LECTURES ON MAPPING CLASS GROUPS, BRAID GROUPS AND FORMALITY

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Abstract. These are the notes of the course "Mapping class groups, braid groups and formality" held in Strasbourg during the second semester of the academic year 2014–2015 (*Master "Mathématiques fondamentales"*, University of Strasbourg). Benjamin Enriquez gave in parallel some lectures on Lie theory and formality of the pure braid group, which are not included here.

Contents

1. Review of surfaces and curves

In this section, we assume a certain familiarity of the student with the basics of algebraic topology. Textbooks in algebraic topology include [Bre93] and [Hat02]. We shall use the following notations:

$$D^n := \{x \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}, \text{the } n\text{-dimensional } \textit{disk};$$

$$S^n := \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}, \text{the } n\text{-dimensional } \textit{sphere}.$$

1.1. Surfaces.

Here is our definition of a "surface."

Definition 1.1. *A* surface Σ *is a 2-dimensional topological manifold, possibly with boundary. The* interior *of* Σ *is* $\mathrm{int}(\Sigma) := \Sigma \setminus \partial\Sigma$ *where* $\partial\Sigma$ *denotes the boundary of* Σ*.*

We shall not review here the definition of a manifold with boundary (see for instance [Bre93] or [Hir76]). We only recall that it involves the notion of "atlas": at any point $x \in \Sigma$, there is a "chart" (U, φ) consisting of an open neighborhood U of x and a homeomorphism $\varphi : U \to \varphi(U)$ onto an open subset $\varphi(U)$ of $\mathbb{R} \times \mathbb{R}_{\geq 0}$. By definition, $x \in \partial\Sigma$ if and only if $\varphi(x) \in \mathbb{R} \times \{0\}$. If two charts (U, φ) and (V, ψ) are given at the same point, then we can consider the "coordinate change"

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \longrightarrow \psi(U \cap V) \tag{1.1}$$

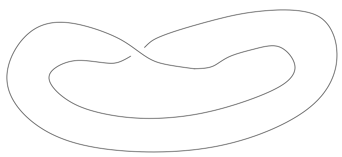
which is a homeomorphism from an open subset of $\mathbb{R} \times \mathbb{R}_{\geq 0}$ to another one.

In these lectures, unless otherwise specified, we assume that a surface Σ has the following properties:

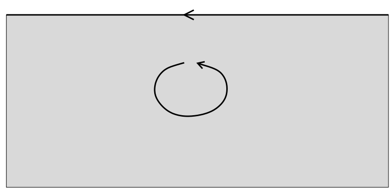
(i) Σ is connected;
(ii) Σ is compact;
(iii) Σ is orientable.

Some comments about the last two conditions are in order:

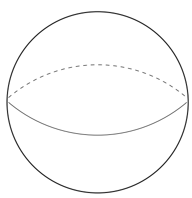
(ii) implies that $\partial\Sigma$ consists of finitely many copies of the circle S^1; if $\partial\Sigma = \varnothing$, then the surface Σ is said to be *closed*;
(iii) means that any coordinate change of Σ should be "orientation-preserving"; this property can be defined using homology but, in dimension 2, it is equivalent to say that the Moebius strip



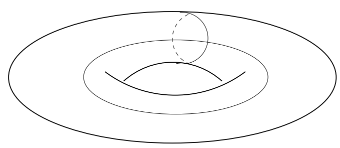
can *not* be embedded in Σ; by condition (i), Σ has exactly two orientations; an orientation on a surface Σ induces an orientation on any connected component of $\partial\Sigma$ (and vice-versa):



Example 1.2. Here are two "elementary" closed surfaces:



the sphere S^2



the torus $S^1 \times S^1$

■

We define three operations which we can perform on a surface:

(i) given a surface Σ, we can *remove a disk* from Σ: the new surface is

$$\Sigma^\circ := \Sigma \setminus \mathrm{int}(D)$$

where $D \subset \mathrm{int}(\Sigma)$ is a closed disk;

(ii) given two surfaces Σ_1 and Σ_2, with a boundary component $\delta_i \subset \partial\Sigma_i$ specified on each, we can do the *gluing*

$$\Sigma_1 \underset{\delta_1=\delta_2}{\cup} \Sigma_2 := (\Sigma_1 \sqcup \Sigma_2)/\sim$$

where $\sim$ is the equivalence relation identifying any $x_1 \in \delta_1$ to $\varphi(x_1) \in \delta_2$, for a fixed homeomorphism $\varphi : \delta_1 \to \delta_2$;

(iii) given two surfaces Σ_1 and Σ_2, we can construct the *connected sum*

$$\Sigma_1 \sharp \Sigma_2 := \big(\Sigma_1 \setminus \mathrm{int}(D_1)\big) \underset{\partial D_1=\partial D_2}{\cup} \big(\Sigma_2 \setminus \mathrm{int}(D_2)\big)$$

where $D_i \subset \mathrm{int}(\Sigma_i)$ is a closed disk.

The above operations are well-defined in the following sense:

(i) the homeomorphism type of Σ° does not depend on the choice of D;
(ii) the homeomorphism type of $\Sigma_1 \cup_{\delta_1=\delta_2} \Sigma_2$ does not depend on the choice of φ;
(iii) consequently, $\Sigma_1 \sharp \Sigma_2$ is also well-defined up to homeomorphism.

Furthermore, these operations can be defined for oriented surfaces:

(i) an orientation on Σ restricts to a unique orientation on Σ°;
(ii) if Σ_1 and Σ_2 are oriented and if φ is orientation-reversing, then there is a unique orientation on $\Sigma_1 \cup_{\delta_1=\delta_2} \Sigma_2$ that is compatible with those of Σ_1 and Σ_2;
(iii) consequently, there is a unique orientation on $\Sigma_1 \sharp \Sigma_2$ that is compatible with those of Σ_1 and Σ_2.

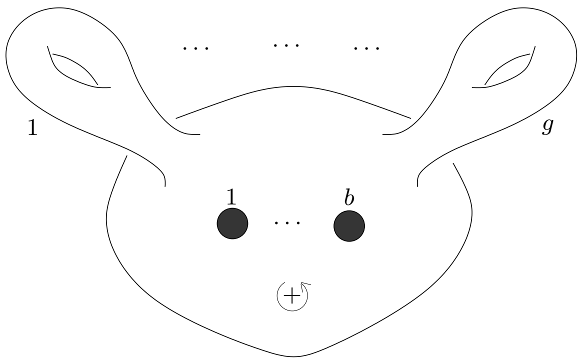
We can now construct infinitely many surfaces out of S^2 and $S^1 \times S^1$ using the above operations.

Definition 1.3. *Set $\Sigma_0 := S^2$ and, for any integer $g \geq 1$, set*

$$\Sigma_g := \underbrace{(S^1 \times S^1)\sharp \cdots \sharp (S^1 \times S^1)}_{g \text{ times}}.$$

Set $\Sigma_{g,0} := \Sigma_g$ and, for any integer $b \geq 1$, let $\Sigma_{g,b}$ be the surface obtained from Σ_g by removing b disks.

Of course, the surface $\Sigma_{g,b}$ is only defined up to homeomorphism, but we can also *fix* a "standard" surface $\Sigma_{g,b} \subset \mathbb{R}^3$ once and for all, and orient it, as shown below:



Theorem 1.4. *For any (connected, compact, orientable) surface S, there exists a unique pair $(g, b) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ such that S is homeomorphic to $\Sigma_{g,b}$.*

The unique integer $g \geq 0$ such that $S \cong \Sigma_{g,b}$ for some $b \geq 0$ is called the *genus* of S.

Sketch of proof. We prove the *unicity*. Let $(g_1, b_1), (g_2, b_2) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ be such that $\Sigma_{g_1,b_1} \cong \Sigma_{g_2,b_2}$: we must show that $g_1 = g_2$ and $b_1 = b_2$. We have $\partial\Sigma_{g_1,b_1} \cong \partial\Sigma_{g_2,b_2}$ and, since the number of connected components is a topological invariant, we obtain $b_1 = b_2$. According to Exercices 1.1–1.2, we have

$$g_i = \frac{1}{2}\big(\mathrm{rank}\ H_1(\Sigma_{g_i,b_i};\mathbb{Z}) - \max(b_i - 1, 0)\big).$$

Since the homology is a topological invariant, we have $H_1(\Sigma_{g_1,b_1};\mathbb{Z}) \simeq H_1(\Sigma_{g_2,b_2};\mathbb{Z})$ and we conclude that $g_1 = g_2$.

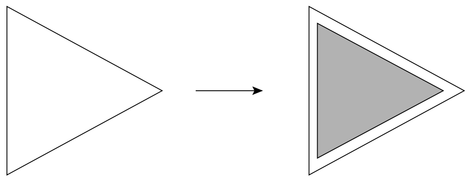
We now sketch the proof of the *existence*. Let S be a surface: we must prove that $S \cong \Sigma_{g,b}$ for some $g, b \geq 0$. We first consider the case where $\partial S = \varnothing$ and we accept the following classical result [Rad25].

Theorem. (Radó 1925) *Any closed surface S has a triangulation.*

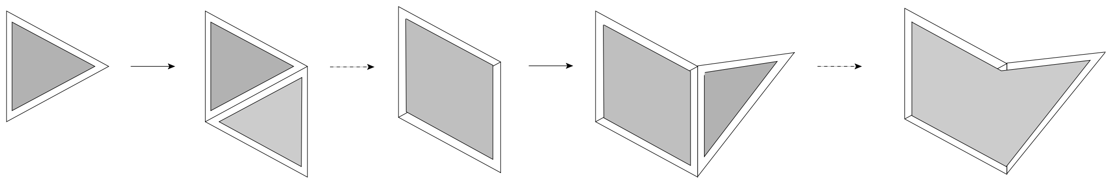
This result may seem obvious at a first glance, but it is actually quite difficult to prove: see for instance [Moi77, §8]. A *triangulation* of S is a homeomorphism $f : K \to S$, whose source K is a topological space consisting of (finitely many) copies of the 2-dimensional simplex

$$\Delta^2 := \{x \in \mathbb{R}^3 : x_1, x_2, x_3 \geq 0, x_1 + x_2 + x_3 = 1\}$$

which are glued one to the other along edges by affine isomorphisms. We call *triangles* the images of those simplices by f and, in the interior of each of these triangles, we color a smaller triangle:



We pick one of these colored triangles, and we merge it to an adjacent colored triangle of our choice. We repeat this process at much as possible, doing some choices at each step:



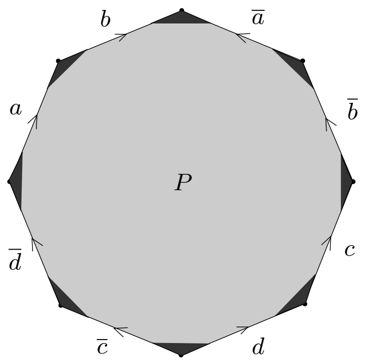
At the end of the process, we get a colored "polygonal" region in S which almost fills S. This shows that the surface S can be obtained from a polygon $P \subset \mathbb{R}^2$ by identifying its edges pairwisely: let

$$\pi : P \longrightarrow S = P/\sim$$

be the corresponding projection. It can happen that the integer

$$n_P := \sharp\, \pi\left(\{\text{vertices of } P\}\right)$$

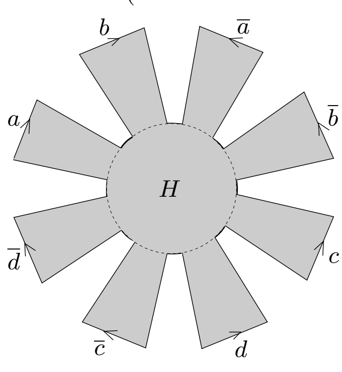
is greater than one. In this case, there is an edge e of P whose two vertices are not identified by $\sim$: so, the same happens for the "twin" edge $\overline{e}$. By collapsing e and simultaneously $\overline{e}$ to their midpoints, we see that n_P can be decreased by one. Hence we can assume that $n_P = 1$: let $\star \in S$ be the common image of all the vertices of P. There is a small closed disk $D \subset S$ such that $\star \in \operatorname{int}(D)$ and $\pi^{-1}(D)$ consists of disjoint neighborhoods of the vertices of P:



The surface

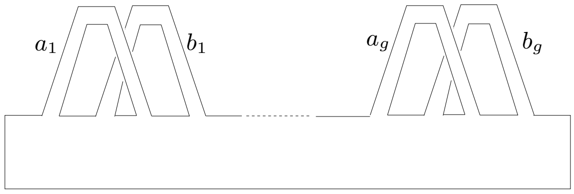
$$H := \big(P \setminus \pi^{-1}\operatorname{int}(D)\big)\,/\sim$$

can now be regarded as a disk with "handles" (one "handle" for each pair of twin edges in P):

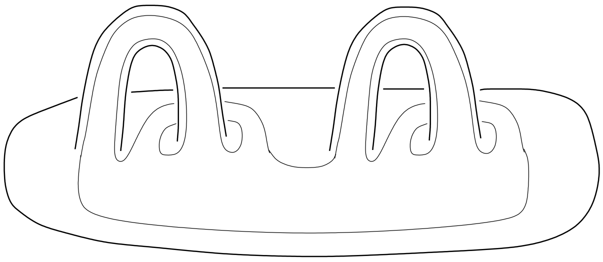


Thus the surface S is obtained by gluing a closed disk to H along its boundary. We pick one of the handle of H – which we call a_1. Because H has just one boundary component, there must be at least one other handle – which we call b_1 – whose attaching intervals "alternate" with those of a_1. Next, if another handle has an attaching interval "under" $a_1 \cup b_1$, we can always "slide" it far away from $a_1 \cup b_1$. Doing this repeatedly, we see that S is obtained from a surface of the type

$N_g :=$



(for some integer $g \geq 0$) by gluing a disk along its unique boundary component. Therefore $S \cong \Sigma_g$ since Σ_g is also obtained from N_g by gluing a disk along its boundary:



The existence in the general case is deduced from the closed case as follows. Assume that S has b boundary components $\delta_1, \dots, \delta_b$. Let S^+ be the surface obtained from S by gluing a closed disk D_i along each boundary component δ_i. Then S^+ is closed so that $S^+ \cong \Sigma_g$ for some $g \geq 0$. For all $i = 1, \dots, b$, let D'_i be the image of D_i under this homeomorphism. Then

$$S = S^+ \setminus \operatorname{int}(D_1 \cup \cdots \cup D_b) \;\cong\; \Sigma_g \setminus \operatorname{int}(D'_1 \cup \cdots \cup D'_b) = \Sigma_{g,b}. \qquad \square$$

A *smooth structure* on a surface Σ is an atlas which is equivalent to the given one and such that any coordinate change (1.1) is a C^∞-diffeomorphism. We accept the following result: see [Hat13] for a proof.

Theorem 1.5. *Any surface has a smooth structure, which is unique up to diffeomorphism.*

Thus, in 2-dimensional topology, one is allowed to freely use the tools of differential topology [Hir76]. For instance, we can define the orientability of a surface Σ by requiring that the Jacobian determinant of any coordinate change (1.1) should be positive.

1.2. **Curves.** Let Σ be a surface. Here is our definition of a "curve" in Σ.

Definition 1.6. *A* closed curve *in a surface Σ is a continuous map $\alpha : S^1 \to \Sigma$. It is* simple *if α is injective.*

In fact, a "closed curve" $\alpha : S^1 \to \Sigma$ will sometimes only refer to the image $\alpha(S^1)$. If we need also to record the image by α of the trigonometric orientation of S^1, then the closed curve is said to be *oriented*. The same closed curve as α with opposite orientation is denoted by $\overline{\alpha}$.

Lemma 1.7. *Any simple closed curved α in the interior of Σ has a closed neighborhood $N(\alpha)$ such that the pair $(N(\alpha), \alpha)$ is homeomorphic to the pair $(S^1 \times [-1,1], S^1 \times \{0\})$.*

Proof. Assuming that Σ and α are smooth, this is the simplest manifestation of the existence theorem of *tubular neighborhoods* for submanifolds [Hir76]. The "triviality" of the tubular neighborhood in this situation is due to the facts that α and Σ are orientable and that $\mathrm{GL}_+(1) = (0, +\infty)$ is contractible. $\square$

Two closed curves α and β are *homotopic* if there is a continuous map $h : S^1 \times [0,1] \to \Sigma$ (called a *homotopy*) such that $h(-,0) = \alpha$ and $h(-,1) = \beta$. A closed curve is *essential* if it is not homotopic to the constant curve or to a boundary component.

Two simple closed curves α and β are *isotopic* if there is a continuous map $h : S^1 \times [0,1] \to \Sigma$ (called an *isotopy*) such that $h(-,0) = \alpha$, $h(-,1) = \beta$ and $h(-,t)$ is simple for every t.

The above definitions can also be formulated in the smooth category if a smooth structure is specified on Σ. It turns out that two smooth simple closed curves are smoothly isotopic if and only if they are isotopic. Furthermore, we will accept the following classical result of Baer [Bae27] (see [Eps66] for a more modern treatment).

Theorem 1.8 (Baer 1927)**.** *Two essential simple closed curves in Σ are isotopic if and only if they are homotopic.*

This is not true anymore if the assumption "essential" is removed. Indeed consider the disk D^2 with its usual orientation. The curve ∂D^2 (with the orientation inherited from D^2) and the curve $-\partial D^2$ (with the opposite orientation) are both homotopic to the constant curve at $0 \in D^2$. But it can be shown that they are not isotopic.[1]

We shall now see two different notions of "intersection invariant" for closed curves in Σ. The first notion is homological and needs the curves to be oriented.

Definition 1.9. *Assume that Σ is oriented. Then the* (homology) intersection form *on Σ is the bilinear map*

$$\omega : H_1(\Sigma;\mathbb{Z}) \times H_1(\Sigma;\mathbb{Z}) \longrightarrow \mathbb{Z}, \ (a,b) \longmapsto \langle Dj_*(a), b\rangle$$

where $j_* : H_1(\Sigma;\mathbb{Z}) \to H_1(\Sigma,\partial\Sigma;\mathbb{Z})$ is induced by the inclusion $j : \Sigma \to (\Sigma,\partial\Sigma)$, $D : H_1(\Sigma,\partial\Sigma;\mathbb{Z}) \to H^1(\Sigma;\mathbb{Z})$ is the Poincaré duality, and $\langle -,-\rangle$ denotes the Kronecker evaluation.

When $a = [\alpha] \in H_1(\Sigma;\mathbb{Z})$ and $b = [\beta] \in H_1(\Sigma;\mathbb{Z})$ are represented by some oriented closed curves α and β, we have the following formula for $\omega(a,b)$. Working in the smooth category, we assume that α and β are *transversal* in the sense that

$$\forall x \in \alpha\cap\beta, \quad T_x\Sigma = \left\langle \vec{\alpha}_x, \vec{\beta}_x \right\rangle$$

and that α,β only meet at double intersection points; then

$$\omega([\alpha],[\beta]) = \sum_{x\in\alpha\cap\beta} \left\{ \begin{array}{ll} +1, & \text{if } (\vec{\alpha}_x,\vec{\beta}_x) \text{ is direct} \\ -1, & \text{otherwise} \end{array} \right\}.$$

(A student who is not familiar with Poincaré duality may accept the above formula as a definition of the form ω.) In particular, it follows that ω is skew-symmetric.

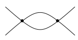
The second notion of "intersection invariant" is homotopic and does not need the curves to be oriented.

Definition 1.10. *Let a,b be homotopy classes of closed curves in Σ. The* (geometric) intersection number *of a,b is*

$$i(a,b) = \min\left\{ \sharp(\alpha\cap\beta) \,|\, \alpha\in a, \beta \in b\right\} \in \mathbb{Z}_{\geq 0}.$$

Two closed curves α,β in Σ are in minimal position *if $\sharp(\alpha\cap\beta) = i([\alpha],[\beta])$.*

The following is very useful to display two simple closed curves in minimal position.

Lemma 1.11. *Two smooth simple closed curves, which are transversal, are in minimal position if and only if they do not show any bigon* .

Proof. See [FM12, Proposition 1.7]. □

To conclude this section, we shall now see that there are only finitely many "types" of simple closed curves in the surface Σ. Here two simple closed curves α and β in Σ are said to have the same *topological type* if there is a self-homeomorphism of Σ carrying α into β.

Lemma 1.12. *Two simple closed curves α and β have the same topological type if, and only if,*

$$\Sigma \setminus \operatorname{int} N(\alpha) \cong \Sigma \setminus \operatorname{int} N(\beta).$$

Proof. Assume that α and β have the same topological type. Then there is a homeomorphism $f : \Sigma \to \Sigma$ such that $f(\alpha) = \beta$. We can assume that $f(N(\alpha)) = N(\beta)$, hence f restricts to a homeomorphism $\Sigma \setminus \operatorname{int} N(\alpha) \to \Sigma \setminus \operatorname{int} N(\beta)$.

Assume that there is a homeomorphism $h : \Sigma \setminus \operatorname{int} N(\alpha) \to \Sigma \setminus \operatorname{int} N(\beta)$. According to Exercice 1.8, we can also assume that h carries the components of $N(\alpha)$ to the components of $N(\beta)$. Hence, by using the identifications $N(\alpha) \cong S^1 \times [-1,1] \cong N(\beta)$, we can extend h to a homeomorphism $\widetilde{h} : \Sigma \to \Sigma$ such that $h(\alpha) = \beta$. □

Proposition 1.13. *There are only finitely many topological types of simple closed curves in Σ.*

This proposition remains valid if we require *orientation-preserving* self-homeomorphisms of Σ in the definition of the topological type of a simple closed curve.

[1] Hint: work in the smooth category, and use the fact that GL(2) is disconnected.

Proof. Let α be an arbitrary simple closed curve in Σ, and consider the possibly *disconnected* surface $\Sigma_\alpha := \Sigma \setminus \operatorname{int} N(\alpha)$. According to Lemma 1.12, it suffices to show that there are only finitely many possibilities for the homeomorphism type of Σ_α.

Assume that Σ_α is not connected: then Σ_α has two connected components S_1 and S_2. Let $g_i \geq 0$ be the genus of S_i and let $b_i \geq 1$ be the number of boundary components of S_i. Then

$$b_1 + b_2 = \sharp\,(\text{boundary component of } \Sigma) + 2$$

which shows that there only finitely many possibilities for b_1, b_2. Next, we have

$$\begin{aligned} \chi(\Sigma) &= \chi(\Sigma_\alpha) + \chi(N(\alpha)) - \chi(\partial N(\alpha)) \\ &= \chi(\Sigma_\alpha) + \chi(S^1 \times [0,1]) - \chi(S^1 \sqcup S^1) \;=\; \chi(\Sigma_\alpha) - \chi(S^1) \;=\; \chi(\Sigma_\alpha). \end{aligned}$$

By Exercice 1.3, we also have

$$\chi(\Sigma_\alpha) = \chi(S_1) + \chi(S_2) = (2 - 2g_1 - b_1) + (2 - 2g_2 - b_2).$$

We deduce that $2g_1 + 2g_2 = 4 - \chi(\Sigma) - b_1 - b_2$, which shows that there are only finitely many possibilities for g_1, g_2. The case where Σ_α is connected can be treated with the same kind of arguments. □

In addition to curves, we will also need "arcs" in the sequel. Assume that $\partial\Sigma \neq \varnothing$. A *proper arc* in Σ is a continuous map $\rho : [0,1] \to \Sigma$ such that $\rho^{-1}(\partial\Sigma) = \{0,1\}$. It is *simple* if ρ is injective. The notions of *homotopy* of arcs and *isotopy* of simple proper arcs are defined as we did for curves, by requiring endpoints of arcs to be fixed at any time.

1.3. **Exercices.**

Exercise 1.1. Let $g \geq 0$ and $b \geq 1$ be integers. Show that $\Sigma_{g,b}$ deformation retracts to a wedge of $2g + b - 1$ circles. Deduce that

$$H_0(\Sigma_{g,b};\mathbb{Z}) \simeq \mathbb{Z}, \quad H_1(\Sigma_{g,b};\mathbb{Z}) \simeq \mathbb{Z}^{2g+b-1}, \quad H_i(\Sigma_{g,b};\mathbb{Z}) = 0 \text{ for } i \geq 2.$$

Exercise 1.2. Let $g \geq 0$ be an integer. Using Exercise 1.1, show that

$$H_0(\Sigma_g;\mathbb{Z}) \simeq \mathbb{Z}, \quad H_1(\Sigma_g;\mathbb{Z}) \simeq \mathbb{Z}^{2g}, \quad H_2(\Sigma_g;\mathbb{Z}) \simeq \mathbb{Z}, \quad H_i(\Sigma_g;\mathbb{Z}) = 0 \text{ for } i \geq 3.$$

Exercise 1.3. Compute the Euler characteristic $\chi(\Sigma_{g,b})$ for any $g, b \geq 0$. Is the Euler characteristic a complete invariant of (compact, connected, orientable) surfaces?

Exercise 1.4. Compute the fundamental group $\pi_1(\Sigma_{g,b})$ for any $g, b \geq 0$. Is the fundamental group a complete invariant of (compact, connected, orientable) surfaces?

Exercise 1.5. Deduce from the classification of surfaces the *Jordan–Schoenflies theorem*: for any simple closed curve $\alpha \subset S^2$, the complement $S^2 \setminus \alpha$ consists of two connected components whose closures in S^2 are closed disks.

Exercise 1.6. Show that the intersection form ω of $\Sigma_{g,b}$ is non-singular if and only if $b \in \{0,1\}$.

Exercise 1.7. Which (compact, connected, orientable) surfaces can be embedded in the plane $\mathbb{R}^2$?

Exercise 1.8. Let Σ be a surface with a least two boundary components δ and δ', and let T be a connected subsurface of Σ such that $\delta \cup \delta' \subset T$. Construct an orientation-preserving self-homeomorphism of Σ which exchanges δ with δ' and is the identity outside T.

Exercise 1.9. A simple closed curve α in the interior of a surface Σ is *non-separating* if the space $\Sigma\setminus\alpha$ is connected. Show that any two non-separating simple closed curves in Σ have the same topological type.

Exercise 1.10. List all the topological types of simple closed curves on the surface $\Sigma_{g,1}$.

* * *

Solution to Exercise 1.1. It is enough to draw such a wedge of circles $\Gamma_{g,b}$ on a picture of $\Sigma_{g,b}$. Since the homology is a homotopy invariant, we have $H_*(\Sigma_{g,b};\mathbb{Z}) \simeq H_*(\Gamma_{g,b};\mathbb{Z})$ and we easily conclude.

Solution to Exercise 1.2. We have $\Sigma_g \cong \Sigma_{g,1} \cup D^2$ where $\Sigma_{g,1}$ and D^2 are glued along their boundary. It follows from the Mayer–Vietoris theorem (in reduced homology with $\mathbb{Z}$ coefficients) that we have a long exact sequence

$$H_2(\Sigma_{g,1}) \oplus H_2(D^2) \longrightarrow H_2(\Sigma_g) \xrightarrow{\partial_*} H_1(S^1) \longrightarrow H_1(\Sigma_{g,1}) \oplus H_1(D^2) \longrightarrow H_1(\Sigma_g) \longrightarrow \widetilde{H}_0(S^1),$$

which simplifies to

$$0 \longrightarrow H_2(\Sigma_g) \xrightarrow{\partial_*} H_1(S^1) \longrightarrow H_1(\Sigma_{g,1}) \longrightarrow H_1(\Sigma_g) \longrightarrow 0.$$

The map $H_1(S^1) \to H_1(\Sigma_{g,1})$ is obviously trivial, and we deduce that

$$H_2(\Sigma_g) \simeq H_1(S^1) \simeq \mathbb{Z}, \quad H_1(\Sigma_g) \simeq H_1(\Sigma_{g,1}) \simeq \mathbb{Z}^{2g}.$$

We also have $H_0(\Sigma_g) \simeq \mathbb{Z}$ since Σ_g is connected, and we have $H_i(\Sigma_g) = 0$ for $i \geq 3$ since Σ_g has a cellular decomposition with only cells of dimension ≤ 2.

Solution to Exercise 1.3. For $b > 0$, we have by Exercise 1.1

$$\chi(\Sigma_{g,b}) = 1 - (2g + b - 1) = 2 - 2g - b$$

and, for $b = 0$, we have by Exercise 1.2

$$\chi(\Sigma_g) = 1 - 2g + 1 = 2 - 2g.$$

Hence $\chi(\Sigma_{g,b}) = 2-2g-b$ for any $g, b \geq 0$. In particular, $\chi(\Sigma_{1,1}) = -1 = \chi(\Sigma_{0,3})$ although $\Sigma_{1,1} \not\cong \Sigma_{0,3}$.

Solution to Exercise 1.4. For $b > 0$, $\Sigma_{g,b}$ deformation retracts to a wedge of circles $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \zeta_1, \ldots, \zeta_{b-1}$ based at a point $\star \in \partial\Sigma_{g,b}$. Hence

$$\pi_1(\Sigma_{g,b}, \star) = F\left(\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g, \zeta_1, \ldots, \zeta_{b-1}\right), \quad \text{the free group on } 2g + b - 1 \text{ generators.}$$

For $b = 0$, we have $\Sigma_g \cong \Sigma_{g,1} \cup D^2$. Hence, by applying the Van Kampen theorem, we get for $g \geq 1$

$$\pi_1(\Sigma_{g,b}, \star) = \left\langle \alpha_1, \beta_1, \ldots, \alpha_g, \beta_g \mid [\beta_1^{-1}, \alpha_1] \cdots [\beta_g^{-1}, \alpha_g] \right\rangle$$

where $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ are appropriately oriented; for $g = 0$, we get $\pi_1(\Sigma_0) = \{1\}$. In particular, note that $\pi_1(\Sigma_0) \simeq \pi_1(\Sigma_{0,1})$ although $\Sigma_0 \not\cong \Sigma_{0,1}$.

Solution to Exercise 1.5. It is enough to show that $S_\alpha := S^2 \setminus \operatorname{int} N(\alpha)$ is the disjoint union of two closed disks.

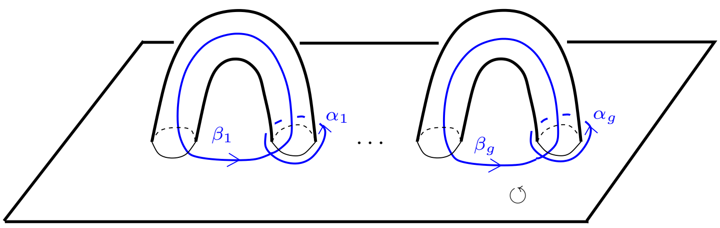
Assume that S_α is connected. Then the two endpoints of the interval $\{(1,0)\} \times [-1,1] \subset S^1 \times [-1,1] \cong N(\alpha)$ can be connected by a path in S_α: thus we obtain a closed curved $\beta \subset S^2$ which we orient in an arbitrary way. Since α and β meet in exactly one point, we have $\omega([\alpha], [\beta]) = \pm 1$ which contradicts the fact that $H_1(S^2; \mathbb{Z})$ is trivial. Therefore S_α has two connected components, which we call S_1 and S_2.

Since S^2 is closed, we must have $S_i \cong \Sigma_{g_i,1}$ for some integer $g_i \geq 0$. Hence

$$2 = \chi(S^2) = \chi(S_\alpha) = \chi(S_1) + \chi(S_2) = (1 - 2g_1) + (1 - 2g_2) = 2 - 2(g_1 + g_2)$$

which implies that $g_1 = g_2 = 0$. Hence $S_1 \cong S_2 \cong \Sigma_{0,1}$, and we have $\Sigma_{0,1} \cong D^2$ since $S^2 = \Sigma_0$ is obtained by gluing two disks along their boundaries.

Solution to Exercise 1.6. Assume that $b \in \{0, 1\}$ and consider the following system of oriented simple closed curves on $\Sigma_{g,b}$:



Set $a_i := [\alpha_i] \in H_1(\Sigma; \mathbb{Z})$ and $b_i := [\beta_i] \in H_1(\Sigma; \mathbb{Z})$ for all $i \in \{1, \ldots, g\}$. Then the matrix of ω in the basis $(a, b) := (a_1, \ldots, a_g, b_1, \ldots, b_g)$ is

$$\Omega := \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

Since we have $\det \Omega = 1$, the form ω is non-singular. Thus ω is a *symplectic form* and (a, b) is a *symplectic basis* of ω.

Assume that $b > 1$. Choose $(b-1)$ boundary components among the b available, and let $z_1, \ldots, z_{b-1} \in H_1(\Sigma_{g,b}; \mathbb{Z})$ be their homology classes. Then $z_i \neq 0$ and $\omega(-, z_i) = 0$ since any element of $H_1(\Sigma_{g,b}; \mathbb{Z})$ can be represented by an oriented closed curve disjoint from the i-th boundary component. Hence the radical of ω is not trivial.

N.B. The matrix of ω in the basis $(a, b, z) := (a_1, \ldots, a_g, b_1, \ldots, b_g, z_1, \ldots, z_{b-1})$ of $H_1(\Sigma_{g,b}; \mathbb{Z})$ is

$$\begin{pmatrix} 0 & I_g & 0 \\ -I_g & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

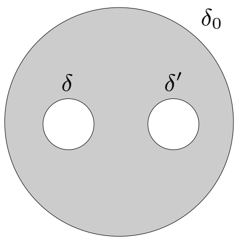
Solution to Exercise 1.7. Let $i : \Sigma \hookrightarrow \mathbb{R}^2$ be an embedding of a surface. Since Σ is compact, we can find a closed disk $D \subset \mathbb{R}^2$ which contains $i(\Sigma)$. We have the following commutative diagram:

$$\begin{array}{ccc} H_1(\Sigma;\mathbb{Z}) \times H_1(\Sigma;\mathbb{Z}) & \xrightarrow{\omega_\Sigma} & \mathbb{Z} \\ {\scriptstyle i_* \times i_*}\downarrow & & \Vert \\ H_1(D;\mathbb{Z}) \times H_1(D;\mathbb{Z}) & \xrightarrow[\omega_D]{} & \mathbb{Z} \end{array}$$

Since $H_1(D;\mathbb{Z}) = 0$, we must have $\omega_\Sigma = 0$. Then, by the previous exercise, the genus of Σ must be zero. Assume now that Σ is closed, i.e. Σ is homeomorphic to a 2-sphere: we can even assume that $\Sigma = S^2$. Fix a point $x \in S^2$ and let $p : \mathbb{R}^2 \to S^2$ be a continuous map which is a homeomorphism onto $S^2 \setminus \{x\}$. Then $f := p \circ i : S^2 \to S^2$ is an embedding whose image does not contain x. We denote by $H_\pm$ the two hemispheres of S^2 and let $E := H_+ \cap H_-$ be the equator. By Exercise 1.5, the simple closed curve $f(E)$ separates S^2 into two closed disks: one of them, called D_1, contains x in its interior while the other one, called D_2, does not. The map f sends $\mathrm{int}(H_+) \sqcup \mathrm{int}(H_-)$ into $D_1 \sqcup D_2$ so, for connectedness reasons, either $\mathrm{int}(H_+)$ is mapped into D_1 and $\mathrm{int}(H_-)$ into D_2, or vice-versa. Assume, for instance, the first possibility: then f is an embedding of H_+ into $D_1 \setminus \{x\}$ which sends the curve $\partial H_+ = E$ to the curve $\partial D_1 = f(E)$. The former is null-homotopic in H_+ while the latter generates $H_1(D_1 \setminus \{x\};\mathbb{Z}) \simeq \mathbb{Z}$ ... contradiction. We conclude that $\Sigma \cong \Sigma_{0,b}$ for some $b > 0$.

Conversely, a disk "with holes" $\Sigma_{0,b}$ (with $b > 0$) can be embedded in $\mathbb{R}^2$ in the obvious way.

Solution to Exercise 1.8. Let α be a proper simple arc in T connecting δ to δ'. There is a closed neighborhood of $\alpha \cup \delta \cup \delta'$ which is homeomorphic to $\Sigma_{0,3}$. Hence we can assume without loss of generality that $\Sigma = T \cong \Sigma_{0,3}$ as shown below:

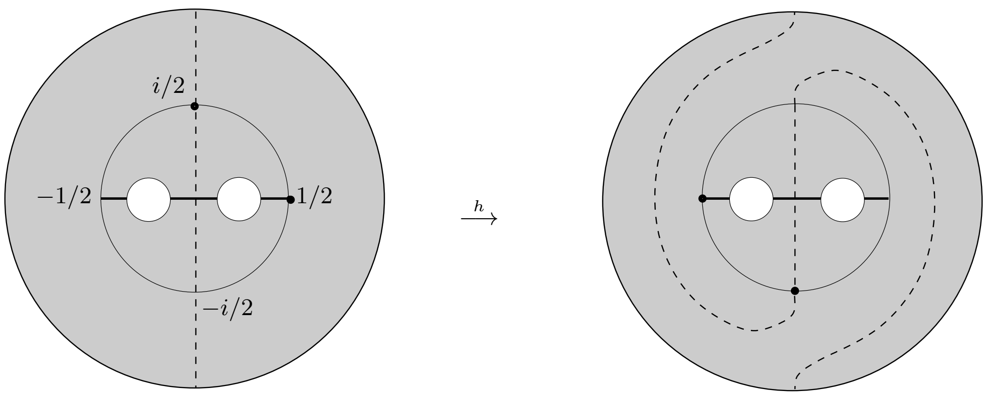


and we need to construct a homeomorphism $f : \Sigma \to \Sigma$ which is the identity on δ_0 and exchanges δ with δ'. To be even more explicit, we can assume that

$$\Sigma = D^2 \setminus \{z \in \mathbb{C} : |z - 1/4| < 1/8 \text{ or } |z + 1/4| < 1/8\} \subset \mathbb{C}.$$

Let $h : D^2 \to D^2$ be the homeomorphism defined by

$$h(z) := \begin{cases} -z & \text{if } |z| \leq 1/2 \\ \exp\big(2i\pi(1 - |z|)\big) \cdot z & \text{if } 1/2 \leq |z| \leq 1. \end{cases}$$



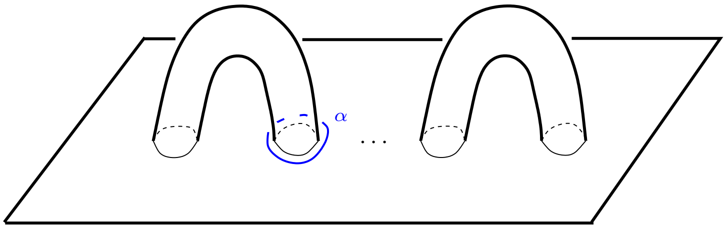
Clearly h exchanges the disk $\{z : |z - 1/4| < 1/8\}$ with the disk $\{z : |z + 1/4| < 1/8\}$ and h is the identity on $S^1 = \partial D^2$. Hence $f := h|_\Sigma$ has the desired properties.

Solution to Exercise 1.9. Assume that $\Sigma \cong \Sigma_{g,b}$ and let $\alpha \subset \mathrm{int}(\Sigma)$ be a non-separating simple closed curve. Then the surface $\Sigma_\alpha := \Sigma \setminus \mathrm{int}\, N(\alpha)$ has $b+2$ boundary components; let g_α be the genus of Σ_α. We have

$$2 - 2g_\alpha - (b+2) = \chi(\Sigma_\alpha) = \chi(\Sigma) = 2 - 2g - b,$$

hence $g_\alpha = g - 1$. We deduce that $\Sigma_\alpha \cong \Sigma_{g-1,b+2}$, i.e. the homeomorphism type of Σ_α does not depend on the choice of α.

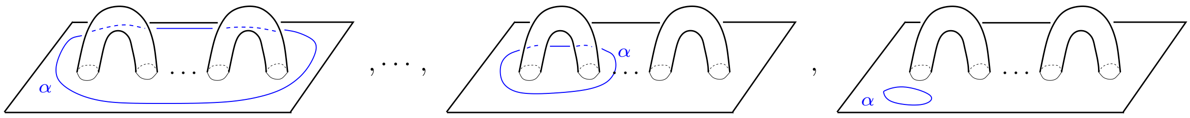
Solution to Exercise 1.10. Let α be an arbitrary simple closed curve on $\Sigma_{g,1}$. According to Exercise 1.9, there is only one possible topological type if we assume that α is non-separating:



Assume that α is separating: let S be the subsurface delimited by α containing $\partial\Sigma_{g,1}$ and let T be the other subsurface. We have $S \cong \Sigma_{h,2}$ and $T \cong \Sigma_{k,1}$ for some $h, k \geq 0$. Moreover,

$$1 - 2g = \chi(\Sigma_{g,1}) = \chi(S) + \chi(T) = (-2h) + (1 - 2k) = 1 - 2(h + k)$$

so that $g = h + k$. Therefore, in the separating case, there are g possibilities of topological types:



$(h,k) = (0,g)$ $\quad$ $(h,k) = (g-1,1)$ $\quad$ $(h,k) = (g,0)$

2. Mapping class groups

We can now turn to the main subject of these lectures, namely the mapping class groups of surfaces. Our exposition is inspired from the books [Bir74], [FM12] and the survey article [Iva02]; it mainly follows the informal notes [Mas09] with some additions and corrections.

2.1. **Definitions of mapping class groups.**

Let Σ be a (compact, orientable, connected) surface, and fix an orientation on Σ. We will consider the following group:

$$\mathrm{Homeo}^{+,\partial}(\Sigma) := \big\{\text{homeomorphisms } \Sigma \to \Sigma \text{ which are orientation-preserving and fix } \partial\Sigma\big\}$$

Recall that, for any two topological spaces X and Y, two continuous maps $a, b : X \to Y$ are *homotopic* if there is a continuous map $H : X \times [0,1] \to Y$ (called an *homotopy*) such that $H(-,0) = a$ and $H(-,1) = b$. Two homeomorphisms $a, b : \Sigma \to \Sigma$ are *isotopic* if there is an homotopy $H : \Sigma \times [0,1] \to \Sigma$ (called an *isotopy*) such that $H(-,t)$ is a homeomorphism for every t. Two homeomorphisms $a, b : \Sigma \to \Sigma$ which fix $\partial\Sigma$ are *isotopic rel* $\partial\Sigma$ if they are related by an isotopy $H : \Sigma \times [0,1] \to \Sigma$ such that $H(-,t)$ fixes $\partial\Sigma$ for every t.

Definition 2.1. *The* mapping class group *of* Σ *is* $\mathcal{M}(\Sigma) := \mathrm{Homeo}^{+,\partial}(\Sigma)/(\textit{isotopy rel } \partial\Sigma)$*.*

Other common notations for the mapping class group include the following ones: $\mathrm{MCG}(\Sigma)$, $\mathrm{Mod}(\Sigma)$ and $\mathcal{M}_{g,b}$, $\Gamma_{g,b}$ if Σ is one of the "standard" surfaces $\Sigma_{g,b}$. The isotopy class of an $f \in \mathrm{Homeo}^{+,\partial}(\Sigma)$ is denoted by $[f] \in \mathcal{M}(\Sigma)$, or simply by $f \in \mathcal{M}(\Sigma)$.

Remark 2.2. Equip the set $\mathrm{Homeo}^{+,\partial}(\Sigma)$ with the *compact-open topology*, i.e. the topology generated by the family of subsets $\{V(K,U)\}_{K,U}$ indexed by compact subsets $K \subset \Sigma$ and open subsets $U \subset \Sigma$ and defined by

$$V(K,U) := \big\{f \in \mathrm{Homeo}^{+,\partial}(\Sigma) : f(K) \subset U\big\}.$$

(See [Bre93] for an exposition of this kind of topology.) Then a continuous map $\rho : [0,1] \to \mathrm{Homeo}^{+,\partial}(\Sigma)$ is the same thing as an isotopy rel $\partial\Sigma$ between $\rho(0)$ and $\rho(1)$. (Observe in particular that, during an isotopy, an orientation-preserving homeomorphism remains orientation-preserving.) Therefore, $\mathcal{M}(\Sigma)$ can also be defined as the set of path-connected components of $\mathrm{Homeo}^{+,\partial}(\Sigma)$. ■

The above definition of a mapping class group has several variations. Here are two variations which give equivalent definitions:

- We could fix a smooth structure on Σ and we could replace homeomorphisms up to isotopy by *diffeomorphisms up to smooth isotopy*, but this would not affect the definition of $\mathcal{M}(\Sigma)$. In other words, any homeomorphism is isotopic to a diffeomorphism (this generalizes Theorem 1.5, see [Hat02]) and any two isotopic diffeomorphisms are smoothly isotopic.
- We could consider homeomorphisms up to *homotopy rel* $\partial\Sigma$ instead of considering them up to isotopy rel $\partial\Sigma$. This would not affect the definition of $\mathcal{M}(\Sigma)$ since Baer used Theorem 1.8 to also show the following: two orientation-preserving homeomorphisms $\Sigma \to \Sigma$ are homotopic rel $\partial\Sigma$ if and only if they are isotopic rel $\partial\Sigma$ (see [Bae28, Eps66]).

Here are two other variations which give non-equivalent definitions of the mapping class group:

- We could allow homeomorphisms not to be the identity on the boundary. Then, generally speaking, the resulting group $\mathcal{M}^{\eth}(\Sigma)$ differs from the above group $\mathcal{M}(\Sigma)$: specifically, we have an exact sequence of groups
$$\mathbb{Z}^b \longrightarrow \mathcal{M}(\Sigma) \longrightarrow \mathcal{M}^{\eth}(\Sigma) \longrightarrow \mathfrak{S}_b \longrightarrow 1,$$
see Exercise 2.10.
- We could allow homeomorphisms not to preserve the orientation: we denote by $\mathcal{M}^{\pm}(\Sigma)$ the resulting group. If the boundary of Σ is non-empty, then any boundary-fixing homeomorphism must preserve the orientation: $\mathcal{M}^{\pm}(\Sigma) = \mathcal{M}(\Sigma)$. If the boundary is empty, then we have a short exact sequence of groups
$$1 \longrightarrow \mathcal{M}(\Sigma) \longrightarrow \mathcal{M}^{\pm}(\Sigma) \longrightarrow \mathbb{Z}_2 \longrightarrow 1,$$
see Exercise 2.11.

2.2. **First examples of mapping class groups.**

We now give two examples of mapping class groups. First, let us prove that the mapping class group of the disk D^2 is trivial. Recall that a topological space X *deformation retracts* to a subspace $R \subset X$ if there exists a continuous map $r : X \to R$ (called the *retraction*) such that $r|_R = \mathrm{id}_R$ and the map $X \to X$ defined by $x \mapsto r(x)$ is homotopic to id_X; if R is a singleton in X, then X is said to be *contractible*.

Proposition 2.3 (Alexander's trick)**.** *The space* $\mathrm{Homeo}^\partial(D^2) = \mathrm{Homeo}^{+,\partial}(D^2)$ *is contractible. In particular, we have* $\mathcal{M}(D^2) = \{1\}$.

Proof. For any homeomorphism $f : D^2 \to D^2$ which is the identity on the boundary, and for all $t \in [0,1]$, we define a homeomorphism $f_t : D^2 \to D^2$ by

$$f_t(x) := \begin{cases} t \cdot f(x/t) & \text{if } 0 \leq |x| \leq t, \\ x & \text{if } t \leq |x| \leq 1. \end{cases}$$

Here D^2 is seen as a subset of $\mathbb{C}$ and $|x|$ denotes the modulus of $x \in \mathbb{C}$. Then, the map

$$H : \mathrm{Homeo}^\partial(D^2) \times [0,1] \longrightarrow \mathrm{Homeo}^\partial(D^2), \ (f,t) \longmapsto f_t$$

is a homotopy between the retraction of $\mathrm{Homeo}^\partial(D^2)$ to $\{\mathrm{id}_{D^2}\}$ and the identity of $\mathrm{Homeo}^\partial(D^2)$. Thus, $\mathrm{Homeo}^\partial(D^2)$ deformation retracts to $\{\mathrm{id}_{D^2}\}$. □

We now compute the mapping class group of the torus $S^1 \times S^1$, which is not trivial.

Proposition 2.4. *Let* (a,b) *be the basis of* $H_1(S^1 \times S^1;\mathbb{Z})$ *defined by* $a := [S^1 \times 1]$ *and* $b := [1 \times S^1]$. *Then, the map*

$$\kappa : \mathcal{M}(S^1 \times S^1) \longrightarrow \mathrm{SL}(2;\mathbb{Z})$$

which sends any isotopy class $[f]$ *to the matrix of* $f_* : H_1(S^1 \times S^1;\mathbb{Z}) \to H_1(S^1 \times S^1;\mathbb{Z})$ *with respect to the basis* (a,b), *is a group isomorphism.*

Proof. The fact that we have a group homomorphism $\kappa : \mathcal{M}(S^1 \times S^1) \to \mathrm{GL}(2;\mathbb{Z})$ follows from the functoriality of the homology. We now check that $\kappa([f]) \in \mathrm{SL}(2;\mathbb{Z})$ for any $[f] \in \mathcal{M}(S^1 \times S^1)$: set

$$\kappa([f]) = \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix}$$

where the integers f_{ij}'s are such that $f_*(a) = f_{11}a + f_{21}b$ and $f_*(b) = f_{12}a + f_{22}b$. Since f preserves the orientation, it also leaves invariant the intersection form ω. So, we have

$$1 = \omega(a,b) = \omega\big(f_*(a), f_*(b)\big) = \omega\big(f_{11}a + f_{21}b, f_{12}a + f_{22}b\big) = f_{11}f_{22} - f_{21}f_{12} = \det \kappa([f]).$$

The surjectivity of κ can be proved as follows. We identify $\mathbb{R}^2/\mathbb{Z}^2$ to $S^1 \times S^1$ via the homeomorphism defined by $(u,v) \mapsto (e^{2i\pi u}, e^{2i\pi v})$ for any $(u,v) \in \mathbb{R}^2$. Any matrix $T \in \mathrm{SL}(2;\mathbb{Z})$ defines a linear isomorphism

$$\mathbb{R}^2 \longrightarrow \mathbb{R}^2, \ \begin{pmatrix} u \\ v \end{pmatrix} \longmapsto T \cdot \begin{pmatrix} u \\ v \end{pmatrix}$$

which leaves $\mathbb{Z}^2$ globally invariant and so induces an (orientation-preserving) homeomorphism $t : \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{R}^2/\mathbb{Z}^2$. It is easily checked that $\kappa([t]) = T$.

To prove the injectivity, let us consider a homeomorphism $f : S^1 \times S^1 \to S^1 \times S^1$ such that $\kappa([f])$ is trivial. Since $\pi_1(S^1 \times S^1)$ is abelian, the Hurewicz theorem implies that f acts trivially at the level of the fundamental group. The canonical projection $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ gives the universal covering of $S^1 \times S^1$. Thus, f can be lifted to a unique homeomorphism $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\widetilde{f}(0,0) = (0,0)$ and, by assumption on f, the homeomorphism $\widetilde{f}$ is $\mathbb{Z}^2$-equivariant. Therefore, the "affine" homotopy

$$H : \mathbb{R}^2 \times [0,1] \longrightarrow \mathbb{R}^2, \ (x,t) \longmapsto t \cdot \widetilde{f}(x) + (1-t) \cdot x$$

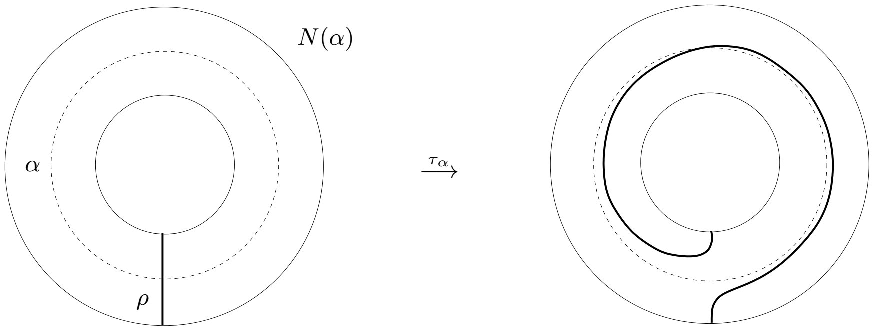
between $\mathrm{id}_{\mathbb{R}^2}$ and $\widetilde{f}$, descends to a homotopy from $\mathrm{id}_{S^1 \times S^1}$ to f. Since homotopy coincides with isotopy in dimension two, we deduce that $[f] = 1 \in \mathcal{M}(S^1 \times S^1)$. □

2.3. **Generation of mapping class groups.** Let Σ be an oriented surface. We now introduce some generators for the mapping class group $\mathcal{M}(\Sigma)$.

Definition 2.5. *Let* α *be a simple closed curve in the interior of* Σ. *We identify a tubular neighborhood* $N(\alpha)$ *of* α *with* $S^1 \times [0,1]$, *in such a way that orientations are preserved. Then, the* Dehn twist *along* α *is the homeomorphism* $\tau_\alpha : \Sigma \to \Sigma$ *defined by*

$$\tau_\alpha(x) = \begin{cases} x & \text{if } x \notin N(\alpha) \\ \big(e^{2i\pi(\theta+r)}, r\big) & \text{if } x = \big(e^{2i\pi\theta}, r\big) \in N(\alpha) = S^1 \times [0,1]. \end{cases}$$

Because of the choice of $N(\alpha)$ and its "parametrization" by $S^1 \times [0,1]$, the homeomorphism τ_α is only defined up to isotopy. Besides, the isotopy class of τ_α only depends on the isotopy class of the curve α. Here is the effect of τ_α on a curve ρ which crosses transversely α in a single point:



Remark 2.6. ⚠ The definition of τ_α does not need the curve α to be oriented, but it requires an orientation on the surface Σ. Since the surface Σ is oriented, there is a notion of "right-hand side" and "left-hand side" on Σ: observe that, in the above picture, the image of the arc ρ by τ_α makes a right-hand turn. Thus our definition of a Dehn twist is the same as in [Bir74], but it is opposite to the definition used in [Iva02] or [FM12] where Dehn twists are "left-handed". ■

It can be checked that $\tau_\alpha = 1 \in \mathcal{M}(\Sigma)$ if α bounds a disk (see Exercise 2.4). Otherwise, $\tau_\alpha \neq 1$ as we now show.

Proposition 2.7. *Let α be a simple closed curve in Σ such that $[\alpha] \neq 1 \in \pi_1(\Sigma)$. Then τ_α has infinite order in $\mathcal{M}(\Sigma)$.*

Proof. Assume that α is isotopic to a boundary curve δ. If Σ is an annulus, then we know from Exercise 2.2 that $\mathcal{M}(\Sigma) \simeq \mathbb{Z}$ generated by τ_α. If Σ is not an annulus, then the curve α regarded in the "double" surface

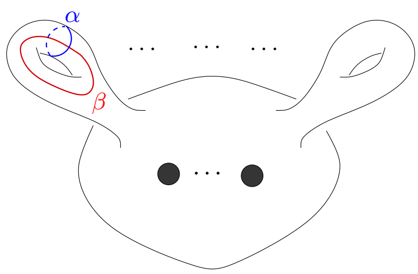
$$\Sigma' := \Sigma \underset{\delta=\delta}{\cup} (-\Sigma)$$

is not anymore isotopic to a boundary curve; furthermore, the inclusion $\Sigma \hookrightarrow \Sigma'$ induces a group homomorphism $\mathcal{M}(\Sigma) \to \mathcal{M}(\Sigma')$; so the fact that τ_α has infinite order in $\mathcal{M}(\Sigma')$ will imply that τ_α has infinite order in $\mathcal{M}(\Sigma)$. Therefore we can assume in the sequel that α is *not* isotopic to a boundary curve, i.e. α is "essential" in the terminology of §1.2.

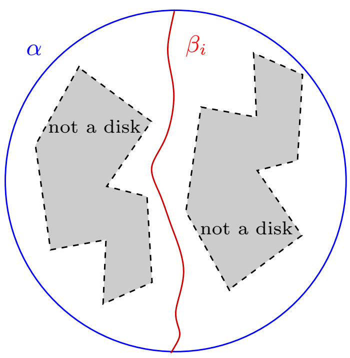
Let now β be an arbitrary simple closed curve in Σ. The following can be deduced from Lemma 1.11 (see [FM12, Proposition 3.2]):

Fact. *For any $k \in \mathbb{Z}$, we have $i\left([\tau_\alpha^k(\beta)], [\beta]\right) = |k| \cdot i([\alpha], [\beta])^2$.*

Hence it is enough to justify that there exists a β such that $i([\alpha], [\beta]) > 0$. Clearly this possibility for α only depends on its topological type. If α is non-separating, then we can choose β as follows:



and Lemma 1.11 implies that $i([\alpha], [\beta]) = 1$. If α is separating, then each of the subsurfaces S_1 and S_2 delimited by α is not a disk nor an annulus. Therefore we can find a proper simple arc $\beta_i \subset S_i$ such that we have the following situation:



If follows from Lemma 1.11 that $\beta := \beta_1 \cup \beta_2$ is a simple closed curve satisfying $i([\alpha],[\beta]) = 2$. □

Here is now a fundamental result about mapping class groups, which dates back to Dehn [Deh38].

Theorem 2.8 (Dehn 1938). *The group $\mathcal{M}(\Sigma)$ is generated by Dehn twists along simple closed curves which are non-separating or which encircle some boundary components.*

In order to prove this, we will need the following result which describes how the mapping class group "grows" when one removes a disk from the surface [Bir69a].

Proposition 2.9 (Birman's exact sequence). *Let Σ° be the surface obtained from Σ by removing a disk D. Then, there is an exact sequence of groups*

$$\pi_1\left(\mathrm{U}(\Sigma)\right) \xrightarrow{\mathrm{Push}} \mathcal{M}(\Sigma^\circ) \xrightarrow{\cup\, \mathrm{id}_D} \mathcal{M}(\Sigma) \longrightarrow 1$$

where $\mathrm{U}(\Sigma)$ denotes the total space of the unit tangent bundle[2] *of Σ. Moreover, the image of the "Push" map is generated by some products of Dehn twists (and their inverses) along curves which are non-separating or which encircle boundary components.*

Sketch of the proof. The argument below needs the notion of "fibration" which we have recalled in Appendix A. Let $\mathrm{Diffeo}^{+,\partial}(\Sigma)$ be the group of orientation-preserving and boundary-fixing diffeomorphisms $\Sigma \to \Sigma$. We equip $\mathrm{Diffeo}^{+,\partial}(\Sigma)$ with the compact-open topology. As we mentioned in §2.1, we have

$$\mathcal{M}(\Sigma) = \pi_0\big(\mathrm{Homeo}^{+,\partial}(\Sigma)\big) = \pi_0\big(\mathrm{Diffeo}^{+,\partial}(\Sigma)\big)$$

so that we can work in the smooth category. Let $\mathrm{Emb}^+(D,\Sigma)$ be the space of orientation-preserving smooth embeddings of the disk D into Σ, and denote by $\iota \in \mathrm{Emb}^+(D,\Sigma)$ the inclusion $D \hookrightarrow \Sigma$. The restriction map $\mathrm{Diffeo}^{+,\partial}(\Sigma) \to \mathrm{Emb}^+(D,\Sigma), f \mapsto f|_D$ is a fibration whose fiber over ι is $\mathrm{Diffeo}^{+,\partial}(\Sigma^\circ)$ (see [Iva02, Theorem 2.6.A], and below). The long exact sequence in homotopy induced by this fibration terminates with

$$\pi_1\big(\mathrm{Diffeo}^{+,\partial}(\Sigma), \mathrm{id}_\Sigma\big) \longrightarrow \pi_1\left(\mathrm{Emb}^+(D,\Sigma),\iota\right) \xrightarrow{P} \mathcal{M}(\Sigma^\circ) \longrightarrow \mathcal{M}(\Sigma) \longrightarrow 1,$$

since any orientation-preserving embedding of D into Σ is isotopic to ι (i.e. $\mathrm{Emb}^+(D,\Sigma)$ has a single path-connected component). The above map P has the following definition. A loop in $\mathrm{Emb}^+(D,\Sigma)$ based at ι is an isotopy $I : D \times [0,1] \to \Sigma$ such that $I(-,0) = I(-,1) = \iota$. By the "isotopy extension theorem" (see [Hir76], for instance), there is an isotopy $\widetilde{I} : \Sigma \times [0,1] \to \Sigma$ starting with $\widetilde{I}(-,0) = \mathrm{id}_\Sigma$ and such that $\widetilde{I}|_{D\times[0,1]} = I$. Then, $[I] \in \pi_1\left(\mathrm{Emb}^+(D,\Sigma),\iota\right)$ is mapped by P to

$$\big[\text{restriction of } \widetilde{I}(-,1) \text{ to } \Sigma^\circ = \Sigma \setminus \mathrm{int}(D)\big].$$

We now give a more concrete description of the map P. For this, let v be a unit tangent vector of D: $v \in T_x\Sigma$ with $\|v\| = 1$ and $x \in \mathrm{int}(D)$. The map $\mathrm{Emb}^+(D,\Sigma) \to \mathrm{U}(\Sigma)$ defined by $f \mapsto (\mathrm{d}_x f)(v)/\|(\mathrm{d}_x f)(v)\|$ is a weak homotopy equivalence (see [Iva02, Theorem 2.6.D]). In particular, it induces an isomorphism

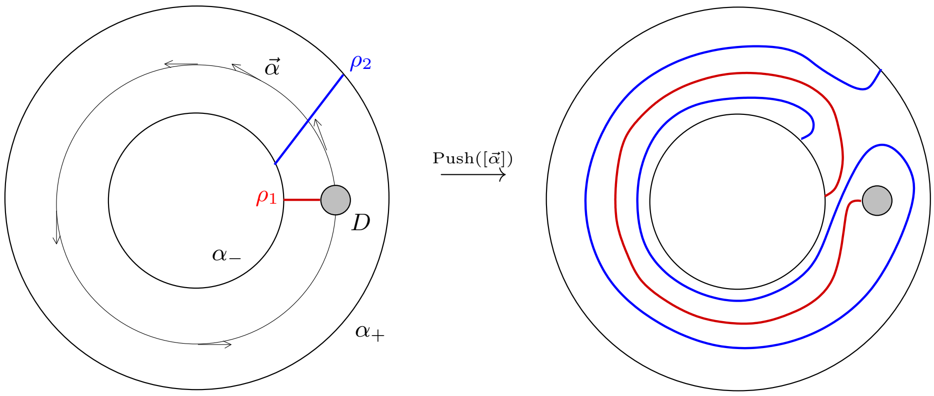
$$\pi_1\left(\mathrm{Emb}^+(D,\Sigma),\iota\right) \simeq \pi_1(\mathrm{U}(\Sigma), v).$$

We denote by "Push" : $\pi_1(\mathrm{U}(\Sigma), v) \to \mathcal{M}(\Sigma^\circ)$ the composition of the homomorphism P with this isomorphism. If $\vec{\alpha}$ is the unit tangent vector field of a smooth simple closed curve α based at x, then $\mathrm{Push}([\vec{\alpha}]) \in \mathcal{M}(\Sigma^\circ)$ can be described as follows. Let $N(\alpha)$ be a tubular neighborhood of α and let α_-, α_+ be the two boundary components of $N(\alpha)$: here we assume that the orientation induced by $N(\alpha)$ on α_+ (respectively, α_-) is the given orientation (respectively, the opposite orientation) on α. Then we have the following formula:

$$\mathrm{Push}([\vec{\alpha}]) = \tau_{\alpha_-}^{-1}\tau_{\alpha_+}. \tag{2.1}$$

This is proved by checking that $\mathrm{Push}([\vec{\alpha}])$ and $\tau_{\alpha_-}^{-1}\tau_{\alpha_+}$ (viewed as elements of $\mathcal{M}(N(\alpha)\setminus \mathrm{int}(D))$) act in the same way on the proper arcs ρ_1 and ρ_2 shown below, and by appealing to Exercise 2.3:

[2] Here, the surface Σ is endowed with an arbitrary smooth structure and a riemannian metric.



Note that the way how Push($[\vec{\alpha}]$) acts on the arc ρ_2 explains the terminology "Push". Next, from the exact sequence of groups

$$\pi_1(S^1, 1) \longrightarrow \pi_1(\mathrm{U}(\Sigma), v) \longrightarrow \pi_1(\Sigma, x) \longrightarrow 1$$

(deduced from the long exact sequence in homotopy for the fibration $\mathrm{U}(\Sigma) \to \Sigma$), we see that $\pi_1(\mathrm{U}(\Sigma), v)$ is generated by the fiber and by unit tangent vector fields of smooth simple closed curves which are non-separating or which are isotopic to components of $\partial\Sigma$. Since the image of the fiber S^1 by the Push map is $\tau_{\partial D}$, we conclude from (2.1) that Push $(\pi_1(\mathrm{U}(\Sigma), v))$ is generated by some products of Dehn twists (and their inverses) along curves which are non-separating or which encircle boundary components. ☐

Remark 2.10. It is known that the path-connected components of the space $\mathrm{Diffeo}^{+,\partial}(\Sigma)$ are contractible when[3] $\chi(\Sigma) < 0$ [EE69, ES70, Gra73]. So, in this case, the above proof produces a short exact sequence of groups

$$1 \longrightarrow \pi_1(\mathrm{U}(\Sigma)) \xrightarrow{\text{Push}} \mathcal{M}(\Sigma^\circ) \xrightarrow{\cup\, \mathrm{id}_D} \mathcal{M}(\Sigma) \longrightarrow 1.$$ ■

We can now proceed to the proof of Dehn's theorem.

Proof of Theorem 2.8. Here we mainly follow the arguments of Ivanov [Iva02, Theorem 4.2.C]. Let g be the genus of Σ and let b be the number of boundary components of Σ. First of all, we deduce from Proposition 2.9 that, if the statement holds at a given g for $b = 0$, then it holds for any $b \geq 0$. So, we can assume that Σ is closed and the proof then goes by induction on $g \geq 0$. For $g = 0$, Exercise 2.1 tells us that there is nothing to prove. The case $g = 1$ is proved using the isomorphism $\kappa : \mathcal{M}(S^1 \times S^1) \to \mathrm{SL}(2; \mathbb{Z})$ introduced in Proposition 2.4. It is well-known that the group $\mathrm{SL}(2; \mathbb{Z})$ is generated by

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad T := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(see [New72, Theorem VII.3]). Hence $\mathrm{SL}(2; \mathbb{Z})$ is also generated by

$$A \quad \text{and} \quad B := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

since we have $ABA = T$. The matrices A and B correspond via κ to the Dehn twists along the simple closed curves $\alpha := S^1 \times \{1\}$ and $\beta := \{1\} \times S^1$ of $S^1 \times S^1$. We deduce that the group $\mathcal{M}(S^1 \times S^1)$ is generated by τ_α and τ_β. Thus, in the sequel, we are allowed to assume that the genus g is at least 2.

Let $f \in \mathcal{M}(\Sigma)$ and let α be a non-separating simple closed curve on Σ. Then, $f(\alpha)$ is another non-separating simple closed curve on Σ. We need the following non-trivial fact due to Lickorish [Lic64]: see [FM12, §4.1.2] or [Iva02, §3.2] for proofs.

Fact. (Connectedness of the complex of curves) *Assume that $g \geq 2$. Then, for any two non-separating simple closed curves ρ and ρ', there exists a sequence of non-separating simple closed curves*

$$\rho = \rho_1 \rightsquigarrow \rho_2 \rightsquigarrow \cdots \rightsquigarrow \rho_r = \rho'$$

such that $i(\rho_j, \rho_{j+1}) = 0$ for all $j = 1, \ldots, r-1$.

We also need the following observation.

---

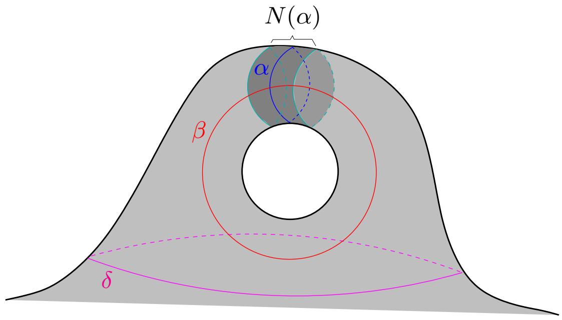
[3] This is not true if $\chi(\Sigma) \geq 0$. For instance, we have the following results in the closed case: $\mathrm{Diffeo}^+(S^2)$ deformation retracts to $\mathrm{SO}(3)$ by [Sma59], while each of the path-connected components of $\mathrm{Diffeo}^+(S^1 \times S^1)$ deformations retracts to $S^1 \times S^1$ by [EE69].

Fact. *If β and γ are two non-separating simple closed curves on Σ such that $i(\beta,\gamma)=0$, then there is a product of Dehn twists T along non-separating simple closed curves such that $T(\beta)=\gamma$.*

Indeed, we can find a third non-separating simple closed curve $\delta \subset \Sigma$ such that $i(\delta,\gamma)=i(\delta,\beta)=1$. Then, by Exercise 2.7, we have $\tau_\delta\tau_\gamma\,\tau_\beta\tau_\delta(\beta)=\tau_\delta\tau_\gamma(\delta)=\gamma$ and we can take $T:=\tau_\delta\tau_\gamma\tau_\beta\tau_\delta$.

The above two facts show that we can find a product of Dehn twists T along non-separating simple closed curves such that $T(\alpha)=f(\alpha)$. Therefore, we are allowed to assume that f preserves α. But, it may happen that f inverses the orientations of α. In this case, we can consider a non-separating simple closed curve β such that $i(\alpha,\beta)=1$ and deduce from Exercise 2.7 that $\tau_\beta\tau_\alpha^2\tau_\beta$ preserves α but inverses its orientations. Therefore, after possible multiplication by $\tau_\beta\tau_\alpha^2\tau_\beta$, we can assume that f preserves α with orientation. Since there is only one orientation-preserving homeomorphism of S^1 up to isotopy, we can assume that f is the identity on α. Furthermore, we can suppose that f is the identity on a tubular neighborhood $N(\alpha)$ of α.

Let $\Sigma':=\Sigma\setminus \operatorname{int} N(\alpha)$ and let f' be the restriction of f to Σ'. The surface Σ' has genus $g':=g-1$ and has 2 boundary components:



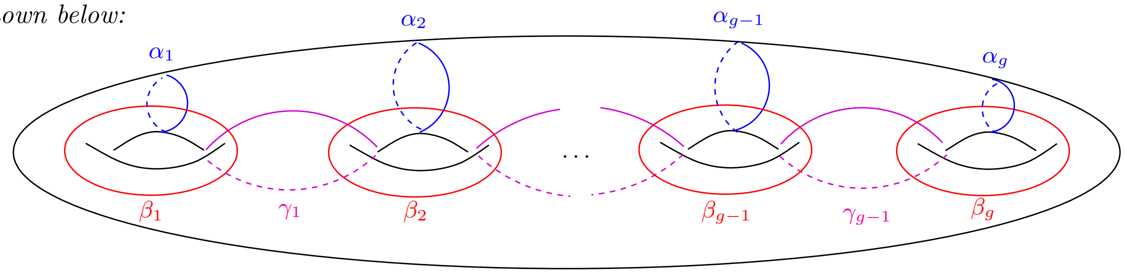
So, by the induction hypothesis, $f'\in\mathcal{M}(\Sigma')$ is a product of Dehn twists along simple closed curves in Σ' which are non-separating or which encircle some boundary components. We can conclude the same thing for $f\in\mathcal{M}(\Sigma)$: indeed a non-separating simple closed curve in Σ' is also non-separating in Σ, and a simple closed curve which encircles some boundary components in Σ' is either isotopic in Σ to α (which is non-separating) or is isotopic to the simple closed curve δ shown above. In the latter case, we use Exercise 2.8 which implies that τ_δ is a product of six Dehn twists along non-separating simple closed curves in Σ (namely α and β). □

The above proof can be improved to show that *finitely* many Dehn twists are enough to generate the mapping class group, and this was already proved by Dehn in the closed case [Deh38]. Much later, Lickorish rediscovered Dehn's result and improved it by reducing the number of generators [Lic64].

Theorem 2.11 (Lickorish 1964)**.** *For $g\geq 1$, the group $\mathcal{M}(\Sigma_g)$ is generated by the Dehn twists along the simple closed curves*

$$\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g,\gamma_1,\dots,\gamma_{g-1} \tag{2.2}$$

shown below:



Afterwards, Humphries showed that $2g+1$ Dehn twists are enough to generate $\mathcal{M}(\Sigma_g)$: specifically, with the above notation, $\mathcal{M}(\Sigma_g)$ is generated by the Dehn twists along

$$\beta_1,\dots,\beta_g,\gamma_1,\dots,\gamma_{g-1},\alpha_1,\alpha_2 \tag{2.3}$$

see [Hum79]. The Dehn twists along the curves (2.2) are called the *Lickorish generators* of $\mathcal{M}(\Sigma_g)$, while those along the curves (2.3) are called the *Humphries generators*.

Remark 2.12. Humphries also proved that $\mathcal{M}(\Sigma_g)$ can not be generated by less than $2g+1$ Dehn twists [Hum79]. Nonetheless, Wajnryb showed that $\mathcal{M}(\Sigma_g)$ is generated by only two elements (one is defined explicitly as a product of $2g$ Dehn twists, while the other one is a Dehn twist composed with the inverse of a Dehn twist: see [Waj96]). ■

2.4. **Presentations of mapping class groups.** Since mapping class groups are generated by Dehn twists, it is natural to look for presentations in terms of Dehn twists. First of all, one can wonder which relations exist between only *two* Dehn twists, and it is intuitively clear that these will depend on how much the two curves intersect each other.

(Disjointness relation) *Let δ and ρ be two simple closed curves on a surface Σ with $i(\delta,\rho)=0$. Then $[\tau_\delta,\tau_\rho]=1$.*

(Braid relation) *Let δ and ρ be two simple closed curves on a surface Σ with $i(\delta,\rho)=1$. Then, $\tau_\delta\tau_\rho\tau_\delta=\tau_\rho\tau_\delta\tau_\rho$.*

The first relation is obvious. To prove the second one, we use Exercise 2.7 and Exercise 2.5:

$$\tau_\rho=\tau_{\tau_\delta\tau_\rho(\delta)}=\tau_\delta\tau_\rho\,\tau_\delta\,(\tau_\delta\tau_\rho)^{-1}.$$

If $i(\delta,\rho)\geq 2$, then τ_δ and τ_ρ generate a free group of rank two: this can be proved using estimates of the geometric intersection number, see [Ish96] or [FM12, §3.5.2]. Thus, there is no relation at all between τ_δ and τ_ρ in this case.

If we now consider *more than two* simple closed curves, then new relations appear between the corresponding Dehn twists. For instance, we can consider a *chain* $\rho_1,\dots,\rho_k$ of simple closed curves, which means that

$$i(\rho_l,\rho_m)=\begin{cases}0 & \text{if } |l-m|>1,\\ 1 & \text{if } |l-m|=1.\end{cases}$$

Each chain induces a relation in the mapping class group.

Lemma 2.13 (k-chain relation)**.** *Let $\rho_1,\dots,\rho_k$ be a chain of simple closed curves in a surface Σ, and consider the subsurface*

$$N:=N(\rho_1)\cup\dots\cup N(\rho_k)\subset\Sigma$$

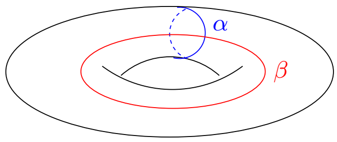
where $N(\rho_i)$ is a (sufficiently small) tubular neighborhood of ρ_i. Then,

- *for k even, $(\tau_{\rho_1}\cdots\tau_{\rho_k})^{2k+2}=\tau_\delta$ where $\delta:=\partial N$.*
- *for k odd, $(\tau_{\rho_1}\cdots\tau_{\rho_k})^{k+1}=\tau_{\delta_1}\tau_{\delta_2}$ where $\delta_1\cup\delta_2:=\partial N$.*

The 2-chain relation is proved in Exercise 2.8. The proof of the k-chain relation for higher k is sketched in [FM12, §9.4.2].

The few relations that we have exhibited so far are enough for a presentation of the mapping class group of $\Sigma_1\cong S^1\times S^1$. Indeed, according to Proposition 2.4, this is equivalent to a presentation of the group $\mathrm{SL}(2;\mathbb{Z})$.

Theorem 2.14. *Set $A:=\tau_\alpha$ and $B:=\tau_\beta$, where $\alpha:=S^1\times\{1\}$ and $\beta:=\{1\}\times S^1$ are shown below:*



Then, we have the presentation

$$\mathcal{M}(S^1\times S^1)=\left\langle A,B\;\middle|\;ABA=BAB,(AB)^6=1\right\rangle. \tag{2.4}$$

Note that the first relation is a braid relation, and that the second one follows from the 2-chain relation.

Proof. Let $\mathrm{PSL}(2;\mathbb{Z})$ be the quotient of $\mathrm{SL}(2;\mathbb{Z})$ by its center, namely the order 2 subgroup generated by $-I_2$. It is well-known that $\mathrm{PSL}(2;\mathbb{Z})$ is a free product $\mathbb{Z}_2*\mathbb{Z}_3$. More precisely, we have

$$\mathrm{PSL}(2;\mathbb{Z})=\left\langle\overline{T},\overline{U}\;\middle|\;\overline{T}^2=1,\overline{U}^3=1\right\rangle \tag{2.5}$$

where $\overline{T}$ and $\overline{U}$ are the classes of the following matrices:

$$T:=\begin{pmatrix}0&1\\-1&0\end{pmatrix}\quad\text{and}\quad U:=\begin{pmatrix}0&1\\-1&-1\end{pmatrix}.$$

(See [New72, §VIII.3] for a one-page proof.) Note that $T^2=-I_2$ and $U^3=I_2$. Hence, using the short exact sequence of groups

$$0\longrightarrow\{\pm I_2\}\longrightarrow\mathrm{SL}(2;\mathbb{Z})\longrightarrow\mathrm{PSL}(2;\mathbb{Z})\longrightarrow 1,$$

we deduce from (2.5) the following presentation of $\mathrm{SL}(2;\mathbb{Z})$:

$$\mathrm{SL}(2;\mathbb{Z})=\left\langle T,U\;\middle|\;T^4=1,U^3=1,[U,T^2]=1\right\rangle.$$

(This can be proved, for instance, using Lemma 3.6 below.) Setting

$$V := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$

and observing that $U = V^{-1}T^2$, we obtain the following equivalent presentation:

$$\mathrm{SL}(2;\mathbb{Z}) = \langle T, V \mid V^6 = 1, T^2 = V^3 \rangle .$$

Finally, setting

$$A := \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

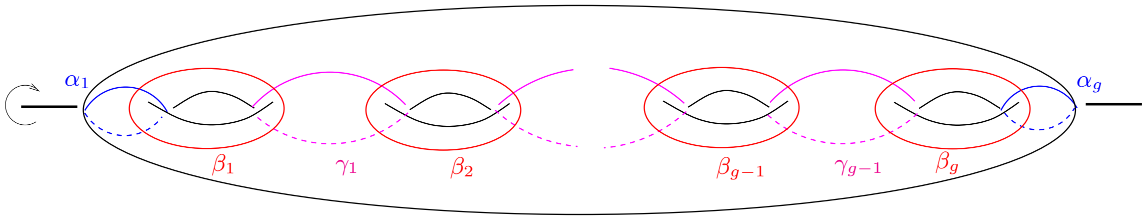
and observing that $T = ABA$ and $V = BA$, we obtain the presentation

$$\mathrm{SL}(2;\mathbb{Z}) = \langle A, B \mid (BA)^6 = 1, (ABA)^2 = (BA)^3 \rangle$$

which is equivalent to (2.4). □

For higher genus, we need more relations and we consider for this the involution h of $\Sigma_g \subset \mathbb{R}^3$ which is a rotation by 180° around an appropriate axis: see below. It can be shown[4] that $h \in \mathcal{M}(\Sigma_g)$ is given by the following word in the Lickorish's generators:

$$h = \tau_{\alpha_1}\tau_{\beta_1}\tau_{\gamma_1}\tau_{\beta_2}\cdots\tau_{\beta_{g-1}}\tau_{\gamma_{g-1}}\tau_{\beta_g}\tau_{\alpha_g}\cdot\tau_{\alpha_g}\tau_{\beta_g}\tau_{\gamma_{g-1}}\tau_{\beta_{g-1}}\cdots\tau_{\beta_2}\tau_{\gamma_1}\tau_{\beta_1}\tau_{\alpha_1}$$

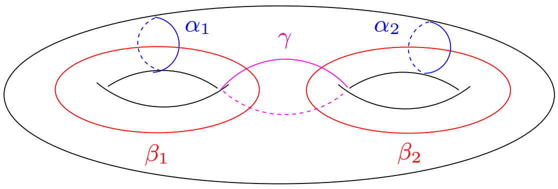


Then, we have the following relations between Lickorish's generators:

(Hyperelliptic relations) *In $\mathcal{M}(\Sigma_g)$, we have $h^2 = 1$ and $[h, \tau_{\alpha_1}] = 1$.*

The first one is obvious, while the second one follows from Exercise 2.5 using the fact that $h(\alpha_1) = \alpha_1$. These additional relations allow for a presentation of $\mathcal{M}(\Sigma_2)$, which has been obtained in [BH71].

Theorem 2.15 (Birman–Hilden 1971)**.** *Set $A := \tau_{\alpha_1}$, $B := \tau_{\beta_1}$, $C := \tau_\gamma$, $D := \tau_{\beta_2}$ and $E := \tau_{\alpha_2}$, where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma$ are the simple closed curves shown below:*



Then, we have the presentation

$$\mathcal{M}(\Sigma_2) = \langle A, B, C, D, E \mid \textit{disjointness, braid, } (ABCDE)^6 = 1, H^2 = 1, [H, A] = 1 \rangle .$$

Here, the word "braid" stands for the 4 possible braid relations between A, B, C, D, E, the word "disjointness" stands for the 6 possible disjointness relations between them and $H := ABCDE^2DCBA$.

Note that the third relation follows directly from the 5-chain relation, while the fourth and fifth relations are the hyperelliptic relations. Birman and Hilden [BH71, Theorem 8] obtained this presentation by means of the 2-fold covering $\Sigma_g \to \Sigma_g/\langle h \rangle \cong S^2$ (which is branched over $2g+2$ points). But, unfortunately, their method do not apply to higher genus.

We know from Theorem 2.11 that mapping class groups are finitely generated. But there exist finitely generated groups which are *not* finitely presented. For instance, the external semi-direct product

$$G := \mathbb{Z}[t, t^{-1}] \rtimes \mathbb{Z}$$

where $\mathbb{Z}$ acts on $\mathbb{Z}[t, t^{-1}]$ by

$$k \cdot \Big(\sum_i z_i t^i\Big) = \sum_i z_i t^{i+k}$$

is finitely generated by $(t, 0)$ and $(0, 1)$, but it can be shown that G has no presentation with finitely many relations [Bau61].

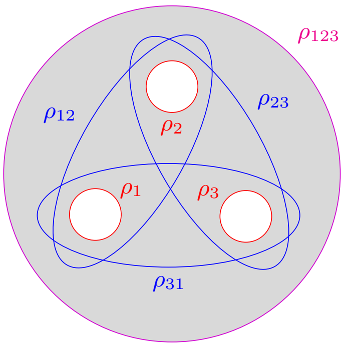
[4] See the proof of [BH71, Equation (8)] for instance.

The first proof that mapping class groups are finitely presented in genus $g \geq 3$ is due to McCool in the boundary case: regarding $\mathcal{M}(\Sigma_{g,b})$ with $b > 0$ as a subgroup of the automorphism group of a free group, he proved that $\mathcal{M}(\Sigma_{g,b})$ is finitely presented by constructing a finite connected 2-dimensional CW-complex whose fundamental group is isomorphic to $\mathcal{M}(\Sigma_{g,b})$ [McC75]. Later, Hatcher and Thurston introduced the complex of "cut systems" on which the mapping class group acts naturally, and they proved by means of Cerf theory that this CW-complex is simply-connected [HT80]. When a group G acts on a simply-connected CW-complex K in such a way that

- K/G has finitely many cells in dimension 0, 1 and 2,
- the stabilizer of any vertex of K is a finitely presented subgroup of G,
- the stabilizer of any edge of K is a finitely generated subgroup of G,

then the group G is finitely presented and there is a general procedure to derive from this a finite presentation of G. Thus, it follows from the work of Hatcher and Thurston that the mapping class group is finitely presented. Instead of the complex of "cut systems" on a surface Σ, one can use the "curve complex" (see the proof of Theorem 2.8) or the "arc complex" to prove with a similar strategy that $\mathcal{M}(\Sigma)$ is finitely presented: see [Iva87] and [Iva02, Theorem 4.3.D] in the former case, and see [FM12, §5.3] in the latter case. Building on the work of Hatcher and Thurston, Wajnryb [Waj83, Waj99] found an explicit finite presentation of $\mathcal{M}(\Sigma_{g,b})$ for $b \in \{0,1\}$ (see also Harer [Har83]). We refer to [Bir88, §1], [Iva02, §4.3] or [FM12, §5.2.1] for a precise statement of Wajnryb's presentation. In a few words, this presentation is given by Humphries' generators subject to the following relations: the disjointness relations, the braid relations, a 3-chain relation, a hyperelliptic relation[5], and an instance of the following relation (which is proved in Exercise 2.9).

(Lantern relation) *In* $\mathcal{M}(\Sigma_{0,4})$, *we have* $\tau_{\rho_{31}}\tau_{\rho_{23}}\tau_{\rho_{12}} = \tau_{\rho_{123}}\tau_{\rho_1}\tau_{\rho_2}\tau_{\rho_3}$.



So far, we have mainly considered presentations of the mapping class groups of *closed* surfaces. Using Birman's exact sequence (Proposition 2.9) and Lemma 3.6 below, it is not difficult to deduce finite presentations of mapping class groups in the boundary case too, but the resulting presentations turn out to be complicated for several boundary components. Nevertheless, Gervais managed to derive from Wajnryb's presentation another finite presentation of $\mathcal{M}(\Sigma_{g,b})$ for any $g > 1, b \geq 0$, and for $g = 1, b > 0$ [Ger01]. Gervais' presentation has more generators than Wajnryb's presentation, but its relations are much more symmetric and they essentially splits into two cases (the disjointness/braid relations and some new "stars" relations).

To conclude, let us mention that the above techniques leading to finite presentations of groups are also useful for the computation of low-dimensional (co)homology groups. Thus Harer used the work of Hatcher and Thurston to compute the second homology group of mapping class groups [HT80]. Later Pitsch explained how to use Hopf's formula to easily deduce this computation from Wajnryb's presentation [Pit99]. See also [FM12, §5.4]. Here, following Harer [Har83], we only explain how the lantern relation can be used to derive the first homology group.

Corollary 2.16. *The abelianization of the mapping class group is*

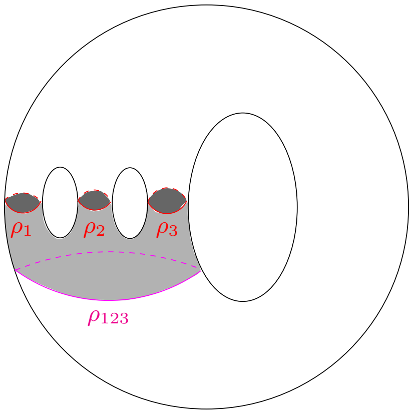
$$\frac{\mathcal{M}(\Sigma_g)}{[\mathcal{M}(\Sigma_g), \mathcal{M}(\Sigma_g)]} \simeq \begin{cases} \mathbb{Z}_{12} & \text{if } g = 1, \\ \mathbb{Z}_{10} & \text{if } g = 2, \\ \{0\} & \text{if } g \geq 3. \end{cases}$$

Proof. We know from Theorem 2.8 that $\mathcal{M}(\Sigma_g)$ is generated by Dehn twists along non-separating simple closed curves. If δ_1 and δ_2 are any two such curves, we know from Exercise 1.9 that there is an orientation-preserving homeomorphism $f : \Sigma_g \to \Sigma_g$ satisfying $f(\delta_1) = \delta_2$. By Exercise 2.5, we get

$$\tau_{\delta_2} = f \circ \tau_{\delta_1} \circ f^{-1}.$$

[5] Specifically, for $b = 0$, this is the relation $[h, \tau_{\alpha_1}] = 1$ written in terms of the Humphries' generators. For $b = 1$, this relation must be omitted.

We deduce that the abelianization of $\mathcal{M}(\Sigma_g)$ is cyclic generated by τ_ρ, where ρ is any non-separating simple closed curve in Σ_g. Then the result in genus $g \in \{1,2\}$ is easily deduced from the presentations of $\mathcal{M}(\Sigma_g)$ given in Theorem 2.14 and Theorem 2.15. In genus $g \geq 3$, there is an embedding of $\Sigma_{0,4}$ in Σ_g such that each of the curves involved in the lantern relation is non-separating in Σ_g:



So, we conclude that $\tau_\rho^4 = \tau_\rho^3$ in the abelianization and the conclusion follows. □

2.5. **Exercises.**

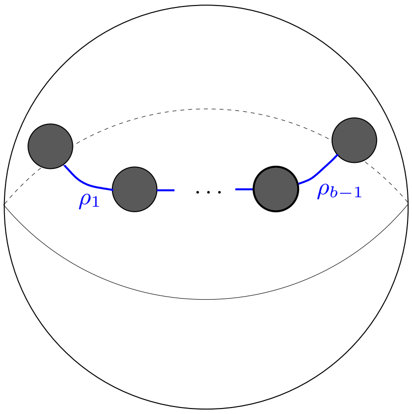
Exercise 2.1. Show that $\mathcal{M}(S^2) = \{1\}$ using the fact that S^2 minus a point is homeomorphic to $\mathbb{R}^2$.

Exercise 2.2. Let $z := \big[S^1 \times \{1/2\}\big]$ be the generator of $H_1(S^1 \times [0,1];\mathbb{Z}) \simeq \mathbb{Z}$, and let ρ be the 1-chain $\{1\} \times [0,1]$ of $S^1 \times [0,1]$. Show that the map

$$\eta : \mathcal{M}(S^1 \times [0,1]) \longrightarrow \mathbb{Z}$$

which sends the isotopy class $[f]$ to the unique integer k_f such that $[f(\rho) - \rho] = k_f \cdot z$, is a group isomorphism. Deduce that $\mathcal{M}(S^1 \times [0,1])$ is freely generated by the Dehn twist along $S^1 \times \{1/2\}$.

Exercise 2.3. Let $b \geq 1$ be an integer, and let $\rho_1, \dots, \rho_{b-1}$ be the following simple proper arcs in $\Sigma_{0,b}$:



Show that any two $f, h \in \mathrm{Homeo}^\partial(\Sigma_{0,b})$ are isotopic rel $\partial\Sigma_{0,b}$ if and only if the simple proper arcs $f(\rho_i), h(\rho_i)$ are isotopic for every i.

Exercise 2.4. Let α be the boundary of a closed disk in an oriented surface Σ. Show that $\tau_\alpha = 1 \in \mathcal{M}(\Sigma)$.

Exercise 2.5. Let Σ be an oriented surface. Show that the conjugate of a Dehn twist in $\mathcal{M}(\Sigma)$ is again a Dehn twist: specifically, for any $f \in \mathcal{M}(\Sigma)$ and any simple closed curve $\alpha \subset \Sigma$, we have $f\,\tau_\alpha\,f^{-1} = \tau_{f(\alpha)}$.

Exercise 2.6. Let α be an oriented simple closed curve on an oriented surface Σ. Show that the action of the Dehn twist τ_α in homology is given by the following formula:

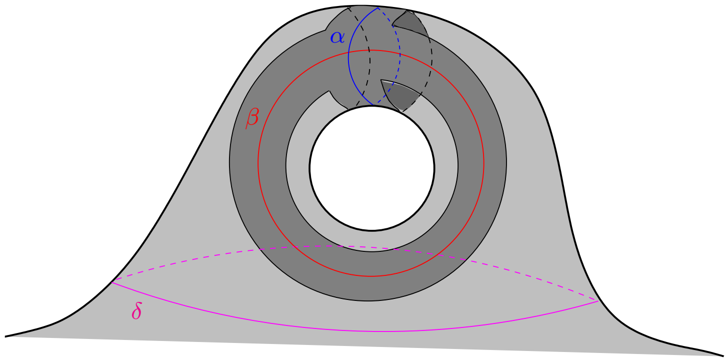
$$\forall x \in H_1(\Sigma;\mathbb{Z}), \quad (\tau_\alpha)_*(x) = x + \omega([\alpha], x) \cdot [\alpha]. \tag{2.6}$$

Exercise 2.7. Let α and β be simple closed curves on an oriented surface Σ such that $i(\alpha,\beta) = 1$, and assume that they are oriented. Show that

$$\tau_\beta \tau_\alpha(\beta) = \begin{cases} \alpha & \text{if } \omega(\alpha,\beta) = +1 \\ \overline{\alpha} & \text{if } \omega(\alpha,\beta) = -1 \end{cases}$$

where $\overline{\alpha}$ denotes the curve α with the opposite orientation.

Exercise 2.8. Let α and β be simple closed curves on an oriented surface Σ such that $i(\alpha, \beta) = 1$, and let $\delta \subset \Sigma$ be a simple closed curve isotopic to the boundary of the subsurface $N(\alpha) \cup N(\beta)$:



Show that $\tau_\delta = (\tau_\alpha \tau_\beta)^6$.

Exercise 2.9. Prove the lantern relation using (a slight variation of) Exercise 2.3.

Exercise 2.10. Let Σ be an oriented surface and denote by $\mathcal{M}^{\not\partial}(\Sigma)$ the "boundary-free" version of the mapping class group of Σ. We number the boundary components of Σ from 1 to b. Show that there is an exact sequence of groups

$$\mathbb{Z}^b \xrightarrow{d} \mathcal{M}(\Sigma) \xrightarrow{c} \mathcal{M}^{\not\partial}(\Sigma) \xrightarrow{s} \mathfrak{S}_b \longrightarrow 1, \tag{2.7}$$

where the homomorphism d maps the i-th canonical vector of $\mathbb{Z}^b$ to the Dehn twist along a curve parallel to the i-th component of $\partial\Sigma$, the map c is the canonical homomorphism and the map s records the way how homeomorphisms permute the components of $\partial\Sigma$. (Hint: to show the exactness at $\mathcal{M}(\Sigma)$, one can use the fact that $\mathrm{Homeo}^+(S^1)$ deformation retracts to the group of rotations of S^1.)

Exercise 2.11. Let Σ be a closed oriented surface and denote by $\mathcal{M}^\pm(\Sigma)$ the "unoriented" version of the mapping class group of Σ. Show that there is a split short exact sequence of groups

$$1 \longrightarrow \mathcal{M}(\Sigma) \xrightarrow{e} \mathcal{M}^\pm(\Sigma) \xrightarrow{f} \mathbb{Z}_2 \longrightarrow 1 \tag{2.8}$$

where e is the canonical homomorphism and f is defined by the condition $f^{-1}(\{0\}) = e(\mathcal{M}(\Sigma))$.

* * *

Solution to Exercise 2.1. Let $f : S^2 \to S^2$ be an orientation-preserving homeomorphism: it suffices to show that f is homotopic to the identity. Fix a point $x \in S^2$. Using rotations in $\mathbb{R}^3$, it is easy to construct an isotopy $I : S^2 \times [0,1] \to S^2$ such that $I(-,0) = \mathrm{id}$ and $I(-,1)$ maps x to $f^{-1}(x)$. Then $f \circ I$ is an isotopy between f and a self-homeomorphism of S^2 which fixes x. Therefore, we can assume without loss of generality that $f(x) = x$.

Since $S^2 \setminus \{x\}$ is homeomorphic to $\mathbb{R}^2$, we are reduced to prove that any self-homeomorphism of $\mathbb{R}^2$ is homotopic to the identity. This is actually true for any continuous map $f : \mathbb{R}^2 \to \mathbb{R}^2$, using the "affine" homotopy

$$H : \mathbb{R}^2 \times [0,1] \longrightarrow \mathbb{R}^2, \ (x,t) \longmapsto t \cdot f(x) + (1-t) \cdot x.$$

Solution to Exercise 2.2. We first check that η is a homomorphism. Let $f, g \in \mathrm{Homeo}^\partial(S^1 \times [0,1])$. Since f fixes the boundary and since all the homology of $S^1 \times [0,1]$ comes from the boundary, f acts trivially in homology. Therefore,

$$[fg(\rho) - \rho] = f_* ([g(\rho) - \rho]) + [f(\rho) - \rho] = [g(\rho) - \rho] + [f(\rho) - \rho]$$

which proves that $k_{fg} = k_f + k_g \in \mathbb{Z}$.

The injectivity of η is proved with the same kind of arguments as in Proposition 2.4. Let $f \in \mathrm{Homeo}^\partial(S^1 \times [0,1])$ be such that $\eta([f]) = 0$. The canonical projection $\mathbb{R} \times [0,1] \to S^1 \times [0,1]$ gives the universal covering of $S^1 \times [0,1]$. Thus, f can be lifted to a unique homeomorphism $\widetilde{f} : \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ such that $\widetilde{f}(0,0) = (0,0)$. Since $\pi_1(S^1 \times [0,1]) \simeq \mathbb{Z}$ is abelian and – as we observed above – since f acts trivially at the level of homology, the map f acts trivially at the level of the fundamental group: so $\widetilde{f}$ is $\mathbb{Z}$-equivariant. The fact that $\widetilde{f}(0,0) = (0,0)$ implies that $\widetilde{f}$ fixes $\mathbb{R} \times \{0\}$; since $k_f = 0$, we also have $\widetilde{f}(0,1) = (0,1)$ so that $\widetilde{f}$ fixes $\mathbb{R} \times \{1\}$. Therefore, the "affine" homotopy

$$H : (\mathbb{R} \times [0,1]) \times [0,1] \longrightarrow \mathbb{R} \times [0,1], \ (x,t) \longmapsto t \cdot \widetilde{f}(x) + (1-t) \cdot x$$

between the identity of $\mathbb{R}\times[0,1]$ and $\widetilde{f}$ is such that $H(-,t)$ is $\mathbb{Z}$-equivariant and fixes the boundary at each time $t\in[0,1]$. Thus H descends to a homotopy rel $\partial(S^1\times[0,1])$ from the identity of $S^1\times[0,1]$ to f. Since homotopy coincides with isotopy in dimension two, we deduce that $[f]=1\in\mathcal{M}(S^1\times[0,1])$.

The Dehn twist along the curve $S^1\times\{1/2\}$ is mapped by η to $+1$. Since $+1$ generates the group $\mathbb{Z}$, we deduce that η is also surjective. This argument also shows that $\mathcal{M}(S^1\times[0,1])$ is generated by that Dehn twist.

Solution to Exercise 2.3. Set $\Sigma:=\Sigma_{0,b}$. Any isotopy of simple proper arcs between $f\rho_i:[0,1]\to\Sigma$ and $h\rho_i:[0,1]\to\Sigma$ is equivalent (by composition with h^{-1}) to an isotopy of simple proper arcs between $h^{-1}f\rho_i:[0,1]\to\Sigma$ and $\rho_i:[0,1]\to\Sigma$. Similarly, any isotopy rel $\partial\Sigma$ between f and h is equivalent (by composition with h^{-1}) to an isotopy rel $\partial\Sigma$ between $h^{-1}f$ and id_Σ. Therefore we can assume without loss of generality that $h=\mathrm{id}_\Sigma$.

It is obvious that, for any $f\in\mathrm{Homeo}^\partial(\Sigma)$ isotopic rel $\partial\Sigma$ to id_Σ, the proper arc $f\rho_i$ is isotopic to ρ_i for every i. To prove the converse, assume that $f\rho_i:[0,1]\to\Sigma$ is isotopic to $\rho_i:[0,1]\to\Sigma$ for every $i\in\{1,\dots,b-1\}$. We can assume that these "individual" isotopies can be unified into a "global" isotopy

$$I:([0,1]\sqcup\cdots\sqcup[0,1])\times[0,1]\longrightarrow\Sigma$$

such that $I(-,0)=\rho_1\sqcup\cdots\sqcup\rho_{b-1}:[0,1]\sqcup\cdots\sqcup[0,1]\to[0,1]$ and $I(-,1)=f\rho_1\sqcup\cdots\sqcup f\rho_{b-1}:[0,1]\sqcup\cdots\sqcup[0,1]\to[0,1]$. As a general fact of differential topology,[6] this isotopy can be extended to an "ambient" isotopy rel $\partial\Sigma$: specifically, there exists an isotopy

$$\widetilde{I}:\Sigma\times[0,1]\longrightarrow\Sigma$$

such that $\widetilde{I}(-,0)=\mathrm{id}_\Sigma$ and $\widetilde{I}(-,1)\circ\big(\rho_1\sqcup\cdots\sqcup\rho_{b-1}\big)=f\circ\big(\rho_1\sqcup\cdots\sqcup\rho_{b-1}\big)$. Thus $\widetilde{I}(-,1)$ and f are two self-homeomorphisms of Σ which coincide on

$$G:=\partial\Sigma\cup\rho_1([0,1])\cup\cdots\cup\rho_{b-1}([0,1]).$$

As a general fact of differential topology,[7] we can also assume (after an isotopy of f) that they coincide on a "regular" neighborhood $N(G)$ of G. Since $\Sigma\setminus\mathrm{int}\,N(G)$ is a closed disk, we deduce from Proposition 2.3 that the restrictions of $\widetilde{I}(-,1)$ and f to $\Sigma\setminus\mathrm{int}\,N(G)$ are isotopic relatively to the boundary. By extending with the identity, we get an isotopy rel $\partial\Sigma$ between $\widetilde{I}(-,1)$ and f. The "concatenation" of this isotopy with $\widetilde{I}$ provides an isotopy between f and id_Σ.

Solution to Exercise 2.4. Let D be the closed disk bounded by α in Σ, and let $\alpha_0\subset\mathrm{int}(D)$ be a simple closed curve parallel to ∂D. Then the inclusion $D\hookrightarrow\Sigma$ induces a group homomorphism $\mathcal{M}(D)\to\mathcal{M}(\Sigma)$. Since $\tau_{\alpha_0}=1\in\mathcal{M}(D)$ by Proposition 2.3, we deduce that $\tau_\alpha=\tau_{\alpha_0}=1\in\mathcal{M}(\Sigma)$.

Alternatively, starting from the defining formula of a Dehn twist, it is not difficult to construct an explicit isotopy between $\tau_\alpha:\Sigma\to\Sigma$ and the identity of Σ.

Solution to Exercise 2.5. Let $\alpha\subset\Sigma$ be a simple closed curve, and let $f:\Sigma\to\Sigma$ be an orientation-preserving homeomorphism fixing $\partial\Sigma$. We claim that

$$f\circ T_\alpha\circ f^{-1}=T_{f(\alpha)}\in\mathrm{Homeo}^{+,\partial}(\Sigma)\tag{2.9}$$

for some appropriate representatives T_α and $T_{f(\alpha)}$ of τ_α and $\tau_{f(\alpha)}$, respectively. Indeed let $N(\alpha)$ be a tubular neighborhood of α, and fix a parametrization $p:S^1\times[-1,1]\to N(\alpha)$. As a tubular neighborhood $N(f(\alpha))$ of $f(\alpha)$, we can take $f(N(\alpha))$ with parametrization $f\circ p$. Then (2.9) can be checked separately on $\Sigma\setminus N(f(\alpha))$ and on $N(f(\alpha))$.

Solution to Exercise 2.6. The abelian group $H_1(\Sigma;\mathbb{Z})$ is generated by the homology classes of oriented simple closed curves in Σ. Thus it is enough to prove (2.6) for the homology class $x:=[\gamma]$ of an oriented simple closed curve $\gamma\subset\Sigma$. By applying an isotopy to γ, we can assume that γ is transversal to α, so that α and γ meet in a finite number n of points. We consider the cyclic order on $\alpha\cap\gamma$ determined by the orientation of γ, which identifies $\alpha\cap\gamma$ to the cyclic group $\mathbb{Z}_n$. As a 1-chain, the oriented simple closed $\tau_\alpha(\gamma)$ is homologous to

$$\sum_{i\in\mathbb{Z}_n}(\gamma_{i,i+1}+\varepsilon_i\cdot\alpha)$$

where $\gamma_{i,i+1}$ is the oriented arc on γ between the points i and $i+1$ and where

$$\varepsilon_i:=\left\{\begin{array}{ll}+1, & \text{if } (\vec{\alpha}_i,\vec{\gamma}_i) \text{ is direct}\\ -1, & \text{otherwise}\end{array}\right\}.$$

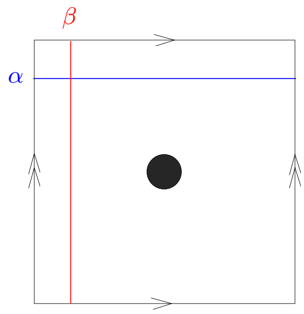
[6]This is the "isotopy extension theorem," see [Hir76].

[7]This is by using the theorem of collar/tubular neighborhoods, see [Hir76].

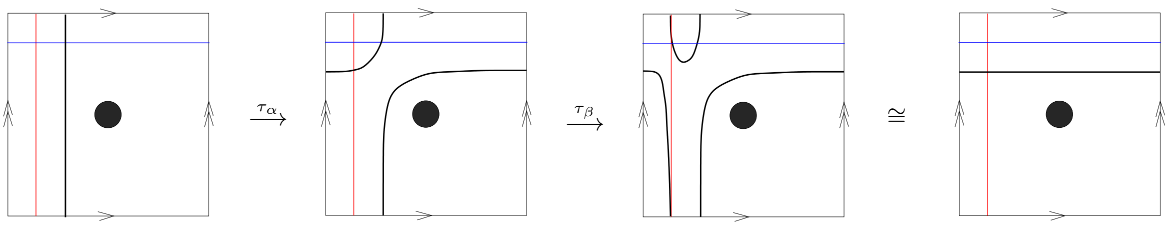
Therefore,

$$(\tau_\alpha)_*(x) = [\tau_\alpha(\gamma)] = \Big[\sum_{i\in\mathbb{Z}_n} (\gamma_{i,i+1} + \varepsilon_i \cdot \alpha)\Big] = \Big[\sum_{i\in\mathbb{Z}_n} \gamma_{i,i+1}\Big] + \Big(\sum_{i\in\mathbb{Z}_n} \varepsilon_i\Big) \cdot [\alpha] = [\gamma] + \omega([\alpha],[\gamma]) \cdot [\alpha].$$

Solution to Exercise 2.7. We can assume without loss of generality that $\Sigma := \Sigma_{1,1}$ is a torus with one hole, and that α, β are the following simple closed curves:

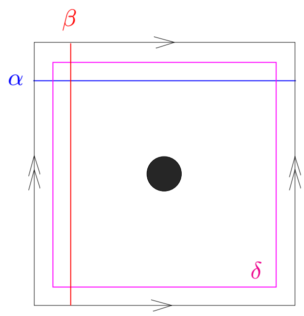


By applying subsequently τ_α and τ_β to a parallel copy of β, we get

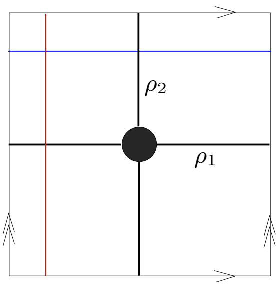


Hence $\tau_\beta\tau_\alpha(\beta) = \alpha$ as *unoriented* curves. The statement about oriented curves is proved by adding some orientations to α and β on the above pictures.

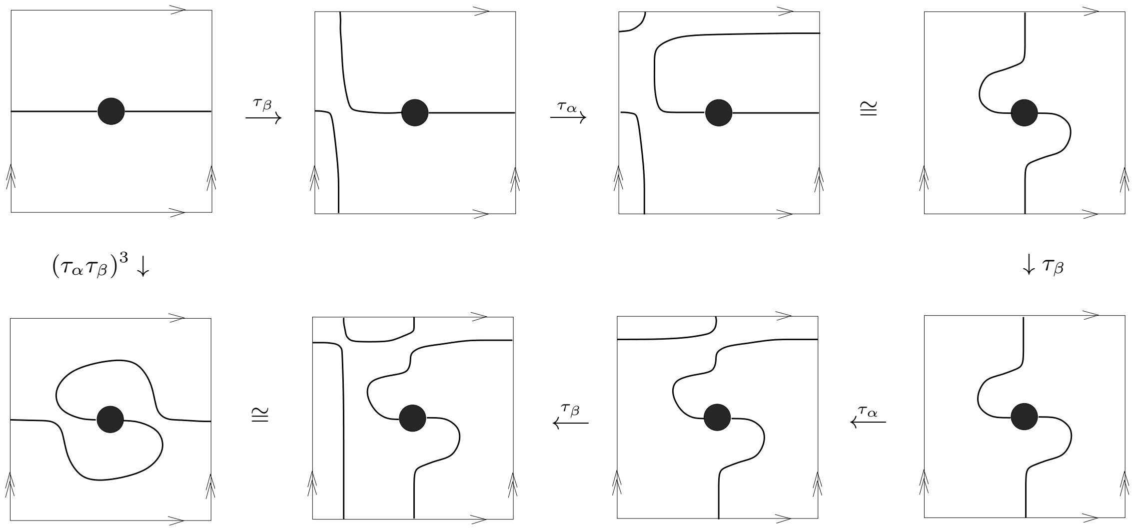
Solution to Exercise 2.8. We can assume without loss of generality that $\Sigma := \Sigma_{1,1}$ is a torus with one hole, and that α, β, δ are the following simple closed curves:



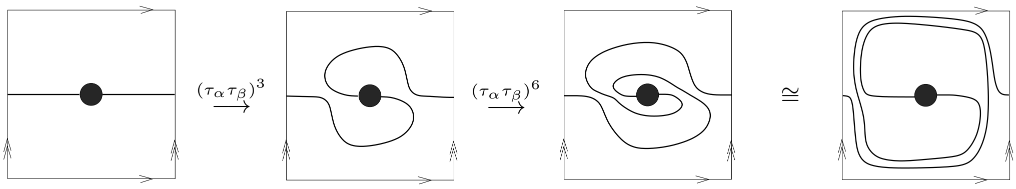
We consider the following two simple proper arcs ρ_1 and ρ_2 in Σ:



Let N be a "regular" neighborhood of $\partial\Sigma \cup \rho_1 \cup \rho_2$. We observe that $\Sigma \setminus \mathrm{int}(N)$ is a closed disk: then, by using the same kind of arguments as in Exercise 2.3, we see that any two $f, h \in \mathrm{Homeo}^\partial(\Sigma)$ are isotopic rel $\partial\Sigma$ if and only if $f(\rho_i)$ is isotopic to $h(\rho_i)$ for every $i \in \{1, 2\}$. Therefore, it is enough to "test" the identity $\tau_\delta = (\tau_\alpha\tau_\beta)^6$ on each of the arcs ρ_1, ρ_2; since $(\tau_\alpha\tau_\beta)^6 = (\tau_\beta\tau_\alpha)^6$ by the braid relation, and for symmetry reasons, it is actually enough to "test" this identity on ρ_1. We have

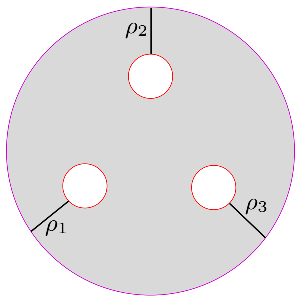


It follows that



We conclude that $(\tau_\alpha \tau_\beta)^6(\rho_1) = \tau_\delta(\rho_1)$ as required.

Solution to Exercise 2.9. We set $\Sigma := \Sigma_{0,4}$ and consider the simple proper arcs $\rho_1, \rho_2, \rho_3 \subset \Sigma$ shown below:



Let N be a "regular" neighborhood of $\partial\Sigma \cup \rho_1 \cup \rho_2 \cup \rho_3$. We observe that $\Sigma \setminus \mathrm{int}(N)$ is a closed disk: hence, by proceeding as in Exercise 2.3, we see that any two $f, h \in \mathrm{Homeo}^\partial(\Sigma)$ are isotopic rel $\partial\Sigma$ if and only if $f(\rho_i)$ is isotopic to $h(\rho_i)$ for every $i \in \{1,2,3\}$. Therefore, it is enough to "test" the lantern relation on each of the arcs ρ_1, ρ_2, ρ_3. This is easily checked: see for instance [FM12, Figure 5.2].

Solution to Exercise 2.10. The fact that the group homomorphism $d : \mathbb{Z}^b \to \mathcal{M}(\Sigma)$ is well defined follows from the disjointness relation, and the surjectivity of s follows from Exercise 1.8. Hence we only have to prove the exactness of the sequence (2.7) at $\mathcal{M}(\Sigma)$ and at $\mathcal{M}^\eth(\Sigma)$.

The fact that $c\big(\mathcal{M}(\Sigma)\big) \subset \ker(s)$ is obvious. To prove the converse inclusion, let $[h] \in \ker(s)$. Then the homeomorphism $h : \Sigma \to \Sigma$ maps every boundary component δ of Σ to itself. Since h preserves the orientation, the self-homeomorphism $h|_\delta$ of $\delta \cong S^1$ is orientation-preserving: hence $h|_\delta$ is isotopic to the identity of δ. There is a neighborhood $N(\delta)$ of δ in Σ which can be identified to $\delta \times [0,1]$ in such a way that $\delta \subset N(\delta)$ corresponds to $\delta \times \{1\}$. (This is the "collaring theorem" of differential topology, see [Hir76].) Using this identification, we easily construct an isotopy $I : \Sigma \times [0,1] \to \Sigma$ such that $I(-,0) = h$ and $I(-,1)|_\delta = \mathrm{id}_\delta$. Doing this construction for every boundary component of Σ, we see that h is isotopic to a self-homeomorphism of Σ fixing $\partial\Sigma$: therefore $[h]$ belongs to the image of c.

To show that $d(\mathbb{Z}^b) \subset \ker(c)$, it is enough to prove that the Dehn twist along the "core" $\alpha := S^1 \times \{1/2\}$ of the annulus $A := S^1 \times [0,1]$ is isotopic to id_A through an isotopy which fixes $S^1 \times \{0\}$ (but which *moves* $S^1 \times \{1\}$). For instance, the isotopy $I : A \times [0,1] \to A$ defined by

$$I\big((e^{2i\pi\theta}, r), t\big) := \big(e^{2i\pi(\theta + r(1-t))}, r\big)$$

has the desired properties. To show the converse inclusion, let $[h] \in \ker(c)$. Then there is an isotopy $I : \Sigma \times [0,1] \to \Sigma$ such that $I(-,0) = \mathrm{id}_\Sigma$ and $I(-,1) = h$, but this isotopy does *not* need to fix $\partial\Sigma$. For every boundary component δ of Σ, the restriction of I to $\delta \times [0,1]$ provides an isotopy between the identity of δ and itself. In the sequel, we identify δ with S^1. It is known that the map

$$S^1 \longrightarrow \mathrm{Homeo}^+(S^1), \ u \longmapsto (\text{multiplication by } u)$$

is a homotopy equivalence (see [Iva02, Corollary 2.7.B] for instance). Since $\pi_1(S^1) \simeq \mathbb{Z}$, we deduce that there exists a (unique) $k \in \mathbb{Z}$ such that the loop of $\mathrm{Homeo}^+(S^1)$ based at id_{S^1} represented by $I|_{\delta\times[0,1]}$ is homotopic to the loop $t \mapsto (z \mapsto ze^{2i\pi kt})$: let H be such a homotopy. As in the previous paragraph, we identify a neighborhood $N(\delta)$ of δ with $\delta \times [0,1]$ in such a way that $\delta \subset N(\delta)$ corresponds to $\delta \times \{1\}$. Using this identification and the homotopy H, we can construct from I a new isotopy $I' : \Sigma \times [0,1] \to \Sigma$ such that $I'(-,0) = \mathrm{id}_\Sigma$, $I'(-,1) = h$ and $I(z,t) = ze^{2i\pi kt}$ for every $(z,t) \in \delta \times [0,1]$. Finally, we can construct from I' a new isotopy $I'' : \Sigma \times [0,1] \to \Sigma$ *fixing* δ such that $I''(-,1) = h$ and $[I''(-,0)] = \tau_{\delta'}^k$ where δ' is a simple closed curve parallel to δ. Doing these constructions for every boundary component of Σ, we deduce that the isotopy class of h rel $\partial\Sigma$ belongs to the image of d.

N.B. The map d is not injective in general. For instance, if Σ is an annulus, the homomorphism $d : \mathbb{Z}^2 \to \mathcal{M}(\Sigma) \simeq \mathbb{Z}$ is given by $(l_1, l_2) \mapsto l_1 + l_2$.

Solution to Exercise 2.11. By definition of f, we have for any self-homeomorphism h of Σ

$$f([h]) = 1 \iff h \text{ reverses the orientation}$$

and it easily follows that f is a group homomorphism. To show the surjectivity of f, we embed Σ in $\mathbb{R}^3$ in such a way that there is an affine plane $H \subset \mathbb{R}^3$ whose corresponding symmetry leaves Σ globally invariant. The restriction s of this symmetry to Σ reverses the orientation, so that $f([s]) = 1$ and f is surjective. This also shows that the homomorphism $\mathbb{Z}_2 \to \mathcal{M}^\pm(\Sigma)$ defined by $1 \mapsto [s]$ is a section of f.

The exactness of (2.8) at $\mathcal{M}^\pm(\Sigma)$ is obvious, and it only remains to justify that e is injective. This follows from the fact that, during an isotopy, an orientation-preserving homeomorphism remains orientation-preserving.

3. Surface braid groups

We now give a brief introduction to surface braid groups, which can also be viewed as mapping class groups of a certain kind. The reader may consult the monographs [Bir74, KT08] for further details.

3.1. **Definition of surface braid groups.** We shall give two equivalent definitions of surface braid groups. Let Σ be an oriented surface and let $n \geq 1$ be an integer. We assume that n distinct points $x_1, \dots, x_n$ have been fixed in the interior of Σ: we call them the *marked points* of Σ.

The first definition is based on the notion of "configuration space". The *configuration space of n ordered points* in Σ is the topological space

$$\mathrm{F}_n(\Sigma) := \big\{ (z_1, \dots, z_n) \in \mathrm{int}(\Sigma)^n : \forall i \neq j,\ z_i \neq z_j \big\}$$

and the *configuration space of n unordered points* in Σ is the quotient space

$$\mathrm{C}_n(\Sigma) := \mathrm{F}_n(\Sigma)/\mathfrak{S}_n,$$

where the symmetric group $\mathfrak{S}_n$ acts on (the left of) $\mathrm{F}_n(\Sigma)$ by permutation of the coordinates:

$$\forall \sigma \in \mathfrak{S}_n,\ \forall (z_1, \dots, z_n) \in \mathrm{F}_n(\Sigma), \quad \sigma \cdot (z_1, \dots, z_n) = \big(z_{\sigma^{-1}(1)}, \dots, z_{\sigma^{-1}(n)}\big).$$

In particular, the n marked points of Σ define some elements

$$x := (x_1, \dots, x_n) \in \mathrm{F}_n(\Sigma) \quad \text{and} \quad \{x\} := \{x_1, \dots, x_n\} \in \mathrm{C}_n(\Sigma).$$

Definition 3.1. *The* surface braid group on n strands in Σ *is*

$$B_n(\Sigma) := \pi_1\big(\mathrm{C}_n(\Sigma), \{x\}\big)$$

and the pure surface braid group on n strands in Σ *is*

$$PB_n(\Sigma) := \pi_1\big(\mathrm{F}_n(\Sigma), x\big).$$

When $\Sigma := D^2$ is a disk, $B_n(\Sigma)$ and $PB_n(\Sigma)$ are simply denoted by B_n and PB_n, respectively, and they are called the *braid group* and the *pure braid group on n strands*. In this case, the marked points $x_1, \dots, x_n$ are assumed to be uniformly distributed (in this order) along the interior of the segment $[-1,1] \times \{0\} \subset D^2 \subset \mathbb{R}^2$.

Lemma 3.2. *The canonical projection $p : \mathrm{F}_n(\Sigma) \to \mathrm{C}_n(\Sigma)$ is a regular covering map, with automorphism group $\mathfrak{S}_n$.*

Proof. We have the following general principle: given a *properly discontinuous* action of a group G on an arc-connected and locally arc-connected topological space Y, which means that

$$\forall y \in Y,\ \exists \text{ neighborhood } V \ni y,\ \forall g \in G \setminus \{1\}, \quad g(V) \cap V = \varnothing,$$

the quotient map $Y \to Y/G$ is a regular covering map with automorphism group G. The fact that $\mathfrak{S}_n$ acts properly discontinuously on $\mathrm{F}_n(\Sigma)$ is easily checked. □

If follows from Lemma 3.2 that we have a short exact sequence of groups

$$1 \longrightarrow PB_n(\Sigma) \xrightarrow{p_\sharp} B_n(\Sigma) \xrightarrow{s} \mathfrak{S}_n \longrightarrow 1. \tag{3.1}$$

Here the map $p_\sharp : \pi_1(\mathrm{F}_n(\Sigma), x) \to \pi_1(\mathrm{C}_n(\Sigma), \{x\})$ is the homomorphism induced by the map p, while $s : B_n(\Sigma) \to \mathfrak{S}_n$ is the canonical map $\pi_1(\mathrm{C}_n(\Sigma), \{x\}) \to \pi_1(\mathrm{C}_n(\Sigma), \{x\})/p_\sharp\pi_1(\mathrm{F}_n(\Sigma), x)$ composed with the isomorphism (depending on x)

$$\pi_1\big(\mathrm{C}_n(\Sigma), \{x\}\big)/p_\sharp\pi_1\big(\mathrm{F}_n(\Sigma), x\big) \xrightarrow[\simeq]{} \mathrm{Aut}(p) = \mathfrak{S}_n$$

that is given by the general theory of covering spaces. Specifically, for any loop $\ell : [0,1] \to \mathrm{C}_n(\Sigma)$ based at $\{x\}$, we have

$$s([\ell]) \cdot x = \widetilde{\ell}(1)$$

where $\widetilde{\ell} : [0,1] \to \mathrm{F}_n(\Sigma)$ is the unique lift of ℓ such that $\widetilde{\ell}(0) = x$.

We now give a second definition of surface braid groups, which corresponds better to one's intuition of what should be a "braid."

Definition 3.3. *A* geometric braid on n strands in Σ *is an embedding of n intervals into the 3-manifold* $\Sigma \times [0,1]$

$$\beta : \{1, \dots, n\} \times [0,1] \longrightarrow \Sigma \times [0,1]$$

such that the following holds:

- *the image of β is contained in* $\mathrm{int}(\Sigma) \times [0,1]$*;*
- *for all $k \in \{1, \dots, n\}$, $\beta(k, 0) = (x_k, 0)$;*
- *there exists $s(\beta) \in \mathfrak{S}_n$ such that, for all $k \in \{1, \dots, n\}$, $\beta(k, 1) = \big(x_{s(\beta)^{-1}(k)}, 1\big)$;*

- *for all $t \in [0,1]$, the hypersurface $\Sigma \times \{t\}$ cuts the image of β into n distinct points.*

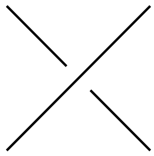
If $s(\beta)$ is the trivial permutation, then β is said to be pure. *Two geometric braids β_0, β_1 are* isotopic *if there exists a continuous map*

$$H : \{1, \ldots, n\} \times [0,1] \times [0,1] \longrightarrow \Sigma \times [0,1]$$

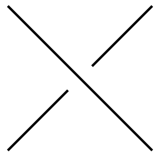
such that $H(-,-,s)$ is a geometric braid for all $s \in [0,1]$, and $H(-,-,s) = \beta_s$ for $s = 0, 1$.

Thus, a geometric braid β on n strands consists of n strings connecting the set of points $\{x\} \times \{0\}$ up to $\{x\} \times \{1\}$ in $\Sigma \times [0,1]$ without going "downwards" (as follows from the third condition).

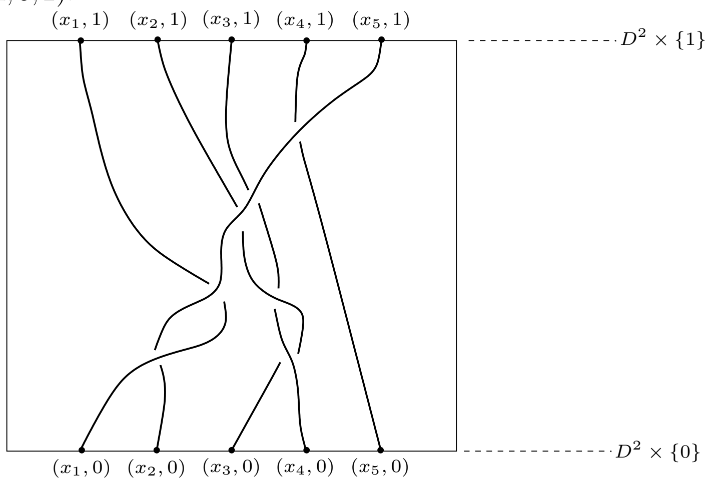
Example 3.4. Assume that $\Sigma := D^2$. Up to isotopy, we can assume that any geometric braid β is smooth and that the composition of β with the projection $D^2 \times I \to \mathbb{R} \times I$ defined by $((x_1, x_2), t) \mapsto (x_1, t)$ has only transverse double points. Thus, we can encode the isotopy class of a geometric braid by a *braid diagram* where those double points are depicted as



or



in order to record the information of the overcrossing/undercrossing strands. For instance, the following diagram represents a geometric braid β on 5 strands whose corresponding permutation $s(\beta)$ is given by $(1,2,3,4,5) \mapsto (1,3,4,5,2)$:



■

We can *multiply* any two geometric braids β_1 and β_2 in the following way. The product $\beta_1 \cdot \beta_2$ is the geometric braid

$$\{1, \ldots, n\} \times [0,1] \ni (k,t) \longmapsto \begin{cases} \frac{1}{2} \times \beta_1(k, 2t) & \text{if } t \in [0, 1/2], \\ \frac{1}{2} \times \beta_2\big(s(\beta_1)^{-1}(k), 2t-1\big) + \frac{1}{2} & \text{if } t \in [1/2, 1], \end{cases}$$

where the map $\frac{1}{2} \times (-) : \Sigma \times [0,1] \to \Sigma \times [0,1]$ in the first case is defined by $(z,t) \mapsto (z, t/2)$ and the map $\frac{1}{2} \times (-) + \frac{1}{2} : \Sigma \times [0,1] \to \Sigma \times [0,1]$ in the second case is defined by $(z,t) \mapsto (z, t/2 + 1/2)$. In a schematical way, we have

$$\boxed{\beta_1} \cdot \boxed{\beta_2} := \begin{array}{|c|}\hline \beta_2 \\ \hline \beta_1 \\ \hline \end{array}.$$

The *trivial* braid on n strands in Σ is defined by $(k,t) \mapsto (x_k, t)$, i.e. it goes straightly from $\{x\} \times \{0\}$ to $\{x\} \times \{1\}$. It is easily verified that the above multiplication law defines on the quotient set

$$B_n^{\text{geo}}(\Sigma) := \{\text{geometric braids on } n \text{ strands in } \Sigma\}/\text{isotopy}$$

a structure of group whose identity element is represented by the trivial braid.

We now justify that the above two definitions of a surface braid group are equivalent. Any loop ℓ in $\mathrm{C}_n(\Sigma)$ based at $\{x\}$ lifts to a unique path $\widetilde{\ell}$ in $\mathrm{F}_n(\Sigma)$ joining x to $s([\ell]) \cdot x$. Hence, we can associate to the loop ℓ the geometric braid $\beta = \beta(\ell)$ defined by

$$\forall k \in \{1, \ldots, n\},\ \forall t \in [0,1],\ \beta(k,t) := \big(k\text{-th coordinate of } \widetilde{\ell}(t), t\big).$$

If we perturb the based loop ℓ by a homotopy, then the path $\widetilde{\ell}$ is changed by a homotopy (fixing endpoints) so that the geometric braid β is only changed by an isotopy. Thus, we have defined a map

$$B_n(\Sigma) \longrightarrow B_n^{\text{geo}}(\Sigma),\ [\ell] \longmapsto [\beta(\ell)] \tag{3.2}$$

which is easily seen to respect the multiplication. To proceed, we observe that we could have replaced in Definition 3.3 the third condition by the following stronger condition:

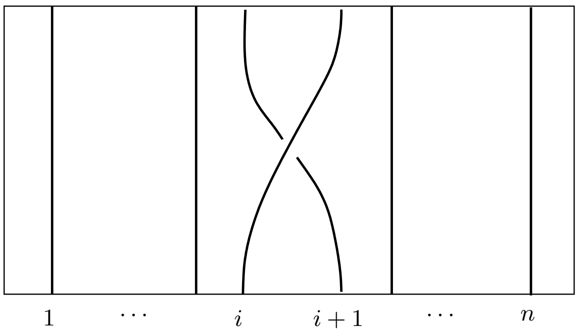
- for all $t \in [0,1]$ and for all $k \in \{1,\dots,n\}$, $\beta(k,t)$ belongs to the hypersurface $\Sigma \times \{t\}$.

Then it can be proved that any geometric braid in the "weak" sense is isotopic to a geometric braid in the "strong" sense, and that if two geometric braids in the "strong" sense are isotopic in the "weak" sense, then so they are in the "strong" sense. It follows that the map (3.2) is a group isomorphism. We also observe that the diagram

$$\begin{array}{ccc} B_n(\Sigma) & \xrightarrow{\ \simeq\ } & B_n^{\mathrm{geo}}(\Sigma) \\ & {\scriptstyle s}\searrow \quad \swarrow{\scriptstyle s} & \\ & \mathfrak{S}_n & \end{array}$$

is commutative. In particular, the subgroup $PB_n(\Sigma)$ of $B_n(\Sigma)$ corresponds to pure geometric braids through (3.2). In the sequel, we will make no difference between the groups $B_n(\Sigma)$ and $B_n^{\mathrm{geo}}(\Sigma)$, and (isotopy classes of) geometric braids will be simply referred to as "braids."

3.2. **Presentations of surface braid groups.** We only consider the case of a disk. (See Remark 3.10 below for the general case.) Let $n \geq 1$ be an integer. For all $i = 1,\dots,n-1$, we denote by $\sigma_i \in B_n$ the braid defined by the following diagram:



These braids can serve as generators of a presentation of B_n, which is due to Artin [Art25] and is considered to be the "canonical" presentation of the braid group.

Theorem 3.5 (Artin 1925)**.** *The braid group B_n has a presentation with generators $\sigma_1,\dots,\sigma_{n-1}$ and with relations*

$$\begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| \geq 2, \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } |i-j| = 1. \end{cases}$$

The proof of Theorem 3.5 given below builds on the short exact sequence (3.1), namely

$$1 \longrightarrow PB_n \longrightarrow B_n \xrightarrow{\ s\ } \mathfrak{S}_n \longrightarrow 1.$$

It consists in "merging" a presentation of $\mathfrak{S}_n$ with a presentation of PB_n using the following lemma.

Lemma 3.6. *Consider a short exact sequence of groups*

$$1 \longrightarrow S \xrightarrow{\ i\ } G \xrightarrow{\ p\ } Q \longrightarrow 1 \tag{3.3}$$

where S and Q are defined by some presentations:

$$S := \langle A \mid B \rangle \quad \text{and} \quad Q := \langle X \mid Y \rangle\,.$$

Let $\mathrm{F}(X)$ be the free group generated by the set X and let $t : \mathrm{F}(X) \to G$ be a homomorphism such that $pt(x) = x \in Q$ for all $x \in X$. For all $y \in Y$, let w_y be a word in A such that $i(w_y) = t(y) \in G$ and, for all $a \in A, x \in X$, let $v_{x,a}$ be a word in A such that $i(v_{x,a}) = t(x)i(a)t(x^{-1}) \in G$. Then the map

$$\varphi : \big\langle A \cup X \ \big|\ B \cup \{w_y y^{-1} | y \in Y\} \cup \{v_{x,a} x a^{-1} x^{-1} | a \in A, x \in X\} \big\rangle \longrightarrow G$$

defined by $\varphi|_A := i|_A$ and $\varphi|_X := t|_X$ is an isomorphism, and so it gives a presentation of G.

A special case of interest is when the short exact sequence (3.3) is *split*, i.e. when there is a group homomorphism $s : Q \to G$ such that $p \circ s = \mathrm{id}_Q$. In this case, the statement can be applied to the map $t : \mathrm{F}(X) \to G$ defined by $x \mapsto s(x)$, and we can assume that w_y is the empty word for each $y \in Y$.

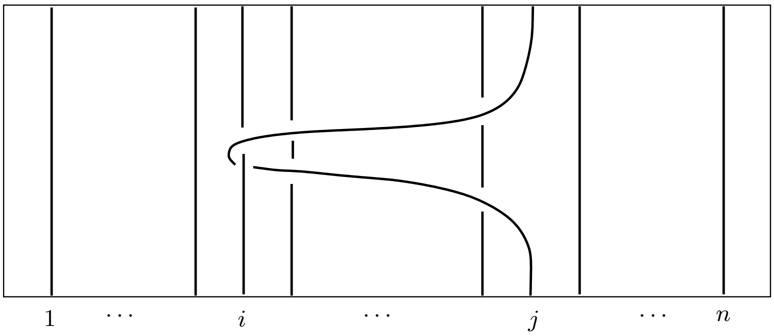
Proof of Lemma 3.6. Denote by G^0 the domain of φ, and consider the subgroup S^0 of G^0 generated by A. Because of the third kind of defining relations for G^0, S^0 is a normal subgroup of G^0 so that we can also consider the quotient group $Q^0 := G^0/S^0$.

We have $\varphi(S^0) = i(S)$ and $\varphi|_{S^0} : S^0 \to i(S)$ is injective because of the first kind of defining relations for G^0, hence an isomorphism $\varphi_S : S^0 \to S$. Moreover, φ induces a homomorphism $\varphi_Q : Q^0 \to Q$ since $\varphi(S^0) = i(S)$. This homomorphism is certainly surjective (since φ is so) and is injective because of the second kind of defining relations for G^0. We sum up the previous constructions in the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & S^0 & \longrightarrow & G^0 & \longrightarrow & Q^0 & \longrightarrow & 1 \\ & & \varphi_S \downarrow \simeq & & \varphi \downarrow & & \varphi_Q \downarrow \simeq & & \\ 1 & \longrightarrow & S & \xrightarrow[i]{} & G & \xrightarrow[p]{} & Q & \longrightarrow & 1 \end{array}$$

whose rows are short exact sequences. We deduce by diagram chasing that φ is an isomorphism. □

For all $i, j \in \{1, \ldots, n\}$ such that $i < j$, let $a_{ij} \in PB_n$ be the pure braid defined by the following diagram:



Note that

$$a_{ij} = \sigma_{j-1}^{-1} \cdots \sigma_{i+1}^{-1} \cdot \sigma_i^2 \cdot \sigma_{i+1} \cdots \sigma_{j-1}. \tag{3.4}$$

Here is the presentation of the pure braid group that we will use to prove Theorem 3.5. This is also due to Artin [Art47].

Theorem 3.7 (Artin 1947)**.** *The pure braid group PB_n has a presentation with generators a_{ij} (for all $1 \leq i < j \leq n$) and with relations*

$$\begin{cases} a_{rs} a_{ij} a_{rs}^{-1} = a_{ij} & \text{if } r < s < i < j \text{ or } i < r < s < j, \\ a_{rs} a_{ij} a_{rs}^{-1} = a_{rj}^{-1} a_{ij} a_{rj} & \text{if } r < s = i < j, \\ a_{rs} a_{ij} a_{rs}^{-1} = [a_{sj}, a_{rj}]^{-1} a_{ij} [a_{sj}, a_{rj}] & \text{if } r < i < s < j, \\ a_{rs} a_{ij} a_{rs}^{-1} = (a_{sj} a_{ij})^{-1} a_{ij} (a_{sj} a_{ij}) & \text{if } r = i < s < j. \end{cases}$$

Here $[x, y]$ denotes the commutator $x^{-1} y^{-1} x y$. Note that the relations given for PB_n are indexed by all possible pairs (r, s) and (i, j) with $r < s$, $i < j$ and $s < j$.

For all $i \in \{1, \ldots, n-1\}$, let $\tau_i \in \mathfrak{S}_n$ be the transposition $(i, i+1)$. Here is the presentation of the symmetric group that we will use to prove Theorem 3.5. This is a classical result which seems to go back to Moore [Moo97].

Theorem 3.8 (Moore 1897)**.** *The symmetric group $\mathfrak{S}_n$ has a presentation with generators τ_i (for all $1 \leq i \leq n-1$) and with relations*

$$\begin{cases} \tau_i^2 = 1 & \\ \tau_i \tau_j = \tau_j \tau_i & \text{if } |i - j| \geq 2, \\ \tau_i \tau_j \tau_i = \tau_j \tau_i \tau_j & \text{if } |i - j| = 1. \end{cases}$$

Assuming Theorem 3.7 and Theorem 3.8 for granted, we can now prove Artin's presentation of the braid group.

Sketch of proof of Theorem 3.5. We apply Lemma 3.6 to the short exact sequence (3.1). The group PB_n has the presentation given by Theorem 3.7, while the group $\mathfrak{S}_n$ has the presentation given by Theorem 3.8. The braid σ_i satisfies $s(\sigma_i) = \tau_i \in \mathfrak{S}_n$. Thus, the group B_n is generated by

$$\{a_{ij} \,|\, 1 \leq i < j \leq n\} \cup \{\sigma_i \,|\, i = 1, \ldots, n-1\}$$

with three kinds of relations: 1) the relations of PB_n, 2) the relations arising from $\mathfrak{S}_n$ and 3) the relations of "conjugation-type." Relations 2) can be choosen as follows:

2.a) $\sigma_i^2 = a_{i,i+1}$,
2.b) $\sigma_i\sigma_j = \sigma_j\sigma_i$ if $|i-j| \geq 2$,
2.c) $\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j$ if $|i-j| = 1$.

As for relations 3), they are of the form

$$\sigma_k a_{ij} \sigma_k^{-1} = \text{word in the } a_{rs}\text{'s}$$

for all $1 \leq i < j \leq n$ and $k \in \{1, \dots, n-1\}$. Among them, we have the relations of the form

$$3.a) \ \sigma_{j-1} a_{ij} \sigma_{j-1}^{-1} = a_{i,j-1}$$

for all i, j such that $j > i+1$. Relations 3) that are no of type 3.a) are declared to be of type 3.b). The relations 3.a) and 2.a) can be used to write each a_{ij} in terms of $\sigma_1, \dots, \sigma_{n-1}$. Thus, we get a new presentation of B_n with generators $\sigma_1, \dots, \sigma_{n-1}$ and with relations 1], 2.b), 2.c) and 3.b], where relations 1] and 3.b] are obtained from 1) and 3.b) respectively by expressing each a_{rs} in terms of the σ_i's. It is a long but straightforward computation to check that each of the relations of type 1] or 3.b] is a consequence of 2.b) and 2.c). The conclusion follows from this check. □

It now still remains to prove Theorem 3.7 and Theorem 3.8. We start with the latter.

Proof of Theorem 3.8. We repeat the proof of [Bur55, Note C] with some slight variations. Let $\mathfrak{S}_n^0$ be the presented group

$$\mathfrak{S}_n^0 := \left\langle \tau_1, \dots, \tau_{n-1} \;\middle|\; \begin{array}{ll} \tau_i^2 = 1 & \\ \tau_i\tau_j = \tau_j\tau_i & \text{if } |i-j| \geq 2 \\ \tau_i\tau_j\tau_i = \tau_j\tau_i\tau_j & \text{if } |i-j| = 1 \end{array} \right\rangle.$$

As it is easily checked, there is a group homomorphism $\varphi : \mathfrak{S}_n^0 \to \mathfrak{S}_n$ defined by $\tau_i \mapsto (i, i+1)$. It is surjective since the transpositions $(1,2), (2,3), \dots, (n-1,n)$ generate $\mathfrak{S}_n$. Since the cardinality of $\mathfrak{S}_n$ is $n!$, the bijectivity of φ will follow from the fact that $\mathfrak{S}_n^0$ is finite with cardinality $|\mathfrak{S}_n^0| \leq n!$. To prove this, we consider the subgroup H_0 of $\mathfrak{S}_n^0$ generated by $\tau_1, \dots, \tau_{n-2}$ and the following n cosets of $\mathfrak{S}_n^0$:

$$H_0,\ \underbrace{H_0\tau_{n-1}}_{=:H_1},\ \underbrace{H_0\tau_{n-1}\tau_{n-2}}_{=:H_2},\ \dots,\ \underbrace{H_0\tau_{n-1}\tau_{n-2}\cdots\tau_1}_{=:H_{n-1}}.$$

We claim the following.

Fact. *For any $i \in \{0, \dots, n-1\}$ and $j \in \{1, \dots, n-1\}$, there is a $k \in \{0, \dots, n-1\}$ such that $H_i\tau_j \subset H_k$.*

Since $1 \in H_0$, we deduce that any element of $\mathfrak{S}_n^0$ belongs to one of the cosets $H_0, \dots, H_{n-1}$. There is an obvious surjection $\mathfrak{S}_{n-1}^0 \to H_i$ for every i. Clearly $\mathfrak{S}_2^0$ is the cyclic group of order 2. Hence, by an induction on $n \geq 2$, we conclude that $\mathfrak{S}_n^0$ is finite with cardinality

$$|\mathfrak{S}_n^0| \leq n \cdot (n-1) \cdots 2 = n!$$

To justify the above fact, it suffices to observe that

$$\big(\tau_{n-1}\cdots\tau_{s+1}\tau_s\big)\tau_j = \begin{cases} \tau_j\big(\tau_{n-1}\cdots\tau_{s+1}\tau_s\big) & \text{if } j < s-1, \\ \tau_{j-1}\big(\tau_{n-1}\cdots\tau_{s+1}\tau_s\big) & \text{if } j > s, \\ \tau_{n-1}\cdots\tau_{s+1}\tau_s\tau_{s-1} & \text{if } j = s-1, \\ \tau_{n-1}\cdots\tau_{s+1} & \text{if } j = s, \end{cases}$$

where the second identity follows from the relation $(\tau_j\tau_{j-1})\tau_j = \tau_{j-1}(\tau_j\tau_{j-1})$. □

Theorem 3.7 will be proved by an induction on $n \geq 1$. On the one hand, there is a homomorphism

$$p : PB_{n+1} \longrightarrow PB_n$$

which consists in "forgetting" the last string of a pure braid on $(n+1)$ strands. On the other hand, there is a group homomorphism

$$i : \pi_1\big(D^2 \setminus \{x_1, \dots, x_n\}, x_{n+1}\big) \longrightarrow PB_{n+1}$$

which sends (the homotopy class of) a loop α to (the isotopy class of) the geometric braid defined by $(k,t) \mapsto (x_k, t)$ for $k \leq n$ and by $(n+1, t) \mapsto (\alpha(t), t)$.

Proposition 3.9. *The sequence of groups*

$$1 \longrightarrow \pi_1\big(D^2 \setminus \{x_1, \dots, x_n\}, x_{n+1}\big) \xrightarrow{\ i\ } PB_{n+1} \xrightarrow{\ p\ } PB_n \longrightarrow 1 \tag{3.5}$$

is exact and it splits.

Proof. We note that $p : PB_{n+1} \to PB_n$ is the group homomorphism $f_\sharp$ induced by the map

$$f : \mathrm{F}_{n+1}(D^2) \longrightarrow \mathrm{F}_n(D^2),\ (z_1, \dots, z_n, z_{n+1}) \longmapsto (z_1, \dots, z_n),$$

and that f has a right inverse. Specifically the continuous map $s : \mathrm{F}_n(D^2) \to \mathrm{F}_{n+1}(D^2)$ defined by $(z_1, \dots, z_n) \mapsto \big(z_1, \dots, z_n, \sqrt{max(|z_1|, \cdots, |z_n|)}\big)$ satisfies $fs = \mathrm{id}_{\mathrm{F}_n(D^2)}$. Besides, we note that i is the group homomorphism induced by the map

$$D^2 \setminus \{x_1, \dots, x_n\} \longrightarrow \mathrm{F}_{n+1}(D^2),\ z \longmapsto (x_1, \dots, x_n, z),$$

whose image is $f^{-1}(x_1, \dots, x_n)$. Thus, we are asked to prove that the sequence

$$1 \longrightarrow \pi_1\big(f^{-1}(x), x^+\big) \longrightarrow \pi_1\big(\mathrm{F}_{n+1}(D^2), x^+\big) \xrightarrow{f_\sharp} \pi_1\big(\mathrm{F}_n(D^2), x\big) \longrightarrow 1 \qquad (3.6)$$

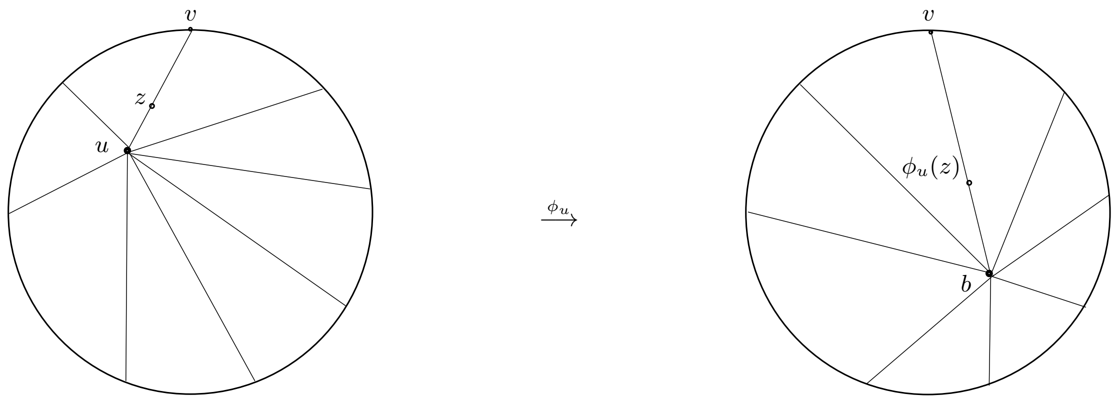
is split exact, where x denotes the base point $(x_1, \dots, x_n)$ of $\mathrm{F}_n(D^2)$ and x^+ denotes the base point $(x_1, \dots, x_{n+1})$ of $\mathrm{F}_{n+1}(D^2)$. It suffices to prove that f is a fiber bundle: then the long exact sequence of homotopy groups induced by f restricts to (3.6), since the existence of a right inverse for f implies that $f_\sharp : \pi_k(\mathrm{F}_{n+1}(D^2)) \to \pi_k(\mathrm{F}_n(D^2))$ is a surjection for any $k \geq 1$.

To show that f is a fiber bundle, let us consider a point $b = (b_1, \dots, b_n) \in \mathrm{F}_n(D^2)$. The minimum of $\big\{|b_i - b_j| > 0 \big| 1 \leq i < j \leq n\big\} \cup \big\{1 - |b_i| > 0 | i = 1, \dots, n\big\}$ is denoted by ε. Then, the product of open disks

$$U := D(b_1, \varepsilon/3) \times \cdots \times D(b_n, \varepsilon/3)$$

(centered at the points b_i's with radius $\varepsilon/3$ in $\mathbb{C}$) is an open neighborhood of b in $\mathrm{F}_n(D^2)$. Let $K := \overline{D(b_1, \varepsilon/3)} \cup \cdots \cup \overline{D(b_n, \varepsilon/3)} \subset \mathrm{int}(D^2)$ and let $\varphi : U \times K \to K$ be a continuous map such that, for all $u \in U$, $\varphi_u := \varphi(u, -) : K \to K$ is a homeomorphism which is the identity on ∂K, and satisfies $\varphi_u(u_i) = b_i$ for all $i = 1 \dots, n$. The existence of such a map φ follows from the following fact, which is easily checked.

Fact. *Let $b \in D(0,1)$. For all $u \in D(0,1)$, let $\phi_u : \overline{D(0,1)} \to \overline{D(0,1)}$ be the map defined by $\phi_u(tu + (1-t)v) = tb + (1-t)v$ for all $v \in S^1 = \partial\overline{D(0,1)}$ and $t \in [0,1]$. Then, ϕ_u is a homeomorphism which is the identity on S^1 and sends u to b. Moreover, the map $\phi : D(0,1) \times \overline{D(0,1)} \to \overline{D(0,1)}$ defined by $(u, z) \mapsto \phi_u(z)$ is continuous.*



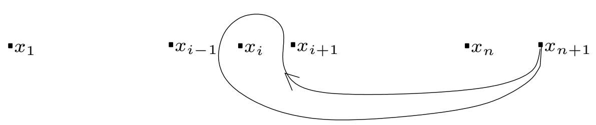
We still denote by $\varphi_u : D^2 \to D^2$ the extension of φ_u by the identity. Then, there is a homeomorphism $f^{-1}(U) \to U \times f^{-1}(b)$ defined by

$$(z_1, \dots, z_n, z_{n+1}) \mapsto \big(z_1, \dots, z_n, (b, \varphi_{(z_1, \dots, z_n)}(z_{n+1}))\big).$$

We conclude that f is a fiber bundle with fiber $D^2 \setminus \{x_1, \dots, x_n\}$. □

We can now conclude the proof of Artin's theorem.

Sketch of proof of Theorem 3.7. The theorem trivially holds for $n = 1$ (and it also holds for $n = 2$ according to Exercice 3.1). Assume that the theorem is valid for n strands. Then, Lemma 3.6 can be applied to the short exact sequence (3.5) in order to get a presentation of PB_{n+1}. For this, we observe that the group $\pi_1\big(D^2 \setminus \{x_1, \dots, x_n\}, x_{n+1}\big)$ is freely generated by $[\ell_1] \dots, [\ell_n]$, where ℓ_i is represented by the following loop:



The image of $[\ell_i]$ in PB_{n+1} is the pure braid $a_{i,n+1}$. Taking advantage of the fact that (3.5) is split, we deduce that PB_{n+1} is generated by

$$\{a_{i,n+1} | i = 1, \dots, n\} \cup \{a_{rs} | 1 \leq r < s \leq n\}$$

and that the relations are those from PB_n together with the relations expressing the conjugate $a_{rs} a_{i,n+1} a_{rs}^{-1}$ in terms of $a_{1,n+1}, \dots, a_{n,n+1}$ for all $1 \leq i \leq n$ and for all $1 \leq r < s \leq n$. The latter are easily computed and the conclusion follows. □

Remark 3.10. The first presentations of surface braid groups for arbitrary surfaces have been obtained by Birman [Bir69c] and Scott [Sco70]. González-Meneses has used the same strategy as the above proof of Theorem 3.5 to compute presentations of surface braid groups for arbitrary closed surfaces [GM01]. Finally, Bellingeri has obtained some presentations of $B_n(\Sigma)$ and $PB_n(\Sigma)$ for arbitrary surfaces Σ in [Bel04]. ■

3.3. **Surface braid groups as mapping class groups.** To conclude this section, we explain how to interpret surface braid groups as mapping class groups of a special kind. Let Σ be an oriented surface with marked points $x_1, \dots, x_n \subset \mathrm{int}(\Sigma)$. We denote $\{x\} := \{x_1, \dots, x_n\}$ and $x := (x_1, \dots, x_n)$. We consider the groups

$$\mathrm{Homeo}^{+,\partial}(\Sigma, \{x\}) := \big\{ f \in \mathrm{Homeo}^{+,\partial}(\Sigma) : f(\{x\}) = \{x\} \big\}$$

$$\text{and} \quad \mathrm{Homeo}^{+,\partial}(\Sigma, x) := \big\{ f \in \mathrm{Homeo}^{+,\partial}(\Sigma) : f(x) = x \big\}$$

which we equip with the compact-open topology.

Definition 3.11. *The* mapping class group *of* Σ *with* marked points $x_1, \dots, x_n$ *is*

$$\mathcal{M}(\Sigma, \{x\}) := \pi_0\big(\mathrm{Homeo}^{+,\partial}(\Sigma, \{x\})\big)$$

and the pure mapping class group *of* Σ *with* marked points $x_1, \dots, x_n$ *is*

$$\mathcal{PM}(\Sigma, x) := \pi_0\big(\mathrm{Homeo}^{+,\partial}(\Sigma, x)\big).$$

For any $[f] \in \mathcal{M}(\Sigma, \{x\})$, we denote by $s([f]) \in \mathfrak{S}_n$ the unique permutation such that $f(x_i) = x_{s([f])(i)}$ for all $i \in \{1, \dots, n\}$. Then, we have the short exact sequence of groups

$$1 \longrightarrow \mathcal{PM}(\Sigma, x) \xrightarrow{\ i_\sharp\ } \mathcal{M}(\Sigma, \{x\}) \xrightarrow{\ s\ } \mathfrak{S}_n \longrightarrow 1 \tag{3.7}$$

where $i_\sharp$ is the map induced by the inclusion $i : \mathrm{Homeo}^{+,\partial}(\Sigma, x) \to \mathrm{Homeo}^{+,\partial}(\Sigma, \{x\})$.

Remark 3.12. We have already met the above groups in an equivalent form. Indeed, when $\partial\Sigma = \varnothing$, the group $\mathcal{M}(\Sigma, \{x\})$ is essentially the "boundary-free" version $\mathcal{M}^{\mathfrak{d}}(\Sigma)$ of the mapping class group considered in §2.1, while $\mathcal{PM}(\Sigma, x)$ corresponds to the image of $\mathcal{M}(\Sigma)$ in $\mathcal{M}^{\mathfrak{d}}(\Sigma)$. Then (3.7) is a portion of the sequence (2.7). ■

The next theorem, which relates surface braid groups to mapping class groups in the general case, is due to Birman [Bir69b].

Theorem 3.13 (Birman 1969)**.** *We have the following commutative diagram with exact rows/columns in the category of groups:*

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
\pi_1\big(\mathrm{Homeo}^{+,\partial}(\Sigma)\big) & \longrightarrow & PB_n(\Sigma) & \xrightarrow{\varpi} & \mathcal{PM}(\Sigma, x) & \longrightarrow & \mathcal{M}(\Sigma) & \longrightarrow & 1 \\
\| & & {\scriptstyle p_\sharp}\downarrow & & \downarrow{\scriptstyle i_\sharp} & & \| & & \\
\pi_1\big(\mathrm{Homeo}^{+,\partial}(\Sigma)\big) & \longrightarrow & B_n(\Sigma) & \xrightarrow{\varpi} & \mathcal{M}(\Sigma, \{x\}) & \longrightarrow & \mathcal{M}(\Sigma) & \longrightarrow & 1 \\
 & & {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} & & & & \\
 & & \mathfrak{S}_n & = & \mathfrak{S}_n & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 1 & & 1 & & & &
\end{array} \tag{3.8}$$

Sketch of proof. It is not difficult to check that the evaluation map

$$e : \mathrm{Homeo}^{+,\partial}(\Sigma) \longrightarrow \mathrm{F}_n(\Sigma), \ f \longmapsto f(x)$$

is a fiber bundle with fiber $e^{-1}(x) = \mathrm{Homeo}^{+,\partial}(\Sigma, x)$: see [Bir69b, Lemma 1.2] or Proposition 3.9 above for similar arguments. We take id_Σ as a base-point for $\mathrm{Homeo}^{+,\partial}(\Sigma)$ and x as a base-point for $\mathrm{F}_n(\Sigma)$. Then the long exact sequence in homotopy groups induced by e terminates with

$$\pi_1\big(\mathrm{Homeo}^{+,\partial}(\Sigma), \mathrm{id}_\Sigma\big) \longrightarrow \underbrace{\pi_1\big(\mathrm{F}_n(\Sigma), x\big)}_{=PB_n(\Sigma)} \xrightarrow{\partial_\sharp} \underbrace{\pi_0\big(\mathrm{Homeo}^{+,\partial}(\Sigma, x)\big)}_{=\mathcal{PM}(\Sigma,x)} \longrightarrow \underbrace{\pi_0\big(\mathrm{Homeo}^{+,\partial}(\Sigma)\big)}_{=\mathcal{M}(\Sigma)} \to 1$$

since, according to Exercise 3.2, the space $\mathrm{F}_n(\Sigma)$ is arc-connected. By definition of the connecting homomorphism $\partial_\sharp$, we have

$$\partial_\sharp([\beta]) = [\widetilde{\beta}(1)]$$

for any pure braid $\beta : [0,1] \to \mathrm{F}_n(\Sigma)$, where $\widetilde{\beta} : [0,1] \to \mathrm{Homeo}^{+,\partial}(\Sigma)$ is an isotopy starting at $\widetilde{\beta}(0) = \mathrm{id}_\Sigma$ and such that $\beta(t) = \widetilde{\beta}(t)(x)$ for any $t \in [0,1]$. We claim that $\partial_\sharp$ is a group anti-homomorphism. Indeed, for any two pure braids $\beta, \beta' : [0,1] \to \mathrm{F}_n(\Sigma)$, let $\widetilde{\beta}' \circ \widetilde{\beta}(1)$ be the path $[0,1] \to \mathrm{Homeo}^{+,\partial}(M)$ defined by $t \mapsto \widetilde{\beta}'(t) \circ \widetilde{\beta}(1)$. Denote by $*$ the concatenation of paths. Then, the path $\widetilde{\beta} * (\widetilde{\beta}' \circ \widetilde{\beta}(1))$ projects to $\beta * \beta'$ by the evaluation map e and starts at id_Σ, so that

$$\partial_\sharp([\beta]\,[\beta']) = \partial_\sharp([\beta * \beta']) = \big[\widetilde{\beta}'(1) \circ \widetilde{\beta}(1)\big] = \big[\widetilde{\beta}'(1)\big]\,\big[\widetilde{\beta}(1)\big] = \partial_\sharp([\beta'])\,\partial_\sharp([\beta]).$$

We define ϖ to be the composition of $\partial_\sharp$ with the group inversion.

Thus, we haved obtained the first row of the diagram (3.8). The second row is obtained in the same way by considering the evaluation map

$$\mathrm{Homeo}^{+,\partial}(\Sigma) \longrightarrow \mathrm{C}_n(\Sigma), \ f \longmapsto f(\{x\})$$

(This is the composition of the above fiber bundle map $e : \mathrm{Homeo}^{+,\partial}(\Sigma) \to \mathrm{F}_n(\Sigma)$ with the covering map $p : \mathrm{F}_n(\Sigma) \to \mathrm{C}_n(\Sigma)$, hence it is a fibration with fiber $\mathrm{Homeo}^{+,\partial}(\Sigma, \{x\})$.) The commutativity of the subdiagram consisting of the first two rows follows from the naturality of the long exact sequence of homotopy groups for fibrations. Finally, the identity $s \circ \varpi = s : B_n(\Sigma) \to \mathfrak{S}_n$ easily follows from the definitions. □

Note that, in general, the group homomorphism $\pi_1(\mathrm{Homeo}^{+,\partial}(\Sigma)) \to PB_n(\Sigma)$ is not trivial, so that ϖ is not necessarily injective. (See [Bir69b, §1] and [Bir74, §4.1] for further details.) Nevertheless, this homomorphism is certainly trivial when $\mathrm{Homeo}^{+,\partial}(\Sigma)$ is contractible. This is for instance the case when $\chi(\Sigma) < 0$ [EE69, ES70, Gra73] or in the following situation.

Example 3.14. Assume that $\Sigma := D^2$ is a disk. Then, by Proposition 2.3, the space $\mathrm{Homeo}^{+,\partial}(\Sigma)$ is contractible. It follows from Theorem 3.13 that $\varpi : PB_n \to \mathcal{PM}(D^2, x)$ and $\varpi : B_n \to \mathcal{M}(D^2, \{x\})$ are isomorphisms. ■

3.4. Exercises.

Exercise 3.1. Show that $\mathrm{F}_2(D^2)$ has the homotopy type of S^1, and deduce from this some presentations of the groups PB_2 and B_2.

Exercise 3.2. Show that, for any oriented surface Σ and for any integer $n \geq 1$, the spaces $\mathrm{F}_n(\Sigma)$ and $\mathrm{C}_n(\Sigma)$ are path-connected.

Exercise 3.3. Let $n \geq 1$ be an integer. Compute the abelianization $PB_n/[PB_n, PB_n]$ of the pure braid group PB_n.

Exercise 3.4. Let $n \geq 1$ be an integer.

(a) Use Proposition 3.9 to show that PB_{n+1} is an iterated semi-direct product

$$PB_{n+1} = F_n \rtimes (F_{n-1} \rtimes (\cdots \rtimes F_1) \cdots)$$

where each $F_i \subset PB_{n+1}$ is a free group of rank $i \geq 1$.

(b) Deduce that each braid $\beta \in PB_{n+1}$ can be written uniquely in the form

$$\beta = \beta_{n+1} \cdot \beta_n \cdots \beta_2$$

where all the strands of the braid β_i are straight vertical except for the i-th strand which "winds around" the $(i-1)$-st strands.

(c) The above decomposition of β is called the *combing* of the braid β: illustrate this with an example for $n = 4$.

Exercise 3.5. Let Σ be an oriented surface with a set of marked points $\{x\} := \{x_1, \dots, x_n\}$. An *arc* in Σ *relative* to $\{x\}$ is an embedded arc $a \subset \mathrm{int}(\Sigma)$ joining two distinct points x_i, x_j and whose interior does not meet $\{x\}$. We choose a closed disk $N \subset \Sigma$ such that $a \subset \mathrm{int}(N)$ and $N \cap \{x\} = \{x_i, x_j\}$, and we fix an orientation-preserving homeomorphism $N \cong D^2 \subset \mathbb{C}$ mapping the arc a to the segment $[-1/2, 1/2]$. Then the *half-twist* along a is the element $\sigma_a \in \mathcal{M}(\Sigma, \{x\})$ represented by the self-homeomorphism of Σ which is the identity outside N and is given on $N \cong D^2$ by

$$z \longmapsto \begin{cases} -z & \text{if } |z| \leq 1/2, \\ \exp(2i\pi(1-|z|)) \cdot z & \text{if } 1/2 \leq |z| \leq 1. \end{cases}$$

(a) Determine the action of σ_a on the set $\{x\}$ and compute σ_a^2.
(b) For any two arcs a and b in Σ relative to $\{x\}$, show that $\sigma_a\sigma_b = \sigma_b\sigma_a$ if $a \cap b = \varnothing$.
(c) Show that $\sigma_a\sigma_b\sigma_a = \sigma_b\sigma_a\sigma_b$ if a and b meet in a single point belonging to $\{x\}$.

Exercise 3.6. Let Σ be an oriented surface with marked points $\{x\} := \{x_1, \dots, x_n\}$, and let $\beta : \{1, \dots, n\} \times [0,1] \to \Sigma \times [0,1]$ be a geometric braid. Show that there is a homeomorphism

$$(\Sigma \times [0,1]) \setminus \beta\big(\{1, \dots, n\} \times [0,1]\big) \xrightarrow{\cong} (\Sigma \setminus \{x_1, \dots, x_n\}) \times [0,1]$$

from the "exterior" of β to the "exterior" of the trivial braid, which fixes $(\Sigma \setminus \{x_1, \dots, x_n\}) \times \{0\}$.

Exercise 3.7. Let $n \geq 2$ be an integer and set $\theta_n := \big(\sigma_1\sigma_2 \cdots \sigma_{n-1}\big)^n \in B_n$.
(a) Draw θ_2, θ_3, and θ_4.
(b) Compute the image of θ_n under the isomorphism $\varpi : B_n \to \mathcal{M}(D^2, \{x\})$.
(c) Deduce that θ_n belongs to the center of B_n.
(d) Show that θ_n has infinite order in B_n.

Exercise 3.8. Consider the group homomorphism

$$\psi : B_3 \longrightarrow \mathcal{M}(S^1 \times S^1),\ \sigma_1 \longmapsto \tau_A,\ \sigma_2 \longmapsto \tau_B$$

where A and B are the simple closed curves $S^1 \times \{1\}$ and $\{1\} \times S^1$ respectively.
(a) Show that ψ induces an isomorphism $B_3/\langle\theta_3\rangle \simeq \mathrm{PSL}(2;\mathbb{Z})$ where $\theta_3 := (\sigma_1\sigma_2)^3$
(b) Using Exercise 3.7, deduce that the center of B_3 is a free cyclic group generated by θ_3.

Exercise 3.9. Let $\mathbb{K}$ be a commutative field and let V be a $\mathbb{K}$-vector space.
(a) A *Yang–Baxter operator* is a linear automorphism $R : V \otimes V \to V \otimes V$ which satisfies the following identity in $\mathrm{End}_{\mathbb{K}}(V^{\otimes 3})$:

$$(R \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes R) \circ (R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R) \circ (R \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes R).$$

Show that R induces a homomorphism

$$\rho_R : B_n \longrightarrow \mathrm{Aut}_{\mathbb{K}}\left(V^{\otimes n}\right)$$

defined for all $i \in \{1, \dots, n-1\}$ by $\rho_R(\sigma_i) := \mathrm{id}_V{}^{\otimes(i-1)} \otimes R \otimes \mathrm{id}_V{}^{\otimes(n-i-1)}$.
(b) Let $F : V \otimes V \to V \otimes V$ be the "flip" defined by $F(v_1 \otimes v_2) := v_2 \otimes v_1$ and let $x \in \mathbb{K} \setminus \{0\}$. Check that xF is a Yang–Baxter operator and compute the representation ρ_{xF} explicitely using the homomorphism $s : B_n \to \mathfrak{S}_n$.
(c) Assume that V is a unitary associative $\mathbb{K}$-algebra, and let $x, y, z \in \mathbb{K} \setminus \{0\}$. Check that the linear map $R_{x,y,z} : V \otimes V \to V \otimes V$ defined by

$$R_{x,y,z}(a_1 \otimes a_2) := x \cdot a_1a_2 \otimes 1 + y \cdot 1 \otimes a_1a_2 - z \cdot a_1 \otimes a_2$$

is a Yang–Baxter operator with inverse $R_{y^{-1},x^{-1},z^{-1}}$ if $x = z$ or if $y = z$. (Hint: decompose $R_{x,y,z}$ as a sum of two terms $R_{x,y,z} = R'_{x,y} - z \cdot \mathrm{id}_{V \otimes V}$.)

* * *

Solution to Exercise 3.1. Since $\mathrm{int}(D^2) \cong \mathbb{C}$, the configuration space $\mathrm{F}_2(D^2)$ is homeomorphic to

$$\mathrm{F}_2(\mathbb{C}) := \big\{(z, z') \in \mathbb{C}^2 : z \neq z'\big\}.$$

The latter is homeomorphic to $\mathbb{C} \times \mathbb{C}^*$ via the map $(z, z') \mapsto (z, z' - z)$. Since $\mathbb{C}$ is contractible and since $\mathbb{C}^*$ deformation retracts to S^1, we deduce that $\mathrm{F}_2(D^2)$ has the homotopy type of S^1. Hence

$$PB_2(D^2) = \pi_1(\mathrm{F}_2(D^2), x) \simeq \pi_1(S^1, 1) \simeq \mathbb{Z}.$$

To be more specific, it results from the previous discussion that the map

$$r : \mathrm{F}_2(\mathbb{C}) \longrightarrow S^1,\ (z, z') \longmapsto (z - z')/|z - z'|$$

is a homotopy equivalence. Furthermore, the composition of r with the loop

$$\ell : [0,1] \longrightarrow \mathrm{F}_2(D^2),\ t \longmapsto (e^{2i\pi t}x_1, e^{2i\pi t}x_2)$$

(where, for concreteness, we assume that $x_1 := -1/2$ and $x_2 := 1/2$ in $D^2 \subset \mathbb{C}$) gives the loop $r \circ \ell : [0,1] \to S^1, t \mapsto e^{2i\pi t}$, which generates $\pi_1(S^1, 1) \simeq \mathbb{Z}$. We deduce that

$$PB_2 = \langle a | \varnothing \rangle \quad \text{where } a := [\ell].$$

To obtain now a presentation of B_2, we shall use the short exact sequence

$$1 \longrightarrow PB_2 \longrightarrow B_2 \xrightarrow{s} \mathfrak{S}_2 \longrightarrow 1.$$

The loop $m : [0,1] \to \mathrm{C}_2(D^2),\ t \mapsto (e^{i\pi t}x_1, e^{i\pi t}x_2)$ defines an element $\sigma := [m] \in B_2$ such that $s(\sigma)$ generates $\mathfrak{S}_2 \simeq \mathbb{Z}_2$. Since $\sigma^2 = a$ and (therefore) $\sigma a \sigma^{-1} = a$ in B_2, we deduce from Lemma 3.6 that

$$B_2 = \left\langle a, \sigma \middle| a^{-1}\sigma^2, [\sigma, a] \right\rangle = \left\langle a, \sigma \middle| a^{-1}\sigma^2 \right\rangle = \langle \sigma | \varnothing \rangle.$$

N.B. Note that $a = a_{12}$ and $\sigma = \sigma_1$ in the notations of §3.2.

Solution to Exercise 3.2. Since $\mathrm{C}_n(\Sigma)$ is a quotient space of $\mathrm{F}_n(\Sigma)$, it suffices to show that $\mathrm{F}_n(\Sigma)$ is path-connected. Consider two points x, x' in $\mathrm{F}_n(\Sigma)$.

Assume that $\{x\} \cap \{x'\} = \varnothing$. Using the fact that Σ is connected (and, so, path-connected) by our convention, we can find for any $i \in \{1, \dots, n\}$ a path $\gamma_i : [0,1] \to \Sigma$ such that $\gamma_i(0) = x_i$, $\gamma_i(1) = x'_i$ and $\gamma_i([0,1]) \cap \gamma_j([0,1]) = \varnothing$ for any $i \neq j$. Then the path $(\gamma_1, \dots, \gamma_n)$ in $\mathrm{F}_n(\Sigma)$ connects x to x'.

Assume now that $\{x\} \cap \{x'\} \neq \varnothing$, and denote by J the subset of the indices $i \in \{1, \dots, n\}$ such that $x_i \in \{x'\}$. Observe that, for any transposition τ of $\{1, \dots, n\}$ and for any $z \in \mathrm{F}_n(\Sigma)$, there is a path connecting z to $\tau \cdot z$: if $\tau(i) = i$, the point z_i remains fixed along this path and, if $\tau(i) \neq i$, z_i is "exchanged" with $z_{\tau(i)}$ inside a small disk $D \subset \mathrm{int}(\Sigma)$ such that $D \cap \{z\} = \{z_i, z_{\tau(i)}\}$. Therefore, we can assume that $x_j = x'_j$ for all $j \in J$. By the previous paragraph, $(x_i)_{i \in \{1,\dots,n\} \setminus J}$ can be connected to $(x'_i)_{i \in \{1,\dots,n\} \setminus J}$ by a path in $\mathrm{F}_{n-|J|}(\Sigma \setminus \{x_j | j \in J\})$: we deduce that x can be connected to x' by a path in $\mathrm{F}_n(\Sigma)$ along which x_j is fixed for any $j \in J$.

Solution to Exercise 3.3. The abelianization of PB_n can be deduced from Artin's presentation. Indeed, every relation of this presentation is of form

$$b a_{ij} b^{-1} = a_{ij}$$

for some $1 \leq i < j \leq n$ and $b \in PB_n$: hence this relation becomes superflous in the abelianization. We conclude that $PB_n/[PB_n, PB_n]$ is a free abelian group whose rank is the number of generators in Artin's presentation, i.e. $\binom{n}{2} = n(n-1)/2$.

Solution to Exercise 3.4. (a) We recall the following general fact: a split short exact sequence

$$1 \longrightarrow S \xrightarrow{\ i\ } G \underset{t}{\overset{q}{\rightleftarrows}} Q \longrightarrow 1$$

determines a semi-direct product decomposition

$$G = S' \rtimes Q'$$

(and vice-versa). Specifically, defining $S' := i(S) \trianglelefteq G$ and $Q' := t(Q) \leq G$, there is for any $g \in G$ a unique $(s, q) \in S' \times Q'$ such that $g = sq$. Thus, by Proposition 3.9, we obtain

$$PB_{n+1} = F_n \rtimes PB'_n$$

where $F_n \simeq \pi_1(D^2 \setminus \{x_1, \dots, x_n\}, x_{n+1})$ and $PB'_n \simeq PB_n$. Note that F_n is a free group of rank n since $D^2 \setminus \{x_1, \dots, x_n\}$ deformation retracts to a wedge of n circles. An element of F_n is a pure braid on $(n+1)$ strands whose n first strings are straight vertical and whose last string winds around the n first strings; an element of PB'_n is a pure braid on $n+1$ strands obtained by "juxtaposing" an arbitrary pure braid on n strand with the trivial braid on one strand. Next, by applying Proposition 3.9 to the copy PB'_n of PB_n in PB_{n+1}, we obtain a semi-direct decomposition

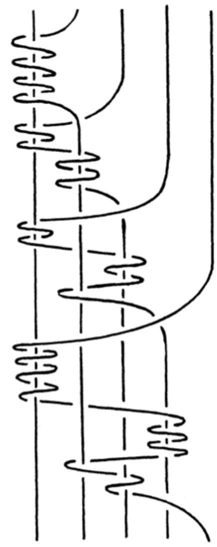
$$PB_{n+1} = F_n \rtimes \left(F_{n-1} \rtimes PB''_{n-1}\right).$$

Here an element of F_{n-1} is a pure braid on $n+1$ strands whose strings are straight vertical except for the n-th string which winds arounds the $(n-1)$ first strings; an element of PB''_{n-1} is a pure braid on $n+1$ strands obtained by "juxtaposing" an arbitrary pure braid on $(n-1)$ strands with the trivial braid on two strands. Thus, by an induction on n, we get an iterated semi-direct decomposition

$$PB_{n+1} = F_n \rtimes (F_{n-1} \rtimes (\cdots \rtimes F_1) \cdots)$$

(b) This follows from the definition a semi-direct product and the discussion carried in (a).

(c) Here is the example given by Artin in his foundational paper [Art47] (after a rotation):



The above braid $\beta \in PB_5$ is decomposed as $\beta_5\beta_4\beta_3\beta_2$ with $\beta_i \in F_i$.

Solution to Exercise 3.5. (a) Let S_a be the self-homeomorphism of Σ representing σ_a as it is given by the statement. Clearly S_a exchanges x_i and x_j since the given self-homeomorphism of D^2 exchanges $-1/2$ and $1/2$, and S_a fixes x_k for any $k \notin \{i,j\}$ since $x_k \notin N$. So the permutation of $\{x\}$ induced by σ_a is the transposition $x_i \leftrightarrow x_j$.

The self-homeomorphism S_a^2 of Σ represents σ_a^2. This homeomorphism fixes $\Sigma \setminus N$ and it is given on $N \cong D^2$ by

$$z \longmapsto -z \longmapsto -(-z) = z \quad \text{if } |z| \in [0, 1/2],$$

$$z \longmapsto \exp(2i\pi(1-|z|))z \longmapsto \exp(2i\pi(1-|z|))^2 z = \exp(2i\pi(2-2|z|))z \quad \text{if } |z| \in [1/2, 1].$$

We deduce that $S_a^2|_N$ represents the Dehn twist along the curve $\{z \in \mathbb{C} : |z| = 3/4\}$. Therefore $\sigma_a^2 \in \mathcal{M}(\Sigma, \{x\})$ is the Dehn twist along ∂N.

(b) Let $N_a \subset \Sigma$ be a closed disk such that $a \subset \operatorname{int}(N_a)$ and $N_a \cap \{x\} = a \cap \{x\}$; let also S_a be a self-homeomorphism of Σ which fixes $\Sigma \setminus N_a$ and represents σ_a. Let N_b and S_b play similar roles for the arc b. Since $a \cap b = \varnothing$, we can assume that $N_a \cap N_b = \varnothing$. Then $S_a S_b$ is the identity on $\Sigma \setminus (N_a \cup N_b)$, it acts the same way as S_a on N_a and its acts the same way as S_b on N_b; the same is true for $S_b S_a$. Therefore $S_a S_b = S_b S_a$ and it follows that $\sigma_a \sigma_b = \sigma_b \sigma_a$.

(c) Assume that the arc a connects x_i to x_j and that the arc b connects x_j to x_k. Let $D \subset \Sigma$ be a closed disk such that $a \cup b \subset \operatorname{int}(D)$ and $D \cap \{x\} = \{x_i, x_j, x_k\}$. Then the inclusion $D \hookrightarrow \Sigma$ induces a group homomorphism $\mathcal{M}(D, \{x_i, x_j, x_k\}) \to \mathcal{M}(\Sigma, \{x\})$, so that it suffices to prove the identity $\sigma_a \sigma_b \sigma_a = \sigma_b \sigma_a \sigma_b$ *in* $\mathcal{M}(D, \{x_i, x_j, x_k\})$. Choose an orientation-preserving homeomorphism $D \cong D^2 \subset \mathbb{C}$ mapping the intervals a and b to $[-1/2, 0]$ and $[0, 1/2]$, respectively. Clearly the group isomorphism

$$B_3 \xrightarrow[\simeq]{\varpi} \mathcal{M}(D^2, \{-1/2, 0, 1/2\}) \xrightarrow[\simeq]{} \mathcal{M}(D, \{x_i, x_j, x_k\})$$

(where ϖ is given by Theorem 3.13) sends the generators σ_1 and σ_2 to σ_a^{-1} and σ_b^{-1}, respectively. Thus the identity $\sigma_a \sigma_b \sigma_a = \sigma_b \sigma_a \sigma_b$ is a consequence of the braid relation $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$.

N.B. The definition of a half-twist corresponds to the construction done in Exercise 1.8 via the isomorphism $\mathcal{M}(\Sigma, \{x\}) \cong \mathcal{M}^{\eth}(\Sigma)$.

Solution to Exercise 3.6. Since the statement is only about the image $\beta\big(\{1, \dots, n\} \times [0,1]\big)$ of β, we can assume that β satisfies the condition

$$\forall t \in [0,1], \quad \beta\big(\{1, \dots, n\} \times \{t\}\big) \subset \Sigma \times \{t\}.$$

Hence there is a loop $\ell : [0,1] \to \mathrm{C}_n(\Sigma)$ such that $\beta = \beta(\ell)$ in the notation of (3.2). We have seen in the proof of Theorem 3.13 that the map

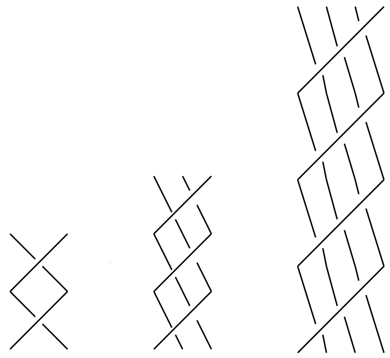
$$\mathrm{Homeo}^{+,\partial}(\Sigma) \longrightarrow \mathrm{C}_n(\Sigma), \ f \longmapsto f(\{x\})$$

is a fibration. Thus, the loop $\ell : [0,1] \to \mathrm{C}_n(\Sigma)$ can be "lifted" to $\mathrm{Homeo}^{+,\partial}(\Sigma)$: specifically, there is an isotopy $\tilde{\ell} : [0,1] \to \mathrm{Homeo}^{+,\partial}(\Sigma)$ such that $\tilde{\ell}(0) = \mathrm{id}_\Sigma$ and $\tilde{\ell}(t)(\{x\}) = \ell(t)$ for all $t \in [0,1]$. Then the map

$$h : (\Sigma \setminus \{x_1, \dots, x_n\}) \times [0,1] \xrightarrow{\cong} (\Sigma \times [0,1]) \setminus \beta\big(\{1,\dots,n\} \times [0,1]\big), \quad (y,t) \longmapsto \big(\widetilde{\ell}(t)(y), t\big)$$

is a homeomorphism since, for any $t \in [0,1]$, it maps the level set $(\Sigma \setminus \{x\}) \times \{t\}$ to the level set $(\Sigma \setminus \ell(t)) \times \{t\}$ by restriction of the self-homeomorphism $\widetilde{\ell}(t)$ of Σ. Since $\tilde{\ell}(0) = \mathrm{id}_\Sigma$, the homeomorphism f fixes the "bottom boundary" $(\Sigma \setminus \{x\}) \times \{0\}$.

Solution to Exercise 3.7. (a) Here are θ_2, θ_3 and θ_4:



(b) Let $D \subset D^2$ be a closed disk containing the set $\{x\}$ in its interior, and let $\delta \subset \mathrm{int}(D^2) \setminus D$ be a simple closed curve parallel to ∂D^2. Consider a tubular neighborhood $N := N(\delta)$ of δ such that $N \subset \mathrm{int}(D^2) \setminus D$. Let $T : D^2 \to D^2$ be the self-homeomorphism which is the identity outside N and is given by

$$S^1 \times [0,1] \ni (e^{2i\pi\theta}, r) \longmapsto \big(e^{2i\pi(\theta+r)}, r\big) \in S^1 \times [0,1]$$

on $N \cong S^1 \times [0,1]$. Thus $[T] \in \mathcal{M}(D^2, \{x\})$ is the Dehn twist τ_δ along δ. We know that T is isotopic to id_{D^2} relatively to ∂D^2 and, by writing down an explicit isotopy, we easily see that

$$\varpi(\theta_n) = \tau_\delta^{-1}.$$

(c) According to (a), it suffices to show that τ_δ belongs to the center of $\mathcal{M}(D^2, \{x\})$. Let T be the self-homeomorphism of D^2 representing τ_δ that we have considered in (a). Any element $y \in \mathcal{M}(D^2, \{x\})$ is represented by a self-homeomorphism of D^2 which fixes ∂D^2: thus y can also be represented by a self-homeomorphism Y of D^2 which fixes a neighborhood of the boundary. Assuming that $N = N(\delta)$ is contained in this neighborhood, we obtain that $TY = YT$. It follows that $t_\delta y = y t_\delta$.

(d) We have $\theta_n \in PB_n$ since $s(\theta_n) = s(t_\delta)^{-1} = 1$. Therefore, it is enough to show that θ_n has infinite order in PB_n. For this, we observe that the canonical homomorphism $PB_n \to PB_{n-1}$ defined by "forgetting" the last string maps θ_n to θ_{n-1}. (This is a consequence of (a) too.) Thus, by an induction on $n \geq 2$, we obtain that there exists a group homomorphism $PB_n \to PB_2$ mapping θ_n to $\theta_2 = \sigma_1^2 = a_{12}$. Since PB_2 is the infinite cyclic group generated by a_{12}, we deduce that θ_n has infinite order in PB_n.

N.B. In fact, it is known that the center of B_n is an infinite cyclic group generated by θ_n for any $n \geq 3$. (See [KT08, Theorem 1.24] for instance; Exercise 3.8 proves this for $n = 3$.) Besides, the group B_n is known to be torsion-free. (See [KT08, Corollary 1.29] for instance.)

Solution to Exercise 3.8. (a) Consider the following composition

$$B_3 \xrightarrow{\ \psi\ } \mathcal{M}(S^1 \times S^1) \xrightarrow{\ \kappa\ } \mathrm{SL}(2;\mathbb{Z}) \twoheadrightarrow \mathrm{PSL}(2;\mathbb{Z}), \qquad \rho : B_3 \dashrightarrow \mathrm{PSL}(2;\mathbb{Z})$$

where κ is the isomorphism defined in Proposition 2.4, and recall from the proof of Theorem 2.14 that

$$\mathrm{PSL}(2;\mathbb{Z}) = \big\langle \overline{T}, \overline{U} \;\big|\; \overline{T}^2 = 1, \overline{U}^3 = 1 \big\rangle$$

where $\overline{T}$ and $\overline{U}$ are the classes of the following matrices:

$$T := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad U := \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$

We have

$$\kappa\psi(\sigma_1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = -TU, \quad \kappa\psi(\sigma_2) = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = -UT$$

so that $\rho(\sigma_1) = \overline{TU}$ and $\rho(\sigma_2) = \overline{UT}$. In particular

$$\rho\big((\sigma_1\sigma_2)^3\big) = (\overline{TU}^2\overline{T})^3 = \overline{TU}^2\overline{T}^2\overline{U}^2\overline{T}^2\overline{U}^2\overline{T} = \overline{TU}^6\overline{T} = \overline{T}^2 = 1,$$

which shows that ρ induces a group homomorphism $\overline{\rho} : B_3/\langle\theta_3\rangle \to \mathrm{PSL}(2;\mathbb{Z})$. It is easily verified that the group homomorphism

$$\mathrm{PSL}(2;\mathbb{Z}) \longrightarrow B_3/\langle\theta_3\rangle,\ \overline{U} \longmapsto \sigma_1^{-1}\sigma_2^{-1},\ \overline{T} \longmapsto \sigma_1\sigma_2\sigma_1$$

is well-defined and is an inverse of $\overline{\rho}$ by using the above presentation of $\mathrm{PSL}(2;\mathbb{Z})$.

(b) The center of $\mathrm{PSL}(2;\mathbb{Z})$ is trivial, as can be checked from the fact that $\mathrm{PSL}(2;\mathbb{Z}) \simeq \mathbb{Z}_2 * \mathbb{Z}_3$. Therefore, the center of $B_3/\langle\theta_3\rangle$ is trivial and it follows that the center $Z(B_3)$ of B_3 is contained in $\langle\theta_3\rangle$. We deduce from Exercise 3.7.(b) that $Z(B_3) = \langle\theta_3\rangle$ and from Exercise 3.7.(c) that $Z(B_3) \simeq \mathbb{Z}$.

Solution to Exercise 3.9. (a) For all $i \in \{1,\dots,n-1\}$, we set $v_i := \mathrm{id}_V^{\otimes(i-1)} \otimes R \otimes \mathrm{id}_V^{\otimes(n-i-1)}$. The braid group B_n is generated by $\sigma_1,\dots,\sigma_{n-1}$ with relations

$$\text{(3.9)} \qquad \begin{cases} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| \geq 2, \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } |i-j| = 1. \end{cases}$$

For all $i \in \{1,\dots,n-2\}$, we have

$$\begin{aligned} v_i \circ v_{i+1} \circ v_i &= \mathrm{id}_V^{\otimes(i-1)} \otimes ((R\otimes \mathrm{id}_V)\circ(\mathrm{id}_V \otimes R)\circ(R\otimes \mathrm{id}_V)) \otimes \mathrm{id}_V^{\otimes(n-i-2)} \\ &= \mathrm{id}_V^{\otimes(i-1)} \otimes ((\mathrm{id}_V \otimes R)\circ(R\otimes \mathrm{id}_V)\circ(\mathrm{id}_V \otimes R)) \otimes \mathrm{id}_V^{\otimes(n-i-2)} \\ &= v_{i+1}\circ v_i \circ v_{i+1} \end{aligned}$$

and, for all $i,j \in \{1,\dots,n-1\}$ such that $i < j-1$, we have

$$v_iv_j = \mathrm{id}_V^{\otimes(i-1)} \otimes R \otimes \mathrm{id}_V^{\otimes(j-i-2)} \otimes R \otimes \mathrm{id}_V^{\otimes(n-j-1)} = v_jv_i.$$

So, there is a unique group homomorphism $\rho_R : B_n \to \mathrm{Aut}_{\mathbb{K}}(V^{\otimes n})$ defined by $\sigma_i \mapsto v_i$.

(b) For all $v_1,v_2,v_3 \in V$, we have

$$v_1\otimes v_2\otimes v_3 \overset{xF\otimes\mathrm{id}}{\longmapsto} x\cdot v_2\otimes v_1\otimes v_3 \overset{\mathrm{id}\otimes xF}{\longmapsto} x^2\cdot v_2\otimes v_3\otimes v_1 \overset{xF\otimes\mathrm{id}}{\longmapsto} x^3\cdot v_3\otimes v_2\otimes v_1$$
$$v_1\otimes v_2\otimes v_3 \overset{\mathrm{id}\otimes xF}{\longmapsto} x\cdot v_1\otimes v_3\otimes v_2 \overset{xF\otimes\mathrm{id}}{\longmapsto} x^2\cdot v_3\otimes v_1\otimes v_2 \overset{\mathrm{id}\otimes xF}{\longmapsto} x^3\cdot v_3\otimes v_2\otimes v_1$$

which shows that xF is a Yang–Baxter operator. The property

$$\forall v_1,\dots,v_n \in V, \quad \rho_F(\beta)(v_1\otimes\cdots\otimes v_n) = v_{\mathrm{s}(\beta^{-1})(1)}\otimes\cdots\otimes v_{\mathrm{s}(\beta^{-1})(n)}$$

is true for $\beta = \sigma_i$ and, so, it is true for any β since B_n is generated by $\sigma_1,\dots,\sigma_{n-1}$. Moreover, we have $\rho_{xF}(\sigma_i) = x\cdot\rho_F(\sigma_i)$. So, we conclude that

$$\begin{aligned} \forall v_1,\dots,v_n \in V, \quad \rho_{xF}(\beta)(v_1\otimes\cdots\otimes v_n) &= x^{|\beta|}\cdot\rho_F(\beta)(v_1\otimes\cdots\otimes v_n) \\ &= x^{|\beta|}\cdot v_{\mathrm{s}(\beta^{-1})(1)}\otimes\cdots\otimes v_{\mathrm{s}(\beta^{-1})(n)} \end{aligned}$$

for any $\beta \in B_n$ whose length in the words $\sigma_1,\dots,\sigma_{n-1}$ is denoted by $|\beta| \in \mathbb{N}$. (This length is well-defined according to the presentation (3.9) of B_n.)

(c) Let us assume, for example, that $x = z$, the case $y = z$ being similar. An easy computation gives

$$R_{x,y,x}\circ R_{y^{-1},x^{-1},x^{-1}} = \mathrm{id}_{V\otimes V} = R_{y^{-1},x^{-1},x^{-1}}\circ R_{x,y,x}$$

and shows that $R_{x,y,x}$ is a linear automorphism. In order to prove that R is a Yang–Baxter operator, we set $R := R_{x,y,x}$ and $R' := R + x\cdot\mathrm{id}_{V\otimes V}$. Thus, we have

$$\begin{aligned} (\mathrm{id}\otimes R)\circ(R\otimes\mathrm{id})\circ(\mathrm{id}\otimes R) = \ &(\mathrm{id}\otimes R')\circ(R'\otimes\mathrm{id})\circ(\mathrm{id}\otimes R') - x^3\cdot\mathrm{id}_{A^{\otimes 3}} \\ &-x\cdot(\mathrm{id}\otimes R')\circ(R'\otimes\mathrm{id}) - x\cdot(R'\otimes\mathrm{id})\circ(\mathrm{id}\otimes R') \\ &-x\cdot(\mathrm{id}\otimes R')^2 + 2x^2\cdot(\mathrm{id}\otimes R') + x^2\cdot(R'\otimes\mathrm{id}) \end{aligned}$$

and

$$\begin{aligned} (R\otimes\mathrm{id})\circ(\mathrm{id}\otimes R)\circ(R\otimes\mathrm{id}) = \ &(R'\otimes\mathrm{id})\circ(\mathrm{id}\otimes R')\circ(R'\otimes\mathrm{id}) - x^3\cdot\mathrm{id}_{A^{\otimes 3}} \\ &-x\cdot(R'\otimes\mathrm{id})\circ(\mathrm{id}\otimes R') - x\cdot(\mathrm{id}\otimes R')\circ(R'\otimes\mathrm{id}) \\ &-x\cdot(R'\otimes\mathrm{id})^2 + 2x^2\cdot(R'\otimes\mathrm{id}) + x^2\cdot(\mathrm{id}\otimes R'). \end{aligned}$$

So, we are reduced to show that

$$\begin{aligned} &(\mathrm{id}\otimes R')\circ(R'\otimes\mathrm{id})\circ(\mathrm{id}\otimes R') - x^2\cdot(R'\otimes\mathrm{id}) - x\cdot(\mathrm{id}\otimes R')^2 \\ \overset{?}{=}\ &(R'\otimes\mathrm{id})\circ(\mathrm{id}\otimes R')\circ(R'\otimes\mathrm{id}) - x^2\cdot(\mathrm{id}\otimes R') - x\cdot(R'\otimes\mathrm{id})^2 \end{aligned}$$

and this is a straightforward computation.

4. Formality of the Torelli group

This last section provides an introduction to the "non semi-simple" part of the mapping class group, namely the Torelli group. The reader may consult Johnson's survey [Joh83b] or the Chapter 6 of [FM12] for the classical aspects of the Torelli group, and the survey article [HM12] for the connections with 3-manifold invariants.

Recall that, with our conventions of §1.1, all surfaces are assumed to be connected, compact and orientable. In this section, we fix an oriented surface Σ *with one or no* boundary component: we denote by n the number of boundary components and by g the genus of Σ. If $n=0$, the bordered surface obtained from Σ by removing a disk is denoted by Σ°; if $n=1$, the closed surface obtained from Σ by gluing a disk is denoted by Σ^+.

4.1. Definition of the Torelli group.

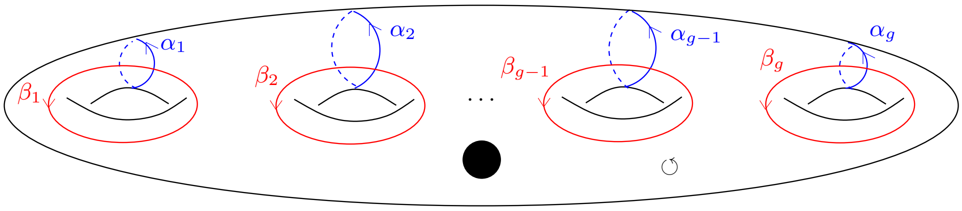
In the sequel we denote $H := H_1(\Sigma;\mathbb{Z})$ and $H^* := \mathrm{Hom}(H,\mathbb{Z})$. (Note that H^* is canonically isomorphic to $H^1(\Sigma;\mathbb{Z})$ by the universal coefficients theorem.) Let $\omega : H \times H \to \mathbb{Z}$ be the homological intersection form defined in §1.2. We have seen in the solution of Exercise 1.6 that ω is a *symplectic* form, in the sense that it is bilinear, skew-symmetric and non-singular:

$$H \xrightarrow{\simeq} H^*,\ x \longmapsto \omega(x,-) \tag{4.1}$$

A group homomorphism $\psi : H \to H$ is said to *preserve* ω if $\omega(\psi(x),\psi(y)) = \omega(x,y)$ for any $x,y \in H$.

Definition 4.1. *The* symplectic group *of H (equipped with the symplectic form ω) is the group of automorphisms of H preserving ω. We denote it by* $\mathrm{Sp}(H)$.

Set $a_i := [\alpha_i] \in H_1(\Sigma;\mathbb{Z})$ and $b_i := [\beta_i] \in H_1(\Sigma;\mathbb{Z})$ for all $i \in \{1,\dots,g\}$, where $\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g$ are the oriented simple closed curves that are shown below:



(4.2)

(The above picture represents the oriented surface Σ if $n=1$, and it represents Σ° if $n=0$.) Then the matrix of ω in the basis $(a,b) := (a_1,\dots,a_g,b_1,\dots,b_g)$ is

$$\Omega := \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

Then, by considering matrix presentations of automorphisms of H, we see that the group $\mathrm{Sp}(H)$ is isomorphic to

$$\mathrm{Sp}(2g;\mathbb{Z}) := \big\{ M \in \mathrm{GL}(2g;\mathbb{Z}) : M^t \cdot \Omega \cdot M = \Omega \big\}.$$

The latter is also referred to as the *symplectic group* or, sometimes, as *Siegel's modular group*.

Proposition 4.2. *The canonical homomorphism*

$$\kappa : \mathcal{M}(\Sigma) \longrightarrow \mathrm{Sp}(2g;\mathbb{Z})$$

which sends any isotopy class $[f]$ to the matrix of $f_* : H \to H$ in the basis (a,b), is surjective.

Proof. If $n=0$, the inclusion $i : \Sigma^\circ \hookrightarrow \Sigma$ induces an isomorphism $i_* : H_1(\Sigma^\circ;\mathbb{Z}) \to H_1(\Sigma;\mathbb{Z})$ sending the basis (a,b) to the basis (a,b); hence we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{M}(\Sigma^\circ) & \xrightarrow{\kappa} & \mathrm{Sp}(2g;\mathbb{Z}) \\ \downarrow & \nearrow_{\kappa} & \\ \mathcal{M}(\Sigma) & & \end{array} \tag{4.3}$$

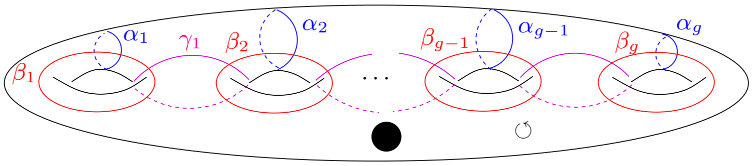
Therefore we can assume that $n=1$. Using a prior work by Klingen [Kli61], Birman observes in [Bir71] that the group $\mathrm{Sp}(2g;\mathbb{Z})$ is generated by the $2g \times 2g$ matices

$$Y_i := \begin{pmatrix} I_g & -A_i \\ 0 & I_g \end{pmatrix},\ U_i := \begin{pmatrix} I_g & 0 \\ A_i & I_g \end{pmatrix},\ Z_j := \begin{pmatrix} I_g & B_j \\ 0 & I_g \end{pmatrix} \tag{4.4}$$

indexed by $i \in \{1,\dots,g\}$ and $j \in \{1,\dots,g-1\}$, where A_i and B_j are the $g \times g$ matrices defined in terms of the elementary matrices E_{kl}'s by

$$A_i := E_{ii} \quad \text{and} \quad B_j := -E_{jj} - E_{j+1,j+1} + E_{j,j+1} + E_{j+1,j}.$$

Consider now the simple closed curves $\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g, \gamma_1, \ldots, \gamma_{g-1}$ corresponding to the Lickorish's generators of the mapping class group:



By using Exercise 2.6, or by a simple direct computation, it is easily checked that

$$\kappa\big(\tau_{\alpha_i}^{-1}\big) = Y_i, \ \kappa\big(\tau_{\beta_i}^{-1}\big) = U_i \quad \text{and} \quad \kappa\big(\tau_{\gamma_j}^{-1}\big) = Z_j \tag{4.5}$$

and we conclude that κ is surjective. □

Proposition 4.2 suggests the following notion.

Definition 4.3. *The* Torelli group *of the surface* Σ *is the subgroup*

$$\mathcal{I}(\Sigma) := \ker \kappa \ \subset \mathcal{M}(\Sigma)$$

of the mapping class group acting trivially in homology.

Observe that the Torelli group $\mathcal{I}(\Sigma)$ is trivial when Σ is a disk or a sphere, since $\mathcal{M}(\Sigma) = \{1\}$ in those two cases (by Proposition 2.3 and Exercise 2.1). Besides, it follows from Proposition 2.4 that $\mathcal{I}(\Sigma)$ is trivial when Σ is a torus. Thus, the "simplest" surface Σ for which $\mathcal{I}(\Sigma) \neq \{1\}$ is the torus with one hole $\Sigma_{1,1}$: see Exercise 4.3.

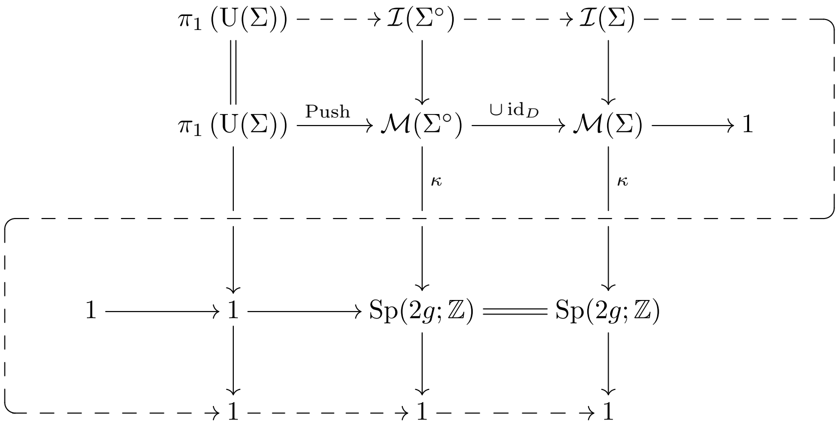
Starting from now, *we assume that* $g \geq 2$. Then the Torelli groups in the closed case and in the bordered case are related as follows.

Proposition 4.4. *If* Σ *is closed, then we have a short exact sequence of groups*

$$1 \longrightarrow \pi_1(\mathrm{U}(\Sigma)) \xrightarrow{\text{Push}} \mathcal{I}(\Sigma^\circ) \xrightarrow{\cup \mathrm{id}_D} \mathcal{I}(\Sigma) \longrightarrow 1$$

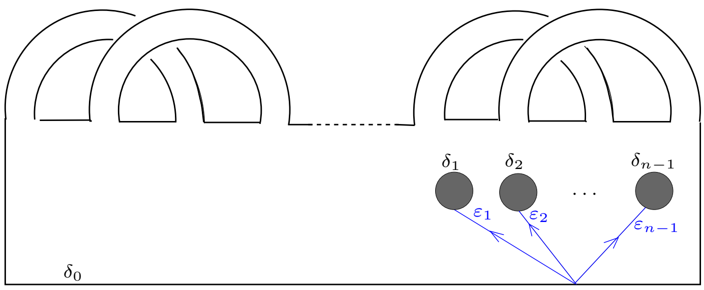
where the maps "Push" and "$\cup\, \mathrm{id}_D$" are as described in Proposition 2.9.

Proof. We have the following commutative diagram:



The third row is obviously exact, the second row is exact by Proposition 2.9, and the dashed arrows are given by the "snake lemma" which makes sense in the present situation. Since $\chi(\Sigma) = 2 - 2g < 0$, the "Push" map is injective by Remark 2.10 and we get the desired short exact sequence. □

Remark 4.5. For a surface S with $n > 1$ boundary components, the Torelli group $\mathcal{I}(S)$ is defined in the following way by Johnson [Joh85a]. Pick one of the boundary component δ_0 and choose a system of oriented simple proper arcs $\varepsilon_1, \ldots, \varepsilon_{n-1}$ connecting δ_0 to each of the other boundary components $\delta_1, \ldots, \delta_{n-1}$:



Then an element $f \in \mathcal{M}(S)$ is declared to belong to $\mathcal{I}(S)$ if and only if $f_* : H_1(S;\mathbb{Z}) \to H_1(S;\mathbb{Z})$ is the identity *and*

$$\forall i \in \{1, \dots, n-1\},\ [f(\varepsilon_i) - \varepsilon_i] = 0 \in H_1(S;\mathbb{Z}).$$

There are other possible definitions of the Torelli group which are not equivalent to the previous one: see [Put07] for a detailed analysis of all the possibilities.

4.2. **Generation of the Torelli group.** We first define two families of elements of the Torelli group, which will be used to generate $\mathcal{I}(\Sigma)$. Recall that we have assumed that $g \geq 2$.

On the one hand, for any simple closed curve ρ in Σ such that $[\rho] = 0 \in H$, we deduce from Exercise 2.6 that $\tau_\rho \in \mathcal{I}(\Sigma)$. Such elements of the Torelli group and their inverses are called *BSCC maps* (for "Bounding Simple Closed Curves"), since the condition $[\rho] = 0$ is equivalent to say that there is a subsurface of Σ with boundary ρ (i.e. ρ is separating.) The *genus* of τ_ρ is the minimum of the genus of the subsurfaces of Σ having this property.

On the other hand, for any simple closed curves ρ, δ in Σ such that $\rho \cap \delta = \varnothing$ and $[\rho] = [\delta] \neq 0$, we deduce from Exercise 2.6 that

$$\tau_\rho \tau_\delta^{-1} \in \mathcal{I}(\Sigma).$$

Such elements of the Torelli group are called *BP maps* (for "Bounding Pair"), since the condition $[\rho] = [\delta]$ is equivalent to say that there is a subsurface of Σ with boundary $\rho \cup \delta$. The *genus* of $\tau_\rho \tau_\delta^{-1}$ is the mininum of the genus of the subsurfaces of Σ having this property.

The following theorem, due to Johnson [Joh79], is the culmination of several results by others whose chronicle is outlined below.

Theorem 4.6 (Johnson 1979)**.** *The Torelli group $\mathcal{I}(\Sigma)$ has the following generating sets, depending on the genus g and the number n of boundary components of Σ:*

	$n = 0$	$n = 1$
$g = 2$	*all BSCC maps of genus* 1	*BSCC maps of genus 1 & all BP maps of genus* 1
$g \geq 3$	*all BP maps of genus* 1	*all BP maps of genus* 1

Outline of the proof. We only consider the closed case ($n = 0$); the case of a surface with non-empty boundary ($n = 1$) can be deduced from this using Proposition 4.4.

The proof of the theorem starts with Birman's paper [Bir71] which we have already referred to in the proof of Proposition 4.2. She did much more than deducing from the paper [Kli61] a finite generating set of $\mathrm{Sp}(2g;\mathbb{Z})$: she also carried out the method proposed there by Klingen to find an explicit finite presentation of $\mathrm{Sp}(2g;\mathbb{Z})$. Thus, after some long computations which are only partly reproduced in [Bir71], she find out a finite set R consisting of 10 types of relations for the system of generators $S := \{Y_i, U_i, Z_j\}_{i,j}$ given at (4.4). The situation can be summed up with the diagram

(4.6)
$$\begin{array}{ccccccccc} & & & & \mathrm{F}(X) & \xrightarrow{\ k\ } & \mathrm{F}(S)/\langle\langle R \rangle\rangle & \longrightarrow & 1 \\ & & & & \downarrow\!\!\downarrow \tau & & \downarrow \simeq & & \\ 1 & \longrightarrow & \mathcal{I}(\Sigma) & \longrightarrow & \mathcal{M}(\Sigma) & \xrightarrow{\ \kappa\ } & \mathrm{Sp}(2g;\mathbb{Z}) & \longrightarrow & 1 \end{array}$$

where $X := \{\alpha_i, \beta_i, \gamma_j\}_{i,j}$ is the set of Lickorish's curves (2.2), the homomorphism τ sends any element of X to the Dehn twist along the corresponding curve, and the homomorphism k is defined by $k(\alpha_i) := Y_i^{-1}$, $k(\beta_i) := U_i^{-1}$ and $k(\gamma_j) := Z_j^{-1}$. This diagram is commutative by (4.5). Let $t : \mathrm{F}(S) \to \mathrm{F}(X)$ be the group homomorphism defined by $t(Y_i) := \alpha_i^{-1}$, $t(U_i) := \beta_i^{-1}$ and $t(Z_j) := \gamma_j^{-1}$. Then, for any word $r \in R$, there are two possibilities for the "lift" $t(r)$:

- either $\tau t(r) = 1$, i.e. the relation r of the symplectic group "survives" in the mapping class group: the word $t(r)$ is a relation between the Lickorish's generators;
- or $\tau t(r) \neq 1$, i.e. $\tau t(r)$ is a non-trivial element of the Torelli group.

For instance, the word $r_i := Y_i U_i Y_i U_i^{-1} Y_i^{-1} U_i^{-1}$ is one of the elements of R found by Birman which corresponds to the relation $Y_i U_i Y_i = U_i Y_i U_i$ in $\mathrm{Sp}(2g;\mathbb{Z})$; we have $t(r_i) := \alpha_i^{-1} \beta_i^{-1} \alpha_i^{-1} \beta_i \alpha_i \beta_i$ so that $\tau t(r_i) = 1$ because of the braid relation

$$\tau_{\alpha_i} \tau_{\beta_i} \tau_{\alpha_i} = \tau_{\beta_i} \tau_{\alpha_i} \tau_{\beta_i} \in \mathcal{M}(\Sigma).$$

The word $s := (Y_1 U_1 Y_1)^4$ is another element of R found by Birman; using the braid relation and the 2-chain relation, we have

$$\tau t(s^{-1}) := (\tau_{\alpha_1} \tau_{\beta_1} \tau_{\alpha_1})^4 = (\tau_{\alpha_1} \tau_{\beta_1} \tau_{\alpha_1})(\tau_{\beta_1} \tau_{\alpha_1} \tau_{\beta_1})(\tau_{\alpha_1} \tau_{\beta_1} \tau_{\alpha_1})(\tau_{\beta_1} \tau_{\alpha_1} \tau_{\beta_1}) = (\tau_{\alpha_1} \tau_{\beta_1})^6 = \tau_{\delta_1} \in \mathcal{M}(\Sigma)$$

where δ_1 is a simple closed curve bounding a subsurface of genus 1: thus we get, this time, a non-trivial element of $\mathcal{I}(\Sigma)$. As a third and last example, consider the word $u := (Y_1U_1Z_1U_2Y_2)^6$ which is another element of R found by Birman; by the 5-chain relation, we have

$$\tau t(u^{-1}) = (\tau_{\alpha_2}\tau_{\beta_2}\tau_{\gamma_1}\tau_{\beta_1}\tau_{\alpha_1})^6 = \tau_{\delta_2} \in \mathcal{M}(\Sigma)$$

where δ_2 is a simple closed curve bounding a genus 2 subsurface of Σ: thus we get another non-trivial element of $\mathcal{I}(\Sigma)$ if $g \geq 3$. It follows from diagram (4.6) that $\mathcal{I}(\Sigma)$ is the subgroup of $\mathcal{M}(\Sigma)$ *normally* generated by the finite set

$$B := \{\tau t(r) \,|\, r \in R\} \setminus \{1\}$$

which Birman fully computes in [Bir71]. The set B consists of the single element $\tau t(s) = \tau t\big((Y_1U_1Y_1)^4\big)$ if $g = 2$, and it consists of four elements including $\tau t(s)$ and $\tau t(u) = \tau t\big((Y_1U_1Z_1U_2Y_2)^6\big)$ if $g \geq 3$.

The next step has been carried out by Powell. He showed that, for $g \geq 3$, any element of B is either a BSCC map of genus ≤ 2 or is a product of BP maps of genus 1 [Pow78]. (We have checked this in the previous paragraph for only two elements of B over four.) Since the conjugate of any BSCC map (respectively, any BP map) is a BSCC map (respectively, a BP map) of the same genus, it follows that $\mathcal{I}(\Sigma)$ is generated by BSCC maps of genus ≤ 2 and BP maps of genus 1 for $g \geq 3$. The same argument shows that $\mathcal{I}(\Sigma)$ is generated by BSCC maps of genus 1 if $g = 2$.

Johnson gave the final touch to the theorem in genus $g \geq 3$. Using the lantern relation, he showed in [Joh79] that any BSCC map of genus 2 is a product of BSCC maps of genus 1 and BP maps of genus 1, and that any BSCC map of genus 1 is itself a product of BP maps of genus 1. It follows that $\mathcal{I}(\Sigma)$ for $g \geq 3$ is generated by BP maps of genus 1. □

Remark 4.7. Putman proved without appealing to Powell's result [Pow78] that $\mathcal{I}(\Sigma)$ is generated by BP maps and BSCC maps: see [Put07, Theorem 1.3]. By adjoining to his result the arguments of Johnson in [Joh79], one gets another proof of Theorem 4.6 which is logically independent of those lengthy computations that are only outlined in [Bir71].

Remark 4.8. It follows from Theorem 4.6 and Exercise 4.5 that, as a *normal* subgroup of $\mathcal{M}(\Sigma)$, the Torelli group $\mathcal{I}(\Sigma)$ is generated by only one or two elements:

	$n = 0$	$n = 1$
$g = 2$	one BSCC map of genus 1	one BSCC map of genus 1 & one BP map of genus 1
$g \geq 3$	one BP map of genus 1	one BP map of genus 1

Theorem 4.6 does not consider the problem of the finite generation/presentation of the Torelli group. We sum up below what is known about this subject.

Theorem 4.9 (Johnson 1983)**.** *In genus $g \geq 3$, the group $\mathcal{I}(\Sigma)$ is generated by a finite number of BP maps.*

About the proof. See [Joh83a] for the original proof and [Joh83b] for a quick outline. Note that the BP maps of the finite generating set provided by Johnson may have genus *greater* than one. □

The generation of the Torelli group in the case $g = 2$ is drastically different than in the case $g \geq 3$.

Theorem 4.10 (McCullough–Miller 1986)**.** *In genus $g = 2$, the group $\mathcal{I}(\Sigma)$ is infinitely generated.*

Outline of the proof. By Proposition 4.4, it is enough to deal with the closed case ($n = 0$). The proof given in [MM86] goes as follows. Let $\widehat{\Sigma}$ be the regular covering of Σ corresponding to the kernel of the group homomorphism

$$p : \pi_1(\Sigma, \star) \longrightarrow \langle s, t \,|\, [s, t] = 1\rangle \simeq \mathbb{Z}^2$$

defined by $p(\alpha_1) := 1$, $p(\beta_1) := 1$, $p(\alpha_2) := s$ and $p(\beta_2) := t$. Here $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ is the system of oriented simple closed curves shown at (4.2) and based at a point $\star \in \Sigma$. The action of the automorphism group $\langle s, t \,|\, [s, t] = 1\rangle$ of the covering $\widehat{\Sigma} \to \Sigma$ induces a structure of R-module on $H_1(\widehat{\Sigma}; \mathbb{Z})$ where $R := \mathbb{Z}[s^{\pm 1}, t^{\pm 1}]$. Thus, there is a group homomorphism

$$\mathcal{I}(\Sigma) \longrightarrow \mathrm{Aut}_R\left(H_1(\widehat{\Sigma}; \mathbb{Z})\right),\ [f] \longmapsto \widehat{f}_* \tag{4.7}$$

where the representative homeomorphism f is assumed to fix $\star$ and $\widehat{f} : \widehat{\Sigma} \to \widehat{\Sigma}$ denotes the unique lift of f fixing a preferred lift $\widehat{\star}$ of $\star$. Now, it turns out that $H_1(\widehat{\Sigma}; \mathbb{Z})$ is a free R-module with basis $\big([\widehat{\alpha}_1], [\widehat{\beta}_1]\big)$ where $\widehat{\alpha}_1, \widehat{\beta}_1$ are lifts of α_1, β_1 respectively. Then, by considering matrix presentations in this basis, one obtains a group homomorphism

$$\kappa_R : \mathcal{I}(\Sigma) \longrightarrow \mathrm{GL}(2; R)$$

which McCullough and Miller explicitly compute on Dehn twists. Thus, using Powell's result that $\mathcal{I}(\Sigma)$ is generated by BSCC maps of genus 1, they deduce that κ_R takes values in the special linear group $\mathrm{SL}(2;R)$. This group has a certain decomposition as an amalgamated free product [BM78]. Using this decomposition, McCullough and Miller are able to show that the image of κ_R is not finitely generated. □

Concerning presentations of the Torelli group, the following is known. Mess proved that $\mathcal{I}(\Sigma)$ is a free group (of infinite rank) for $g = 2$ and $n = 0$ [Mes92]. Putman also obtained in [Put09] some interesting *infinite* presentations of $\mathcal{I}(\Sigma)$ for any $g \geq 2$ and $n \in \{0, 1\}$, whose generators are given by BSCC maps, BP maps and commutators of the kind considered in Exercise 4.4. But it is not known whether the Torelli group is finitely presented in genus $g \geq 3$.

4.3. **The Johnson homomorphisms.** Starting from now, we assume that Σ is an oriented surface of genus $g \geq 2$ with a single boundary component. All the results and constructions below exist in the closed case too; but their statements are often more technical, and the proofs in the closed case are derived from the bordered case (with the notable exception of Hain's results in §4.5). Therefore we have preferred to omit the closed case in what follows.

Let $\pi := \pi_1(\Sigma, \star)$ be the fundamental group of Σ based at a point $\star \in \partial\Sigma$. The homotopy class of the boundary curve is denoted by $\zeta := [\partial\Sigma] \in \pi$. The following shows that mapping class groups naturally embed into automorphism groups of free groups.

Theorem 4.11 (Dehn, Nielsen & Baer 20's)**.** *The group homomorphism*

$$\rho : \mathcal{M}(\Sigma) \longrightarrow \mathrm{Aut}(\pi), \ \ [f] \longmapsto f_\sharp$$

is injective and its image consists of all $\psi \in \mathrm{Aut}(\pi)$ satisfying $\psi(\zeta) = \zeta$.

About the proof. The fact that ρ is a group homomorphism follows from the functoriality of $\pi_1(-)$. By definition of $\mathcal{M}(\Sigma)$, we obviously have

$$\rho\big(\mathcal{M}(\Sigma)\big) \subset \{\psi \in \mathrm{Aut}(\pi) : \psi(\zeta) = \zeta\}$$

The proof of the converse inclusion, which is much more involved, can be found in [ZVC80] for instance.

To prove the injectivity, assume that $f \in \mathcal{M}(\Sigma)$ is such that $f_\sharp = \mathrm{id}_\pi$. Since Σ deformation retracts to a bouquet of circles, it is a $K(\pi, 1)$-space. Thus, for any pointed topological space (X, x), the map

$$\frac{\big\{\text{continuous maps } g : (X, x) \to (\Sigma, \star)\big\}}{\text{homotopy}} \longrightarrow \mathrm{Hom}\big(\pi_1(X, x), \pi\big), \quad [g] \longmapsto g_\sharp \tag{4.8}$$

is a bijection. Taking $(X, x) = (\Sigma, \star)$, we deduce that there is a homotopy between f and id_Σ (which is not necessarily relative to the boundary). Since homotopy coincides with isotopy in dimension 2, we deduce from (2.7) that $[f] = \tau_\gamma^k \in \mathcal{M}(\Sigma)$ for some $k \in \mathbb{Z}$ and where γ is a simple closed curve parallel to $\partial\Sigma$. It is easily checked that $(\tau_\gamma)_\sharp$ is the conjugation by ζ, so that $(\tau_\gamma)_\sharp^l$ is non trivial for any $l \neq 0$. We deduce that $k = 0$ and that $[f] = 1 \in \mathcal{M}(\Sigma)$. □

The Dehn–Nielsen–Baer representation ρ can be "approximated" step-by-step by considering a succession of nilpotent quotients of the group π. Specifically, we consider the lower central series of π

$$\pi = \Gamma_1\pi \supset \Gamma_2\pi \supset \cdots \supset \Gamma_k\pi \supset \Gamma_{k+1}\pi \supset \cdots$$

defined inductively by $\Gamma_{k+1}\pi := \big[\pi, \Gamma_k\pi\big]$ for all $k \geq 1$. Since $\Gamma_{k+1}\pi$ is a characteristic subgroup of π, there is a canonical homomorphism $\mathrm{Aut}(\pi) \to \mathrm{Aut}(\pi/\Gamma_{k+1}\pi)$ and we define a representation ρ_k of the mapping class group by the following composition:

$$\mathcal{M}(\Sigma) \xrightarrow{\ \rho\ } \mathrm{Aut}(\pi) \longrightarrow \mathrm{Aut}(\pi/\Gamma_{k+1}\pi), \qquad \rho_k : \mathcal{M}(\Sigma) \dashrightarrow \mathrm{Aut}(\pi/\Gamma_{k+1}\pi)$$

Defining $J_k\mathcal{M}(\Sigma) := \ker \rho_k$ for every $k \geq 0$, we obtain a sequence of subgroups

$$\mathcal{M}(\Sigma) = J_0\mathcal{M}(\Sigma) \supset J_1\mathcal{M}(\Sigma) \supset \cdots \supset J_k\mathcal{M}(\Sigma) \supset J_{k+1}\mathcal{M}(\Sigma) \supset \cdots$$

which is called the *Johnson filtration* of the mapping class group $\mathcal{M}(\Sigma)$. Note that $J_1\mathcal{M}(\Sigma) = \mathcal{I}(\Sigma)$ and, for any $k \geq 1$, we will sometimes write $J_k\mathcal{I}(\Sigma)$ instead of $J_k\mathcal{M}(\Sigma)$. We now give two important properties of the Johnson filtration.

Lemma 4.12. *We have*

(i) $\big[J_k\mathcal{M}(\Sigma), J_l\mathcal{M}(\Sigma)\big] \subset J_{k+l}\mathcal{M}(\Sigma)$ *for any integers* $k, l \geq 0$,
(ii) $\bigcap_{k\geq 0} J_k\mathcal{M}(\Sigma) = \{1\}$.

Proof. We refer to [Mor91, Corollary 3.3] for the proof of (i). To prove (ii), consider an $f \in \mathcal{M}(\Sigma)$ such that $f \in J_k\mathcal{M}(\Sigma)$ for all $k \geq 0$. Let $x \in \pi$. Then, by assumption,

$$\forall k \geq 1, \quad f_\sharp(x)x^{-1} \in \Gamma_k\pi.$$

Since π is a free group, it is *residually nilpotent* i.e. $\bigcap_{k\geq 1} \Gamma_k\pi = \{1\}$. It follows that $f_\sharp(x) = x$ for all $x \in \pi$. We deduce from Theorem 4.11 that $f = 1 \in \mathcal{M}(\Sigma)$. □

As an application of Lemma 4.12, we obtain the following.

Proposition 4.13. *The Torelli group $\mathcal{I}(\Sigma)$ is residually nilpotent.*

Proof. Using Lemma 4.12.(i), we obtain by an induction on $k \geq 1$ that

$$\Gamma_k\mathcal{I}(\Sigma) \subset J_k\mathcal{M}(\Sigma). \tag{4.9}$$

Then it follows from Lemma 4.12.(ii) that $\bigcap_{k\geq 1} \Gamma_k\mathcal{I}(\Sigma) \subset \{1\}$. □

Proposition 4.13 shows that it is important to compute the graded object associated to the lower central series of $\mathcal{I}(\Sigma)$, namely

$$\operatorname{Gr}^\Gamma \mathcal{I}(\Sigma) := \bigoplus_{k\geq 1} \frac{\Gamma_k\mathcal{I}(\Sigma)}{\Gamma_{k+1}\mathcal{I}(\Sigma)}.$$

As a general fact, $\operatorname{Gr}^\Gamma \mathcal{I}(\Sigma)$ is a graded Lie algebra.[8] The conjugation action of $\mathcal{M}(\Sigma)$ on $\mathcal{I}(\Sigma)$ induces an action of the symplectic group $\operatorname{Sp}(H)$ on $\operatorname{Gr}^\Gamma \mathcal{I}(\Sigma)$:

$$\forall M \in \operatorname{Sp}(H), \forall f \in \Gamma_k\mathcal{I}(\Sigma),\ M\cdot[f] := [mfm^{-1}] \in \frac{\Gamma_k\mathcal{I}(\Sigma)}{\Gamma_{k+1}\mathcal{I}(\Sigma)}, \text{ where } m \in \mathcal{M}(\Sigma) \text{ is such that } m_* = M$$

Clearly this action preserves the graded Lie algebra structure of $\operatorname{Gr}^\Gamma \mathcal{I}(\Sigma)$.

The degree one part of $\operatorname{Gr}^\Gamma \mathcal{I}(\Sigma)$, i.e. the abelianisation of $\mathcal{I}(\Sigma)$, will be seen in §4.4 while its rationalization $\big(\operatorname{Gr}^\Gamma \mathcal{I}(\Sigma)\big) \otimes \mathbb{Q}$ will be considered in §4.5. Before that, observe that the inclusion (4.9) induces a canonical map

$$\operatorname{Gr}^\Gamma \mathcal{I}(\Sigma) \longrightarrow \operatorname{Gr}^J \mathcal{I}(\Sigma) := \bigoplus_{k\geq 1} \frac{J_k\mathcal{I}(\Sigma)}{J_{k+1}\mathcal{I}(\Sigma)}. \tag{4.10}$$

Furthermore, using Lemma 4.12.(i), we can also give to $\operatorname{Gr}^J \mathcal{I}(\Sigma)$ the structure of a graded Lie algebra with $\operatorname{Sp}(H)$-action as we did for $\operatorname{Gr}^\Gamma \mathcal{I}(\Sigma)$: clearly, (4.10) is a homomorphism of graded Lie algebras and it is $\operatorname{Sp}(H)$-equivariant.

Remark 4.14. ⚠ Although it is induced by an injection of filtered groups, namely the inclusion (4.9), the graded homomorphism (4.10) is *not* injective. Indeed, it is not injective in degree 1 as will follow from the results of §4.4.

Thus, we are now interested in the Lie algebra $\operatorname{Gr}^J \mathcal{I}(\Sigma)$. We will show that it embeds in a Lie algebra of derivations. Let $H := H_1(\Sigma;\mathbb{Z})$ and let $\mathfrak{L}(H)$ be the Lie algebra *freely* generated by H:

$$\mathfrak{L}(H) = \bigoplus_{k\geq 1} \mathfrak{L}_k(H) \quad \text{where } \mathfrak{L}_1(H) = H,\ \mathfrak{L}_2(H) = \Lambda^2 H,\ \dots$$

The natural action of $\operatorname{Sp}(H)$ on H extends to an action of $\operatorname{Sp}(H)$ on the graded Lie algebra $\mathfrak{L}(H)$: for instance, for any $M \in \operatorname{Sp}(H)$ and for all $h_1, h_2, h_3 \in H$, we have $M\cdot[h_1,[h_2,h_3]] = [M(h_1),[M(h_2),M(h_3)]]$. Recall that a *derivation* of $\mathfrak{L}(H)$ is a $\mathbb{Z}$-linear map $\delta : \mathfrak{L}(H) \to \mathfrak{L}(H)$ such that

$$\forall x, y \in \mathfrak{L}(H),\ \delta([x,y]) = [\delta(x),y] + [x,\delta(y)];$$

a derivation δ is *positive* if it maps $H = \mathfrak{L}_1(H)$ to $\mathfrak{L}_{\geq 2}(H)$; a derivation δ is *symplectic* if $\delta(\omega') = 0$ where $\omega' \in \Lambda^2 H$ is the bivector dual to the intersection form $\omega : H \times H \to \mathbb{Z}$ (see Exercise 4.2). The set $\operatorname{Der}\big(\mathfrak{L}(H)\big)$ of derivations of $\mathfrak{L}(H)$ equipped with the Lie bracket $[\delta_1,\delta_2] := \delta_1 \circ \delta_2 - \delta_2 \circ \delta_1$ is a Lie algebra, and its subset $\operatorname{Der}^+_\omega\big(\mathfrak{L}(H)\big)$ consisting of positive symplectic derivations is a Lie subalgebra. Note that the canonical action of $\operatorname{Sp}(H)$ on $\mathfrak{L}(H)$ induces an $\operatorname{Sp}(H)$-action on the Lie algebra $\operatorname{Der}\big(\mathfrak{L}(H)\big)$:

$$\forall M \in \operatorname{Sp}(H),\ \forall \delta \in \operatorname{Der}\big(\mathfrak{L}(H)\big),\ M\cdot\delta := \big(M\cdot(-)\big)\circ\delta\circ\big(M^{-1}\cdot(-)\big),$$

which, by Exercise 4.2.(a), leaves $\operatorname{Der}^+_\omega\big(\mathfrak{L}(H)\big) \subset \operatorname{Der}\big(\mathfrak{L}(H)\big)$ globally invariant. Finally, $\operatorname{Der}^+_\omega\big(\mathfrak{L}(H)\big)$ is a *graded* Lie algebra whose homogeneous elements of degree $k > 0$ are the derivations mapping H to $\mathfrak{L}_{k+1}(H)$.

[8] If no ring of coefficients is specified, a "Lie algebra" $\mathfrak{g}$ will refer here to a Lie algebra over $\mathbb{Z}$. It is said to be "graded" if $\mathfrak{g} = \oplus_{k>0}\mathfrak{g}_k$ is graded as a $\mathbb{Z}$-module and $[\mathfrak{g}_k,\mathfrak{g}_l] \subset \mathfrak{g}_{k+l}$ for any $k, l > 0$.

Theorem 4.15 (Johnson 80's, Morita 90's)**.** *There is a canonical graded Lie algebra homomorphism*

$$\tau : \mathrm{Gr}^J \mathcal{I}(\Sigma) \longrightarrow \mathrm{Der}^+_\omega\big(\mathfrak{L}(H)\big)$$

which is injective and $\mathrm{Sp}(H)$*-equivariant.*

Sketch of proof. We first define, for any integer $k \geq 1$, a group homomorphism

$$\tau_k : J_k\mathcal{I}(\Sigma) \longrightarrow \mathrm{Hom}\big(H, \mathcal{L}_{k+1}(H)\big) \quad \text{such that} \quad \ker \tau_k = J_{k+1}\mathcal{I}(\Sigma). \tag{4.11}$$

Let $f \in J_k\mathcal{I}(\Sigma)$. For all $x \in \pi$, we set

$$\tau_k(f)([x]) := \rho_{k+1}(f)([x]) \cdot [x]^{-1} \in \frac{\Gamma_{k+1}\pi}{\Gamma_{k+2}\pi}$$

where the x on the left-hand side represents an element of $\pi/\Gamma_2\pi \simeq H$ and the x on the right-hand side represents an element of $\pi/\Gamma_{k+1}\pi$. Using the commutator identities, it can be checked that the right-hand side only depends on the class of x modulo $\Gamma_2\pi$ and that the resulting map $H \to \Gamma_{k+1}\pi/\Gamma_{k+2}\pi$ is actually a group homomorphism. We now check that the resulting map $\tau_k : J_k\mathcal{I}(\Sigma) \to \mathrm{Hom}\big(H, \Gamma_{k+1}\pi/\Gamma_{k+2}\pi\big)$ is a group homomorphism (where $\mathrm{Hom}\big(H, \Gamma_{k+1}\pi/\Gamma_{k+2}\pi\big)$ has the operation induced by the commutative multiplication in $\Gamma_{k+1}\pi/\Gamma_{k+2}\pi$): let $f, h \in J_k\mathcal{I}(\Sigma)$, then

$$\begin{aligned}
\forall x \in \pi, \quad \tau_k(fh)([x]) &= \rho_{k+1}(fh)([x]) \cdot [x]^{-1} \\
&= \big(\rho_{k+1}(f) \circ \rho_{k+1}(h)\big)([x]) \cdot [x]^{-1} \\
&= \rho_{k+1}(f)\big(\rho_{k+1}(h)([x])\big) \cdot [x]^{-1} \\
&= \rho_{k+1}(f)\big(\tau_k(h)([x]) \cdot [x]\big) \cdot [x]^{-1} \\
&= \underbrace{\rho_{k+1}(f)\big(\tau_k(h)([x])\big)}_{=\tau_k(h)([x])} \cdot \underbrace{\rho_{k+1}(f)\big([x]\big) \cdot [x]^{-1}}_{=\tau_k(f)([x])} \\
&= \tau_k(f)([x]) \cdot \tau_k(h)([x])
\end{aligned}$$

where, in the last identity, we use the fact that $\rho_{k+1}(f) \in \mathrm{Aut}(\pi/\Gamma_{k+2}\pi)$ is the identity on $\Gamma_{k+1}\pi/\Gamma_{k+2}\pi$. Since π is a free group, there is a canonical isomorphism of graded Lie algebras between $\mathrm{Gr}^\Gamma \pi$ and $\mathfrak{L}(H)$: this is the unique isomorphism which is given in degree one by the Hurewicz isomorphism between $\pi/\Gamma_2\pi$ and H. In particular, we may identify $\Gamma_{k+1}\pi/\Gamma_{k+2}\pi$ with $\mathfrak{L}_{k+1}(H)$.

Therefore, we have managed to construct a homomorphism τ_k as in (4.11) for any integer $k \geq 1$. Taking the direct sum over all $k \geq 1$, we obtain an injective group homomorphism

$$\tau : \mathrm{Gr}^J \mathcal{I}(\Sigma) \longrightarrow \mathrm{Hom}(H, \mathfrak{L}_{\geq 2}H).$$

Since the Lie algebra $\mathfrak{L}(H)$ is generated by its degree one part, any derivation of $\mathfrak{L}(H)$ is determined by its restriction to H. Therefore, we can identify the abelian groups $\mathrm{Der}^+(\mathfrak{L}(H))$ and $\mathrm{Hom}(H, \mathfrak{L}_{\geq 2}H)$ to obtain an injective group homomorphism

$$\tau : \mathrm{Gr}^J \mathcal{I}(\Sigma) \longrightarrow \mathrm{Der}^+\big(\mathfrak{L}(H)\big).$$

We now check that this map τ takes values in the Lie subalgebra of symplectic derivations. Let $(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$ be a system of simple closed curves as shown in (4.2), and consider their homology classes $a_i := [\alpha_i]$, $b_i := [\beta_i]$. Then $(a_1, b_1, \dots, a_g, b_g)$ is a symplectic basis of H so that

$$\omega' = \sum_{i=1}^{g} a_i \wedge b_i.$$

Let $f \in J_k\mathcal{I}(\Sigma)$ and set $\delta := \tau_k(f) \in \mathrm{Der}^+(\mathfrak{L}(H))$. We must show that

$$\sum_{i=1}^{g} \big[\delta(a_i), b_i\big] + \sum_{i=1}^{g} \big[a, \delta(b_i)\big] = 0. \tag{4.12}$$

By connecting the curves α_i, β_i's to $\star$ by some arcs, we can promote them to loops based at $\star$ which we denote by the same letters. Then, for an appropriate choice of those arcs, the homotopy class of the boundary curve decomposes as

$$\zeta = \big[\alpha_g, \beta_g^{-1}\big] \cdots \big[\alpha_1, \beta_1^{-1}\big] \in \Gamma_2\pi.$$

Then (4.12) can be deduced from the fact that $f(\zeta) = \zeta$ using commutators identities.

We now verify that the map τ is $\mathrm{Sp}(H)$-equivariant. Let $m \in \mathcal{M}(\Sigma)$ and $f \in J_k\mathcal{I}(\Sigma)$. For any $x \in \pi$, we have

$$\begin{aligned}
\tau_k(mfm^{-1})([x]) &= \rho_{k+1}(mfm^{-1})([x]) \cdot [x]^{-1} \\
&= \rho_{k+1}(mf)\big([m_\sharp^{-1}(x)]\big) \cdot [x]^{-1}
\end{aligned}$$

$$\begin{aligned} &= \rho_{k+1}(m)\big(\rho_{k+1}(f)([m_\sharp^{-1}(x)]) \cdot \rho_{k+1}(m^{-1})([x]^{-1})\big) \\ &= \rho_{k+1}(m)\big(\rho_{k+1}(f)([m_\sharp^{-1}(x)]) \cdot [m_\sharp^{-1}(x)]^{-1}\big) \ = \ \rho_{k+1}(m)\big(\tau_k(f)([m_\sharp^{-1}(x)])\big). \end{aligned}$$

This shows that $\tau_k(mfm^{-1}) = \big(m_* \cdot (-)\big) \circ \tau_k(f) \circ \big(m_*^{-1} \cdot (-)\big) = m_* \cdot \tau_k(f)$ if $\tau_k(f)$ is regarded as a derivation of $\mathfrak{L}(H)$, which proves the $\mathrm{Sp}(H)$-equivariance.

Finally, the fact that $\tau : \mathrm{Gr}^J \mathcal{I}(\Sigma) \to \mathrm{Der}^+_\omega\big(\mathfrak{L}(H)\big)$ preserves the Lie brackets is proved in [Mor91, Propositions 3.4 & 3.5] to which we refer. □

The map $\tau_k : J_k\mathcal{I}(\Sigma) \to \mathrm{Hom}(H, \mathfrak{L}_{k+1}(H))$ introduced in the proof of Theorem 4.15 is called the *k-th Johnson homomorphism*. It has been introduced by Johnson for $k = 1$ in [Joh80a] and for arbitrary $k \geq 1$ in [Joh83b]. The general properties of the Johnson homomorphisms have been studied by Morita in [Mor91, Mor93].

The *first* Johnson homomorphism $\tau_1 : \mathcal{I}(\Sigma) \to \mathrm{Hom}(H, \mathfrak{L}_2(H))$ deserves a special attention. There is a canonical isomorphism $H \to H^*$ defined by $h \mapsto \omega(h, -)$, so that we can identify the target of τ_1 with the $\mathbb{Z}$-module

$$\mathrm{Hom}(H, \mathfrak{L}_2(H)) \simeq H^* \otimes \mathfrak{L}_2(H) \simeq H \otimes \mathfrak{L}_2(H).$$

Furthermore, there is a group homomorphism $\Lambda^3(H) \to H \otimes \mathfrak{L}_2(H)$ defined by

$$h_1 \wedge h_2 \wedge h_3 \longmapsto h_1 \otimes [h_2, h_3] + h_3 \otimes [h_1, h_2] + h_2 \otimes [h_3, h_1].$$

It is injective since it can be inserted into the following commutative diagram

$$\begin{array}{ccc} \Lambda^3 H & \xrightarrow{\ 3\times\ } & \Lambda^3 H_{\mathbb{Q}} \\ \downarrow & & \uparrow \\ H \otimes \mathfrak{L}_2(H) & \longrightarrow & H_{\mathbb{Q}} \otimes \mathfrak{L}_2(H_{\mathbb{Q}}) \end{array}$$

where $H_{\mathbb{Q}} := H \otimes \mathbb{Q}$ and the map $H_{\mathbb{Q}} \otimes \mathfrak{L}_2(H_{\mathbb{Q}}) \to \Lambda^3 H_{\mathbb{Q}}$ is defined by $u \otimes [v, w] \longmapsto u \wedge v \wedge w$. Therefore, we can regard $\Lambda^3 H$ as a submodule of $H \otimes \mathfrak{L}_2(H)$. It can be verified by a direct computation that, for any BP map $\tau_\gamma \tau_\delta^{-1}$ of genus 1,

$$\tau_1\big(\tau_\gamma \tau_\delta^{-1}\big) = \pm[\gamma] \wedge [\rho'] \wedge [\rho''] \in \Lambda^3 H \tag{4.13}$$

where ρ' and ρ'' are simple oriented closed curves on the subsurface of genus 1 delimited by $\delta \cup \gamma$ and they are such that $i(\rho', \rho'') = 1$: see [Joh80a, Lemma 4.B]. Furthermore, τ_1 vanishes on any BSCC map: see Exercise 4.9. It follows from Theorem 4.6 that τ_1 takes values in $\Lambda^3 H$. Thus, to sum up our discussion, the first Johnson homomorphism is an $\mathrm{Sp}(H)$-equivariant homomorphism

$$\tau_1 : \mathcal{I}(\Sigma) \longrightarrow \Lambda^3 H.$$

The formula (4.13) can also be used to show that τ_1 is surjective: see [Joh80a, Theorem 1].

4.4. **The abelianization of the Torelli group.** In contrast with the mapping class group (see Corollary 2.16), the Torelli group has an interesting abelianization which we now survey. As in the previous subsection, we consider an oriented surface Σ of genus $g \geq 2$ with a single boundary component and we set $H := H_1(\Sigma; \mathbb{Z})$. To describe the abelianization of $\mathcal{I}(\Sigma)$, we will need the set

$$\Omega := \Big\{ H \otimes \mathbb{Z}_2 \xrightarrow{\ q\ } \mathbb{Z}_2 : \forall x, y \in H \otimes \mathbb{Z}_2,\ q(x+y) - q(x) - q(y) = \omega(x, y) \Big\}$$

where $\omega : (H \otimes \mathbb{Z}_2) \times (H \otimes \mathbb{Z}_2) \to \mathbb{Z}_2$ is the symmetric $\mathbb{Z}_2$-bilinear form obtained by taking the mod 2 reduction of the intersection form $\omega : H \times H \to \mathbb{Z}$. In other words, Ω is the set of *quadratic forms* whose polar form is the mod 2 reduction of ω.

Lemma 4.16. *The set Ω is an affine space over the $\mathbb{Z}_2$-vector space $H \otimes \mathbb{Z}_2$, the action being given by*

$$\forall x \in H \otimes \mathbb{Z}_2,\ \forall q \in \Omega, \quad q + \vec{x} := q + \omega(x, -). \tag{4.14}$$

Proof. Since $\omega : H \times H \to \mathbb{Z}$ is non-singular, its mod 2 reduction is non-singular too, i.e. the map

$$H \otimes \mathbb{Z}_2 \longrightarrow \mathrm{Hom}_{\mathbb{Z}_2}(H \otimes \mathbb{Z}_2, \mathbb{Z}_2),\ x \longmapsto \omega(x, -) \tag{4.15}$$

is an isomorphism. The action (4.14) is transitive because, for any $q, q' \in \Omega$, the map $q - q' : H \otimes \mathbb{Z}_2 \to \mathbb{Z}_2$ is a homomorphism and the surjectivity of (4.15) implies that there is an $x \in H \otimes \mathbb{Z}_2$ such that $\omega(x, -) = q - q'$. The action (4.14) is free because, for any $q \in \Omega$ and $x \in H \otimes \mathbb{Z}_2$ such that $q + \vec{x} = q$, we have $\omega(x, -) = 0$ and the injectivity of (4.15) implies that $x = 0$. □

Consequently, we can consider affine functions on Ω and, furthermore, we can consider the space

$$\mathrm{Cubic}(\Omega,\mathbb{Z}_2) := \left\{\Omega \xrightarrow{c} \mathbb{Z}_2 : c \text{ is a sum of triple products of affine functions}\right\}$$

of *cubic functions* on Ω. The (formal) *third differential* of a $c \in \mathrm{Cubic}(\Omega,\mathbb{Z}_2)$ is the map

$$\mathrm{d}^3 c : (H\otimes\mathbb{Z}_2)\times(H\otimes\mathbb{Z}_2)\times(H\otimes\mathbb{Z}_2) \longrightarrow \mathbb{Z}_2$$

defined by

$$\begin{aligned}\mathrm{d}^3c(x,y,z) \quad := \quad & c(q+\vec{x}+\vec{y}+\vec{z}) + \big(c(q+\vec{y}+\vec{z})+c(q+\vec{x}+\vec{z})+c(q+\vec{x}+\vec{y})\big)\\ & +\big(c(q+\vec{x})+c(q+\vec{y})+c(q+\vec{z})\big)+c(q)\end{aligned}$$

where $q\in\Omega$ is an arbitrary point. It can be checked that the map d^3c is trilinear and does not depend on the choice of q (because the function $c:\Omega\to\mathbb{Z}_2$ is cubic) and that d^3c is alternate (because we are here working in characteristic 2). Therefore the map d^3c defines an element

$$\mathrm{d}^3c \in \mathrm{Hom}_{\mathbb{Z}_2}\big(\Lambda^3(H\otimes\mathbb{Z}_2),\mathbb{Z}_2\big) \simeq \Lambda^3(H\otimes\mathbb{Z}_2).$$

Here the isomorphism $\Lambda^3(H\otimes\mathbb{Z}_2)\to\mathrm{Hom}_{\mathbb{Z}_2}\big(\Lambda^3(H\otimes\mathbb{Z}_2),\mathbb{Z}_2\big)$ is induced by the non-singular $\mathbb{Z}_2$-bilinear form $\Lambda^3(H\otimes\mathbb{Z}_2)\times\Lambda^3(H\otimes\mathbb{Z}_2)\to\mathbb{Z}_2$ defined by

$$\big(v_1\wedge v_2\wedge v_3, w_1\wedge w_2\wedge w_3\big)\longmapsto \det\begin{pmatrix}\omega(v_1,w_1) & \omega(v_1,w_2) & \omega(v_1,w_3)\\ \omega(v_2,w_1) & \omega(v_2,w_2) & \omega(v_2,w_3)\\ \omega(v_3,w_1) & \omega(v_3,w_2) & \omega(v_3,w_3)\end{pmatrix}\in\mathbb{Z}_2.$$

Theorem 4.17 (Johnson 1985)**.** *There is a group homomorphism $\beta:\mathcal{I}(\Sigma)\to\mathrm{Cubic}(\Omega,\mathbb{Z}_2)$ such that the following diagram is commutative:*

(4.16)
$$\begin{array}{ccc}\mathcal{I}(\Sigma) & \xrightarrow{\beta} & \mathrm{Cubic}(\Omega,\mathbb{Z}_2)\\ {\scriptstyle\tau_1}\downarrow & & \downarrow{\scriptstyle\mathrm{d}^3}\\ \Lambda^3 H & \xrightarrow[-\otimes\mathbb{Z}_2]{} & \Lambda^3H\otimes\mathbb{Z}_2\end{array}$$

For $g\geq 3$, this diagram induces an isomorphism

$$(\tau_1,\beta):\frac{\mathcal{I}(\Sigma)}{[\mathcal{I}(\Sigma),\mathcal{I}(\Sigma)]}\xrightarrow{\simeq}\Lambda^3H\times_{\Lambda^3H\otimes\mathbb{Z}_2}\mathrm{Cubic}(\Omega,\mathbb{Z}_2)$$

between the abelianization of the Torelli group and the corresponding fibered product.

It follows that, for $g\geq 3$, the abelianization of the Torelli group is (non-canonically) isomorphic to

$$\Lambda^3 H \;\oplus\; \bigoplus_{i=0}^{2}\Lambda^i(H\otimes\mathbb{Z}_2)$$

since the space of quadratic functions on Ω is (non-canonically) isomorphic to $\bigoplus_{i=0}^2\Lambda^i(H\otimes\mathbb{Z}_2)$. Since $\mathrm{rank}\,\Lambda^3H=\binom{2g}{3}$ and $\dim_{\mathbb{Z}_2}\Lambda^i(H\otimes\mathbb{Z}_2)=\binom{2g}{i}$, we deduce that the group $\mathcal{I}(\Sigma)$ can not be generated by less than

$$\binom{2g}{3}+\binom{2g}{2}+\binom{2g}{1}+\binom{2g}{0}=\frac{4}{3}g^3+\frac{5}{3}g+1$$

elements: this is in sharp contrast with the mapping class group for which we can find a system of generators whose cardinality is linear in g. Note that the second part of Theorem 4.17 does not hold in genus $g=2$ since $\mathcal{I}(\Sigma)$ is not finitely generated in this case (Theorem 4.10).

About the proof of Theorem 4.17. The theorem is proved by Johnson in [Joh85c] by combining all his prior works on the Torelli group with a bit of 3-dimensional topology. We only mention here, in a very rough way, how 3-manifolds and their topological invariants arise in Johnson's proof.

We can associate to any $f\in\mathcal{M}(\Sigma)$ a topological 3-manifold which is closed (i.e. compact without boundary), connected and oriented: this is the *mapping torus* of f defined by

$$\mathrm{T}_f := \left(\frac{\Sigma\times[-1,1]}{\sim}\right)\cup_\partial\left(S^1\times D^2\right).$$

Here the equivalence relation $\sim$ identifies $(f(x),1)$ with $(x,-1)$ for all $x\in\Sigma$, and the "solid torus" $S^1\times D^2$ is glued as follows along its boundary: the meridian $\{1\}\times\partial D^2$ is glued along the circle $(\{\star\}\times[-1,1])/\sim$ while the longitude $S^1\times\{1\}$ is glued along $\partial\Sigma\times\{1\}$. If now $f\in\mathcal{I}(\Sigma)$, then the inclusion $\iota : \Sigma \hookrightarrow \mathrm{T}_f$ defined by $x \mapsto (x,1)$ induces an isomorphism between $H = H_1(\Sigma;\mathbb{Z})$ and $H_1(\mathrm{T}_f;\mathbb{Z})$. The intersection of closed immersed surfaces in T_f defines a trilinear alternate form

$$H_2(\mathrm{T}_f;\mathbb{Z}) \times H_2(\mathrm{T}_f;\mathbb{Z}) \times H_2(\mathrm{T}_f;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

(in a way similar to the definition of the intersection form of an oriented surface in §1.2): thus we have assigned to any $f \in \mathcal{I}(\Sigma)$ an element of

$$\mathrm{Hom}\left(\Lambda^3 H_2(\mathrm{T}_f;\mathbb{Z}),\mathbb{Z}\right) \simeq \Lambda^3\,\mathrm{Hom}\left(H_2(\mathrm{T}_f;\mathbb{Z}),\mathbb{Z}\right) \simeq \Lambda^3 H^2(\mathrm{T}_f;\mathbb{Z}) \simeq \Lambda^3 H_1(\mathrm{T}_f;\mathbb{Z}) \simeq \Lambda^3 H$$

where the second isomorphism is given by the universal coefficients theorem and the third isomorphism is by Poincaré duality. Therefore we have obtained a map $\mathcal{I}(\Sigma) \to \Lambda^3 H$, and this map turns out to coincide with the first Johnson homomorphism τ_1 [Joh83b].

We now explain the new homomorphism $\beta : \mathcal{I}(\Sigma) \to \mathrm{Cubic}(\Omega, \mathbb{Z}_2)$ that appears in the statement of the theorem. This is actually a "compilation" of many homomorphisms which have been first introduced by Birman & Craggs [BC78] before being studied in great detail by Johnson [Joh80b]. Thus the map β is called the *Birman–Craggs homomorphism* and, similarly to the first Johnson homomorphism, it can be described using the "mapping torus" construction [Tur83]. Specifically, for any $f \in \mathcal{I}(\Sigma)$, the cubic function $\beta(f) : \Omega \to \mathbb{Z}_2$ is the map which assigns to any spin structure σ on T_f the Rochlin invariant of the spin 3-manifold (T_f, σ). This definition of β would need several explanations. Let us only clarify a few points: (i) when it is not empty, the set $\mathrm{Spin}(M)$ of spin structures on an oriented manifold M constitutes an affine space over the $\mathbb{Z}_2$-vector space $H^1(M;\mathbb{Z}_2)$; (ii) the Rochlin invariant of a closed spin 3-manifold is defined as an element of $\mathbb{Z}_{16}$ (using 4-dimensional topology) but, when the first homology group with integer coefficients of the manifold is torsion-free, the Rochlin invariant belongs to the subset $\{0,8\} \subset \mathbb{Z}_{16}$ so that it defines an element of $\mathbb{Z}_2$; (iii) the inclusion $\iota : \Sigma \hookrightarrow \mathrm{T}_f$ induces an affine isomorphism $\mathrm{Spin}(\mathrm{T}_f) \to \mathrm{Spin}(\Sigma)$; (iv) there is an affine canonical correspondence between $\mathrm{Spin}(\Sigma)$ and the space of quadratic forms Ω [Joh80c].

The facts that the Rochlin function $\mathrm{Spin}(M) \to \mathbb{Z}_{16}$ is cubic and that its third differential is given by the trilinear intersection form on $H_2(M;\mathbb{Z})$ is true for any closed oriented 3-manifold M [Tur83]. In particular, we get the commutative diagram (4.16). But it then remains to prove the second part of the theorem which, of course, is the most difficult one. □

4.5. **The Malcev Lie algebra of the Torelli group.** We conclude these lecture notes by mentioning Hain's results on the formality of the Torelli group. We first need some general definitions about the notion of "formality".

Recall that the *Malcev Lie algebra* of a discrete group G is a filtered Lie $\mathbb{Q}$-algebra $\mathfrak{M}(G)$, which is usually defined as the primitive part of the I-adic completion of the group $\mathbb{Q}$-algebra of G:

$$\mathfrak{M}(G) := \mathrm{Prim}\Big(\varprojlim_k \mathbb{Q}[G]/I^k\Big) \quad \text{where } I \text{ is the kernel of the augmentation } \mathbb{Q}[G] \to \mathbb{Q}.$$

The graded Lie $\mathbb{Q}$-algebra $\mathrm{Gr}\,\mathfrak{M}(G)$ associated to $\mathfrak{M}(G)$ is isomorphic to $(\mathrm{Gr}^\Gamma G) \otimes \mathbb{Q}$ in a canonical way and, of course, the latter is generated by its degree 1 part. Hence there is always a canonical graded Lie $\mathbb{Q}$-algebra homomorphism

$$\mathfrak{L}\left(\tfrac{G}{[G,G]} \otimes \mathbb{Q}\right) \longrightarrow\!\!\!\!\to (\mathrm{Gr}^\Gamma G) \otimes \mathbb{Q} \simeq \mathrm{Gr}\,\mathfrak{M}(G)$$

which is surjective and whose kernel is denoted by $\mathfrak{R}(G) = \bigoplus_{k \geq 2} \mathfrak{R}_k(G)$.

Definition 4.18. *A finitely generated group G is* 1-formal *if $\mathfrak{M}(G)$ is isomorphic to the degree-completion of $\mathrm{Gr}\,\mathfrak{M}(G) \simeq (\mathrm{Gr}^\Gamma G) \otimes \mathbb{Q}$ and if the ideal $\mathfrak{R}(G)$ is generated by $\mathfrak{R}_2(G)$.*

Thus, for a 1-formal group G, all the information captured by the Malcev Lie algebra is contained in the graded Lie $\mathbb{Q}$-algebra $(\mathrm{Gr}^\Gamma G) \otimes \mathbb{Q}$ and the latter has a finite presentation with only quadratic relations. Examples of 1-formal groups include finitely-generated free groups, fundamental groups of closed surfaces and pure braid groups.

As in the previous subsections, we now consider an oriented surface Σ of genus $g \geq 2$ with a single boundary component. The following result, whose proof is out of reach to us in these notes, has been proved in [Hai97].

Theorem 4.19 (Hain 1997)**.** *The Torelli group $\mathcal{I}(\Sigma)$ is 1-formal if $g \geq 6$.*

In genus $g \in \{3,4,5\}$, Hain has also proved that $\mathfrak{M}(\mathcal{I}(\Sigma))$ is isomorphic to the degree-completion of $\mathrm{Gr}\,\mathfrak{M}(\mathcal{I}(\Sigma))$ and that $\mathfrak{R}(\mathcal{I}(\Sigma))$ is generated by $\mathfrak{R}_2(\mathcal{I}(\Sigma)) \oplus \mathfrak{R}_3(\mathcal{I}(\Sigma))$. Recall that $\mathcal{I}(\Sigma)$ is not finitely generated if $g = 2$ (Theorem 4.10) so that Definition 4.18 does not apply in this case. In the sequel, we will mostly consider the case $g \geq 6$.

In order to get explicit quadratic presentations of $\mathrm{Gr}\,\mathfrak{M}(\mathcal{I}(\Sigma))$, one still needs to compute $\mathfrak{R}_2(\mathcal{I}(\Sigma))$. By definition, $\mathfrak{R}_2(\mathcal{I}(\Sigma))$ is a subspace of

$$\mathfrak{L}_2\left(\frac{\mathcal{I}(\Sigma)}{[\mathcal{I}(\Sigma),\mathcal{I}(\Sigma)]}\otimes\mathbb{Q}\right)$$

and, according to Theorem 4.17, we have an isomorphism

$$\tfrac{\mathcal{I}(\Sigma)}{[\mathcal{I}(\Sigma),\mathcal{I}(\Sigma)]}\otimes\mathbb{Q}\xrightarrow[\simeq]{\tau_1\otimes\mathbb{Q}}\Lambda^3 H_{\mathbb{Q}}\quad\text{where } H_{\mathbb{Q}}:=H\otimes\mathbb{Q}.$$

Therefore we can regard $\mathfrak{R}_2(\mathcal{I}(\Sigma))$ as a subspace of $\mathfrak{L}_2(\Lambda^3 H_{\mathbb{Q}})$ and, since τ_1 is $\mathrm{Sp}(H)$-equivariant, $\mathfrak{R}_2(\mathcal{I}(\Sigma))$ is actually an $\mathrm{Sp}(H_{\mathbb{Q}})$-submodule of $\mathfrak{L}_2(\Lambda^3 H_{\mathbb{Q}})$. The following has been firstly proved in the analogous case of a closed surface by Hain [Hai97], and it has been subsequently extended to the case of a bordered surface by Habegger and Sorger (unpublished); the proof needs the representation theory of the symplectic group $\mathrm{Sp}(H_{\mathbb{Q}})\simeq\mathrm{Sp}(2g;\mathbb{Q})$.

Proposition 4.20 (Hain 1997, Habegger–Sorger 2000)**.** *If $g\geq 6$, then the $\mathrm{Sp}(H_{\mathbb{Q}})$-module $\mathfrak{R}_2(\mathcal{I}(\Sigma))$ is spanned by the following elements r_1, r_2 of $\mathfrak{L}_2(\Lambda^3 H_{\mathbb{Q}})$:*

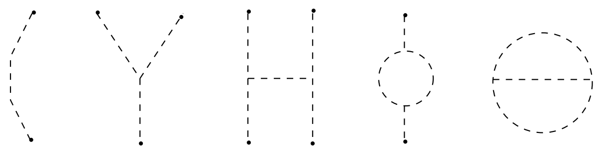
$$\left\{\begin{array}{l} r_1 := [\alpha_1\wedge\alpha_2\wedge\beta_2, \alpha_3\wedge\alpha_4\wedge\beta_4] \\ r_2 := [\alpha_1\wedge\alpha_2\wedge\beta_2, \alpha_g\wedge\omega'] \end{array}\right.$$

where $(\alpha_1,\beta_1,\dots,\alpha_g,\beta_g)$ denotes a symplectic basis of H and $\omega' := \sum_{i=1}^{g}\alpha_i\wedge\beta_i$.

Thus, we get the following quadratic presentation of the Malcev Lie algebra of the Torelli group:

$$\mathrm{Gr}\,\mathfrak{M}(\mathcal{I}(\Sigma))\simeq\frac{\mathfrak{L}(\Lambda^3 H_{\mathbb{Q}})}{\big\langle\langle r_1,r_2\rangle_{\mathrm{Sp}(H_{\mathbb{Q}})}\big\rangle_{\mathrm{ideal}}}\qquad(\text{for } g\geq 6)\tag{4.17}$$

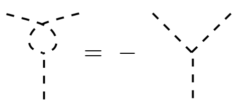
The next step would be to deduce from the presentation (4.17) a diagrammatic description of $\mathrm{Gr}\,\mathfrak{M}(\mathcal{I}(\Sigma))$. One expects something similar to the description of $\mathrm{Gr}\,\mathfrak{M}(PB_n)$ in terms of "chord diagrams". On this purpose, we introduce the following notion: a *Jacobi diagram* is a finite graph whose vertices have valency 1 (*external* vertices) or 3 (*internal* vertices). Each internal vertex is *oriented*, in the sense that its incident edges are cyclically ordered. A Jacobi diagram is *colored* by $H_{\mathbb{Q}}$ if a map from the set of its external vertices to the vector space $H_{\mathbb{Q}}$ is specified. A *strut* is a Jacobi diagram with only two external vertices and no internal vertex. Examples of connected Jacobi diagrams (the strut, the Y graph, the H graph, the Phi graph and the Theta graph) are shown below:



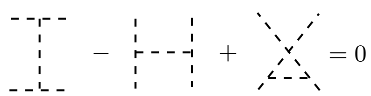
(Here and in the sequel, the vertex orientations are given by the counter-clockwise orientation.) We consider the following $\mathbb{Q}$-vector space:

$$\mathcal{A}(H_{\mathbb{Q}}) := \frac{\mathbb{Q}\cdot\left\{\begin{array}{c}\text{Jacobi diagrams without strut component}\\ \text{and with external vertices colored by } H_{\mathbb{Q}}\end{array}\right\}}{\text{AS, IHX, multilinearity}}.$$

The "AS" and "IHX" relations are diagrammatic analogues of the antisymmetry and Jacobi identities in Lie algebras:



AS

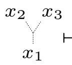


IHX

The "multilinearity" relation states that a Jacobi diagram D with one external vertex v colored by $q_1\cdot h_1+q_2\cdot h_2$ (with $q_1,q_2\in\mathbb{Q}$ and $h_1,h_2\in H_{\mathbb{Q}}$) is equivalent to the linear combination $q_1\cdot D_1+q_2\cdot D_2$ where D_i is the Jacobi diagram D with the vertex v colored by h_i. The *degree* of a Jacobi diagram is the number of its internal vertices. Thus, $\mathcal{A}(H_{\mathbb{Q}})$ is a graded vector space:

$$\mathcal{A}(H_{\mathbb{Q}})=\bigoplus_{d=0}^{\infty}\mathcal{A}_d(H_{\mathbb{Q}}).$$

The degree 0 part $\mathcal{A}_0(H_\mathbb{Q})$ is 1-dimensional spanned by the empty diagram $\varnothing$, while the degree 1 part $\mathcal{A}_1(H_\mathbb{Q})$ is isomorphic to $\Lambda^3 H_\mathbb{Q}$ via the map



$$\longmapsto x_1 \wedge x_2 \wedge x_3.$$

There is an interesting operation $\star$ in $\mathcal{A}(H_\mathbb{Q})$: for any $H_\mathbb{Q}$-colored Jacobi diagrams D and E whose sets of external vertices are denoted by V and W respectively, we set

$$D \star E := \sum_{\substack{V' \subset V,\ W' \subset W \\ \beta\,:\, V' \xrightarrow{\simeq} W'}} \frac{1}{2^{|V'|}} \cdot \prod_{v \in V'} \omega\Big(\mathrm{color}(v), \mathrm{color}\big(\beta(v)\big)\Big) \cdot (D \cup_\beta E)$$

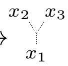
where the sum is taken over all ways of identifying a subset V' of V with a subset W' of W, and $D \cup_\beta E$ is obtained from $D \sqcup E$ by gluing each vertex $v \in V'$ to $\beta(v) \in W'$. Clearly $\star$ is $\mathrm{Sp}(H_\mathbb{Q})$-equivariant if the group $\mathrm{Sp}(H_\mathbb{Q})$ acts on $\mathcal{A}(H_\mathbb{Q})$ in the obvious way, i.e. by acting on the colors, and it is easily verified that $(\mathcal{A}(H_\mathbb{Q}), \star)$ is an associative algebra. Let $[-,-]_\star$ be the Lie bracket defined by $[D,E]_\star := D \star E - E \star D$. The subspace of $\mathcal{A}(H_\mathbb{Q})$

$$\mathcal{A}^c(H_\mathbb{Q}) = \bigoplus_{k \geq 1} \mathcal{A}^c_k(H_\mathbb{Q})$$

spanned by non-empty connected Jacobi diagrams is preserved by $[-,-]_\star$: we call $(\mathcal{A}^c(H_\mathbb{Q}), [-,-]_\star)$ the *Lie algebra of symplectic Jacobi diagrams* [HM09].

Proposition 4.21. *For $g \geq 3$, there is a unique $\mathrm{Sp}(H_\mathbb{Q})$-equivariant homomorphism of graded Lie $\mathbb{Q}$-algebras*

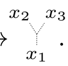
$$Y : \frac{\mathfrak{L}(\Lambda^3 H_\mathbb{Q})}{\big\langle \langle r_1, r_2 \rangle_{\mathrm{Sp}(H_\mathbb{Q})} \big\rangle_{\mathrm{ideal}}} \longrightarrow \mathcal{A}^c(H_\mathbb{Q})$$

that is defined by $x_1 \wedge x_2 \wedge x_3 \mapsto$ 

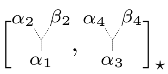
in degree 1.

Note that the image of Y is the Lie subalgebra $a(H_\mathbb{Q})$ of $\mathcal{A}^c(H_\mathbb{Q})$ generated by $\mathcal{A}^c_1(H_\mathbb{Q}) = \mathcal{A}_1(H_\mathbb{Q})$. According to (4.17), Y provides an $\mathrm{Sp}(H_\mathbb{Q})$-equivariant surjective homomorphism of graded Lie algebras $\mathrm{Gr}\,\mathfrak{M}(\mathcal{I}(\Sigma)) \to a(H_\mathbb{Q})$ for $g \geq 6$. But, unfortunately, it is not known whether Y is injective (although it is known to be so in degree 2 [HM09]).

Proof of Proposition 4.21. Clearly there is a unique graded Lie algebra map $\widetilde{Y} : \mathfrak{L}(\Lambda^3 H_\mathbb{Q}) \to \mathcal{A}^c(H_\mathbb{Q})$ defined in degree 1 by

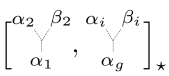
$$x_1 \wedge x_2 \wedge x_3 \longmapsto$$


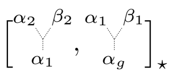
Since $\widetilde{Y}_1$ is $\mathrm{Sp}(H_\mathbb{Q})$-equivariant and since $[-,-]_\star$ is $\mathrm{Sp}(H_\mathbb{Q})$-equivariant, we deduce that $\widetilde{Y}$ is $\mathrm{Sp}(H_\mathbb{Q})$-equivariant. Thus, the proposition will follow from the facts that $\widetilde{Y}_2(r_1) = \widetilde{Y}_2(r_2) = 0$. We obviously have

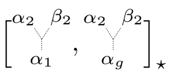
$$\widetilde{Y}_2(r_1) =$$


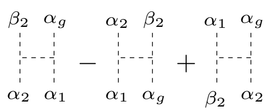
$$= 0,$$

and we have

$$\widetilde{Y}_2(r_2) = \sum_{i=1}^{g}$$


$$=$$


$$+$$


$$=$$


$$\overset{\mathrm{IHX}}{=} 0.$$

□

The algebra $\mathcal{A}(H_\mathbb{Q})$ and the Lie algebra $\mathcal{A}^c(H_\mathbb{Q})$ originate from the theory of finite-type invariants for 3-manifolds. In particular the following is proved in [HM09] without using Hain's results. Here $\widehat{\mathcal{A}}(H_\mathbb{Q})$ denotes the degree-completion of the vector space $\mathcal{A}(H_\mathbb{Q})$.

Theorem 4.22 (Habiro–Massuyeau 2009)**.** *Let $g \geq 2$. There is an injective map $Z : \mathcal{I}(\Sigma) \to \widehat{\mathcal{A}}(H_{\mathbb{Q}})$ which is*

(i) *multiplicative in the sense that $Z(f \cdot h) = Z(f) \star Z(h)$ for all $f, h \in \mathcal{I}(\Sigma)$,*

(ii) *filtration-preserving in the sense that $Z\big(\Gamma_k \mathcal{I}(\Sigma)\big) \subset \varnothing + \widehat{\mathcal{A}}_{\geq k}(H_{\mathbb{Q}})$ for all $k \geq 1$.*

Furthermore, Z induces at the graded level an $\mathrm{Sp}(H_{\mathbb{Q}})$-equivariant homomorphism of graded Lie $\mathbb{Q}$-algebras

$$\mathrm{Gr}\, Z : \big(\,\mathrm{Gr}^{\Gamma} \mathcal{I}(\Sigma)\big) \otimes \mathbb{Q} \longrightarrow \mathcal{A}^c(H_{\mathbb{Q}})$$

which, in degree 1, is given by the isomorphism $\tau_1 \otimes \mathbb{Q} : (\mathcal{I}(\Sigma)/[\mathcal{I}(\Sigma), \mathcal{I}(\Sigma)]) \otimes \mathbb{Q} \to \Lambda^3 H_{\mathbb{Q}} \simeq \mathcal{A}^c_1(H_{\mathbb{Q}})$.

Again, the image of $\mathrm{Gr}\, Z$ is the Lie subalgebra $a(H_{\mathbb{Q}})$ of $\mathcal{A}^c(H_{\mathbb{Q}})$ generated by $\mathcal{A}^c_1(H_{\mathbb{Q}})$. It is expected that $\mathrm{Gr}\, Z$ is injective or, at least, it is expected to be so in the "stable range".

About the proof. The map Z is derived from a 3-dimensional TQFT which has been constructed in [CHM08] using the Le–Murakami–Ohtsuki invariant of [LMO98]. Thus we call Z the *LMO homomorphism.* The injectivity of Z is proved by showing that all the Johnson homomorphisms $\tau_1, \tau_2, \tau_3, \ldots$ are explicitly determined by Z, and using Lemma 4.12.(ii). The multiplicativity of Z is the manifestation of the functoriality of the TQFT constructed in [CHM08]. The fact that Z is filtration-preserving is proved by using a kind of surgery calculus in 3-manifolds (the so-called "calculus of claspers" developped by Goussarov and Habiro). □

4.6. **Exercises.**

Exercise 4.1. Show that $\mathrm{Sp}(2; \mathbb{Z}) = \mathrm{SL}(2; \mathbb{Z})$.

Exercise 4.2. Let H be a finitely generated free abelian group and let $\omega : H \times H \to \mathbb{Z}$ be a symplectic form. Let $\omega' \in \Lambda^2 H$ be the bivector corresponding to $\omega \in \Lambda^2 H^*$ through the isomorphism

$$w : H \xrightarrow{\simeq} H^*, h \longmapsto \omega(h, -)$$

(a) Show that an automorphism ψ of H preserves the bilinear form ω if and only if $(\Lambda^2\psi)(\omega') = \omega'$.

(b) Deduce that any $\psi \in \mathrm{Sp}(H)$ satisfies $\det(\psi) = 1$.

Exercise 4.3. Let $\Sigma := \Sigma_{1,1}$ be a torus with one disk removed, and let $\delta \subset \mathrm{int}(\Sigma)$ be a simple closed curve parallel to $\partial\Sigma$. Show that $\mathcal{I}(\Sigma)$ is the infinite cyclic group generated by the Dehn twist along δ.

Exercise 4.4. Let Σ be an oriented surface with at most one boundary component. Let δ, ρ be simple closed curves such that $\omega([\delta], [\rho]) = 0$ for some arbitrary orientations of these curves. Show that the commutator $[\tau_\delta, \tau_\rho]$ belongs to $\mathcal{I}(\Sigma)$ and that it is not trivial in general.

Exercise 4.5. Let Σ be an oriented surface with at most one boundary component.

(a) Show that any two BSCC maps of the same genus are conjugate in $\mathcal{M}(\Sigma)$.

(b) Show that any two BP maps of the same genus are conjugate in $\mathcal{M}(\Sigma)$.

Exercise 4.6. Let $k \geq 2$ be an integer. Show that any BP map of genus k is a product of BP maps of genus 1.

Exercise 4.7. Let $k \geq 3$ be an integer. Using the lantern relation, show that any BSCC map of genus k is a product of BSCC maps of genus 1 and 2.

Exercise 4.8. Let $\omega : H \times H \to \mathbb{Z}$ be a symplectic form on a free abelian group H, and let $w : H \to H^*$ be the isomorphism introduced in Exercise 1.6. Show that the isomorphism

$$\mathrm{Der}^+\big(\mathfrak{L}(H)\big) \xrightarrow[\simeq]{\text{restriction}} \mathrm{Hom}(H, \mathfrak{L}_{\geq 2}(H)) \simeq H^* \otimes \mathfrak{L}_{\geq 2}(H) \xrightarrow[\simeq]{w^{-1} \otimes \mathrm{id}} H \otimes \mathfrak{L}_{\geq 2}(H)$$

maps $\mathrm{Der}^+_\omega\big(\mathfrak{L}(H)\big)$ onto the kernel of the Lie bracket $[-,-] : H \otimes \mathfrak{L}_{\geq 2}(H) \to \mathfrak{L}_{\geq 3}(H)$, and prove that it is $\mathrm{Sp}(H)$-equivariant.

Exercise 4.9. Show that the first Johnson homomorphism τ_1 vanishes on every BSCC map.

* * *

Solution to Exercise 4.1. Consider an arbitrary 2×2 matrix with entries in $\mathbb{Z}$:

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

We compute

$$M^t\, \Omega\, M \;=\; \begin{pmatrix} a & b \\ c & d \end{pmatrix}^t \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} c & d \\ -a & -b \end{pmatrix}$$
$$= \begin{pmatrix} 0 & ad-bc \\ cb-ad & 0 \end{pmatrix}.$$

It follows that
$$M^t\,\Omega\,M = \Omega \Longleftrightarrow ad-bc=1.$$

Solution to Exercise 4.2. (a) The bilinear form ω being skew-symmetric, it can be seen as an element $\omega \in \mathrm{Hom}(\Lambda^2 H,\mathbb{Z}) \simeq \Lambda^2 H^*$. Specifically, let $(\alpha,\beta) := (\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g)$ be a symplectic basis of H, which means that

$$\omega(\alpha_i,\alpha_j)=\omega(\beta_i,\beta_j)=0 \quad \text{and} \quad \omega(\alpha_i,\beta_j) = -\omega(\beta_j,\alpha_i)=\delta_{ij}. \tag{4.18}$$

Then
$$\omega = \sum_{i=1}^g \alpha_i^*\wedge\beta_i^* \ \in \Lambda^2 H^*$$
where $(\alpha_1^*,\dots,\alpha_g^*,\beta_1^*,\dots,\beta_g^*)$ denotes the basis of H^* dual to the basis $(\alpha_1,\dots,\alpha_g,\beta_1,\dots,\beta_g)$ of H. It follows from (4.18) that the isomorphism $w: H \to H^*$ defined by $h\mapsto \omega(h,-)$ sends α_i to β_i^* and β_i to $-\alpha_i^*$. Therefore
$$\omega' = \sum_{i=1}^g (-\beta_i)\wedge\alpha_i = \sum_{i=1}^g \alpha_i\wedge\beta_i.$$
Let $\psi\in\mathrm{Aut}(H)$ and denote by
$$M = \left(\begin{array}{c|c} C & E \\ \hline D & F \end{array}\right)$$
the matrix of ψ in the basis (α,β). Then
$$\begin{aligned}
&(\Lambda^2\psi)(\omega')\\
=\ & \sum_k \psi(\alpha_k)\wedge\psi(\beta_k)\\
=\ & \sum_{k,i,j} (c_{ik}\alpha_i + d_{ik}\beta_i)\wedge(e_{jk}\alpha_j + f_{jk}\beta_j)\\
=\ & \sum_{k,i,j} c_{ik}e_{jk}\alpha_i\wedge\alpha_j + \sum_{k,i,j} d_{ik}f_{jk}\beta_i\wedge\beta_j + \sum_{k,i,j}(c_{ik}f_{jk}-d_{jk}e_{ik})\alpha_i\wedge\beta_j\\
=\ & \sum_k\sum_{i<j}(c_{ik}e_{jk}-c_{jk}e_{ik})\alpha_i\wedge\alpha_j + \sum_k\sum_{i<j}(d_{ik}f_{jk}-d_{jk}f_{ik})\beta_i\wedge\beta_j + \sum_{k,i,j}(c_{ik}f_{jk}-d_{jk}e_{ik})\alpha_i\wedge\beta_j.
\end{aligned}$$
We deduce that
$$\begin{aligned}
(\Lambda^2\psi)(\omega')=\omega' \ &\Longleftrightarrow\ \big(CE^t - EC^t = 0, DF^t - FD^t = 0, CF^t - ED^t = I_g\big)\\
&\Longleftrightarrow\ M\,\Omega\,M^t = \Omega\\
&\Longleftrightarrow\ \Omega = M^{-1}\,\Omega\,(M^t)^{-1}\\
&\Longleftrightarrow\ \Omega^{-1} = M^t\,\Omega^{-1}\,M\\
&\Longleftrightarrow\ -\Omega = M^t\,(-\Omega)\,M \ \Longleftrightarrow\ M\in\mathrm{Sp}(2g;\mathbb{Z}) \ \Longleftrightarrow\ \psi\in\mathrm{Sp}(H).
\end{aligned}$$
(b) Let $\psi\in\mathrm{Sp}(H)$. We deduce from (a) that
$$\begin{aligned}
\Lambda^{2g}H \ni (\Lambda^{2n}\psi)(\omega'\wedge\cdots\wedge\omega') &= (\Lambda^2\psi)(\omega')\wedge\cdots\wedge(\Lambda^2\psi)(\omega')\\
&= \omega'\wedge\cdots\wedge\omega'
\end{aligned}$$
Besides, using a symplectic basis (α,β) of H, we have
$$\begin{aligned}
\Lambda^{2g}H \ni \omega'\wedge\cdots\wedge\omega' &= \sum_{i_1,\dots,i_g=1}^g \alpha_{i_1}\wedge\beta_{i_1}\wedge\cdots\wedge\alpha_{i_g}\wedge\beta_{i_g}\\
&= \sum_{\sigma\in\mathfrak{S}_n} \alpha_{\sigma(1)}\wedge\beta_{\sigma(1)}\wedge\cdots\wedge\alpha_{\sigma(g)}\wedge\beta_{\sigma(g)}\\
&= n!\cdot\alpha_1\wedge\beta_1\wedge\alpha_2\wedge\beta_2\wedge\cdots\wedge\alpha_g\wedge\beta_g.
\end{aligned}$$
We deduce that
$$\begin{aligned}
\Lambda^{2g}H \ni \omega'\wedge\cdots\wedge\omega' &= (\Lambda^{2n}\psi)(\omega'\wedge\cdots\wedge\omega')\\
&= n!\cdot\psi(\alpha_1)\wedge\psi(\beta_1)\wedge\psi(\alpha_2)\wedge\psi(\beta_2)\wedge\cdots\wedge\psi(\alpha_g)\wedge\psi(\beta_g)
\end{aligned}$$

$$
\begin{aligned}
&= n!\det(\psi)\cdot\alpha_1\wedge\beta_1\wedge\alpha_2\wedge\beta_2\wedge\cdots\wedge\alpha_g\wedge\beta_g\\
&= \det(\psi)\cdot\omega'\wedge\cdots\wedge\omega'
\end{aligned}
$$

which implies that $\det(\psi)=1$.

Solution to Exercise 4.3. It is clear that $\tau_\delta\in\mathcal{I}(\Sigma)$ since the group $H_1(\Sigma;\mathbb{Z})$ is generated by classes of oriented curves in Σ and any such curve can be made disjoint from δ by a homotopy. We also know from Proposition 2.7 that τ_δ has infinite order. Therefore, it suffices to show that $\mathcal{I}(\Sigma)$ is generated by τ_δ. For this, we use Birman's exact sequence (see Proposition 2.9):

$$\pi_1(\mathrm{U}(\Sigma^+))\xrightarrow{\text{Push}}\mathcal{M}(\Sigma)\xrightarrow{\cup\,\mathrm{id}_D}\mathcal{M}(\Sigma^+)\longrightarrow 1.$$

The subgroup $\mathcal{I}(\Sigma)$ of $\mathcal{M}(\Sigma)$ is mapped by "$\cup\,\mathrm{id}_D$" to $\mathcal{I}(\Sigma^+)=\{1\}$ since Σ^+ is a torus. Therefore, $\mathcal{I}(\Sigma)$ is contained in the image of "Push", which we denote by I. Let α and β be simple oriented closed curves in Σ^+ which meet in a single point belonging to $\mathrm{int}(D)$. Since $\pi_1(\Sigma^+)$ is generated by $[\alpha]$ and $[\beta]$, the group $\pi_1(\mathrm{U}(\Sigma^+))$ is generated by $[\vec{\alpha}]$, $[\vec{\beta}]$ and the class f of the fiber $\mathrm{U}(1)\cong S^1$. If we look back at the proof of Proposition 2.9, we see that

$$\text{Push}([\vec{\alpha}])\overset{(2.1)}{=}\tau_{\alpha_-}^{-1}\tau_{\alpha_+}=1$$

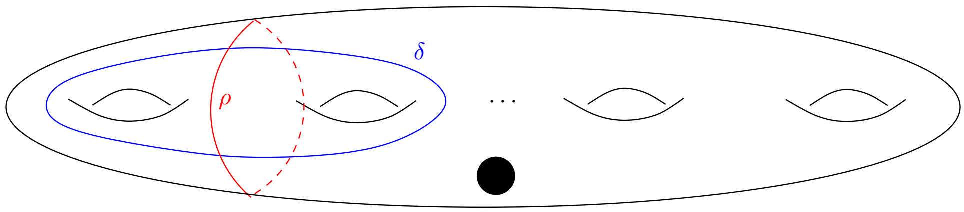
since α_- is isotopic to α_+ in Σ ; the same phenomenon happens for β ; finally it follows easily from the definition of "Push" that $\text{Push}(f)=\tau_\delta$. We conclude that $\langle\tau_\delta\rangle\subset\mathcal{I}(\Sigma)\subset I\subset\langle\tau_\delta\rangle$ so that $\langle\tau_\delta\rangle=\mathcal{I}(\Sigma)$.

Solution to Exercise 4.4. Set $H:=H_1(\Sigma;\mathbb{Z})$ and let $d,r:H\to H$ be the actions of τ_δ,τ_ρ in homology. It follows from Exercise 2.6 that, for any $x\in H$,

$$
\begin{aligned}
dr(x) &= d\big(x+\omega([\rho],x)\cdot[\rho]\big)\\
&= d(x)+\omega([\rho],x)\cdot d([\rho])\\
&= \big(x+\omega([\delta],x)\cdot[\delta]\big)+\omega([\rho],x)\cdot([\rho]+0\cdot[\delta]) \;=\; x+\omega([\delta],x)\cdot[\delta]+\omega([\rho],x)\cdot[\rho].
\end{aligned}
$$

A similar computation for $rd(x)$ gives the same result. Hence $[d,r]=1\in\mathrm{Aut}(H)$ and it follows that $[\tau_\delta,\tau_\rho]$ belongs to the Torelli group.

Let δ,ρ be the following simple closed curves in Σ:



It follows from Lemma 1.11 that $i(\delta,\rho)=2$ since δ and ρ do not delimitate bigons: therefore τ_δ and τ_ρ generate a free subgroup of $\mathcal{M}(\Sigma)$ of rank 2, and in particular $[\tau_\delta,\tau_\rho]\neq 1\in\mathcal{M}(\Sigma)$. Besides $\omega([\delta],[\rho])=0$ since the two intersection points of δ and ρ have opposite signs. We conclude that $[\tau_\delta,\tau_\rho]$ is a non-trivial element of $\mathcal{I}(\Sigma)$.

Solution to Exercise 4.5. We only consider (b) since (a) can be solved by the same kind of arguments. Denote by $g\geq 0$ the genus of Σ and by $n\in\{0,1\}$ the number of components of $\partial\Sigma$. Let $\rho,\delta\subset\Sigma$ and let $\rho',\delta'\subset\Sigma$ be simple closed curves defining some BP maps

$$p:=\tau_\delta\tau_\rho^{-1},\quad p':=\tau_{\delta'}\tau_{\rho'}^{-1}$$

of genus $k\geq 1$. Let $S_1\cong\Sigma_{k,2}$ and $S_2\cong\Sigma_{g-k,2+n}$ be the subsurfaces of Σ delimited by the curves δ and ρ; let S_1' and S_2' play the same role for δ' and ρ'. Since $S_1\cong S_1'$ and $S_2\cong S_2'$, we easily construct an $f\in\mathrm{Homeo}^{+,\partial}(\Sigma)$ mapping S_i to S_i' for each $i\in\{1,2\}$, and such that $f(\delta)=\delta'$ and $f(\rho)=\rho'$. We deduce that

$$p'=\tau_{f(\delta)}\tau_{f(\rho)}^{-1}=\big(f\tau_\delta f^{-1}\big)\big(f\tau_\rho f^{-1}\big)^{-1}=f\tau_\delta\tau_\rho^{-1}f^{-1}=fpf^{-1}.$$

Solution to Exercise 4.6. Let Σ be an oriented surface with at most one boundary component. Let $\rho,\delta\subset\Sigma$ be simple closed curves defining a BP map $p:=\tau_\delta\tau_\rho^{-1}$ of genus $k\geq 2$. Thus ρ and δ delimitate in Σ a subsurface $S\cong\Sigma_{k,2}$. We can decompose S into k subsurfaces of genus 1

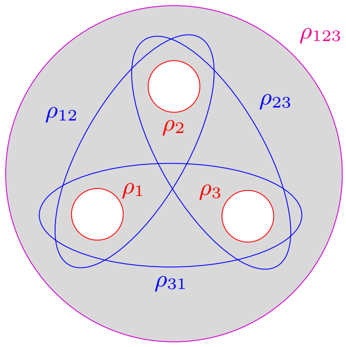
$$S=S_1\cup S_2\cup\cdots\cup S_{k-1}\cup S_k$$

in such a way that $\partial S_1=\rho\cup\epsilon_1$, $\partial S_2=\epsilon_1\cup\epsilon_2$, $\ldots$, $\partial S_{k-1}=\epsilon_{k-2}\cup\epsilon_{k-1}$, $\partial S_k=\epsilon_{k-1}\cup\delta$ where $\epsilon_1,\ldots,\epsilon_{k-1}\subset\mathrm{int}(S)$ are pairwise-disjoint simple closed curves. We deduce that

$$p=\tau_\delta\tau_\rho^{-1}=(\tau_\delta\tau_{\epsilon_1}^{-1})(\tau_{\epsilon_1}\tau_{\epsilon_2}^{-1})\cdots(\tau_{\epsilon_k}\tau_{\epsilon_{k-1}}^{-1})(\tau_{\epsilon_{k-1}}\tau_\rho^{-1})$$

is a product of k BP maps of genus 1.

Solution to Exercise 4.7. Let Σ be an oriented surface with at most one boundary component and consider a BSCC map $d := \tau_\delta \in \mathcal{M}(\Sigma)$ of genus $k \geq 3$. Thus δ is a simple closed curve bounding a subsurface $S \cong \Sigma_{k,1}$ of Σ. We can embed in S the disk with 3 holes



in such a way that we have $\rho_{123} = \delta = \partial S$, each of ρ_1 and ρ_2 bounds a subsurface of genus 1, and ρ_3 bounds a subsurface of genus $k-2$. By the lantern relation, we have

$$\tau_{\rho_{123}} = \tau_{\rho_1}^{-1}\tau_{\rho_2}^{-1}\tau_{\rho_3}^{-1}\tau_{\rho_{31}}\tau_{\rho_{23}}\tau_{\rho_{12}}.$$

This shows that the initial BSCC map d is a product of 6 BSCC maps of genus 1, 2, $k-2$ and $k-1$. Therefore, we can conclude by an induction on $k \geq 3$.

Solution to Exercise 4.8. Let $\Theta : \mathrm{Der}^+\big(\mathfrak{L}(H)\big) \to H \otimes \mathfrak{L}_{\geq 2}(H)$ be the isomorphism under study, and let $(\alpha_1, \beta_1, \dots, \alpha_g, \beta_g)$ be a symplectic basis of H. For any $\delta \in \mathrm{Der}^+\big(\mathfrak{L}(H)\big)$, we compute $\Theta(\delta)$:

$$\delta \longmapsto \delta|_H \longmapsto \sum_{i=1}^{g} \alpha_i^* \otimes \delta(\alpha_i) + \sum_{i=1}^{g} \beta_i^* \otimes \delta(\beta_i) \longmapsto -\sum_{i=1}^{g} \beta_i \otimes \delta(\alpha_i) + \sum_{i=1}^{g} \alpha_i \otimes \delta(\beta_i)$$

Hence the Lie bracket of $\Theta(\delta)$ is

$$-\sum_{i=1}^{g} \big[\beta_i, \delta(\alpha_i)\big] + \sum_{i=1}^{g} \big[\alpha_i, \delta(\beta_i)\big] = \sum_{i=1}^{g} \big[\delta(\alpha_i), \beta_i\big] + \sum_{i=1}^{g} \big[\alpha_i, \delta(\beta_i)\big] = \delta\Big(\sum_{i=1}^{g} [\alpha_i, \beta_i]\Big) = \delta(\omega')$$

where $\omega' \in \Lambda^2 H \simeq \mathfrak{L}_2(H)$ is the bivector dual to ω. We deduce that the Lie bracket of $\Theta(\delta)$ is trivial if and only if δ is a symplectic derivation.

We now show that Θ is $\mathrm{Sp}(H)$-equivariant. Since Θ is defined as a composition $\Theta_3\Theta_2\Theta_1$ of three isomorphisms, it suffices to verify that each of them is $\mathrm{Sp}(H)$-equivariant. The fact that the restriction homomorphism $\Theta_1 : \mathrm{Der}^+\big(\mathfrak{L}(H)\big) \to \mathrm{Hom}(H, \mathfrak{L}_{\geq 2}(H))$ is $\mathrm{Sp}(H)$-equivariant is obvious, if $\mathrm{Sp}(H)$ acts on $\mathrm{Hom}(H, \mathfrak{L}_{\geq 2}(H))$ by $M \cdot f := \big(M \cdot (-)\big) \circ f \circ M^{-1}$. Also, the fact that $\Theta_2 : \mathrm{Hom}(H, \mathfrak{L}_{\geq 2}(H)) \to H^* \otimes \mathfrak{L}_{\geq 2}(H)$ is $\mathrm{Sp}(H)$-equivariant is obvious, if $\mathrm{Sp}(H)$ acts on $H^* \otimes \mathfrak{L}_{\geq 2}(H)$ by $M \cdot (u \otimes v) := (u \circ M^{-1}) \otimes (M \cdot v)$. To justify now that $\Theta_3 : H^* \otimes \mathfrak{L}_{\geq 2}(H) \to H \otimes \mathfrak{L}_{\geq 2}(H)$ is an $\mathrm{Sp}(H)$-equivariant isomorphism, it remains to prove that $w : H \to H^*$ is $\mathrm{Sp}(H)$-equivariant. This is checked as follows:

$$\forall M \in \mathrm{Sp}(H), \forall h \in H, \quad w(M \cdot h) = w(M(h)) = \omega(M(h), -) = \omega(h, M^{-1}(-)) = M \cdot w(h).$$

Solution to Exercise 4.9. Let Σ be an oriented surface of genus $g \geq 2$ with one boundary component, and let $\delta \subset \Sigma$ be a simple closed curve bounding a subsurface $S \subset \Sigma$. We give S the orientation induced by Σ and we give δ the orientation induced by S. We pick a base point $\star \in \partial\Sigma$ and set $\pi := \pi_1(\Sigma, \star)$. We are asked to show that

$$\tau_1(\tau_\delta) = 0 \in \mathrm{Hom}(H, \mathfrak{L}_2(H)) \quad \text{or, equivalently,} \quad \rho_2(\tau_\delta) = 1 \in \mathrm{Aut}(\pi/\Gamma_3\pi).$$

Let $x \in \pi$ and let ξ be an oriented closed curve based at $\star$ representing x. We can assume that δ and ξ are transversal and that $\delta \cap \xi$ consists of $2n$ double points. (There is an even number of intersection points since δ is null-homologous.) We number these intersection points $p_1, \dots, p_{2n}$ in the order that they are encountered as one runs along ξ in the positive direction. In the computation below, we use the following notations: for any oriented closed curve γ and for any simple points $p, q \in \gamma$, we denote by γ_{pq} the path where one runs along γ from p to q in the positive direction; the concatenation of paths is denoted by $*$; when one runs along a path γ in the negative direction, it is denoted by $\overline{\gamma}$. Then, with these notations, we have

$$\begin{aligned} & (\tau_\delta)_\sharp(x) \cdot x^{-1} \\ = \; & [\xi_{\star p_1} * \delta_{p_1p_1} * \xi_{p_1p_2} * \overline{\delta}_{p_2p_2} * \cdots * \delta_{p_{2n-1}p_{2n-1}} * \xi_{p_{2n-1}p_{2n}} * \overline{\delta}_{p_{2n}p_{2n}} * \xi_{p_{2n}\star}] \cdot [\overline{\xi}_{\star p_{2n}} \overline{\xi}_{p_{2n}\star}] \end{aligned}$$

$$
\begin{aligned}
&= \prod_{i=1}^{n}[\xi_{\star p_{2i-1}} * \delta_{p_{2i-1}p_{2i-1}} * \xi_{p_{2i-1}p_{2i}} * \overline{\delta}_{p_{2i}p_{2i}} * \overline{\xi}_{p_{2i}\star}] \\
&= \prod_{i=1}^{n}[\xi_{\star p_{2i-1}} * \delta_{p_{2i-1}p_{2i-1}} * \xi_{p_{2i-1}p_{2i}} * \delta_{p_{2i}p_{2i-1}} * \overline{\delta}_{p_{2i-1}p_{2i}} * \overline{\delta}_{p_{2i}p_{2i-1}} * \overline{\delta}_{p_{2i-1}p_{2i}} * \overline{\xi}_{p_{2i}\star}] \\
&= \prod_{i=1}^{n}[\xi_{\star p_{2i-1}} * \delta_{p_{2i-1}p_{2i-1}} * \xi_{p_{2i-1}p_{2i}} * \delta_{p_{2i}p_{2i-1}} * \overline{\delta}_{p_{2i-1}p_{2i-1}} * \overline{\delta}_{p_{2i-1}p_{2i}} * \overline{\xi}_{p_{2i}\star}] \\
&= \prod_{i=1}^{n}[\xi_{\star p_{2i-1}} * \delta_{p_{2i-1}p_{2i-1}} * \overline{\xi}_{p_{2i-1}\star} * \xi_{\star p_{2i-1}} * \xi_{p_{2i-1}p_{2i}} * \delta_{p_{2i}p_{2i-1}} * \overline{\delta}_{p_{2i-1}p_{2i-1}} * \overline{\delta}_{p_{2i-1}p_{2i}} * \overline{\xi}_{p_{2i}\star}] \\
&= \prod_{i=1}^{n} y_i \cdot [\xi_{\star p_{2i}} * \delta_{p_{2i}p_{2i-1}} * \overline{\delta}_{p_{2i-1}p_{2i-1}} * \overline{\delta}_{p_{2i-1}p_{2i}} * \overline{\xi}_{p_{2i}\star}] \\
&= \prod_{i=1}^{n} y_i \cdot [\xi_{\star p_{2i}} * \delta_{p_{2i}p_{2i-1}} * \overline{\xi}_{p_{2i-1}\star} * \xi_{\star p_{2i-1}} * \overline{\delta}_{p_{2i-1}p_{2i-1}} * \overline{\xi}_{p_{2i-1}\star} * \xi_{\star p_{2i-1}} * \overline{\delta}_{p_{2i-1}p_{2i}} * \overline{\xi}_{p_{2i}\star}] \\
&= \prod_{i=1}^{n} y_i \cdot z_i \cdot y_i^{-1} \cdot z_i^{-1}
\end{aligned}
$$

where $y_i := [\xi_{\star p_{2i-1}} * \delta_{p_{2i-1}p_{2i-1}} * \overline{\xi}_{p_{2i-1}\star}]$ and $z_i := [\xi_{\star p_{2i}} * \delta_{p_{2i}p_{2i-1}} * \overline{\xi}_{p_{2i-1}\star}]$. Since δ is null-homologous, y_i is a commutator in π and the above computation shows that $(\tau_\delta)_\sharp(x) \cdot x^{-1} \in \Gamma_3\pi$. We deduce that $(\tau_\delta)_\sharp \in \mathrm{Aut}(\pi)$ is the identity modulo $\Gamma_3\pi$.

N.B. In fact, Johnson proved that $\ker \tau_1 = J_2\mathcal{I}(\Sigma)$ is generated by BSCC maps [Joh85b]. It is not known whether this group is finitely generated.

Appendix A. Fibrations

The concept of "fibration" is a basic notion in homotopy theory. In this appendix, refering to the textbooks [Bre93, Hat02] for further details, we only review the definition and the behaviour of fibrations with respect to homotopy groups.

Definition A.1. *A map $f : E \to B$ is a* fibration[9] *if it has the* homotopy lifting property *with respect to any pair of CW-complexes (X, Y). Thus, for any homotopy $u : X \times I \to B$, for every lift $w : Y \times I \to E$ of $u|_{Y\times I}$ and for every lift $u_0 : X \to E$ of $u(-, 0)$ such that $w(-, 0) = u_0|_Y$, there is a lift $\widetilde{u} : X \times I \to E$ of u such that $\widetilde{u}(-, 0) = u_0$ and $\widetilde{u}|_{Y\times I} = w$:*

$$\begin{array}{ccc} X \times \{0\} \cup Y \times I & \xrightarrow{u_0 \cup w} & E \\ \big\downarrow & \overset{\widetilde{u}}{\nearrow} & \big\downarrow f \\ X \times I & \xrightarrow[u]{} & B. \end{array} \tag{A.1}$$

It turns out that f is a fibration if and only if it has the homotopy lifting property with respect to $(X, Y) = (D^n, \partial D^n)$ for all $n \geq 0$ (since CW-complexes are constructed by attachement of disks along their boundaries). Furthermore, f is a fibration if and only if it has the homotopy lifting property with respect to $(X, Y) = (D^n, \varnothing)$ for all $n \geq 0$ (since the pair $(D^n \times I, D^n \times \{0\} \cup \partial D^n \times I)$ is homeomorphic to the pair $(D^n \times I, D^n \times \{0\})$ as it is easily checked).

For example, if $E = B \times F$ and if $f : E \to B$ is the cartesian projection, then f is clearly a fibration. More generally, we have the following notion.

Definition A.2. *A map $f : E \to B$ is a* fiber bundle *with* fiber *F if, for all $b \in B$, there is a neighborhood $U \ni b$ and a homeomorphism $h : U \times F \to f^{-1}(U)$ such that $f \circ h : U \times F \to U$ is the cartesian projection.*

Thus, a fiber bundle $f : E \to B$ with fiber F is locally "shaped" as the cartesian projection $B \times F \to B$. It follows that fiber bundles are fibrations. For instance, covering maps are fiber bundles with the peculiarity that their fiber is discrete.

Fibrations behave very well with respect to homotopy groups. Specifically, they have the following property.

Theorem A.3. *Let $f : E \to B$ be a fibration, fix some base-points $e_0 \in E$, $b_0 \in B$ such that $f(e_0) = b_0$. We set $F := f^{-1}(b_0)$ and denote by $i : F \to E$ the inclusion. Then we have a long exact sequence*

$$\cdots \to \pi_n(F) \xrightarrow{i_\sharp} \pi_n(E) \xrightarrow{f_\sharp} \pi_n(B) \xrightarrow{\partial_\sharp} \pi_{n-1}(F) \to \cdots \to \pi_0(F) \xrightarrow{i_\sharp} \pi_0(E) \xrightarrow{f_\sharp} \pi_0(B)$$

where the homotopy groups of F, E, B are based at e_0, e_0, b_0 respectively.

We refer to [Bre93, Hat02] for the proof and the definition of the "connecting" homomorphism $\partial_\sharp : \pi_n(B) \to \pi_{n-1}(F)$. In these notes, we only need the definition for $n = 1$. Then the homomorphism $\partial_\sharp : \pi_1(B, b_0) \to \pi_0(F, e_0)$ is defined by

$$\partial_\sharp([u]) = \big[\widetilde{u}(1)\big]$$

for any path $u : [0, 1] \to B$ such that $u(0) = u(1) = b_0$ and where $\widetilde{u} : [0, 1] \to E$ is a lift of u such that $\widetilde{u}(0) = e_0$. (The existence of $\widetilde{u}$ is ensured by (A.1) with X a singleton and $Y := \varnothing$.)

[9] Fibrations in our sense are sometimes called "Serre fibrations" in the litterature.

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