A NOTE ON THE CATEGORY OF PARTIAL
DIFFERENTIAL EQUATIONS *)

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ABSTRACT. The category of partial differential equations as introduced
by A. M. Vinogradov is shown to be comonadic in the case of a fixed
base manifold of independent variables.

KEY WORDS. Comonad, Eilenberg - Moore category, nonlinear partial
differential equation.

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The theory of partial differential equations was made categorical by
A. M. Vinogradov in the late seventies. The category was first described
in [4]. From e.g. [5,6,7] we see that the categorical approach is useful
for both the theory and practice of differential equations.

In this note we would like to contribute to a better understanding of
the category itself, at least in the case of a fixed base manifold of
independent variables. This category is given an alternative description
here, as the Eilenberg - Moore category of a (rather well-known) comonad.

We use only very fundamental facts about comonads. More detailed
information is available in [2], in dual form: Algebraic theories =
= monads are comonads in the opposite category.

1. THE COMONAD. The endofunctor of the comonad we use is the familiar
∞-jet prolongation functor \( J^\infty \) for fibered manifolds, so first we need to
have a workable base category of \( \infty \)-dimensional fibered manifolds in which
\( J^\infty \) could act. Perhaps in the simplest way it is obtained when admitting

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\( n = \infty \) in the standard definition of an \( n \)-dimensional smooth manifold, with the following agreements:

1.1. \( \mathbb{R}^\infty \) is considered with the product topology.

1.2. A map \( f: U \to \mathbb{R}^\nu \) of an open set \( U \subseteq \mathbb{R}^\infty \) is regarded as smooth whenever all its components \( f^i: U \to \mathbb{R} \) are smooth; a map \( f: U \to \mathbb{R} \) is regarded as smooth whenever \( U \) admits an open covering \( U = \cup_i U_i \) such that every \( f^i|_{U_i} \) smoothly depends on only a finite number of variables.

No topological requirements (as \( T_2 \), countable basis etc.) are supposed. Here \( \infty = \aleph_0 \).

From now on, \( \mathcal{M} \) will denote the category of the \( v \)-dimensional manifolds, \( v \leq \infty \), with smooth maps (= whose every coordinate expression is smooth) as morphisms. Obviously, \( \mathcal{M} \) has finite products.

We define a submanifold \( M \) of a dimension \( m \leq \infty \) and a codimension \( k \leq \infty \) in an \( n \)-dimensional manifold \( N \), \( n = m + k \), as a subspace \( M \subseteq N \), locally homeomorphic to \( \mathbb{R}^m \times \mathbb{O} \to \mathbb{R}^m \times \mathbb{R}^k = \mathbb{R}^n \). We define an \( n \)-dimensional fibered manifold \( N \) with the \( m \)-dimensional base manifold \( M \) and \( k \)-dimensional fibres, \( n = m + k \leq \infty \), as a smooth map \( p: N \to M \) such that it is a factoring map of topological spaces, locally homeomorphic to the projection \( \mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^k \to \mathbb{R}^m \).

Let \( \mathcal{M}_M \) denote the category of all fibered manifolds over a fixed finite dimensional base manifold \( M \), all morphisms being over \( M \).

Our basic technical tool is an equalizer. See it in [2].

1.3. Proposition. Let \( f, g: N \to P \) be two morphisms of \( \mathcal{M} \), resp. \( \mathcal{M}_M \). Let \( E = \{ x \in N \mid fx = gx \} \) be a submanifold in \( N \). Then

\[
\begin{array}{ccc}
E & \longrightarrow & N \\
\text{\hspace{1cm}} & \overset{f}{\longrightarrow} & \text{\hspace{1cm}} P \\
\text{\hspace{1cm}} & \overset{g}{\longrightarrow} & \text{\hspace{1cm}}
\end{array}
\]

is an equalizer in \( \mathcal{M} \) resp. \( \mathcal{M}_M \).

Our definition of an \( r \)-jet \( j^r_x \gamma \), \( r \leq \infty \), of a local cross section \( \gamma \) of an \( n \)-dimensional fibered manifold \( \gamma: Y \to M \), \( n \leq \infty \), is formally the same as its standard version for \( n < \infty \), see [3,4,5,6,7] hence omitted. The same concerns the so called standard coordinates \( \ldots, \bar{x}^i, \ldots, \bar{y}^k, \ldots, \bar{y}^k_{i_1}, \ldots, \bar{y}^k_{i_s}, \ldots \).
on $j^r Y$, $x^r$ being the coordinates on $M$ and $y^r$ being the coordinates in the fibres of $Y$. It is essential that the transformation law for standard coordinates in $j^r Y$ is smooth in the sense of 1.2, even if $Y$ is $\omega$-dimensional, so that we have well-defined functors $j^r : M^r \to M^r$ for $r \leq \omega$.

1.4. The functors $j^r : M^r \to M^r$ preserve finite products and the equalizers of 1.3. In local coordinates it is evident.

We have also natural transformations $\pi : \omega^r 0 : j^r \to \text{Id}$ and $\nu : \omega^r 0 : j^r \to j^r j^r$ defined by $\pi Y : j^r Y \to Y$, $j^r x \gamma \mapsto \gamma(\pi)$ and $\nu Y : j^r Y \to j^r j^r Y$, $j^r x \gamma \mapsto j^r j^r x \gamma$. They satisfy the easily verifiable conditions of commutativity of the diagrams

which are precisely the same as those needed for $j^r$ together with $\pi, \nu$ constituting a comonad. Thus, $j^r = (j^r, \pi, \nu)$ is a comonad in $M^r$. For the reader's convenience we finish this section with the explicit description of the Eilenberg–Moore category $M^r$.

The objects of $M^r$ called $j^r$-coalgebras, are pairs $(Y, \xi)$ with $Y \in M^r$ and $\xi : j^r Y \to j^r Y$ over $M$ such that

```
    Y    j^r Y
     \downarrow \quad \downarrow \pi Y
    id \quad \gamma Y
      \downarrow \quad \downarrow \gamma Y
    Y
```

commute. The morphisms $(Y_1, \xi_1) \to (Y_2, \xi_2)$, called $j^r$-homomorphisms, are maps $j^r ; Y_1 \to Y_2$ over $M$ such that

```
    j^r f
    \downarrow \quad \downarrow \xi_1 \quad \xi_2
    j^r Y_1 \quad j^r Y_2
    f^r \quad \gamma_1 \quad \gamma_2
    Y_1 \quad Y_2
    \downarrow \quad \downarrow \gamma_1 \quad \gamma_2
    Y_1 \quad Y_2
```

commutes.
1.5. It follows from the definitions, that every \((\mathcal{J}^\infty Y, \mathcal{U} Y)\) is a \(\mathcal{J}^\infty\)-coalgebra. It is called a cofree coalgebra because of its universal property: For any \((A, \alpha) \in M^J_M\) and any \(f:A \to Y\) the composition \(f^\# = \mathcal{J}^\infty f \circ \alpha\) is the only \(\mathcal{J}^\infty\)-homomorphism \((A, \alpha) \to (\mathcal{J}^\infty Y, \mathcal{U} Y)\) such that \(\sigma_\circ f^\# = f\).

2. Differential equations. In this section we identify the \(\mathcal{J}^\infty\)-coalgebras with infinitely prolonged systems of partial differential equations whose manifold of independent variables is \(M\). In their definition we slightly differ from [4,5,6].

2.1. An \(r\)-th order system, \(r < \infty\), of partial differential equations, henceforth simply an equation, say

\[
\mathcal{J}^l(\ldots, \mathcal{J}^k Y_{i_1} \ldots, \mathcal{J}^k Y_{i_s} \ldots) = \mathcal{J}^l(\ldots, \mathcal{J}^k x_{i_1} \ldots, \mathcal{J}^k x_{i_s} \ldots)
\]

is written in arrows as an equalizer

\[
E \xrightarrow{e} \mathcal{J}^n Y \xrightarrow{\mathcal{J}^n f} Z \xrightarrow{g} Z
\]

in \(M^J_M\) in the sense of 1.3. Here \(i, i_1, \ldots, i_s = 1, \ldots, \dim M; s \leq r\); \(l = 1, \ldots, \dim Z\); \(k = 1, \ldots, \dim Y\).

A solution of such an equation, say \(y^k = \gamma^k(\ldots, \mathcal{J}^k x_{i_1} \ldots, \ldots)\), is represented by a local cross section \(\gamma\) of \(Y\) such that \(\mathcal{J}^n f \circ \mathcal{J}^n \gamma = \mathcal{J}^n g \circ \mathcal{J}^n \gamma\) i.e. such that \(\mathcal{J}^n \gamma\) factors through \(e: E \to \mathcal{J}^n Y\).

An infinite prolongation of such an equation is, by definition, the equation together with all its differential consequences, i.e. the system

\[
\frac{d^s}{dx^{i_1} \ldots dx^{i_s}} f^l = \frac{d^s}{dx^{i_1} \ldots dx^{i_s}} g^l \quad 0 \leq s < \infty
\]

Here

\[
\frac{d}{dx^i} = \frac{\partial}{\partial x^i} + \sum \frac{\partial}{\partial y^k_{i_1} \ldots i_s} \frac{\partial}{\partial y^k_{i_1} \ldots i_s}
\]

is the so called total derivative with respect to \(x^i\).
2.2. Expressed in arrows, the infinite prolongation of 2.1 is the equalizer of $j^\infty f \circ \iota^\infty_\gamma$ and $j^\infty g \circ \iota^\infty_\gamma$, if it exists:

$$
\begin{array}{cccccc}
E^\infty & \rightarrow & j^\infty Y & \rightarrow & j^\infty Y & \rightarrow & j^\infty Z \\
\uparrow \iota^\infty_\gamma & & \downarrow j^\infty Y & & \downarrow j^\infty Y & & \downarrow j^\infty g \\
E^\infty & \rightarrow & j^\infty Y & \rightarrow & j^\infty Y & \rightarrow & j^\infty Y
\end{array}
$$

Here $\iota^\infty_\gamma : j^\infty Y \rightarrow j^\infty j^\infty Y$. It is easily verified that

2.3. $E$ and $E^\infty$ have the same solutions in the above sense.

The infinitely prolonged equations are the objects of the Vinogradov category. We show how they can be converted into $j^\infty$-coalgebras. By 1.3, 2.1 and 2.2 there is a unique arrow $e^\infty$ completing the diagram

$$
\begin{array}{ccc}
ej^\infty E & \rightarrow & j^\infty j^\infty Y \\
\uparrow e^\infty & & \downarrow j^\infty j^\infty Y \\
E^\infty & \rightarrow & j^\infty Y
\end{array}
$$

The so obtained square is easily checked to be universal, via the universality of $e^\infty$ and $j^\infty e$. Consequently, it is also preserved by $j^\infty$, by 1.3 and 2.1.22 of [2], and the existence of $e$ in

$$
\begin{array}{ccc}
ej^\infty E & \rightarrow & j^\infty j^\infty Y \\
\downarrow e^\infty & & \downarrow j^\infty j^\infty Y \\
ej^\infty j^\infty E & \rightarrow & j^\infty j^\infty j^\infty Y
\end{array}
$$

follows.

2.4. Proposition. $(E^\infty, \sim)$ is a $j^\infty$-coalgebra.

Proof: The front square of the last diagram reads: $(E^\infty, \sim)$, if it were a coalgebra, would be a subcoalgebra of the cofree coalgebra $(j^\infty j^\infty Y, \iota Y)$, by $e^\infty$. 

In this situation it is known (3.1.10 of [2]) that \((E^\infty, \tilde{e})\) is indeed a \(j^\infty\)-coalgebra, if only \(e^\infty, j^\infty j^\infty e^\infty\) are both monomorphisms, but this is the case.

2.5. From the other side, a \(j^\infty\)-coalgebra \((E, e)\) is an equation \(\mu E = j^\infty e\) via the (absolute) Beck equalizer

\[
\begin{array}{c}
E \xrightarrow{e} j^\infty E \xrightarrow{\mu E} j^\infty j^\infty E
\end{array}
\]

This equation is infinitely prolonged \(=\) isomorphic to its infinite prolongation. Indeed, it holds

\[
eq (j^\infty \mu E \circ \mu E, j^\infty j^\infty e \circ \mu E) = \eq (\mu j^\infty E \circ \mu E, \mu j^\infty E \circ j^\infty e) = \eq (\mu E, j^\infty e)
\]

because \(\mu j^\infty e\) is a monomorphism.

Natural question is, what is the interpretation of the solutions of differential equations in terms of the \(M^j_M\). We start with the following observation: The isomorphism \(j^\infty \text{id} : M \to j^\infty M\) converts \(M\) into a \(j^\infty\)-coalgebra. Since \((j^\infty Y, \mu Y)\) is cofree, it follows that the \(j^\infty\)-homomorphisms \(M \to j^\infty Y\) are just \(\infty\)-jet prolongations \(j^\infty Y\) of global sections \(\gamma : M \to Y\) (over \(M\)).

From 3.1.10 of [2] again we deduce that

2.5. Morphisms \(M \to (E^\infty, \tilde{e})\) in \(M^j_M\) are just global solutions of the equation \(E^\infty\), i.e., of the equation \(E\), in view of 2.3. Consequently, \(j^\infty\)-homomorphisms are the right morphisms between equations in the sense that they transform solutions to solutions, via composition.

3. Cartan distribution. Hence \(M^j_M\) and \(DE\) of [5,6] both satisfy conditions 1 - 4 of [5,6] on a category of differential equations to be reasonable. We shall show in this section that, actually, \(M^j_M = DE_M = DE\) restricted to a fixed base manifold \(M\). An object of \(DE_M\) is roughly speaking a manifold \(E \in M\), together with a Frobenius distribution on it, is interpreted as an equation together with its Cartan distribution, consisting of all tangent planes to (formal) solutions = 1-jets of formal solutions. A morphism of \(DE_M\) = the Cartan distribution preserving differential operator (a map) between underlying manifolds.
In our terms, Cartan distribution is simply $e_1 : E \to j^1 E$ if $e_r$ denotes the composition $E \xrightarrow{j^r E} j^{r+1} E \xrightarrow{j^{r+1} e_1} j^{r+1} E$, $r < \infty$, for a coalgebra $(E, e) \in M^j_M$. A map $f : E \to E'$ between two $j^\infty$-coalgebras $(E, e), (E', e')$ preserves the Cartan distribution, if $j^1 f \circ e_1 = e'_1 \circ f$. Thus, to identify $M^j_M$ with $DE_M$ it is necessary and sufficient to prove

3.1. Proposition. A map $f : E \to E'$ is a $j^\infty$-homomorphism if and only if $j^1 f \circ e_1 = f \circ e_1$.

Proof. With the help of

\[
\begin{array}{c}
E \xrightarrow{j^r E} j^{r+1} E \xrightarrow{j^{r+1} e_1} j^{r+1} E, \\
E' \xrightarrow{j^{r+1} e'_1} j^{r+1} E'.
\end{array}
\]

we easily prove by induction, that $j^r f \circ e_r = e'_r \circ f \quad \forall r < \infty$ if $j^1 f \circ e_1 = e'_1 \circ f$. The equality $j^\infty f \circ e = e' \circ f$ then follows from the fact that $j^\infty E' = \lim j^r E'$ in $M_M$. This proves the "if" part, the "only if" part being evident.

The restriction to fixed $M$ means that the independent variables are prescribed for the whole category and undergo no transformations by morphisms. Nevertheless, this constraint is unimportant for many aspects of $[4, 5, 6, 7]$. For instance, in $M^j_M$ there is an analog of the universal linearization operator $\mathcal{V}$, namely the vertical bundle functor $V$ of $[3]$, $1.6.1$. Because of its commutation property $\mathcal{V}j^r = j^r \mathcal{V}$ it admits an extension to $V : M^j_M \to M^j_M$ as $(E, e) \mapsto (VE, VE \xrightarrow{\mathcal{V}e} Vj^\infty E \equiv j^\infty VE)$. The natural projection $\tau E : E \to E'$ then gives a natural transformation of functors $V \xrightarrow{\tau} \text{Id}$ in $M^j_M$.

In $[4, 6]$ the universal linearization is used to compute infinitesimal symmetries of equations. An infinitesimal symmetry turns out to be a special vertical vector field on $E$, in our terms
3.2. Proposition. An infinitesimal symmetry, $\Phi$, of an equation $(E, e) \in M^3_M$ is a section of the vertical bundle $\tau_E: VE \rightarrow E$, which is simultaneously $j^\infty$-homomorphism, i.e. for which

\[
\begin{array}{ccc}
E & \xrightarrow{\Phi} & VE \\
\downarrow{e_1} & & \downarrow{VE_1} \\
J^1E & \xrightarrow{\Phi} & J^1VE
\end{array}
\]

commutes.

Proof. The diagram is that of 3.1. Expressed in local coordinates it gives the condition of [6], Proposition 11.

4. Concluding remarks

4.1. The result of Kock [1] that $j^\infty: M_M \rightarrow M_M$ admits an extension to $j^\infty: M_M \rightarrow M_M^\infty$ possessing left adjoint $p^\infty: M_M^\infty \rightarrow M_M$, where $M_M \rightarrow M_M^\infty$ can be proved in purely classical terms as well. Objects of $M_M^\infty$ are pairs $\varepsilon = (E_0, E)$ of a fibered manifold $E$ and its fibered submanifold $E_0$, and morphisms $\varepsilon \rightarrow \varepsilon'$ are, locally certain $\infty$-jets of maps of pairs $(E_0, E) \rightarrow (E_0', E')$, with respect to derivations in directions transversal to $E_0 \rightarrow E$. Hence isomorphism classes in $M_M^\infty$ are naturally identified with "infinitesimal parts of fibered manifolds".

4.2. From $p^\infty \rightarrow j^\infty$ and Yoneda lemma it follows, that there exist natural transformations $\circ: Id \rightarrow p^\infty$, $\delta: p^\infty \rightarrow p^\infty$ such that

\[
\begin{array}{ccc}
\tilde{M}_M^\infty(X, e) & \xrightarrow{\tau_X} & \tilde{M}_M^\infty(p^\infty X, e) \\
\downarrow{\circ} & & \downarrow{\delta} \\
\tilde{M}_M^\infty(p^\infty X, e) & \equiv & \tilde{M}_M^\infty(X, j^\infty e)
\end{array}
\]

commute for all $X, e \in M_M^\infty$. Then, as can be easily seen, $p^\infty = (p^\infty, \circ, \delta)$ is
a monad in $\mathcal{M}_1^\infty$ and moreover

$$(\mathcal{M}_1^\infty)^P \cong (\mathcal{M}_1^\infty)^J.$$ 

Thus, there is a category of "infinitesimal parts of differential equations" which is both monadic and comonadic.

4.3. There is a natural question (of P. Michor) whether the category of differential equations is cartesian closed. The answer is not, although the condition of being a $j^\infty$-homomorphism is a differential one. It is obstructed by the fixed $M$. As for the full category $\mathcal{DE}$ of [5] the question is opened.

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