CONFIGURATION SPACES

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§1. Introduction.

This a a talk about configuration spaces and as such is not directly relevant to the main theme of the conference. However it should illustrate the kind of topological reasoning which lies behind some of the results mentioned in Segal's talk on algebraic K-theory. In particular I will sketch a proof of the Barratt-Quillen -Priddy theorem that $B\Sigma_\infty$ is homology isomorphic to $(\Omega^\infty S^\infty)_0$, and also give a simpler formulation of the Atiyah-Singer proof of the Bott periodicity theorem in [1], incidentally removing from it all the analysis.

The configuration space C(M) of a smooth manifold with boundary M is the set of finite subsets of M (think of a finite subset of M as a configuration of particles on M) topologised so that the particles cannot collide. Thus C(M) is the disjoint union $\coprod_{k\geq 0} C_k(M)$, where $C_k(M)$ is the configurations with k particles, i.e. the quotient of the ordered configuration space $OC_k(M) = \{(m_1,\ldots,m_k): m_i \in M, m_i \neq m_j \text{ if } i \neq j\}$ by the action of the symmetric group Σ_k. The configuration space $C_k(\mathbb{R}^n)$ is an approximation to the classifying space $B\Sigma_k$. For $OC_k(\mathbb{R}^n)$ is just $\mathbb{R}^{nk}$ with certain hyperplanes of codimensionnn removed, so its homotopy groups vanish up to dimension n-2. Thus $\lim_{n\to\infty} OC_k(\mathbb{R}^n)$, where the limit is taken with respect to the usual inclusions $\mathbb{R}^n \longrightarrow \mathbb{R}^{n+1}$, is a contractible space on which Σ_k acts freely. The quotient $\lim_{n\to\infty} C_k(\mathbb{R}^n)$ is therefore homotopic to $B\Sigma_k$.

Now, any finite subset $s \subseteq \mathbb{R}^n$ gives rise to a map $\varphi(s): \mathbb{R}^n \longrightarrow \mathbb{R}^n \cup \{\infty\}$. For let $\{\delta_x\}_{x\in s}$ be a collection of disjoint open discs with centres at the points x of s and with radius $\epsilon(x)$ (choose $\epsilon(s) > 0$ so that it is as large as possible but ≤ 1 say). Then define $\varphi(s)$ by:

$$\varphi(x)y = \infty \qquad \text{if } y \text{ is not in any } \delta_x,$$

$$\varphi(s)y = (\epsilon(x)-\|y-x\|)^{-1}(y-x) \text{ if } y \text{ is in } \delta_x.$$

This $\varphi(s)$ actually has compact support (in the sense that $\varphi^{-1}(\mathbb{R}^n)$ has compact closure), so φ maps $C_k(\mathbb{R}^n)$ to $\mathrm{Map}_k(\mathbb{R}^n, \mathbb{R}^n \cup \{\infty\})$, the space of maps $\mathbb{R}^n \longrightarrow \mathbb{R}^n \cup \{\infty\}$ with compact support and degree k. Notice that the spaces $\mathrm{Map}_k(\mathbb{R}^n, \mathbb{R}^n \cup \{\infty\})$ are homotopic for different k, and, in fact, a e all homotopic to $(\Omega^n S^n)_0$, the space of basepoint preserving maps with degree 0 from the n-sphere S^n to itself (identifying S^n with $\mathbb{R}^n \cup \{\infty\}$). Also, we can form $\lim_{k\to\infty} C_k(\mathbb{R}^n)$ by mapping $C_k(\mathbb{R}^n)$ to $C_{k+1}(\mathbb{R}^n)$ by adding a particle from infinity in a standard way. Altogether we get a map $\varphi: \lim_{k\to\infty} C_k(\mathbb{R}^n) \longrightarrow (\Omega^n S^n)_0$, and this, by a theorem of Segal [6], induces an isomorphism on integral homology. Taking the limit now over n gives the Barratt-Quillen-Priddy theorem that $\lim_{n,k\to\infty} C_k(\mathbb{R}^n) \simeq \lim_{k\to\infty} B\Sigma_k \longrightarrow (\Omega^\infty S^\infty)_0$ is a homology isomorphism, or equivalently that $(B\Sigma_\infty)^{ab} \longrightarrow (\Omega^\infty S^\infty)_0$ is a homotopy equivalence, where ab denotes Quillen's construction abelianising the fundamental group.

Segal's theorem is a special case of a more general theorem valid for any connected manifold M. Let $\Gamma_k(M)$ be the space of possibly infinite, compactly supported vector fields on M, that is, the space of sections of the fibre bundle over M whose fibre at $x \in M$ is the sphere $T_x \cup \{\infty\}$ obtained by compactifying the tangent space T_x at x. Then, as before, there is a map $\varphi: C_k(M) \longrightarrow \Gamma_k(M)$. Because it is always possible to move the particles away from the boundary ∂M of M, φ may be defined so that it takes values in $\Gamma_k(M, \partial M)$, the sections which equal ∞ on ∂M.

Theorem 1. $\varphi: C_k(M) \longrightarrow \Gamma_k(M, \partial M)$ induces an isomorphism on integral homology groups up to a dimension tending to ∞ with k.

Taking the limit as $k \longrightarrow \infty$ gives the previous result when $M = \mathbb{R}^n$. Notice that because homology commutes with direct limits it is enough to prove this for compact M. Also, the result for $\mathbb{R}^n$ follows immediately from that for D^n, where D^n is the unit disc in $\mathbb{R}^n$, since $C(\mathbb{R}^n)$ is homotopic to $C(D^n)$. Therefore from now on we will consider only compact manifolds.

§2. Gromov's method. Theorem 1 is proved by a very general method which was first used systematically by Gromov [3]. Let $\mathfrak{M}$ be the category of smooth, compact manifolds of a fixed dimension n with embeddings as morphisms, and suppose that F is a contravariant functor from $\mathfrak{M}$ to (spaces). Associated to F there is a fibre bundle $E_F(M)$ on each $M \in \mathrm{Obj}(\mathfrak{M})$ which has fibre $F(D_x)$ at the point x of M. (Here M is supposed to have a Riemannian metric, and D_x is the unit disci n the tangent space to M at x, so that $E_F(M)$ is associated to the tangent bundle on M.) The exponential map gives an embedding $\exp_x: D_x \longrightarrow M$ for each x, and one defines $\varphi: F(M) \longrightarrow \Gamma_F(M)$ (where $\Gamma_F(M)$ is the space of continuous sections of $E_F(M)$) by $\varphi(f)(x) = F(\exp_x)f$, for f in F(M). Then we have

Theorem 2. Suppose that

(i) for all embeddings $j: D^n \longrightarrow D^n$, $F(j): F(D^n) \longrightarrow F(D^n)$ is a homotopy equivalence;

(ii) for any square
$$\begin{array}{ccc} M_1 \cup M_2 & \longleftarrow & M_1 \\ \uparrow & & \uparrow \\ M_2 & \longleftarrow & M_1 \cap M_2 \end{array}$$
in $\mathfrak{M}$ the corresponding square
$$\begin{array}{ccc} F(M_1 \cup M_2) & \longrightarrow & F(M_1) \\ \downarrow & & \downarrow \\ F(M_2) & \longrightarrow & F(M_1 \cap M_2) \end{array}$$
is homotopy Cartesian.

Then $\varphi: F(M) \longrightarrow \Gamma_F(M)$ is a homotopy equivalence for all $M \in \mathrm{Obj}(\mathfrak{M})$. (A square
$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Z & \xrightarrow{g} & Y \end{array}$$
is called Cartesian if W is the fibre product $\{(x,y); f(x) = g(z)\}$ of X and Z over Y. Similarly it is called homotopy Cartesian if W is homotopic to the homotopy-theoretic fibre product of X and Z over Y, that is if $W = \{(x,z,\gamma): \gamma$ is a path in Y with $\gamma(0) = f(x)$, $\gamma(1) = g(z)\}$.)

The proof of this theorem is trivial. For it follows immediately from (i) that it holds when $M = D^n$. The general case follows by induction over the number of discs in a finite covering of M, because condition (ii), which holds also for the functor Γ_M, implies that if φ is an equivalence for $M_1 \cap M_2$, M_1 and M_2 it is one for $M_1 \cup M_2$ too.

There is another version of this theorem where the square in (ii) satisfy the condition that no component of $\partial(M_1 \cap M_2)$ lies entirely in the interior of M_1. In this case one argues by induction over the number of handles in a handle decomposition of M, where the handles have index < dim(M), and proves that φ is an equivalence for all open M, i.e. for M for which $M - \partial M$ has no compact component.

The interest of this theorem lies in the fact that there are many functors F satisfying these conditions. An example of the kind considered by Gromov in his thesis [3] is the functor which assigns to each manifold the space of all its symplectic structures, and the theorem yields an existence statement: and open manifold M of dimension 2k has a symplectic structure iff the obvious algebraic condition is satisfied, namely, if there is some (not necessarily closed or smooth) 2-form ρ on M such that ρ^k has no zeroes.

To apply Gromov's theorem one needs a criterion for a square to be homotopy Cartesian. A well-known result is

Lemma. A Cartesian square
$$\begin{array}{ccc} W & \longrightarrow & X \\ & & \downarrow f \\ Z & \xrightarrow{g} & Y \end{array}$$
is homotopy Cartesian if f is a fibration.

Proof. We must show that the inclusion of $W = \{(x,z): f(x) = g(z)\}$ in $W' = \{(x,z,\gamma): \gamma$ is a path in Y with $\gamma(0) = f(x)$, $\gamma(1) = g(z)$ is a homotopy equivalence. However, in order to retract W' to W it is enough to lift the homotopy $W' \times I \longrightarrow Y: ((x,z,\gamma),t) \longrightarrow \gamma(t)$ to X with initial lifting $((x,z,\gamma),0) \longrightarrow x$, and this may be done since f is a fibration.

In fact it suffices here that f be a quasifibration, that is a map such that for all $y \in Y$ the inclusion of the actual fibre $f^{-1}(y)$ at y into the homotopy fibre $F(y,f) = \{(x,\gamma):\gamma$ is a path in Y with $\gamma(0) = y$, $\gamma(1) = f(x)\}$ is an equivalence. For the purposes of homotopy theory these maps are just as good as fibrations, and for instance have a long exact homotopy sequence. They often arise in the following form. The base space Y is filtered by an increasing sequence of closed subspaces $Y_1 \subseteq Y_2 \subseteq Y_3 \cdots$ such that, over each difference $Y_k - Y_{k-1}$, f is the product fibration $(Y_k-Y_{k-1})\times F \longrightarrow (Y_k-Y_{k-1})$. (Notice that this means that all the fibres $f^{-1}(y)$ are homeomorphic to F.) Also for each k there is an open neighborhood $\mathcal{U}_k$ of Y_k in Y_{k+1} and a deformation retraction r_t of $\mathcal{U}_k$ to Y_k which may be lifted to a deformation retraction $\tilde{r}_t$ of $f^{-1}(\mathcal{U}_k)$ to $f^{-1}(Y_k)$. Since all the fibres may be identified with F the maps $\tilde{r}_1 : f^{-1}(y) \longrightarrow f^{-1}(r_1(y))$ give rise to a collection of maps $F \longrightarrow F$, called the attaching maps of f. If they are all homotopy equivalences f is a quasifibration (see [2]). If they are all homology equivalences f can be called a "homology fibration" (see [4,5]).

§3. The application to configuration spaces.

In order to apply all this to configuration spaces we must introduce a new functor of the correct variance, for $M \longrightarrow C(M)$ is obviously covariant. Thus we consider $\tilde{C}(M)$, the configuration space of particles on M which are annihilated and created on the boundary of M. More formally, $\tilde{C}(M)$ is the quotient of C(M) by the relation $s \sim s'$ iff $s \cap (M-\partial M) = s' \cap (M-\partial M)$. Clearly any embedding $N \longrightarrow M$ gives rise to a restriction map $\tilde{C}(M) \longrightarrow \tilde{C}(N)$: $s \longrightarrow s \cap N$, so that $\tilde{C}$ is a contravariant functor.

It is not difficult to see that $\tilde{C}(D^n)$ is homotopic to S^n. For, by expanding radially from the centre of D^n, it is possible to push all but at most one particle in each configuration out to the boundary where it vanishes. This retracts $\tilde{C}(D^n)$ to its subspace consisting of configurations of at most one particle, and this is just $D^n/\partial D^n \cong S^n$. This argument also shows that $\tilde{C}$ satisfies condition (i) of Theorem 2. Notice, too, that the fibre $\tilde{C}(D_x)$ of the bundle $E_{\tilde{C}}(M)$ is $D_x/\partial D_x \cong T_x \cup \{\infty\}$, so that the elements of $\Gamma_{\tilde{C}}(M)$ may be considered to be possibly infinite vector fields on M.

Now consider the restriction map $r : \tilde{C}(M) \longrightarrow \tilde{C}(N)$ induced by an inclusion $N \longrightarrow M$. Filter $\tilde{C}(N)$ by the sets $\tilde{C}_k$ consisting of $\leq k$ particles. Then $r^{-1}(\tilde{C}_k - \tilde{C}_{k-1})$ is just the product $(\tilde{C}_k - \tilde{C}_{k-1}) \times F$, where F is the space of configurations in the closure $\overline{M-N}$ of M-N which are annihilated on $\partial(\overline{M-N}) \cap \partial M$ but not the rest of $\partial(\overline{M-N})$. We may choose $\mathcal{U}_k \subseteq \tilde{C}_{k+1}$ to consist of configurations with at least one particle near ∂N, so that $\mathcal{U}_k$ retracts to $\tilde{C}_k$ by pushing this particle out to ∂N. The attaching map on the fibre is then just the map $F \longrightarrow F$ which adds a particle to each configuration at some point m on $\partial(\overline{M-N}) \cap N$. Clearly this map will not be an equivalence if $\partial(\overline{M-N})$ lies entirely in the interior of M since F is then $C(\overline{M-N})$, a configuration space with no annihilations. However it is an equivalence if each component of ∂N meets ∂M, for then the added particle may be moved along near ∂N until it reaches ∂M where it disappears. (Notice that we can assume that all particles except the extra one have been cleared away from a neighborhood of $\partial N \cap (\overline{M-N})$ so that no collisions will occur.)

Thus the functor $\tilde{C}$ satisfies the (modified) condition (ii) in Theorem 2. It follows that $\varphi:\tilde{C}(M) \longrightarrow \Gamma_{\tilde{C}}(M)$ is an equivalence for all connected manifolds M with non-empty boundary. In order to obtain a theorem about C(M) one adds an annulus $\partial M \times I$ to M along ∂M and considers the commutative diagram

$$\begin{array}{ccc} \tilde{C}(M \cup_{\partial M} \partial M \times I) & \xrightarrow[\simeq]{\varphi} & \Gamma_{\tilde{C}}(M \cup_{\partial M} \partial M \times I) \\ \tilde{C}(\partial M \times I) & \xrightarrow[\simeq]{\varphi} & \Gamma_{\tilde{C}}(\partial M \times I) \end{array}$$

The actual fibres of the restriction maps r and R are $C(M) = \coprod_{k \geq 0} C_k(M)$ and $\Gamma_{\tilde{C}}(M, \partial M)$. They are not equivalent because, although

R is a fibration, as we saw above r is not a quasifibration. However, by "stablising the fibre with respect to the attaching maps" one alters r to a map $r':X \longrightarrow \tilde{C}(\partial M \times I)$ with fibre $Z \times \lim_{k\to\infty} C_k(M)$, where this limit is formed with respect to the attaching maps, that is the maps which add a particle to the configurations in $C_k(M)$ at some point m on ∂M. The attaching maps of r' are essentially the same as those of r, but now they are homology isomorphisms. This implies that r' is a homology fibration in the sense mentioned above. Thus φ induces a homology isomorphism between the fibres of r' and R. Theorem 1 now follows. The details of this argument may be found in [4].

§4. Bott periodicity.

I shall conclude by describing a variant of the Atiyah-Singer proof [1] of the Bott periodicity theorem to show how closely it is related to the preceding argument. Let U_∞ be the stable unitary group $\lim_{n\to\infty} U_n$. We shall construct a quasifibration $f:H \longrightarrow U_\infty$ with H contractible and fibre $Z \times BU_\infty$. This is enough to prove the complex Bott periodicity theorem. The real case can be treated similarly.

Let H_n be the $n \times n$ Hermitian matrices with all eigenvalues in $[0,1]$ (so H_n is linearly contractible) and define $f_n:H_n \longrightarrow U_n$ by $f_n(h) = \exp(2\ ih)$. The fibre $f_n^{-1}(u)$ of f_n at $u \in U_n$ may be identified with the grassmannian of all subspaces of ker(u-1) by the map $h \longrightarrow \ker(h-1) \subseteq \ker(u-1)$. The f_n are compatible with the usual inclusions $H_n \longrightarrow H_{n+1}$ and $U_n \longrightarrow U_{n+1}$ (given by $h \longrightarrow h\oplus 0$ and $u \longrightarrow u\oplus 1$) so that one has $f_\infty:H_\infty \longrightarrow U_\infty$ with all fibres homeomorphic to $\coprod_{n\geq 0} Gr_n(C^\infty) \cong \coprod_{n\geq 0} BU_n$. Filtering the base U_∞ by the U_n one sees that the attaching maps come from inclusions $BU_m \longrightarrow BU_{n+r}$. All one needs to do now is to stabilise so that the fibres become $Z \times BU_\infty$ and the attaching maps homotopy equivalences. To do this, take the standard $V = C^{\infty}$ with basis $\{e_i\}_{-S<i<\infty}$. Let V_k be the subspace spanned by e_i for $i < k$. Then $Z \times BU_{\infty}$ can be thought of as the space of all subspaces W of V such that $V_p \subset W \subset V_q$ for some $-\infty < p \leq q < \infty$. Define $k_o : V \longrightarrow V$ by

$$\begin{aligned} k_o(e_i) &= e_i && (i<0) \\ &= 0 && (i\geq 0), \end{aligned}$$

and let H be the Hermitian operators $V \longrightarrow V$ which have eigenvalues in $[0,1]$and have the form $k_O + k$, where k is represented with respect to the basis $\{e_i\}$ by a matrix with finitely many non-zero terms. Similarly let U be the unitary operators $V \longrightarrow V$ of the form $I + v$, where the matrix for v has finitely many non-zero terms. Define $f : H \longrightarrow U$ by $h \longrightarrow \exp(2\ ih)$. Then for each $u \in U$ $f^{-1}(u) \simeq Z \times BU_{\infty}$, where the identification is as above but with V replaced by its subspace $\ker(u-1)$. The attaching maps arise from inclusions $\ker(u-1) \longrightarrow \ker(u'-1)$. Thus they are equivalences, and f is a quasifibration.

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