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A NOTE ON THE CATEGORY OF PARTIAL DIFFERENTIAL EQUATIONS *)

Michal Marvan

ABSTRACT. The category of partial differential equations as introduced by A. M. Vinogradov is shown to be comonadic in the case of a fixed base manifold of independent variables.

KEY WORDS. Comonad, Eilenberg - Moore category, nonlinear partial differential equation.

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The theory of partial differential equations was made categorical by A. M. Vinogradov in the late seventies. The category was first described in [4]. From e.g. [5,6,7] we see that the categorical approach is useful for both the theory and practice of differential equations.

In this note we would like to contribute to a better understanding of the category itself, at least in the case of a fixed base manifold of independent variables. This category is given an alternative description here, as the Eilenberg - Moore category of a (rather well-known) comonad.

We use only very fundamental facts about comonads. More detailed information is available in [2], in dual form: Algebraic theories = = monads are comonads in the opposite category.

1. THE COMONAD. The endofunctor of the comonad we use is the familiar ∞-jet prolongation functor j^∞ for fibered manifolds, so first we need to have a workable base category of ∞-dimensional fibered manifolds in which j^∞ could act. Perhaps in the simplest way it is obtained when admitting

*) *This paper is in final form and no version of it will be submitted for publication elsewhere.*

$n = \infty$ in the standard definition of an n-dimensional smooth manifold, with the following agreements:

1.1. R^{∞} is considered with the product topology.

1.2. A map $f: U \to R^{\nu}$ of an open set $U \subseteq R^{\infty}$ is regarded as smooth whenever all its components $f^i: U \xrightarrow{f} R^{\nu} \xrightarrow{pr_i} R$ are smooth; a map $f: U \to R$ is regarded as smooth whenever U admits an open covering $U = \bigcup_i U_i$ such that every $f|_{U_i}$ smoothly depends on only a finite number of variables.

No topological requirements (as T_2, countable basis etc.) are supposed. Here $\infty = \aleph_0$.

From now on, M will denote the category of the ν-dimensional manifolds, $\nu \leq \infty$, with smooth maps (= whose every coordinate expression is smooth) as morphisms. Obviously, M has finite products.

We define a submanifold M of a dimension $m \leq \infty$ and a codimension $k \leq \infty$ in an n-dimensional manifold N, $n = m + k$, as a subspace $M \hookrightarrow N$, locally homeomorphic to $R^m \times 0 \hookrightarrow R^m \times R^k = R^n$. We define an n-dimensional fibered manifold N with the m-dimensional base manifold M and k-dimensional fibres, $n = m + k \leq \infty$, as a smooth map $p: N \to M$ such that it is a factoring map of topological spaces, locally homeomorphic to the projection $R^n = R^m \times R^k \to R^m$. Let M_M denote the category of all fibered manifolds over a fixed *finite dimensional* base manifold M, all morphisms being over M.

Our basic technical tool is an equalizer. See it in [2].

1.3. Proposition. *Let* $f, g: N \to P$ *be two morphisms of* M, *resp.* M_M. *Let* $E = \{x \in N \ ; \ fx = gx\}$ *be a submanifold in* N. *Then*

$$E \hookrightarrow N \underset{g}{\overset{f}{\rightrightarrows}} P$$

is an equalizer in M *resp.* M_M.

Our definition of an r-jet $j^r_x \gamma$, $r \leq \infty$, of a local cross section γ of an n-dimensional fibered manifold $Y \to M$, $n \leq \infty$, is formally the same as its standard version for $n < \infty$, see [3,4,5,6,7] hence omitted. The same concerns the so called standard coordinates $\ldots, x^i, \ldots, y^k, \ldots, y^k_{i_1 \ldots i_s}, \ldots$

on $j^r Y$, x^i being the coordinates on M and y^k being the coordinates in the fibres of Y. It is essential that the transformation law for standard coordinates in $j^r Y$ is smooth in the sense of 1.2, even if Y is ∞-dimensional, so that we have well-defined functors $j^r : \mathsf{M}_M \to \mathsf{M}_M$ for $r \le \infty$.

1.4. The functors $j^r : \mathsf{M}_M \to \mathsf{M}_M$ preserve finite products and the equalizers of 1.3. In local coordinates it is evident.

We have also natural transformations $\pi := \pi^{\infty,0} : j^\infty \to Id$ and $\iota := \iota^{\infty,\infty} : j^\infty \to j^\infty j^\infty$ defined by $\pi Y : j^\infty Y \to Y$, $j^\infty_x \gamma \mapsto \gamma(x)$ and $\iota Y : j^\infty Y \to j^\infty j^\infty Y$, $j^\infty_x \gamma \mapsto j^\infty_x j^\infty \gamma$. They satisfy the easily verifiable conditions of commutativity of the diagrams

$$\begin{array}{ccccc} & & j^\infty & & \\ & \swarrow^{\mathrm{id}} & \downarrow \iota & \searrow^{\mathrm{id}} & \\ j^\infty & \xleftarrow[\pi j^\infty]{} & j^\infty j^\infty & \xrightarrow[j^\infty \pi]{} & j^\infty \end{array} \qquad \begin{array}{ccc} j^\infty & \xrightarrow{\iota} & j^\infty j^\infty \\ \downarrow \iota & & \downarrow \iota j^\infty \\ j^\infty j^\infty & \xrightarrow[j^\infty \iota]{} & j^\infty j^\infty j^\infty \end{array}$$

which are precisely the same as those needed for j^∞ together with π, ι constituting a comonad. Thus, $\mathrm{j}^\infty = (j^\infty; \pi, \iota)$ is a comonad in M_M. For the readers convenience we finish this section with the explicit description of the Eilenberg - Moore category $\mathsf{M}^{\mathrm{j}^\infty}_M$.

The objects of $\mathsf{M}^{\mathrm{j}^\infty}_M$, called j^∞-coalgebras, are pairs (Y, ζ) with $Y \in \mathsf{M}_M$ and $\zeta : Y \to j^\infty Y$ over M such that

$$\begin{array}{ccc} Y & \xrightarrow{\zeta} & j^\infty Y \\ & \searrow^{\mathrm{id}} & \downarrow \pi Y \\ & & Y \end{array} \qquad \begin{array}{ccc} Y & \xrightarrow{\zeta} & j^\infty Y \\ \downarrow \zeta & & \downarrow \iota Y \\ j^\infty Y & \xrightarrow[j^\infty \zeta]{} & j^\infty j^\infty Y \end{array}$$

commute. The morphisms $(Y_1, \zeta_1) \to (Y_2, \zeta_2)$, called j^∞-homomorphisms, are maps $f : Y_1 \to Y_2$ over M such that

$$\begin{array}{ccc} Y_1 & \xrightarrow{f} & Y_2 \\ \downarrow \zeta_1 & & \downarrow \zeta_2 \\ j^\infty Y_1 & \xrightarrow[j^\infty f]{} & j^\infty Y_2 \end{array}$$

commutes.

1.5. It follows from the definitions, that every $(j^\infty Y, \iota Y)$ is a j^∞-co-algebra. It is called a cofree coalgebra because of its universal property: For any $(A,\alpha) \in M_M^{j^\infty}$ and any $f:A \to Y$ the composition $f^\# = j^\infty f \circ \alpha$ is the only j^∞-homomorphism $(A,\alpha) \to (j^\infty Y, \iota Y)$ such that $\pi \circ f^\# = f$.

2. Differential equations. In this section we identify the j^∞-coalgebras with infinitely prolonged systems of partial differential equations whose manifold of independent variables is M. In their definition we slightly differ from [4,5,6].

2.1. An r-th order system, $r < \infty$, of partial differential equations, henceforth simply an equation, say

$$f^l(\ldots,x^i,\ldots,y^k,\ldots,y^k_{i_1\ldots i_s},\ldots) = g^l(\ldots,x^i,\ldots,y^k,\ldots,y^k_{i_1\ldots i_s},\ldots)$$

is written in arrows as an equalizer

$$E \xrightarrow{\ e\ } j^r Y \overset{f}{\underset{g}{\rightrightarrows}} Z$$

in M_M in the sense of 1.3. Here $i, i_1,\ldots,i_s = 1,\ldots,\dim M$; $s \le r$; $l = 1,\ldots,\dim Z$; $k = 1,\ldots,\dim Y$.

A solution of such an equation, say $y^k = \gamma^k(\ldots,x^i,\ldots,)$, is represented by a local cross section γ of Y such that $f \circ j^r\gamma = g \circ j^r\gamma$ i.e. such that $j^r\gamma$ factors through $e:E \to j^r Y$.

An infinite prolongation of such an equation is, by definition, the equation together with all its differential consequences, i.e. the system

$$\frac{d^s}{dx^{i_1}\ldots dx^{i_s}} f^l = \frac{d^s}{dx^{i_1}\ldots dx^{i_s}} g^l \qquad 0 \le s < \infty$$

Here

$$\frac{d}{dx^i} = \frac{\partial}{\partial x^i} + \sum y^k_{i_1\ldots i_s i} \frac{\partial}{\partial y^k_{i_1\ldots i_s}}$$

is the so called total derivative with respect to x^i.

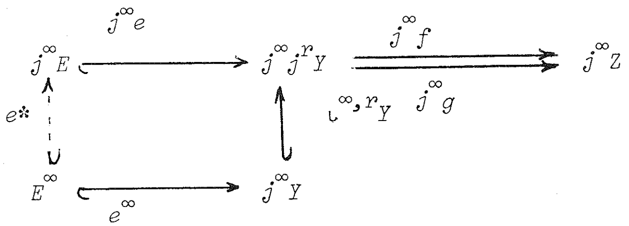
2.2. Expressed in arrows, the infinite prolongation of 2.1 is the equalizer of $j^{\infty}f \circ \iota^{\infty,r}$ and $j^{\infty}g \circ \iota^{\infty,r}$, if it exists:

$$E^{\infty} \xhookrightarrow{e^{\infty}} j^{\infty}Y \underset{\iota^{\infty,r}Y}{\overset{\iota^{\infty,r}Y}{\rightrightarrows}} j^{\infty}j^{r}Y \underset{j^{\infty}g}{\overset{j^{\infty}f}{\rightrightarrows}} j^{\infty}Z$$

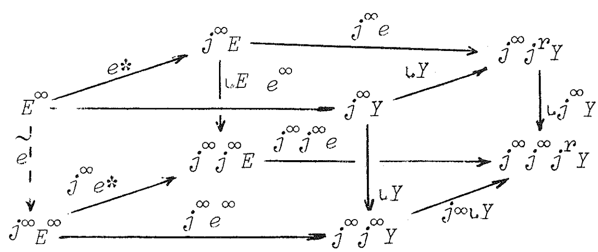
Here $\iota^{\infty,r}: j^{\infty}_{x}Y \mapsto j^{\infty}_{x}j^{r}Y$. It is easily verified that

2.3. E and E^{∞} have the same solutions in the above sense.

The infinitely prolonged equations are the objects of the Vinogradov category. We show how they can be converted into j^{∞}- coalgebras. By 1.3, 2.1 and 2.2 there is a unique arrow e^{*} completing the diagram



The so obtained square is easily checked to be universal, via the universality of e^{∞} and $j^{\infty}e$. Consequently, it is also preserved by j^{∞}, by 1.3 and 2.1.22 of [2], and the existence of $\tilde{e}$ in



follows.

2.4. Proposition. *$(E^{\infty}, \tilde{e})$ is a j^{∞}-coalgebra.*

Proof: The front square of the last diagram reads: $(E^{\infty}, \tilde{e})$, if it were a coalgebra, would be a subcoalgebra of the cofree coalgebra $(j^{\infty}Y, \iota Y)$, by e^{∞}.

In this situation it is known (3.1.10 of [2]) that $(E^{\infty}, \tilde{e})$ is indeed a j^{∞}-coalgebra, if only $e^{\infty}, j^{\infty} j^{\infty} e^{\infty}$ are both monomorphisms, but this is the case.

2.5. From the other side, a j^{∞}-coalgebra (E, e) is an equation $\iota E = j^{\infty} e$ via the (absolute) Beck equalizer

$$E \xrightarrow{e} j^{\infty} E \underset{j^{\infty} e}{\overset{\iota E}{\rightrightarrows}} j^{\infty} j^{\infty} E$$

This equation is infinitely prolonged = isomorphic to its infinite prolongation. Indeed, it holds

$$eq(j^{\infty} \iota E \circ \iota E,\ j^{\infty} j^{\infty} e \circ \iota E) = eq(\iota j^{\infty} E \circ \iota E, \iota j^{\infty} E \circ j^{\infty} e) = eq(\iota E, j^{\infty} e)$$

because $\iota j^{\infty} e$ is a monomorphism.

Natural question is, what is the interpretation of the solutions of differential equations in terms of the $M^{j^{\infty}}_{M}$. We start with the following observation: The isomorphism $j^{\infty} \mathrm{id} : M \to j^{\infty} M$ converts M into a j^{∞}-coalgebra. Since $(j^{\infty} Y, \iota Y)$ is cofree, it follows that the j^{∞}-homomorphisms $M \to j^{\infty} Y$ are just ∞-jet prolongations $j^{\infty} \gamma$ of global sections $\gamma \colon M \to Y$ (over M). From 3.1.10 of [2] again we deduce that

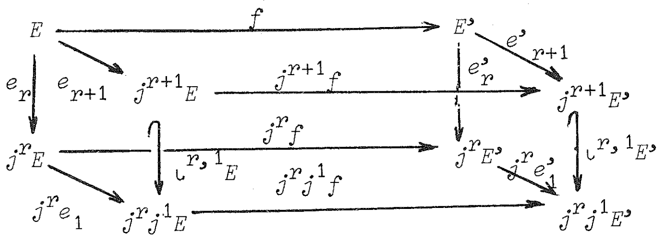
2.5. Morphisms $M \to (E^{\infty}, \tilde{e})$ in $M^{j^{\infty}}_{M}$ are just global solutions of the equation E^{∞} i.e. of the equation E, in view of 2.3. Consequently, j^{∞}-homomorphisms are the right morphisms between equations in the sense that they transform solutions to solutions, via composition.

3. Cartan distribution. Hence $M^{j^{\infty}}_{M}$ and DE of [5,6] both satisfy conditions 1 - 4 of [5,6] on a category of differential equations to be reasonable. We shall show in this section that, actually, $M^{j^{\infty}}_{M} = DE_M =$ $= DE$ restrictied to a fixed base manifold M. An object of DE_M = roughly speaking a manifold $E \in M_M$ together with a Frobenius distribution on it, is interpreted as an equation together with its Cartan distribution, consisting of all tangent planes to (formal) solutions = 1-jets of formal solutions. A morphism of DE_M = the Cartan distribution preserving differential operator (a map) between underlying manifolds.

In our terms, Cartan distribution is simply $e_1 : E \to j^1 E$ if e_r denotes the composition $E \xrightarrow{e} j^\infty E \xrightarrow{\pi^{\infty,r}} j^r E$, $r < \infty$, for a coalgebra $(E, e) \in M^{j^\infty}_M$. A map $f : E \to E'$ between two j^∞-coalgebras $(E, e), (E', e'_\infty)$ preserves the Cartan distribution, if $j^1 f \circ e_1 = e'_1 \circ f$. Thus, to identify $M^{j^\infty}_M$ with DE_M it is necessary and sufficient to prove

3.1. Proposition. *A map $f : E \to E'$ is a j^∞-homomorphism if and only if $j^1 f \circ e'_1 = f \circ e_1$.*

Proof. With the help of



we easily prove by induction, that $j^r f \circ e_r = e'_r \circ f$ $\forall\, r < \infty$ if $j^1 f \circ e_1 = e' \circ f$. The equality $j^\infty f \circ e = e' \circ f$ then follows from the fact that $j^\infty E' = \lim j^r E'$ in M_M. This proves the "if" part, the "only if" part being evident.

The restriction to fixed M means that the independent variables are prescribed for the whole category and undergo no transformations by morphisms. Nevertheless, this constraint is unimportant for many aspects of [4,5,6,7]. For instance, in $M^{j^\infty}_M$ there si an analog of the universal linearization operator l, namely the vertical bundle functor V of [3], 1.6.1. Because of its commutation property $Vj^r \cong j^r V$ it admits an extension to $V : M^{j^\infty}_M \to M^{j^\infty}_M$ as $(E, e) \mapsto (VE, VE \xrightarrow{Ve} Vj^\infty E \cong j^\infty VE)$. The natural projection $\tau E : \to E$ then gives a natural transformation of functors $V \xrightarrow{\tau} \mathrm{Id}$ in $M^{j^\infty}_M$.

In [4,6] the universal linearization is used to compute infinitesimal symmetries of equations. An infinitesimal symmetry turns out to be a special vertical vector field on E, in our terms

3.2. Proposition. *An infinitesimal symmetry, φ, of an equation $(E,e) \in M^{j^\infty}_M$ is a section of the vertical bundle $\tau E : VE \to E$, which is simultaneously j^∞-homomorphism, i.e. for which*

$$\begin{array}{ccc}
E & \xrightarrow{\ \varphi\ } & VE \\
 & & \big\downarrow Ve_1 \\
e_1 \big\downarrow & & Vj^1E \\
 & & \wr\!\wr \\
j^1E & \xrightarrow{\ j^1\varphi\ } & j^1VE
\end{array}$$

commutes.

Proof. The diagram is that of 3.1. Expressed in local coordinates it gives the condition of [6], Proposition 11.

4. Concluding remarks

4.1. The result of Kock [1] that $j^\infty : M_M \to M_M$ admits an extension to $j^\infty : M^\infty_M \to M^\infty_M$ possessing left adjoint $p^\infty : M^\infty_M \to M^\infty_M$, where $M_M \hookrightarrow M^\infty_M$ can be proved in purely classical terms as well. Objects of M^∞_M are pairs $\varepsilon = (E_0, E)$ of a fibered manifold E and its fibered submanifold E_0, and morphisms $\varepsilon \to \varepsilon'$ are, locally certain ∞-jets of maps of pairs $(E_0, E) \to (E'_0, E')$, with respect to derivations in directions transversal to $E_0 \hookrightarrow E$. Hence isomorphism classes in M^∞_M are naturally identified with "infinitesimal parts of fibered manifolds".

4.2. From $p^\infty \dashv j^\infty$ and Yoneda lemma it follows, that there exist natural transformations $o : \mathrm{Id} \to p^\infty$, $\delta : p^\infty p^\infty \to p^\infty$ such that

$$\begin{array}{ccccc}
 & M^\infty_M(X,\varepsilon) & & & \\
 {\scriptstyle o^*}\nearrow & & \nwarrow{\scriptstyle \pi_*} & & \\
M^\infty_M(p^\infty X,\varepsilon) & \cong & M^\infty_M(X, j^\infty\varepsilon) & &
\end{array}
\qquad
\begin{array}{ccc}
M^\infty_M(p^\infty p^\infty X,\varepsilon) & \cong & M^\infty_M(X, j^\infty j^\infty \varepsilon) \\
\delta^* \big\uparrow & & \big\uparrow \iota_* \\
M^\infty_M(p^\infty X,\varepsilon) & \cong & M^\infty_M(X, j^\infty\varepsilon)
\end{array}$$

commute for all $X, \varepsilon \in M^\infty_M$. Then, as can be easily seen, $\mathrm{p}^\infty = (p^\infty, o, \delta)$ is a monad in M^{∞}_{M}, and moreover

$$(M^{\infty}_{M})^{p^{\infty}} \cong (M^{\infty}_{M})^{j^{\infty}}.$$

Thus, there is a category of "infinitesimal parts of differential equations" which is both monadic and comonadic.

4.3. There is a natural question (of P. Michor) whether the category of differential equations is cartesian closed. The answer is not, although the condition of being a j^{∞}-homomorphism is a differential one. It is obstructed by the fixed M. As for the full category DE of [5] the question is opened.

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M. Marvan

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Department of Mathematics, Faculty of Science
University of J. E. Purkyně
Janačkovo nam. 2a
662 95 Brno
Czechoslovakia