of all morphisms of the mathematical objects of a given species A, and then recognizing that these new mathematical objects all belong to a common non-trivial species C which is independent of A. (Lawvere, 1969, 281)

What Lawvere is now after are these non-trivial species C that are *independent* of previously given mathematical objects of species A. The independence is here guaranteed by the fact that the non-trivial species C are defined entirely in terms of adjoints. At this stage, category theory is presented as being fully autonomous, at least at the *conceptual* level.

It should be emphasized that his work on the category of categories and the category of sets prepared Lawvere remarkably well for this qualitative shift, for, in the first case, he had to consider categories in a purely abstract fashion, something which no one had done systematically before, and, in the second case, he tried to substitute infinite operations by elementary adjunctions in the context of sets. This can be seen directly by looking at his papers on the category of categories and on the elementary theory of the category of sets respectively.

In the paper on the category of categories, Lawvere introduces four abstract finite categories: the first one has one object and one morphism and is denoted by $\mathit{1}$; the second one has two objects and one non-trivial morphism and is denoted by $\mathit{2}$; the third one has three objects and three non-trivial morphisms and is denoted by $\mathit{3}$; and the last one has four objects and six non-trivial morphisms and is denoted by $\mathit{4}$. These categories are abstract in the sense that the objects and the morphisms, apart from the trivial identity morphisms, have no identity other than being part of the category. They have to be considered as being abstract data. This abstract character is illustrated by the following pictorial representations:



These categories are used to represent objects, morphisms, composition of morphisms and associativity of morphisms respectively. More specifically, a functor $\mathit{1} \xrightarrow{F} \mathcal{C}$ picks an object of $\mathcal{C}$, a functor $\mathit{2} \xrightarrow{G} \mathcal{C}$ picks a morphism of $\mathcal{C}$, etc. Of course, these categories are related by obvious functors. Thus, these simple abstract categories, these basic forms, are used to represent the basic properties of categories themselves. We should also point out that these geometric patterns are obviously related to simplicial sets and that this connection is underlying contemporary research in higher-dimensional categories. See, for instance, Leinster 2002 on Street's approach.

The other key element is the operation of exponentiation $\mathcal{D}^{\mathcal{C}}$ which can be described as an adjoint. In the proper context, this operation contributes to the reduction of higher-order infinitary operations to finitary algebraic operations. With these data, Lawvere introduced the notion, a new abstract form, of a Cartesian closed category. A *Cartesian closed category* $\mathcal{C}$ is a category with the following three adjunctions:

1. The unique functor $\mathcal{C} \longrightarrow \mathbf{1}$ has a *right adjoint*; this amounts to the claim that $\mathcal{C}$ has a terminal object;
2. The diagonal functor $\Delta : \mathcal{C} \longrightarrow \mathcal{C} \times \mathcal{C}$, which to every object X of $\mathcal{C}$ assigns the pair $\langle X, X \rangle$ and acts on morphisms in the obvious fashion, has a *right adjoint*; this amounts to the claim that $\mathcal{C}$ has binary products, e.g. for each pair $\langle X, Y \rangle$, there is an object $X \times Y$ of $\mathcal{C}$ together with morphisms $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$, called the *projections*, satisfying the following universal property: for each object Z and every morphisms $Z \xrightarrow{f} X$ and $Z \xrightarrow{g} Y$, there is a unique morphism $Z \xrightarrow{h} X \times Y$ such that the following diagram

$$\begin{array}{ccccc} & & Z & & \\ & {}^{f}\swarrow & \downarrow h & \searrow^{g} & \\ X & \xleftarrow{p_X} & X \times Y & \xrightarrow{p_Y} & Y \end{array}$$

   commutes.
3. Let X be a fixed object of $\mathcal{C}$. Then, each functor $\mathcal{C} \xrightarrow{X \times (-)} \mathcal{C}$ has a *right adjoint*; the right adjoint is the exponentiation functor $\mathcal{C} \xrightarrow{(-)^X} \mathcal{C}$. It can also be characterized by a universal property that we will not state here.

One striking feature of these adjoints is that they do not relate a category of structures to another, different, category of structures. The functors are here between a category $\mathcal{C}$ and categories of the form $\mathcal{C}^{\mathcal{D}}$, where the category $\mathcal{D}$ is usually entirely abstract and finite. For instance, in condition ii. above, the diagonal functor can be described as a functor $\Delta : \mathcal{C} \longrightarrow \mathcal{C}^2$, where the category $\mathbf{2}$ denotes the abstract category with two objects and no other morphisms than the identity morphisms. These three simple axioms are surprisingly powerful. Many important properties follow directly from them. For instance, it is easy to show that a Cartesian closed category has all finite products and various exponential laws are obtained directly from the adjoint situation. In other words, a Cartesian closed category can be characterized as having all finite products and all its objects are exponentiable.

Using the notion of a Cartesian closed category, Lawvere then introduced the notion of an *hyperdoctrine*, also characterized by the existence of certain adjoints, although in this case, there is an important underlying structure, nowadays seen as being a fibration. (See for instance Jacobs 1999.) The notion of an elementary topos came quickly afterwards and can be defined by following the same general principle. Indeed, a topos $\mathcal{E}$ is a Cartesian closed category with all finite limits and a subobject classifier. As we have just seen being Cartesian closed is characterized by the existence of certain adjoints and so is the notion of having finite limits (in fact, one only needs to add the existence of equalizers, and this amounts to the existence of a right adjoint to an appropriately defined diagonal functor (see Mac Lane, 1998, p. 88.)). The existence of a subobject classifier is also equivalent to the existence of an adjoint situation.

In 1970, it became clear to every category theorist that category *theory* contained *all* the resources to define new mathematical forms, e.g. abelian categories, Quillen's model categories, Barr's regular and exact categories, Lawvere-Tierney's elementary toposes, which in turn provided new contexts to do mathematics entirely within these contexts and to solve problems in various fields. Furthermore, these new forms could be directly linked to well-known forms, in particular *logical* systems. Indeed, following Lawvere, various mathematicians including Mitchell, Bénabou, Reyes, Makkai and especially Joyal, showed how certain categories — regular, coherent, Heyting, Boolean, geometric —, were equivalent in a precise technical sense to logical systems and how various theorems of logic could be translated into theorems about categories and vice-versa, theorems about categories could be translated into theorems about logical systems, e.g. completeness results. (See, for instance, Makkai & Reyes 1977 or Johnstone 2002, vol. 2, chap. D.) This correspondence is still being explored and exploited, especially in theoretical computer science. (See for instance Scott 2000.)

4 What is category theory?

Let us first recapitulate the different functions that category theory has played in the history that I have sketched. There are, first, the heuristic roles:

1. To provide a conceptually consistent framework, i.e. give a general definition of the notion of natural transformation;
2. To provide a language to state precisely what certain mathematical theories are, e.g. homology and cohomology theories, homological algebra, etc.;
3. To provide a language useful in the statement of certain theorems and their proofs;
4. To provide a language that suggests new definitions and new proofs, e.g. dualities in algebraic topology;
5. To provide a language that unifies various mathematical notions and theories.

To many mathematicians, even today, this is what category theory is all about. It is, in a nutshell, a useful tool to organize, present and develop certain areas of mathematics that are usually considered as being already given in a different setting. The main point here is that these applications do not use constructions *on* and *in* categories, they do not use categorical concepts, that is concepts defined directly in terms of categories, functors, natural transformations and their properties. They do not use category *theory*. The main concepts, categories, functors and natural transformations, and the language, arrows, commutative diagrams, are seen as useful prosthetics to the mind. But they are believed to be just that: extraordinarily useful *for us*, visual creatures that we are. At the end of the day, it is believed that the mathematics done with these aids do not depend *essentially* on categories, functors, etc. This position is sometimes pushed even further when it is claimed that categories themselves depend essentially on other, simpler, concepts. This dependence is not without reminding me of the debate in philosophy of mind on the dependence of the mind on the body, of the spirit on the material, or, to put it in a metaphorical language that brings us closer to our problem, of the form on the substance, although the dependence relation is sometimes presented as being a relation of *cognitive* dependence — to *understand* the notion *X*, one has to *understand* the notion *Y* — or *logical* dependence — to *define* the notion *X*, one has to *define* the notion *Y* — , the two relations being sometimes identified. The key here is of course the notion of *abstraction* as a cognitive/logical process and whether once a notion has been abstracted and that an entirely autonomous context has been provided for it, it can be considered in and for itself. I believe that this process of abstraction started in the early nineteen forties with Eilenberg and Mac Lane, but that it is only in the nineteen sixties and seventies that an autonomous context for the notions involved emerged, first by considering constructions of categories, second by defining mathematical concepts and doing parts of mathematics directly in categories, third by using categorical means to introduce structures in categories, or equivalently structured categories, fourth by seeing that these structures could be presented and understood by purely categorical means, in particular adjoint functors, fifth by establishing connections between fundamental categorical structures and logical frameworks. This process of abstraction is still going on with the developments of (weak) higher-dimensional categories and their applications.

Once category *theory* was developed and used, in particular when the central theoretical role played by adjoint functors was understood, a fascinating process of reversal of perspective, a gestalt switch, took place: what was seen as a useful tool in organizing and guiding mathematical thought became a theoretical framework that revealed the basic or fundamental principles underlying mathematical concepts, theories and theorems. Thus, Stone duality theorem is indeed more perspicuously presented in the context of categories and functors — it is organized neatly and the basic consequences of the result are transparent — but once it is seen as a special case of a very general adjoint situation, a *theoretical* understanding of the phenomenon becomes available. Category theory is not *applied* to Stone's theorem, it is the latter that becomes a specific instance of a general, universal conceptual situation.

Although it might in the end be more obscure than what I have said so far, I dare at this stage put forward a slogan that, I believe, sums up the core of what I have been presenting: *category theory is the architectonic of mathematics*. Category theory is, indeed, as in the philosophical sense of the expression "architectonic", the systematization of mathematical knowledge. Mathematical knowledge *is* systematic. Mathematics *is* a conceptual system. That much is indubitable. Of course, the set theoretical modeling of mathematical knowledge is also systematic. However, I believe that the standard set theoretical foundations for mathematics do not reflect this fact in the same way that category theory does and the differences are crucial. What category theory reveals is how the fundamental constructions of mathematics, even the logical operations underlying mathematical thinking, are related to one another systematically, how they arise from one another according to simple and general basic principles. It is not that set theory does not reveal general principles of construction, the fact is, these principles are simply subsumed under the more general and abstract principles of category theory. Furthermore, category theory provides an *intrinsic* classification of mathematical concepts as well as the means to see how the various mathematical categories, no pun intended, are related to one another.

One should also keep in mind the original meaning of the expression "architectonic": pertaining to constructions. It is, of course, *conceptual* constructions that are at issue. Indeed, category *theory* is about various fundamental and general conceptual constructions to which or rather within which other, more specific, mathematical concepts can find their place. Category theory provides *an overall conceptual frame* for mathematics. This frame, it must be said, has no "starting point" or "basement". One should imagine that we are building a space station, not a skyscraper or any other similar building that has to stand on solid grounds. Building in space must be done according to general principles, according to general laws of physics and engineering, but the construction does not have to have a definite orientation, an up and a down, a foundation in a *geocentric* sense of that expression. Perhaps category theory is forcing us to make a conceptual Copernican revolution.

I will go even one step further: *category theory is the architectonic of concepts* or *of conceptual systems in general*. (And, yes, I guess one could say that it is the architectonic of Reason… and it is extraordinarily tempting to go back to Kant. I will leave this task to others since I do not believe that it would be illuminating. I don't think using Kant here would be of any use, even though it could be fun, for category theory and its properties are more clear, at least to me, that Kant can be). The claim is simply that category theory can illuminate fields other than pure mathematics and that it will play in these fields a role similar to the one it now plays in pure mathematics.

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