for example [Steenrod, 1951, p. 21].) In still another point of view, one uses collections of smooth functions on open subsets. (Compare [de Rham].) All of these definitions are equivalent.

In conclusion here are three problems for the reader. The first two of these will play an important role in later sections.

Problem 1-A. Let $M_1 \subset R^A$ and $M_2 \subset R^B$ be smooth manifolds. Show that $M_1 \times M_2 \subset R^A \times R^B$ is a smooth manifold, and that the tangent manifold $D(M_1 \times M_2)$ is canonically diffeomorphic to the product $DM_1 \times DM_2$. Note that a function $x \mapsto (f_1(x), f_2(x))$ from M to $M_1 \times M_2$ is smooth if and only if both $f_1 : M \to M_1$ and $f_2 : M \to M_2$ are smooth.

Problem 1-B. Let P^n denote the set of all lines through the origin in the coordinate space R^{n+1}. Define a function

$$q : R^{n+1} - \{0\} \to P^n$$

by $q(x) = Rx$ = line through x. Let F denote the set of all functions $f : P^n \to R$ such that $f \circ q$ is smooth.

a) Show that F is a smoothness structure on P^n. The resulting smooth manifold is called the *real projective space* of dimension n.

b) Show that the functions $f_{ij}(Rx) = x_i x_j / \Sigma x_k^2$ define a diffeomorphism between P^n and the submanifold of $R^{(n+1)^2}$ consisting of all symmetric $(n+1) \times (n+1)$ matrices A of trace 1 satisfying $AA = A$.

c) Show that P^n is compact, and that a subset $V \subset P^n$ is open if and only if $q^{-1}(V)$ is open.

Problem 1-C. For any smooth manifold M show that the collection $F = C^\infty(M, R)$ of smooth real valued functions on M can be made into a ring, and that every point $x \in M$ determines a ring homomorphism $F \to R$ and hence a maximal ideal in F. If M is compact, show that every maximal ideal in F arises in this way from a point of M. More generally, if there is a countable basis for the topology of M, show that every ring homomorphism $F \to R$ is obtained in this way. (Make use of an element $f \geq 0$ in F such that each $f^{-1}[0, c]$ is compact.) Thus the smooth manifold M is completely determined by the ring F. For $x \in M$, show that any R-linear mapping $X : F \to R$ satisfying $X(fg) = X(f)\,g(x) + f(x)\,X(g)$ is given by $X(f) = Df_x(v)$ for some uniquely determined vector $v \in DM_x$.