SPIN STRUCTURES ON MANIFOLDS

by J. Milnor

Let M be an oriented, Riemannian manifold. Then the tangent bundle of M has the rotation group $SO(n)$ as structural group. If it is possible to replace $SO(n)$ by the 2-fold covering group Spin (n) as structural group, then one says that M can be given a " spin structure ". The object of this note will be to make this concept precise, and to discuss the related concept of " spin cobordism ".

Let me take this opportunity to point out an error in a previous paper. The definition of spinor cobordism group which was proposed in my paper " A survey of cobordism theory " [7, §2, Example 4] is erroneous. A corrected version of this definition will be given at the end of the present paper.

Let ξ denote a principle fibre bundle with structural group $SO(n)$. Here n can be any positive integer. The value $n = +\infty$ is also acceptable. The total space of ξ will be denoted by $E(\xi)$ and the base space by B. We will always assume that B is a CW-complex, or a manifold.

Definition: A *spin structure* on ξ is a pair (η, f) consisting of

(1) A principal bundle η over B with the spinor group Spin (n) as structural group; and

(2) A map $f \colon E(\eta) \to E(\xi)$ such that the following diagram is commutative.

$$\begin{array}{ccccc} E(\eta) \times \mathrm{Spin}\,(n) & \xrightarrow{\text{right transl.}} & E(\eta) & \searrow & \\ \downarrow f \times \lambda & & \downarrow & & B \\ E(\xi) \times SO(n) & \xrightarrow{\text{right transl.}} & E(\xi) & \nearrow & \end{array}$$

Here λ denotes the standard homomorphism from Spin (n) to $SO(n)$.

This definition must be qualified as follows. *A second spin structure (η', f') on ξ should be identified with (η, f) if there exists an isomorphism g from η' to η so that $f \circ g = f'$.*

Note that the definition makes sense even in the special cases $n = 2$ and $n = 1$. It is to be understood that Spin (2) is the 2-fold covering group of the circle $SO(2)$; and that Spin (1) is a cyclic group of order 2.

As an example, the tangent bundle of the 2-sphere S^2 has a unique spin structure (η, f) where $E(\eta)$ is a 3-sphere. The tangent bundle of the circle S^1 has two distinct spin structures.

The above definition is straightforward, but is rather cumbersome. An elegant variant was suggested to the author by M. Hirsch:

Alternative definition 1: A *spin structure* on ξ is a cohomology class[1] $\sigma \varepsilon H^1 (E(\xi); J_2)$ whose restriction to each fibre is a generator of the cyclic group H^1 (Fibre; J_2). (Note: This last clause disappears in the special case of an $SO(1)$-bundle.)

The idea is the following: Any such cohomology class determines a 2-fold covering of $E(\xi)$. This 2-fold covering space is to be taken as the total space $E(\eta)$. The condition on $\sigma \mid$ Fibre guarantees that each fibre is covered by a copy of Spin(n), the unique 2-fold covering of $SO(n)$. With this interpretation it is not difficult to show that the alternative definition is completely equivalent to the original. Henceforth we will use the two definitions interchangeably.

It is known that an $SO(n)$-bundle can be given a spin structure if and only if its Stiefel-Whitney class w_2 is zero. (Compare Borel and Hirzebruch [2, pg. 350].)

Lemma: *If $w_2(\xi) = 0$ then the number of distinct spin structures on ξ is equal to the number of elements in $H^1(B; J_2)$.*

Proof: If B is connected, then this follows from the exact sequence

$$0 \to H^1(B; J_2) \to H^1(E(\xi); J_2) \to H^1(SO(n); J_2) \to H^2(B; J_2)$$

which can be extracted from the spectral sequence of the fibration ξ. The general case follows easily.

[These facts suggest an analogy between the concept of " spin structure " for $SO(n)$-bundles and the concept of " orientation " for $O(n)$-bundles. Thus an $O(n)$-bundle can be oriented if and

[1]) The notation J_2 is used for the integers modulo 2.

only if $w_1 = 0$. If $w_1 = 0$ then the number of distinct orientations is equal to the number of elements in $H^0(B;\ J_2)$.]

Now a word of warning. It may happen that two spin structures (η, f) and (η', f') on ξ are distinct, even though the corresponding spinor group bundles η and η' are isomorphic. As an illustration, consider the following.

Example: Let ε^n denote the trivial $SO(n)$-bundle over the real projective plane P^2. Since $H^1(P^2;\ J_2) \cong J_2$ this bundle can be given two distinct spin structures. For $n = 1$ or for $n = 2$ the two corresponding Spin(n)-bundles are distinct from each other. However for $n > 2$ it can be shown that the two Spin(n)-bundles are isomorphic: in fact both are trivial.

[This example suggests the conjecture that if (η, f) and (η', f') are two spin structures on the same $SO(n)$-bundle, with $n >$ dim B, then η is necessarily isomorphic to η'. The analagous statement for orientations of an $O(n)$-bundle is known to be true.]

Now we will restrict attention to tangent bundles.

Definition: A *spin manifold* will mean an oriented Riemannian manifold M, together with a spin structure on the tangent bundle of M.

To be more explicit let FM denote the space of oriented orthonormal n-frames on M. Then we will think of the spin structure as being a cohomology class $\sigma \in H^1(FM;\ J_2)$ whose restriction to each fibre is non-trivial (if $n > 1$).

The notation (M, σ) will be used for such a spin manifold. However if M happens to be simply connected, so that σ is uniquely determined, then we will simply say that M is a spin manifold.

Suppose that V is a k-dimensional submanifold of M with a specified field of normal $(n - k)$-frames. Then $FV \subset FM$; hence any spin structure σ on M determines a spin structure $\sigma \mid FV$ on V. In particular this is true if V is the boundary ∂M.

Definition: A closed spin manifold (V, σ_1) will be called a *spin boundary* if there exists a compact spin manifold (M, σ) with $\partial M = V$ and $\sigma \mid FV = \sigma_1$.

As an illustration consider the following.

Example: The 2-dimensional disk D^2 has a unique spin structure σ. Restricting to the boundary $\partial D^2 = S^1$ we obtain the non-zero cohomology class

$$\sigma_1 \in H^1\,(FS^1;\; J_2) \cong J_2.$$

Thus (S^1, σ_1) is a spin boundary. On the other hand if we take the zero cohomology class in $H^1(FS^1;\; J_2)$ (this is permissible since $n = 1$) we obtain a different spin manifold $(S^1, 0)$. It can be shown that $(S^1, 0)$ is not a spin boundary.

Similarly one can define the relation of cobordism between closed n-dimensional spin manifolds. The corresponding cobordism group will be denoted by Ω_n^{spin}. Here is a list of the first eight spin cobordism groups.

$\Omega_0^{\text{spin}} \cong J$ (infinite cyclic) by definition.

$\Omega_1^{\text{spin}} \cong J_2$ generated by $(S^1, 0)$.

$\Omega_2^{\text{spin}} \cong J_2$ generated by the torus with a suitable spin structure.

$\Omega_3^{\text{spin}} = 0$.

$\Omega_4^{\text{spin}} \cong J$ generated by a Kummer surface K^4. (Compare [5 pg. 127].)

$\Omega_5^{\text{spin}} = 0$.

$\Omega_6^{\text{spin}} = 0$. [1])

$\Omega_7^{\text{spin}} = 0$.

$\Omega_8^{\text{spin}} \cong J \oplus J$ generated by the quaternion projective plane and by a manifold L^8 such that $L^8 + L^8 + L^8 + L^8$ is spin cobordant to $K^4 \times K^4$. Alternatively, as second generator, one could take the almost parallelizable manifold M_0^8 of reference [4].

It follows that in dimensions 4 and 8 the spin cobordism class of a manifold (V, σ) is completely determined by the Pontrjagin numbers of V. In dimension 4 the Pontrjagin number $p_1\,[V^4]$ is subject only to Rohlin's relation

$$p_1\,[V^4] \equiv O \qquad (\text{mod } 48).$$

1) Compare Wall [8, p. 428].

In dimension 8 the two Pontrjagin numbers are subject only to the Borel-Hirzebruch relation

$$7p_1^2[V^8] \equiv 4p_2[V^8] \pmod{5760}.$$

(Compare references [1, Corollary 2], [3, §3.1] and [4].)

The computation of these eight groups is similar to the usual computations in cobordism theory. Thus one first shows that Ω_n^{spin} is isomorphic to the stable homotopy group $\pi_{n+k}\, M(\mathrm{Spin}(k))$ of a suitable Thom complex; and then determines these homotopy groups by a formidable computation. No details will be given.

For $n \geqq 6$ the spin cobordism group can also be interpreted as the cobordism group for the class of 2-connected oriented Riemannian manifolds. In fact if $n \geqq 6$ then:

Assertion 1: Any closed n-dimensional spin manifold is spin cobordant to a 2-connected manifold.

Assertion 2: If a 2-connected n-manifold is a spin boundary, then it bounds a 2-connected manifold. Proofs are easily given using the technique of surgery (=spherical modification) which is described in references [6], [9].

In conclusion let me mention two other variant definitions for the concept of spin structure, which may prove useful for special purposes. We will assume that $n \geqq 2$.

Let ξ be an $SO(n)$-bundle over a CW-complex B. The k-skeleton of B will be denoted by B^k.

Alternative definitions 2: A *spin structure* on ξ is a homotopy class of cross-sections of $\xi \mid B^1$ which can be extended to cross-sections of $\xi \mid B^2$.

It can be shown that every " spin structure " in this sense determines a spin structure in the original sense, and conversely. No details will be given.

Now let the group Spin(n) act on a high dimensional sphere S^N in such a way that the cyclic subgroup $J_2 \subset \mathrm{Spin}(n)$ acts freely on S^N. [Such an action can be obtained by using a spinor representation of the group Spin(n).] We will assume that $N > \dim B$.

Then the quotient group $SO(n) = \mathrm{Spin}(n)/J_2$ acts on the quotient space $\mathrm{P}^N = S^N/J_2$. Hence to every $SO(n)$-bundle ξ over B there corresponds an associated bundle ξ' having the projective space P^N as fibre.

Alternative definition 3: A *spin structure* on ξ is a homotopy class of cross-sections of the associated bundle ξ'.

Again it can be seen that this definition is equivalent to the original definition.

REFERENCES

[1] Atiyah, M. and F. Hirzebruch, Riemann-Roch theorems for differentiable manifolds, *Bull. Amer. Math. Soc.*, 65 (1959), 276-281.

[2] Borel, A. and F. Hirzebruch, Characteristic classes and homogeneous spaces II, *Amer. Journ. Math.*, 81 (1959), 315-382.

[3] —— Characteristic classes and homogeneous spaces III, *Amer. Journ. Math.*, 82 (1960), 491-504.

[4] Kervaire, M. and J. Milnor, Bernoulli numbers, homotopy groups, and a theorem of Rohlin, *Proc. Intern. Congress Math.*, Edinburgh 1958 (1960), 454-458.

[5] Milnor, J., On simply connected 4-manifolds, *Symposium Internacional de Topologia Algebraica, Mexico* (1958), 122-128.

[6] —— A procedure for killing homotopy groups of differentiable manifolds, *A.M.S. Symposia in Pure Math. III Differential geometry, Providence* (1961), 39-55.

[7] —— A survey of cobordism theory, *L'Enseignement Math.*, 8 (1962), 16-23.

[8] Wall, C. T. C., Killing the middle homotopy groups of odd dimensional manifolds, *Trans. Amer. Math. Soc.*, 103 (1962), 421-433.

[9] Wallace, A. H., Modifications and cobounding manifolds, *Canadian J. Math.*, 12 (1960), 503-528.

Princeton University.