Strings Moving on Group Manifolds: The WZW Model

Contents

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Although string theory, as derived from the Polyakov action is completely solvable, due to the fact that it represents a 2D theory of D free bosons, there are many more stringy systems where things are not so easy. In a general background one does not even know how to solve the equations of motions, due to the lack of symmetry of the action.

There is however a class of theories, where strings behave more or less as free strings although they are interpreted as strings moving on a curved manifold. These models are the Wess-Zumino-Witten models, and describe the propagation of strings on a group manifold.

It is intended to be readable for people who have some notion of general relativity, and have followed some introductory course on string theory. The last requirement will be easily met, as this piece is intended as a way to get a better note for such a course. Meaning that if you are reading this, the chance is big that you have followed C. Muñoz's introductory course on string theory. Anyway, it is recommended that one first reads the appendices A, B and C.

1 The Non-linear Sigma model

Here we'll give, as a kind of introduction, some explication on the non-linear Sigma model as used in string theory[1]. As is readily known, the Polyakov formulation of a bosonic string is given by

$$S_g \;=\; \int_{\Sigma_g} d^2\sigma\,\sqrt{-h}h^{\alpha\beta}\eta_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu \;. \tag{1}$$

Due to the reparametrization invariance and the Weyl invariance of this theory, this is equavalent to a 2D theory of D free bosons, and as such there is no problem in the quantization of the theory (See for example [5], part I). From the interpretation of the eigenmodes of the closed string, one knows that the massless spectrum contains a rank 2 symmetric tensor excitation, a rank 2 antisymmetric excitation and a scalar, the so called dilaton.

From the point of view from the Polyakov action, these are all modes that can be emitted and absorbed by the closed string and that appear from the Target point of view as massless particle. A legitimate question is whether we could take an arbitrary number of strings and have them interact with themselves and calculate the fields, as seen by a targetspace, low-energy observer. Up to the moment of writing, this is an undoable task, so that one has to invent some other things. One of the things one could do is to suppose that we have such fields and see whether we can have a test string moving in such a background. This then leads to the question: "How can we modify the Polyakov action as to incorporate the effect of the massless excitations?" The answer is found in the Non-linear Sigma model [4].

The NLS is defined by

$$S_{nls} \;=\; \int d^2\sigma\left[\sqrt{-h}h^{\alpha\beta}G(X)_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu \;+\; \varepsilon^{\alpha\beta}B(X)_{\mu\nu}\partial_\alpha X^\mu\partial_\beta X^\nu \;+\; \sqrt{-h}\Phi(X)R^{(2)}(h)\right] \;, \tag{2}$$

Some remarks about the form of this action are in order: First of all the G, B and Φ are just couplings: Although they depend on the X's, their explicit forms are determined on the forehand. Note that this doesn't mean that they are arbitrary, since the NLS has to give rise to a consistent quantum theory. Secondly, mentioned here just for the obvious clarity, is that

[1]Sigma models were first introduced by Gell-Mann and Lévy (Nuov. Cim. **16**(1960), 705) as a toy model in which one could study theories with chiral symmetries and partially conserved axial currents. The rationale for studying such a model lied in the axial anomaly, related to the π^0 lifetime calculations. See for example the discussion in [6].

the B term is not coupled to a $\sqrt{-h}$ since the ε is already a tensorial density of degree 1, just like $\sqrt{-h}$. The last of the remarks would have been about the strange way of coupling the dilaton to the curvature of the world sheet, had it not been for the fact that the reason lies outside the scope of this piece of paper (See [12]).

After gauge fixing the difeomorphism invariance of the theory, we end, classically, up with a Weyl invariant theory. We also need to have conformal invariance: We are dealing with a test string, which classically is conformal invariant. Calculating the β equation for the NLS at the first loop in string perturbation theory[2] and imposing that the theory is conformal invariant at this order, leads, apparently, to

$$0 \;=\; R + \frac{1}{12}H_{\mu\nu\kappa}H^{\mu\nu\kappa} \;+\; 4(\nabla\Phi)^2 \;-\; 4\nabla_\mu\partial^\mu\Phi \;+\; 2\frac{c-D}{3\alpha'}\;, \tag{3}$$

$$0 \;=\; R_{\mu\nu} \;+\; \frac{1}{4}H_{\mu\kappa\rho}H_\nu{}^{\kappa\rho} \;-\; 2\nabla_\mu\nabla_\nu\Phi\;, \tag{4}$$

$$0 \;=\; \nabla_\kappa\left(e^{-2\Phi}H^\kappa{}_{\mu\nu}\right)\;, \tag{5}$$

where $H = dB$ and all the connections and curvatures are defined as to be calculated using G as the metric.

Seeing the great similarity of Eqs. (3,4, 5) with the Einstein field equations for gravity coupled to matter, it will come as no surprise that one can derive them from an action, we take $c = D$ in order to make life easier,

$$S_d \;=\; \int d^dx\sqrt{-g}e^{-2\phi}\left(R(g) \;+\; \tfrac{1}{12}H^2 \;-\; 4(\partial\phi)^2\right)\;. \tag{6}$$

Using the results found in appendix A we can use the H, being a 3-form, as the generator of a torsion part for the connection, and rewrite the above action as

$$S_d \;=\; \int d^dx\sqrt{-g}e^{-2\phi}\left(R(\hat{\omega}^{\pm}) \;-\; 4(\partial\phi)^2\right)\;, \tag{7}$$

where

$$\hat{\omega}^{\pm} \;\equiv\; \omega \;\pm\; \tfrac{1}{2}H\;. \tag{8}$$

The case in which the dilaton vanishes, the case we will be interested in, one can see that Eqs. (3,4,5) can be written simply, using Eq.(71), as one equation

$$R(\hat{\omega}^{\pm})_{\mu\nu} \;=\; 0\;. \tag{9}$$

Having this result, and having read appendix B, one sees that one can use groups in order to solve the β-equations. How to do this, will be explained in the next section.

2 The Wess-Zumino-Witten Model

In the preceding section we saw that one can reformulate the β-equations in terms of a generalized connection. Now it is known that a group manifold is a paralizable manifold, i.e. there exists a connection on a group manifold such that the curvature is nil, meaning that

[2] It is not intended to give the derivation of these equations: The reader is kindly referred to the literature [4, 12].

one could imagine a consistent string propagation on a group manifold. It is the aim of this section to find an action, which represents just this.

The Polyakov action, in the conformal gauge, can be written in terms of groups as a $U(1)^D$ model by defining

$$U \;\equiv\; \exp\left(T_i X^i\right) \;, \tag{10}$$

where the T's generate the Abelian $U(1)^D$, i.e. $[T_i, T_j] = 0$. Iff we then define $Tr(T_i T_j) = \eta_{ij}$ we can formulate the Polyakov action as

$$S_{Polyakov} \;=\; \int dz d\bar{z} Tr\left(\partial U\, \bar{\partial} U^{-1}\right) \;. \tag{11}$$

Although a mere reformulation, it has some extremely nice properties. As a starter it has a global $U(1)^D \otimes U(1)^D$ symmetry. This can easily be seen by making the following transformation

$$U(z,\bar{z}) \;\rightarrow\; hU(z,\bar{z})g \;, \tag{12}$$

where h and g are elements of $U(1)^D$. Now, since the group acts independently from the left and from the right, this kind of symmetry is called a Chiral symmetry.

Having the above form of the Polyakov action, it is not hard to imagine a generalization to non-Abelian groups: We simply take the T's to generate some non-abelian group, G say.

$$[T_a, T_b] \;=\; f_{ab}{}^{c} T_c \tag{13}$$

and write

$$S_G \;=\; \int d^2 z\, Tr\left(\partial U \bar{\partial} U^{-1}\right) \;=\; -\int d^2 z\, Tr\left(U^{-1}\partial U\, U^{-1}\bar{\partial} U\right) \;, \tag{14}$$

where we have taken advantage of the group structure to rewrite the action in a form in which we can say some things about the geometry of the group manifold. As before, the trace-operation is used to introduce some kind of Lorentz metric, and as such it has to be $dim(G)$-dimensional and invertible. Since it has to be $dim(G)$ dimensional, it is implied that one has to take the T's in the adjoint representation[3]. This then implies that we are considering semisimple algebras only and that the Lorentz-like metric is actually the Killing form. Later on, we will enlarge the possible Lie algebras to the class of reductive algebras.

Decomposing the left-invariant fields as, See appendix B,

$$U^{-1}\partial_\alpha U \;=\; \sigma^a_\alpha T_a \;\equiv\; e^a{}_\mu \partial_\alpha X^\mu T_a \tag{15}$$

we see that the above action can be written as

$$S_G \;=\; \int d^2\sigma\, G(X)_\mu \partial_\alpha X^\mu \partial_\beta X^\nu \;, \tag{16}$$

where we have defined

$$G(X)_{\mu\nu} \;\equiv\; \eta_{ab} e^a{}_\mu e^b{}_\nu \;. \tag{17}$$

This then means that the above action can be identified with a NLS in a straightforward manner. Now, as can be seen in appendix B, a group manifold is flat, so one might think that the above model always leads to a solution of the β functions. This is however erroneous: As can be seen in the same appendix, we only consider the connection as defined by the Vielbein,

[3]Note that the adjoint representation for a given group is the only faithful $dim(G)$ representation.

i.e. the ordinary Levi-Cività connection, as to define the curvature of the manifold. This in general does not lead to vanishing Ricci curvature[4]. So how do we get an action which leads to an automatic solution of the β-equations?

E. Witten solved the above problem by introducing an antisymmetric coupling, by means of a Wess-Zumino coupling

$$S_{WZ} \;=\; \int_{\Sigma^1_g} Tr(\Lambda^3 U^{-1} dU) \;, \tag{18}$$

How does one identify this with the anti-symmeric part in the NLS? Well, the part under the integral is a 3-form defined on a 3D manifold and as such it is a closed form. A theorem by Poincaré then states that locally we always can find a 2-form, B say, such that

$$Tr(\Lambda^3 U^{-1} dU) \;=\; dB \;, \tag{19}$$

and another theorem, due to Stokes, then states that

$$\int_{\mathcal{M}} dB \;=\; \int_{\partial\mathcal{M}} B \;. \tag{20}$$

This then means that iff $\partial\Sigma^1_g \equiv \Sigma_g$, we can identify, locally, Eq. (18) with the B term in the NLS Eq.(2).

The real question is whether the composed action leads to a solution to the β-equations: It was shown by E. Witten [14] that with an appropriate choice for the constant multiplying Eq. (18), the theory is quantically conformal invariant. Combining this with the fact that it can be identified with a NLS, it follows that it solves the β-equations. The demonstration of this lies also outside the scope of this work, and we will content ourselves with the explicit form of the WZW-model:

$$S_{WZW} \;=\; \tfrac{1}{2\pi}\int d^2z\, g^{-1}dg\, g^{-1}dg \;+\; \tfrac{1}{6\pi}\textstyle\int \wedge^3 g^{-1}dg \;. \tag{21}$$

As was said before, the WZW-model as defined above, only works for the semisimple Lie algebras, due to the fact that the trace operation introdces the Killing form as the metric. A way around this restriction was found by C. Nappi and (once again) E. Witten [11]: The Killing form, or rather the metric we want to use on the Lie algebra, has to be bi-invariant and invertible in order to guarantee that the WZW model has the chiral invariance. Now on semisimple Lie algebras there is a natural candidate for such a metric, but this does not mean that there are no non-semisimple Lie algebras on which we could define some kind of a metric. If we then decompose the left invariant forms as in Eq. (87), and introduce some metric η_{ab} on the Lie algebra, we can write down a generalization of the WZW-model

$$S \;=\; \tfrac{1}{2\pi}\textstyle\int d^2z\, \eta_{ab}\sigma^a\otimes\sigma^b \;+\; \tfrac{1}{6\pi}\textstyle\int d^3y\, f_{abc}\sigma^a\wedge\sigma^b\wedge\sigma^c \;, \tag{22}$$

If we then impose chiral invariance, one finds that

$$\begin{array}{rcl} 0 &=& \eta_{dc}f_{ab}{}^d \;+\; \eta_{bd}f_{ac}{}^d \;, \\ f_{abc} &=& f_{ab}{}^d\eta_{dc} \;, \end{array} \tag{23}$$

to which one has to add $det(\eta) \neq 0$.

[4]An obvious exception to this rule is the $U(1)^D$ model.

2.1 Conserved charges and Kač-Moody algebras

To every global symmetry we can associate an conserved current. In this case we have two conserved currents, one depending on z and the other one depending on $\bar{z}$. Since the discussion for either one of the currents is the same as for the other, we will only look at the holomorphic currents, i.e. the currents depending on z. The conserved holomorphic current is

$$J(z) \;=\; U^{-1}\partial U \;, \tag{24}$$

and from it we can define the conserved charge

$$\mathcal{Q} \;\equiv\; \tfrac{1}{2\pi i} \oint_{\mathcal{C}_z} dz\, J(z) \;. \tag{25}$$

Due to the equations of motion however, one sees that also $z^n J(z)$ will lead to a conserved charge: The system leads to an infitnite number of conserved charges! The charges are just the coefficients of $J(z)$ in its Laurent series, as one can easily see by constructing them explicitly. Having a look at appendix C, one can see that it is advantageous to use the Operator Product expansion whilst dealing with infinite dimensional algebras, a thing we are dealing with here.

Seeing that the conformal dimension of $J(z)$ is 1, the most general algebra we can think of after decomposing J on a basis of $Lie(G)$, is[5]

$$\underbrace{J_a(z)J_b(w)} \;=\; \frac{\eta_{ab}}{(z-w)^2} \;+\; \frac{h_{ab}{}^c\, J_c(w)}{z-w} \;. \tag{26}$$

Imposing associativity, closely related to the Jacobi identity, upon the above OPE, one sees that the h's have to satisfy the Jacobi identity, and that the η has to satisfy

$$\eta_{ec}h_{ab}{}^e \;+\; \eta_{be}h_{ac}{}^e \;=\; 0 \;. \tag{27}$$

Expanding the J's as usual for a conformal dimension 1 field, one can see that the above relation leads to

$$[J_a^m, J_b^n] \;=\; h_{ab}{}^c J_c^{m+n} \;+\; m\eta_{ab}\delta_{m,-n} \;, \tag{28}$$

which is known to the world as a Kač-Moody algebra (E.g. see [2] , part III, for a technical introduction. [3] is a nice exposure on the history of Kač-Moody algebras.). Looking at the zero-mode part of the above algebra, one sees that it forms a Lie algebra, which necessarily must be $Lie(G)$. This then means that the h's are the structure constants for $Lie(G)$.

Resuming, we see that the WZW-model has an infinite number of conserved charges whose algebra is described by

$$\underbrace{J_a(z)J_b(w)} \;=\; \frac{\eta_{ab}}{(z-w)^2} \;+\; \frac{f_{ab}{}^c J_c(w)}{z-w} \;\ldots \tag{29}$$

3 Operator constructions

In this section we'll have a look at the conformal structure of the WZW-models, that is to say we'll mimic the construction of the Virasoro algebra in terms of the creation- and annihilation operators in the Polyakov case.

[5] Witten [14] has shown, by using a form of lightcone/canonical quantization that this is the correct symmetry algebra.

At this point it should be clear that the $U^{-1}\partial U$ play the same role as the left moving fields in the Polyakov approach to string theory. Decomposing the left moving fields in term of the eigenmodes, we can use the corresponding operators to write down the Virasoro algebra. From the general theory of Kač-Moody algebras we know that the only form the central extension of the conformal algebra in 2D can have the form (See [5] for a physical argument.)

$$\underbrace{T(z)T(w)} \;=\; \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{(\partial T)(w)}{z-w} \;. \tag{30}$$

The problem of creating a Virasoro algebra out of a general Chiral algebra, generated by the currents $J_i(z)$, was first tackled by Sugawara.

Having the J's, remember that they are conformal dimension 1 operators, it is straightforward to construct a conformal dimension 2 operator, such as the stress tensor. We define

$$T(z) \;\equiv\; L^{ab} : J_a J_b : (z) \;. \tag{31}$$

If the above Ansatz is to be a true stress tensor, it has to give the correct result for the conformal dimension of the J's, namely 1. If one then imposes just that, one finds that

$$\begin{aligned} L^{bc} f_{ac}{}^{d} + L^{cd} f_{ac}{}^{b} &= 0 \;, \\ 2L^{bc}\eta_{ac} + L^{de} f_{ad}{}^{h} f_{he}{}^{b} &= \delta^{b}{}_{a} \;, \end{aligned} \tag{32}$$

so that L is an invariant matrix under the group G. Using this fact in the last equation, one obtains that L has to satisfy

$$L^{ab}\left(2\eta_{bc} + K_{bc}\right) \;=\; \delta^{a}{}_{c} \;, \tag{33}$$

where we have defined the Killing form for a group G, Eq. (86).

Now we are in a position to calculate the Virasoro algebra: Using the equations in appendix C, one finds that

$$c \;=\; 2Tr\left(L\eta\right) \;. \tag{34}$$

In general $c \neq 0$, so that we are still dealing with an anomalous theory: This can however be overcome by adding some extra string models and claiming them to 'internal'. A remark is at order here: In the Polyakov case, the BRST-ghosts arrising from gauge fixing the 2D gauge symmetries (See [5], part I), lead to a $c = -26$ Virasoro algebra. Adding then D free scalars to our system, i.e. just making the direct sum of the two Virasoro algebras, one sees that in order for the total system to have have $c = 0$, i.e. not to be anomalous, one needs to add $D = 26$ free scalars (A free scalar has $c = 1$).

4 Some examples

Here we'll consider two examples in order to see what kind of solutions to backgrounds one can get, by using the the WZW-model. One will be a WZW-model defined on some simple Lie-group[6], leading to a Taub-NUT like space. The other one will be a WZW-model based of the 4D Heisenberg group leading to a *pp-wave*.

[6] As always, the semi-simple group will be $SU(2)$.

4.1 The WZW-model on the 4D Heisenberg algebra

Let us examine the algebra H_4, which originates from the dynamics of a single one-dimensional harmonic oscillator. H_4 is generated by $\{\alpha,\, \alpha^\dagger,\, N=\alpha^\dagger\alpha,\, I\}$ and the commutation relations

$$\begin{aligned}
[\alpha,\alpha^\dagger] &= I\,,\\
[N,\alpha^\dagger] &= \alpha^\dagger\,,\\
[N,\alpha] &= -\alpha\,.
\end{aligned} \tag{35}$$

It is non-semisimple so that its Killing form is degenerate. A non-degenerate solution to eq.(23) exists and is given by

$$\eta_{ab} = \begin{pmatrix} 0 & a & 0 & 0\\ a & 0 & 0 & 0\\ 0 & 0 & b & -a\\ 0 & 0 & -a & 0 \end{pmatrix},\ \eta^{ab} = \begin{pmatrix} 0 & \frac{1}{a} & 0 & 0\\ \frac{1}{a} & 0 & 0 & 0\\ 0 & 0 & 0 & -\frac{1}{a}\\ 0 & 0 & -\frac{1}{a} & -\frac{b}{a^2} \end{pmatrix}. \tag{36}$$

A general element of H_4 is written as

$$g = e^{iq\alpha+i\bar{q}\alpha^\dagger}e^{iuN+ivI}\,, \tag{37}$$

where the first term on the right hand side of eq.(37) is an element of the coset space $H_4/U(1)\times U(1)\sim C$ parametrized by the complex coordinates $q,\bar{q}$. By using the relations

$$\begin{aligned}
e^{iq\alpha+i\bar{q}\alpha^\dagger} &= e^{iq\alpha}e^{i\bar{q}\alpha^\dagger}e^{\frac{q\bar{q}}{2}}\,,\\
e^{-iuN}\alpha e^{iuN} &= e^{iu}\alpha\,,\\
e^{-i\bar{q}\alpha^\dagger}\alpha e^{i\bar{q}\alpha^\dagger} &= \alpha + i\bar{q}I\,,
\end{aligned} \tag{38}$$

we find that

$$g^{-1}dg = ie^{iu}dq\alpha + ie^{-iu}d\bar{q}\alpha^\dagger + iduN + (idv + \frac{1}{2}qd\bar{q} - \frac{1}{2}\bar{q}dq)I\,, \tag{39}$$

so that the σ^a's in eq.(87) are given by

$$\begin{aligned}
\sigma^1 &= e^{iu}dq\,,\\
\sigma^2 &= e^{-iu}d\bar{q}\,,\\
\sigma^3 &= du\,,\\
\sigma^4 &= dv - \frac{i}{2}qd\bar{q} + \frac{i}{2}\bar{q}dq\,.
\end{aligned} \tag{40}$$

The terms that are being integrated over in (22) are calculated to be

$$\begin{aligned}
\sigma^a_\alpha\sigma^{b\alpha}\eta_{ab} &= 2a\partial_\alpha q\partial^\alpha\bar{q} - 2a(\partial_\alpha v - \frac{i}{2}q\partial_\alpha\bar{q} + \frac{i}{2}\bar{q}\partial_\alpha q)\partial^\alpha u + b\partial_\alpha u\partial^\alpha u\,,\\
\epsilon^{\alpha\beta\gamma}\sigma^a_\alpha\sigma^b_\beta\sigma^c_\gamma f_{abc} &= 6ia\epsilon_{\alpha\beta\gamma}\partial^\gamma(u\partial^\alpha q\partial^\beta\bar{q})\,,
\end{aligned} \tag{41}$$

and the WZW action (22) is written as

$$\begin{aligned}
\mathcal{S}_{WZW} = -\frac{1}{2\pi}\int d^2\sigma \Big(&2a\partial_\alpha q\partial^\alpha\bar{q} - 2a(\partial_\alpha v - \frac{i}{2}q\partial_\alpha\bar{q} + \frac{i}{2}\bar{q}\partial_\alpha q)\partial^\alpha u\\
&+ b\partial_\alpha u\partial^\alpha u + 2ia\epsilon_{\alpha\beta}u\partial^\alpha q\partial^\beta\bar{q}\Big)\,.
\end{aligned} \tag{42}$$

By regarding this action as a σ-model action, Eq. (2) but then in the conformal gauge, we can read off the background space-time metric and the antisymmetric field. In the coordinate base $[dq, d\bar{q}, du, dv]$ they are given, up to multiplicative factors, by

$$\begin{aligned} G_{\mu\nu} &= \begin{pmatrix} 0 & \frac{1}{2} & -\frac{i}{4}\bar{q} & 0 \\ \frac{1}{2} & 0 & \frac{i}{4}q & 0 \\ -\frac{i}{4}\bar{q} & \frac{i}{4}q & \beta^2 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & 0 \end{pmatrix}, \\ B_{q\bar{q}} &= \frac{i}{2}u, \end{aligned} \tag{43}$$

where $\beta^2 = \frac{b}{2a}$, and thus, the background space-time line element is given by

$$ds^2 = dq d\bar{q} - (dv - \frac{i}{2} q d\bar{q} + \frac{i}{2}\bar{q} dq) du + \beta^2 du^2 . \tag{44}$$

By introducing polar coordinates $q = Re^{i\theta}$, $\bar{q} = Re^{-i\theta}$, the line element turns out to be

$$ds^2 = dR^2 + R^2 d\theta^2 - (dv - R^2 d\theta) du + \beta^2 du^2 . \tag{45}$$

The signature of this metric is manifest in the orthonormal base

$$\begin{aligned} e^0 &= \frac{1}{2\beta}(dv - R^2 d\theta), \\ e^1 &= dR, \\ e^2 &= R d\theta, \\ e^3 &= \beta du - e^0, \end{aligned} \tag{46}$$

where the metric is $\eta_{\mu\nu} = (-1, +1, +1, +1)$. The only non-vanishing components of the Ricci tensor in the above base are

$$R_{00} = R_{33} = R_{03} = \frac{1}{2\beta^2} . \tag{47}$$

In the same base, the antisymmetric two form field $B = \frac{1}{2} B_{\mu\nu} dx^\mu \wedge dx^\nu$ is written as

$$B = u e^1 \wedge e^2 , \tag{48}$$

and thus

$$H = dB = \frac{1}{\beta} e^3 \wedge e^1 \wedge e^2 + \frac{1}{\beta} e^0 \wedge e^1 \wedge e^2 . \tag{49}$$

The non-vanishing components of H are then

$$H_{123} = H_{012} = \frac{1}{\beta} . \tag{50}$$

Employing eqs.(47), (50) in the one-loop beta-function equations Eqs. (3,4,5), one can finds that the dilaton is constant and that the central charge is four ($c = 4$).

Now that we have seen that it leads to a solution of the β-equations, it would be nice to see to what kind of background it corresponds. As can be see from Eq. (44) the metric has a null Killing, i.e. a direction on which the metric does not depend and whose length is zero. This means that the above metric is a member of a generic class of metrics known as *pp*-waves[7], and it describes the propagation of a 2-plane parametrized by q and $\bar{q}$.

The same result can be obtained non-perturbatively as follows: By using the metric, Eq. (36), and the Killing form for $\mathcal{H}_4$ in Eq. (33) one finds

$$L^{ab} = \frac{1}{2a}\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & -1 & -\frac{b+2}{a} \end{pmatrix} . \tag{51}$$

The central charge is given by Eq. (34)

$$c = 2L^{ab}\eta_{ab} = 4 , \tag{52}$$

4.2 The WZW-model on $SU(2)$

We will parametrize an element of $SU(2)$ by means of the well-known Euler angles [1], i.e.

$$g \equiv \exp(\chi T_3)\exp(\theta T_1)\exp(\phi T_3) , \tag{53}$$

where we take the commutation relations to be

$$[T_i, T_j] = \epsilon_{ijk}T_k , \tag{54}$$

and the topological structure of $SU(2)$ states that

$$0 \leq \chi < 4\pi , \ 0 \leq \theta < \pi , \ 0 \leq \phi < 2\pi . \tag{55}$$

This then means that θ, ϕ are the usual angular coordinates describing the unit radius sphere S^2, and χ describes some circle S^1; The parametrization expresses the fact that topologically $SU(2)$ is S^3, which due to a mapping devised by Hopf, is equal to a S^2 fibered by an S^1. With this parametrization one finds the Maurer-Cartan forms to be

$$\begin{cases} \sigma^1 = \cos(\phi)\, d\theta + \sin(\phi)\sin(\theta)\, d\chi , \\ \sigma^2 = -\sin(\phi)\, d\theta + \cos(\phi)\sin(\theta)\, d\chi , \\ \sigma^3 = d\phi + \cos(\theta)\, d\chi . \end{cases} \tag{56}$$

As is obvious from the form of the commutation relations we have taken the metric on $su(2)$ to be just Kronecker's delta, so that we find that

$$\begin{aligned} ds^2 &= \delta_{ij}\sigma^i \otimes \sigma^j = d\chi^2 + d\theta^2 + d\phi^2 + 2\cos(\theta)\, d\chi d\phi \\ &= (d\chi + \cos(\theta)\, d\phi)^2 + d\theta^2 + \sin^2(\theta)\, d\phi^2 . \end{aligned} \tag{57}$$

This form for the metric is related to so-called Taub-NUT spaces, albeit in a compactified form. Usually Taub-NUT metrics will contain a factor

$$(dt + N\, f(r,t)\cos(\theta)\, d\phi)^2 , \tag{58}$$

[7] PP-waves is shorthand for *plane-fronted waves with parallel rays.*

where N is the so-called Taub-NUT charge which has the interpretation of a gravitational instanton; In this case one can see that it actually classifies the Hopf-fibration. For example, had N been nil, we would have had a metric describing $S^2 \otimes S^1$ which is a trivial fiberbundle.

The anti-symmetric contribution can then be seen to be

$$\tfrac{1}{6}\int \epsilon_{ijk}\sigma^i\sigma^j\sigma^k \; = \; \int d\left[\cos(\theta)\, d\chi d\phi\right] \;, \tag{59}$$

so that

$$B_{\chi\phi} \; = \; \cos(\theta) \;. \tag{60}$$

Putting the metric as displayed in Eq. (57) into some computer program for analytical manipuplation[8] and one finds that, on the base $[\alpha, \gamma, \beta]$,

$$R_{\mu\nu} \; = \; -\tfrac{1}{2}\begin{pmatrix} 1 & \cos(\theta) & 0 \\ \cos(\theta) & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;, \tag{61}$$

and the Ricci curvature is

$$R \; = \; -\frac{3}{2} \;. \tag{62}$$

One can show that the one-loop β-functions are satisfied.

As far as the operator construction is concerned we can be brief: The general metric is proportional to the Killing metric, i.e.

$$g_{ab} \; = \; k\delta_{ab} \quad , \quad K_{ab} \; = \; -2\delta_{ab} \;. \tag{63}$$

Putting this into Eq. (33) we can calculate the matrix L, leading to, due to eq. (34),

$$c \; = \; \frac{3k}{k-1} \;. \tag{64}$$

Epilouge

In this work, we have given a rather coarse introducion to the world of WZW-models. Due to lack of space, time and, mostly, knowledge on behalf of the author, some extremely interesting topics related to the WZW model have been left out. To name but a few: Reduction of the WZW-model to Toda theories (A whole topic in itself), the gauged WZW- models, T-duality in WZW-models and in case of WZW-models based on semi-simple Lie algebras its relation to the Weyl group acting on the principle root system, the appearence of quantum groups in the study of the conformal blocks, the interplay between the Sugawara construction and the chiral symmetry as resulting in the Knizhnik-Zamolodchikov equation.......

An apparent drawback is also that one has not dealt with the supersymmetric WZW-models. This can be done, perhaps as an exercise for next years course..

Seeing that during the course not only the string action, but also the brane action were covered, the question about an analogous formulation of branes on groups comes only natural.

[8]Here all calculations we made using **GRTensorII** running under MapleV, but there are similar packages for Mathematica or Axiom.

In this respect, there are quite a few things to be said: Seeing that the anti-symmetric coupling exists in odd dimensions only, leads to the fact that, restricting ourselves to 10 dimensional algebras, we are dealing with type IIB string theory. This ought to come as no surprise: Type IIB is the only string theory which has two 2-forms, related by S-duality, representing the fact that the theory not only contains strings but also D-(1)-branes. Therefore, looking upon the generalization to arbitrary dimensions of the WZW-model, it is only natural that we stay in the type IIB world. The last remark in this work is that since the introduction of D-branes some 3 years ago, about 4 articles have appeared dealing with the subject of D-branes moving on groups, but without giving an action.

A General Relativity: Conventions *et cetera*.

In this appendix, we will give our conventions whilst introducing the needed elements of general relativity. As a first remark, we will always use a metric of signature $(1, D-1)$, meaning that the Lorentz metric defined on the Minkowski space reads $\eta \equiv diag(+,-,\ldots,-)$.

Assuming that the reader has at least some basic notion of general relativity, we define the action of the covariant derivative as

$$\nabla_\mu A_\nu \;=\; \partial_\mu A_\nu \,-\, \Gamma^\rho{}_{\mu\nu} A_\rho \;, \tag{65}$$

$$\nabla_\mu A^\nu \;=\; \partial_\mu A^\nu \,+\, \Gamma^\nu{}_{\mu\rho} A^\rho \;, \tag{66}$$

where the Christoffel's second symbol is defined in the usual manner, i.e.

$$\Gamma^\rho{}_{\mu\nu} \;=\; \tfrac{1}{2} g^{\rho\sigma} \left(\partial_\nu g_{\mu\sigma} \,+\, \partial_\mu g_{\nu\sigma} \,-\, \partial_\sigma g_{\mu\nu} \right) \;. \tag{67}$$

With these conventions we then define the Riemannian curvature as

$$R(\Gamma)^\alpha{}_{\beta\mu\nu} \;=\; \partial_\mu \Gamma^\alpha{}_{\beta\nu} \,-\, \partial_\nu \Gamma^\alpha{}_{\beta\mu} \,+\, \Gamma^\alpha{}_{\lambda\mu}\Gamma^\lambda{}_{\beta\nu} \,-\, \Gamma^\alpha{}_{\lambda\nu}\Gamma^\lambda{}_{\beta\mu} \;, \tag{68}$$

the Ricci- and scalar curvature as

$$R(\Gamma)_{\mu\nu} \;=\; R(\Gamma)^\alpha{}_{\mu\alpha\nu} \qquad , \qquad R(\Gamma) \;=\; R(\Gamma)^\mu{}_\mu \;. \tag{69}$$

Thus far the 'classical' theory of general relativity[9].

Torsion: Soon after the introduction of GR, it became clear that metric compatibility allowed for a non-symmetric part in the (Levi-Cività) connection[10]. Although a lot more can be said about torsion and its application to physics (See e.g. [13]), here we will take what we need: We extend the Levi-Civita connection to incorporate a totally anti-symmetric three tensor, denoted T, i.e.

$$\hat{\Gamma}^\rho_{\mu\nu} \;\equiv\; \Gamma^\rho{}_{\mu\nu} \,-\, \tfrac{1}{2} g^{\rho\sigma} T_{\mu\nu\sigma} \;. \tag{70}$$

Calculating then the Ricci curvature as before, but expanding it in terms of the Ricci curvature, calculated by using g, one can see that it reads

$$R(\hat{\Gamma})_{\mu\nu} \;=\; R(\Gamma)_{\mu\nu} \,-\, \nabla_\lambda T^\lambda{}_{\mu\nu} \,+\, T_{\mu\kappa\lambda} T_\nu{}^{\kappa\lambda} \;, \tag{71}$$

[9] Classical here not meant as to differentiate it from a true quantum theory of gravity, but rather classical in the sense that it is the form in which it was originally defined by Einstein, before E. Cartan laid his hands on the theory

[10] Propably the first one to note this was E. Cartan, when he was formalizing geometry in terms of forms (See below). It became a physical object when Kibble and Utiyama [9] tried to find a gauge theory of gravity.

showing that the Ricci curvature with torsion is not symmetric in general.

Vielbein: One of the axioms of GR is that locally the laws of special reltivity hold. This then means that we can find coordinates, locally i.e. defined on an open chart, such that the metric in these coordinates is just the Lorentz metric. The connection between these so-called Lorentz coordinates and the true coordinates, is given by the Vielbein. We will denote the Lorentz coordintes by x^a and the vielbein as $e_\mu{}^a$. Seeing that it is a mere change of coordinates one sees that the invariant length reads

$$ds^2 \;=\; \eta_{ab} dx^a dx^b \;=\; \eta_{ab} e_\mu{}^a e_\nu{}^b dx^\mu dx^\nu \;, \tag{72}$$

leading to the identification

$$g_{\mu\nu} \;=\; \eta_{ab} e_\mu{}^a e_\nu{}^b \;. \tag{73}$$

Note that this, due to the fact that Vielbeins have to be invertible, implies

$$g^{\mu\nu} e_\mu{}^a e_\nu{}^b \;=\; \eta^{ab} \;. \tag{74}$$

In order to incorporate the fact that we can switch from covariant- to Lorentz-coordinates, we will extend the definition of covariant derivatives to include an Lorentz term, denoted $\omega_\mu{}^{ab}$, as follows: On an object with Lorentz indices we define

$$\nabla_\mu N^a \;=\; \partial_\mu N^a \;-\; \omega_\mu{}^a{}_b N^b \;, \tag{75}$$

$$\nabla_\mu N_a \;=\; \partial_\mu N_a \;+\; \omega_{\mu a}{}^b N_b \;. \tag{76}$$

Applying then the fact that the connection has to be metric compatible, we obtain

$$0 \;=\; \nabla_\kappa g_{\mu\nu} \;=\; 2 e_{a(\mu} \partial_\kappa e_{\nu)}{}^a \;\Rightarrow\; \nabla_\mu e_\mu{}^a \;=\; 0 \;, \tag{77}$$

one determines the ω by

$$\omega_\mu{}^{ab} \;=\; e^{b\nu} \partial_\mu e_\nu{}^a \;-\; \Gamma^\rho{}_{\kappa\nu} e^{b\nu} e_\rho{}^a \;. \tag{78}$$

Cartan's structure equations[11]: If we define the forms

$$\begin{cases} e^a \;\equiv\; e_\mu{}^a dx^\mu & : \text{ coordinate 1-form} \\ \omega^a{}_b \;\equiv\; \Gamma^a{}_{cb} e^c & : \text{ connection 1-form} \\ T^a \;\equiv\; \tfrac{1}{2} T^a{}_{bc} e^b \wedge e^c & : \text{ Torsion 2-form} \\ R^a{}_b \;=\; \tfrac{1}{2} R^a{}_{bcd} e^c \wedge e^d & : \text{ Curvature 2-form} \end{cases} \tag{79}$$

one can see that

$$de^a \;+\; \omega^a{}_b \wedge e^b \;=\; T^a \;, \tag{80}$$

is nothing but the definition of Γ. The above equation is known as Cartan's first structure equation. In the same way we can find an equation for the curvature 2-form. It is known as Cartan's second structure equation and reads

$$R^a{}_b \;=\; d\omega^a{}_b \;+\; \omega^a{}_c \wedge \omega^c{}_b \;. \tag{81}$$

[11]For more founded introduction to the Cartan equations the reader is kindly referred to the literature, e.g. [10].

The power of the above equations is the speed with which the curvature can be calculated.

The Bianchi identities can also be expressed in the language of forms. They read

$$\begin{aligned} 0 &= dT^a + \omega^a{}_b \wedge T^b - R^a{}_b \wedge e^b \, , \\ 0 &= dR^a{}_b + \omega^a{}_c \wedge R^c{}_b - R^a{}_c \wedge \omega^c{}_b \, . \end{aligned} \tag{82}$$

B Groups, Lie algebras and geometry

It is understood that the reader has at least some notion about group theory, especially about the Lie algebra associated to a given group and their interrelation. Then apart from nomencalture, very little will be needed:

- **Jacobi Identity**: The Jacobi identity reads

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = 0 \, , \tag{83}$$

  for any triplet of element in the Lie algebra. The main importance of the Jacobi identity lies in the fact that it represents associativity, as one can see by expanding the above equation NOT supposing associativity.

  In terms of the structure constants, see below, the Jacobi identity is expressed as

$$0 = f_{bc}{}^d f_{ad}{}^e + f_{ca}{}^d f_{bd}{}^e + f_{ab}{}^d f_{cd}{}^e \, . \tag{84}$$

- **Lie data**: The data needed to reconstruct the Lie algebra. The Lie data then consist of a basis of the vector space spanned by the generators, T_a ($a = 1 \ldots dim\, Lie(G)$), and the structure constants $f_{ab}{}^c$, defining the commutator-relations.

- **Cartan Subalgebra (CSA)**: The CSA is the maximally commuting subalgebra of a Lie algebra. The dimension of the CSA defines the rank of the Lie algebra, i.e. $dim\, CSA(G) \equiv rank(G)$.

- **Casimir Operator**: A Casimir operator for a given Lie algebra, is a quadratic operator, made out of the generators, such that it commutes with all the elements of the Lie algebra. Note that the Casimir is neither an element of the Lie algebra nor the group.

- **Killing form**: The Killing form, K, is defined by

$$K(X,Y) = Tr\,(ad(X)ad(Y)) \, , \tag{85}$$

  and expressed in terms of a base for Lie(G), it reads

$$K_{ab} = f_{ac}{}^d f_{bd}{}^c \, . \tag{86}$$

- **Semi- and Simple Lie algebras**: A subclass of the Lie algebras, which were classified by E. Cartan in the mid twenties. The are characterized by the fact that the Killing form is not degenerate (This is known as Cartan's second criterion), which can then be used as a metric for the vectorspace spanned by the Lie algebra. In Cartan's classification there are 7 possibilities for the simple Lie algebras[12]: $A_{n\geq 1}$, $B_{n\geq 1}$, $C_{n\geq 1}$, $D_{n\geq 3}$, $E_{n\geq 6}$, F_4 and G_2. Some of these algebras are old companions of physicists, e.g. $SU(2) \sim A_1$, $SU(3) \sim A_2$. For a better introduction to these Lie algebras, the reader is referred to the vast literature available.

- **Reductive algebras**: A Lie algebra is called reductive, iff one can find a non-degenrate, bi-invariant form it. This is closely related to a sufficient number of Casimirs for the algebra. An example of a reductive but not semi-simple algebra is the 4-dimensional Poincaré group: Due to the translation part in the algebra, the Killing form is degenerate, showing that it is not semi-simple. There are however 2 Casimir operators, the mass-squared and the Lorentz product of the Pauli-Lubanski vector [6]. One can see that the sum of these two Casimirs gives rise to a non-degenerate form on the algebra, rendering the Poincaré group reductive.

This ought to be sufficient for the reader to understand the terminology, used in the paper, if not see [2].

B.1 The geometry of groups

As is known from the general theory of groups, a Lie group is not only a group, but also a smooth manifold. As such we could intend to introduce some connection on these manifolds, and see what the curvature of these manifolds is. This is exactly what we are going to do in this appendix.

In Cartan's formulation of geometry, curvature and torsion of a manifold are defined by Cartan's structure equations Eqs. (80 ,81).

On a group there is a set of preferred directions which are the so-called left(right)-invariant vector fields. This is due to the fact that a group G is a manifold on which the group G acts transitively, i.e. without fixed points and covering the whole manifold. This then means that the value of any form in any point of the group-manifold is related to the value of the forms in the identity by making a group transformation. the natural candidates for these Lie algebra valued one-forms are $g^{-1}dg$, where the $g : R^{dim(G)} \rightarrow G$. As is readily acknowledged g is nothing else but the exponential of an element of $\Omega^0(M) \otimes Lie(G)$, so it only natural that the $g^{-1}dg$ will be an element in $\Omega^1 \otimes Lie(G)$. Seeing this, we expand

$$g^{-1}dg \;=\; \sigma^a T_a \;, \tag{87}$$

where the σ's are elements of $\Omega^1(M)$, i.e. one-forms. we can then calculate $d(g^{-1}dg) = dg^{-1} \wedge dg = -g^{-1}dg \wedge g^{-1}dg$, which is an identity nothing else to it. Expanding it on the base of $Lie(G)$ we find the first Cartan identity fo the σ's:

$$d\sigma^c \,+\, \frac{1}{2} f_{ab}{}^{c} \sigma^a \wedge \sigma^b \;=\; 0 \;, \tag{88}$$

Plugging the above equation into Eq. (81), we see that for the chosen connection

$$R^a{}_b \;=\; 0 \;. \tag{89}$$

This means that with our chosen torsionfull connection, a given group manifold is flat: In more fancy language it is said that a group manifold is paralizable.

[12] As the term "semi- simple Lie algbras" indicates, they are directsums of simple Lie algebras.

What would have happened, had we chosen not to include the torsion in our connection? Well, looking at Eq. (71), although it is but a special case it is exactly the case at hand, one sees that then the curvature vanishes if and only iff the structure constants are trivial. But in that case our group is $U(1)^d$, which is just Minkowski/Euclidean space.

C Virasoro and CFT techniques

As is known from the treatment of the Polyakov string, or for that matter any field theory, whilst quantizing, the products of operators must be regularized, i.e. normal ordered. Since the Vacuum Expectation Values (VEV) are defined by time ordering, one makes use of Wick's identities in order to express the time ordered product in terms of contractions and normal ordered products. It is intention of this appendix, to explain the machinery as used in Conformal Field Theory, to do just this.

The straightforward thing to do in case of strings, seeing that we have a natural candidate for a time-like coordinate τ, is to define our VEV's in terms of τ-ordered products. If we then, as is usual in CFT, map the worldsheet onto the complex plane (See figure 1), one sees that τ becomes the radial parameter. This then means that on the complex plane, the VEV's have to be defined using RADIAL ordering: The more to the center a point is, the more history it is!

Another point worth stressing is that our field-operators, are now defined on the complex plane, enabling us to use the machinery of complex functions. We define the radially orderd product of two fields by

$$\mathcal{R}\left(A(z)B(w)\right) \;=\; \left\{ \begin{array}{rl} A(z)B(w) & if \mid z \mid > \mid w \mid \,, \\ (-)^{ab}B(w)A(z) & if \mid w \mid > \mid z \mid \,, \end{array} \right. \tag{90}$$

where $a = 0, 1$ if A is a worldsheet boson, fermion. The convention, which we will adopt for the rest of this work, will be that we will not write the $\mathcal{R}$, but assume that every product of fields will be radially ordered.

If we define the expansion of a product of fields in terms of their Laurent series, we can identify the contraction and the normal ordered product of these operators, i.e.

$$A(z)B(w) \;=\; \underbrace{A(z)B(w)} \;+\; :AB:(w) \;+\; \mathcal{O}(z-w) \;, \tag{91}$$

so that the contraction is defined as the singular part in the expansion and the normal ordered product is the part independent in z. We will never pay attention to the $\mathcal{O}(z-w)$ parts, and from now on it is to be understood that they are always present, although they won't be mentioned again.

Inverting the above equation, we see that this means that the normal ordered product of two fields is given by

$$:AB:(w) \;=\; \frac{1}{2i\pi}\oint_{C_w} \frac{dz}{z-w}\, A(z)B(w) \;, \tag{92}$$

where C_w is some contour around w. A few extremely handy equations can be derived from the above definitions, here are a few

$$\underbrace{A(z) : BC : (w)} \;=\; \frac{1}{2i\pi}\oint_{C_w} \frac{dx}{x-w}\left(\underbrace{A(z)B(x)}\, C(w) + (-)^{ab} B(x)\, \underbrace{A(z)C(w)} \right) \;, \tag{93}$$

Figure 1: The mapping from the worldsheet to the complex plane: τ will become the radial parameter.

$$:[A,B\}:(z) \equiv :AB:(z) - (-)^{ab} :BA:(z)$$
$$= -\underbrace{A(z)B(w)} + (-)^{ab}\underbrace{B(w)A(z)} , \quad (94)$$

$$::[A,B\}:C: = :A:BC:: -(-)^{ab} :B:AC:: . \quad (95)$$

How does this help us with the often tedious calculations needed for the Virasoro algebra? Well: The virasoro algebra can be written in terms of the contraction of some fields only. In order to see this, remember that the Virasoro algebra is given by

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n,-m} , \quad (96)$$

where c is the so-called Conformal Anomaly. Note that iff $c = 0$ the Virasoro algebra reduces to the De Witt algebra, i.e. the algebra describing the classical 2D conformal group [5]. Defining then the field

$$T(z) = \sum_{n=-\infty}^{\infty} \frac{L_n}{z^{n+2}} , \quad (97)$$

one can see, by changing some contours of integration, that the Virasoro algebra can be written as

$$\underbrace{T(z)T(w)} = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{(\partial T)(w)}{z-w} . \quad (98)$$

The great advantage of using the contraction over the algebra is paramount: It is easier and faster to calculate residues than changing indices, resumming and decomposing commutators. Note that the results one obtains are the same: The only thing we did is to redefine things such that our life is made easier.

As in the study of every other symmetry, there is a class of fields that transform under an "irriducible representation" of the conformal group[13]: The primary fields. A Primary field, is a field which transforms as

$$\phi(z,\bar{z}) \to \left(\frac{dz'}{dz}\right)^{\Delta} \left(\frac{d\bar{z}'}{d\bar{z}}\right)^{\bar{\Delta}} \phi(z',\bar{z}') , \quad (99)$$

where Δ ($\bar{\Delta}$) is the (anti-)holomorphic conformal dimension. In physical systems, the conformal dimensions of the objects are easily found to be their scaling dimensions. For the Polyakov action, Eq. (1), one can see that

$$\Delta(\partial X^\mu) = 1 , \ \Delta(\bar{\partial} X^\mu) = 0 , \ \bar{\Delta}(\partial X^\mu) = 0 , \ \bar{\Delta}(\bar{\partial} X^\mu) = 1 . \quad (100)$$

Note that this does not determine the $\bar{\Delta}$ for the holomorphic part of X^μ, and vise versa, so that the conformal dimensions for the X's is undetermined.

After quantization, the fields need to have the same conformal dimension as before quantization (Otherwise, the theory would be anomalous), and we can define the conformal dimension (See [8]) by

$$\underbrace{T(z)\phi(w)} = \frac{\Delta\phi(w)}{(z-w)^2} + \frac{(\partial\phi)(w)}{z-w} . \quad (101)$$

[13] Well, seeing that the algebra is infinite dimensional, there is no chance of having finite dimensional representations. What we have is something resembling highest weight representations, but which is not finite dimensional. In the CFT/Mathematics Jibberish, one speaks of Verma modules.

Introducing then the convention that every Δ-field is expanded as

$$\phi(z) \;=\; \sum_n \frac{\phi_n}{z^{n+\Delta}} \;, \tag{102}$$

one can see that Eq. (101), is nothing else than

$$[L_n, \phi_m] \;=\; \{(\Delta - 1)\, n - m\}\, \phi_{n+m} \;, \tag{103}$$

a rule which will be familiar from the quantization of the Polyakov string, when we replace the ϕ's by the a^μ's and take $\Delta = 1$. Note that if we apply the above rules to the stress tensor, $T(z)$, we see that it is an anomalous principle field.

C.1 Polyakov's string seen as a CFT

In the Polyakov string, after quantization, we know that the creation and annihilation operators satisfy (See e.g. [5].)

$$[a^\mu_n, a^\nu_n] \;=\; m\eta^{\mu\nu}\delta_{m,-n} \;. \tag{104}$$

We also know that the are the coefficients for the Laurent expansion for ∂X^μ, i.e.

$$\partial X^\mu(z) \;=\; \sum_n \frac{a^\mu_n}{z^{n+1}} \;. \tag{105}$$

Using this field we can write the above commutation relations as

$$\underbrace{\partial X^\mu(z)\partial X^\nu(w)} \;=\; \frac{\eta^{\mu\nu}}{(z-w)^2} \;. \tag{106}$$

Updating the classical stres tensor to an operator by stating that all products of operators at the same position are normal ordered,

$$T(z) \;=\; \eta_{\mu\nu} :\partial X^\mu \partial X^\nu: (z) \;, \tag{107}$$

we can calulate the CFT information needed

$$\underbrace{T(z)\partial X^\mu(w)} \;=\; \frac{\partial X^\mu(w)}{(z-w)^2} + \frac{\partial^2 X^\mu(w)}{z-w} \;, \tag{108}$$

$$\tag{109}$$

$$\underbrace{T(z)T(w)} \;=\; \frac{D}{2(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} \;, \tag{110}$$

clearly showing that $c = D$.

C.2 A Weird example

Although there are numerous other examples, we will illustrate the above methods with an example resembling Polyakov's string: Instead of commuting fields, we will use anti-commuting fields, whose action will be quadratic in derivatives.

The action we will take as our parting point is

$$S \;=\; \tfrac{1}{2} \int d^2\sigma \sqrt{-h}\, h^{\alpha\beta} \left(\partial_\alpha \Theta \partial_\beta \Xi \;-\; \partial_\alpha \Xi \partial_\beta \Theta\right) \;, \tag{111}$$

where Θ and Ξ are two Grassmann field, i.e. they anticommute. The form of the above action ensures that the stress tensor is symmetric

$$T_{\alpha\beta} \equiv -\frac{2}{\sqrt{-h}}\frac{\delta S}{\delta h^{\alpha\beta}} \;=\; 2\partial_{(\alpha}\Theta\partial_{\beta)} \;-\; h_{\alpha\beta}\partial_\gamma\Theta\partial^\gamma\Xi \; . \tag{112}$$

One can easily see that the trace of this stress tensor vanishes, showing that the action is classically conformal invariant. Since the action is conformally invariant, we can assign conformal dimensions to the derivatives of the fields: The conformal dimensions are 1. This however does not imply that Θ and Ξ have conformal dimension 0.

Going over to the complex plane and imposing the conformal gauge we see that the action reads

$$S \;=\; \int d^2z\,(\partial\Theta\bar{\partial}\Xi \;-\; \partial\Xi\bar{\partial}\Theta) \;\; . \tag{113}$$

The holomorphic part of the stress tensor then reads

$$T(z) \;=\; \partial\Theta\partial\Xi - \partial\Xi\partial\Theta \; , \tag{114}$$

In order to quantize this system, we need to introduce the canonical momenta, which we choose to do by left-functional derivation[14]

$$\Pi_a \;\equiv\; \frac{\delta^L S}{\delta^L(\partial_\tau\Psi^a)} \; , \tag{115}$$

Where we have defined $\Psi^1 = \Theta$ and $\Psi^2 = \Xi$. Explicitly one finds that

$$\begin{cases} \Pi_1 \;=\; \partial_\tau\Xi \; , \\ \Pi_2 \;=\; -\partial_\tau\Theta \; . \end{cases} \tag{116}$$

All that remains to do, is to make an expansion of the fields in z and $\bar{z}$ and to impose the Equal Time Commutation Relations

$$\left\{\Pi_a(\tau,\sigma),\Psi^b(\tau,\sigma')\right\} \;=\; -2i\pi\delta(\sigma-\sigma')\,\delta_a{}^b \; , \tag{117}$$

$$\{\Pi_a(\tau,\sigma),\Pi_b(\tau,\sigma')\} \;=\; 0 \; , \tag{118}$$

$$\left\{\Psi^a(\tau,\sigma),\Psi^b(\tau,\sigma')\right\} \;=\; 0 \; . \tag{119}$$

Since the derivatives of the fields have conformal dimension 1, we expand the fields as

$$\partial\Psi^a \;=\; \sum_n A^a_n\, z^{-n-1} \; , \tag{120}$$

$$\Psi^a(z) \;=\; B^a \;+\; A^a_0\,\log z \;+\; \sum_{n\neq 0}\tfrac{1}{n}A^a_{-n}\,z^n \tag{121}$$

A small calculation then shows that the eigenmode operators satisfy

$$\left\{A^a_n, A^b_m\right\} \;=\; \tfrac{n}{2}\delta_{n,-m}g^{ab} \quad , \quad \left\{A^a_n, B^b\right\} \;=\; ig^{ab}\delta_{n,0} \; , \tag{122}$$

[14]Note that for Grassmannian fields there are two ways of defining derivatives. Only left derivation leads to the equality between derivation and integration, though.

where $g^{12} = g^{21} = -1$ and the other components are nil.

Defining then, as by now should be obvious, the fields

$$\partial\Psi^a(z) \ = \ \sum_n \frac{A_n^a}{z^{n+1}} \ , \tag{123}$$

we can rewrite the above anti-commutation realtions to

$$\underbrace{\partial\Psi^a(z)\partial\Psi^b(w)} \ = \ \frac{g^{ab}}{(z-w)^2} \ . \tag{124}$$

Upgrading the stress tensor to an properly defined quantized operator we can calculate the Lie data of the CFT system: $\partial\Psi$ behaves as a genuine dimension 1 principle field and the conformal anomaly results in

$$c = -2 \ . \tag{125}$$

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