From action-angle coordinates to geometric quantization and back

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Outline

1. Quantization: The general picture
2. Bohr-Sommerfeld leaves of regular fibrations and action-angle coordinates
3. Quantization via sheaf cohomology
4. Singular polarizations and quantization using singular action-angle coordinates
5. Action-angle coordinates for Poisson manifolds and Geometric Quantization

Today I will talk about

Today I will talk about my work with Mark Hamilton on geometric quantization of real polarizations with singularities in dimension 2.

I will also mention the case of Lagrangian foliations (joint work with Fran Presas) and my project with Mark Hamilton and Romero Solha about integrable systems with singularities in higher dimensions.

Classical vs. Quantum: a love & hate story

1. Classical systems
2. Observables $C^{\infty}(M)$
3. Bracket $\{f, g\}$

1. Quantum System
2. Operators in $\mathcal{H}$ (Hilbert)
3. Commutator
   $[A, B]_h = \frac{2\pi i}{h}(AB - BA)$

Quantization via real polarizations (mathematically speaking)

- Let (M^{2n}, ω) be a symplectic manifold such that $[\omega]$ is integral. (This condition allows to "copy" a successful pre-quantization scheme for $T^*(M)$ due to Segal).
- $(\mathbb{L}, \nabla)$ the complex line bundle with a connection ∇ over M that satisfies $curv(\nabla) = \omega$.
- The symplectic manifold (M^{2n}, ω) is called prequantizable and the pair $(\mathbb{L}, \nabla)$ is called *a prequantum line bundle* of (M^{2n}, ω).
- A real polarization $\mathcal{P}$ is a foliation whose leaves are lagrangian submanifolds.
- Integrable systems provide natural examples of real polarizations. If the manifold M is compact the "moment map": $F : M^{2n} \to \mathbb{R}^n$ has singularities that correspond to *equilibria*.
- Flat sections along $\mathcal{P}$ are given by the equation $\nabla_X s = 0$ for any vector field X tangent to $\mathcal{P}$.

Bohr-Sommerfeld leaves

Definition

A Bohr-Sommerfeld leaf is a leaf of a polarization that admits sections globally defined along it.

Example

Consider $M = S^1 \times \mathbb{R}$ and $\omega = dt \wedge d\theta$. Take as $\mathbb{L}$ the trivial bundle with connection 1-form $\Theta = t d\theta$. Now, let $\mathcal{P} =< \frac{\partial}{\partial \theta} >$

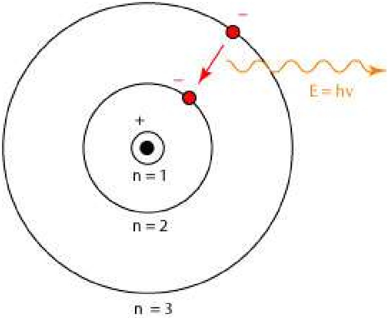
Then flat sections satisfy,

$$\nabla_X \sigma = X(\sigma) - i < \theta, X > \sigma.$$

Thus $\sigma(t, \theta) = a(t).e^{it\theta}$.

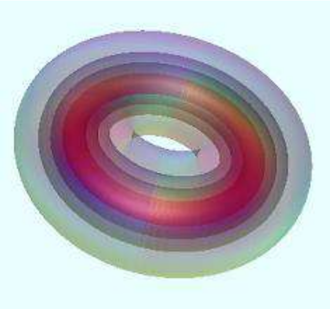
Then Bohr-Sommerfeld leaves are given by the condition $t = 2\pi k, k \in \mathbb{Z}$.

An image is worth more than a hundred words...



The theorem of Liouville-Mineur-Arnold and Bohr-Sommerfeld leaves

For integrable systems, we can mimic this construction whenever the connection 1-form can be chosen in action-angle coordinates.



The orbits of an integrable system in a neighbourhood of a compact orbit are tori. In action-angle coordinates (p_i, θ_i) the foliation is given by the fibration $\{p_i = c_i\}$ and the symplectic structure is Darboux $\omega = \sum_{i=1}^{n} dp_i \wedge d\theta_i$.

The computation above asserts that only "integral" Liouville tori are Bohr-Sommerfeld leaves.

Bohr-Sommerfeld leaves: continued...

Theorem (Guillemin-Sternberg)

If the polarization is a regular fibration with compact leaves over a simply connected base B, then the Bohr-Sommerfeld set is discrete and is given by the condition,

$$BS = \{p \in M, (f_1(p), \ldots, f_n(p)) \in \mathbb{Z}^n\}$$

where $f_1, \ldots, f_n$ are global action coordinates on B.

1. This result connects with Arnold-Liouville theorem for integrable systems in a neighbourhood of a compact connected invariant manifold, where a set of "action-angle" coordinates is constructed.
2. In the case of toric manifolds the base B may be identified with the image of the moment map by the toric action.

Bohr-Sommerfeld leaves and Delzant polytopes

Theorem (Delzant)

Toric manifolds are classified by Delzant's polytopes. More specifically, the bijective correspondence between these two sets is given by the image of the moment map:

$$\begin{array}{ccc} \{\textit{toric manifolds}\} & \longrightarrow & \{\textit{Delzant polytopes}\} \\ (M^{2n}, \omega, \mathbb{T}^n, F) & \longrightarrow & F(M) \end{array}$$

The image of a toric manifold is a Delzant Polytope.

Sniatycki result: Quantizing is counting Bohr-Sommerfeld leaves

- It would make sense to "quantize" these systems counting Bohr-Sommerfeld leaves.
- In the case the polarization is an integrable system with global action-angle coordinates, Bohr-Sommerfeld leaves are just "integral" Liouville tori.

This is the content of Snyaticki theorem:

Theorem (Sniatycki)

If the leaf space B^n is Hausdorff and the natural projection $\pi : M^{2n} \to B^n$ is a fibration with compact fibers, then quantization dimension is given by the sheer count of Bohr-Sommerfeld leaves.

But how exactly?

Quantization: The cohomological approach

- Following the idea of Kostant, in the case there are no global sections we define the quantization of $(M^{2n}, \omega, \mathbb{L}, \nabla, P)$ as

$$\mathcal{Q}(M) = \bigoplus_{k \geq 0} H^k(M, \mathcal{J}).$$

- $\mathcal{J}$ is the sheaf of flat sections, i.e.: the space of local sections σ of $\mathbb{L}$ such that $\nabla_X \sigma = 0$, for all sections X of P.

Then quantization is given by precisely:

Theorem (Sniatycki)

If the leaf space B^n is a Hausdorff manifold and the natural projection $\pi : M^{2n} \to B^n$ is a fibration with compact fibres, then $\mathcal{Q}(M^{2n}) = H^n(M^{2n}, \mathcal{J})$, and the dimension of $H^n(M^{2n}, \mathcal{J})$ is the number of Bohr-Sommerfeld leaves.

What is this cohomology?

1. Define the sheaf: $\Omega^i_{\mathcal{P}}(U) = \Gamma(U, \wedge^i \mathcal{P})$..
2. Define $\mathcal{C}$ as the sheaf of complex-valued functions that are locally constant along $\mathcal{P}$. Consider the natural (fine) resolution

$$0 \to \mathcal{C} \xrightarrow{i} \Omega^0_{\mathcal{P}} \xrightarrow{d_{\mathcal{P}}} \Omega^1_{\mathcal{P}} \xrightarrow{d_{\mathcal{P}}} \Omega^1_{\mathcal{P}} \xrightarrow{d_{\mathcal{P}}} \Omega^2_{\mathcal{P}} \xrightarrow{d_{\mathcal{P}}} \cdots$$

The differential operator $d_{\mathcal{P}}$ is the restriction of the exterior differential to the directions of the distributions (as in foliated cohomology).

3. Use this resolution to obtain a fine resolution of $\mathcal{J}$ by twisting the previous resolution with the sheaf $\mathcal{J}$ ($\mathcal{S}$ is the sheaf of sections).

$$0 \to \mathcal{J} \xrightarrow{i} \mathcal{S} \xrightarrow{\nabla_{\mathcal{P}}} \mathcal{S} \otimes \Omega^1_{\mathcal{P}} \xrightarrow{\nabla_{\mathcal{P}}} \mathcal{S} \otimes \Omega^2_{\mathcal{P}} \to \cdots$$

Via an abstract De Rham theorem, fine resolutions compute the cohomology of the sheaf.

When is this computation useful?

1. This point of view of fine resolutions is useful when the foliation is regular (example Snyaticki case) to compute sheaf cohomology. In this case, we obtain:
   - $H^i(M, \mathcal{J}) = 0$ when $i \neq n$.
   - $H^n(M, \mathcal{J}) =_{b \in BS} \mathbb{C}_b$.
2. It also works when the fibers are not torus but cylinder (partial action-angle coordinates). Then cohomology captures non-vanishing cycles on Lagrangian leaves with no-holonomy around them (Bohr-Sommerfeld leaves). If an invariant complete leaf is $\mathbb{T}^k \times \mathbb{R}^{n-k}$.
   - $H^i(M, \mathcal{J}) = 0$ when $i \neq k$.
   - $H^k(M, \mathcal{J}) =_{b \in BS} \mathbb{C}_b$

Applications to the general case of Lagrangian foliations

This fine resolution approach can be useful to compute the case of some Lagrangian foliations (which do not come from integrable systems).

Classification of foliations on the torus is given by Kneser-Denjoy-Schwartz theorem.

The case of the torus: irrational slope.

Consider $X_\eta = \eta\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$, with $\eta \in \mathbb{R} \setminus \mathbb{Q}$. This vector field descends to a vector field in the quotient torus that we still denote X_η. Denote by $\mathcal{P}_\eta$ the associated foliation in $\mathbb{T}^2$.

Theorem (Presas-Miranda)

Let $(\mathbb{T}^2, \omega)$ be the 2-torus with a symplectic structure ω of integer class. Let $\mathcal{P}_\eta$ the irrational foliation of slope η in this manifold. Then $\mathcal{Q}(T^2, \mathcal{J})$ is always infinite dimensional.
However, if we compute the limit case of the foliated cohomology ($\omega = 0$), we obtain that $\mathcal{Q}(\mathbb{T}^2, \mathcal{J}) = \mathbb{C} \bigoplus \mathbb{C}$ if the irrationality measure of η is finite and $\mathcal{Q}(\mathbb{T}^2, \mathcal{J})$ is infinite dimensional if the irrationality measure of η is infinite (i.e. it is a Liouville number).

This generalizes a result El Kacimi for foliated cohomology and reinterprets old results of Haefliger.

“Quantization Computation kit” for regular foliations

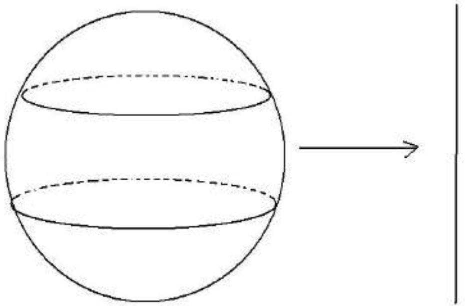
Quantization of general regular Lagrangian foliations for surfaces (joint with Presas):

We use

1. The classification of line fields (Lagrangian foliations on the torus): the Kneser-Denjoy-Schwartz theorem.
2. the local symplectic normal forms: Darboux-Caratheodory to find a connection 1-form using these “local action-angle coordinates.
3. Mayer-Vietoris formula Consider $M \leftarrow U \sqcup V \stackrel{\leftarrow}{\leftarrow} U \cap V$, then the following sequence is exact,
   $0 \to \mathcal{S} \otimes \Omega^*_{\mathcal{P}}(M) \xrightarrow{r} \mathcal{S} \otimes \Omega^*_{\mathcal{P}}(U) \oplus \mathcal{S} \otimes \Omega^*_{\mathcal{P}}(V) \xrightarrow{r_0 - r_1} \mathcal{S} \otimes \Omega^*_{\mathcal{P}}(U \cap V) \to 0.$

Toric Manifolds

What happens if we go to the edges and vertexes of Delzant's polytope? Consider the case of rotations of the sphere.



There are two leaves of the polarization which are singular and correspond to fixed points of the action.

Quantization of toric manifolds

Theorem (Hamilton)

For a $2n$-dimensional compact toric manifold and let BS_r be the set of regular Bohr-Sommerfeld leaves,

$$\mathcal{Q}(M) = H^n(M;\mathcal{J}) \cong \bigoplus_{l \in BS_r} \mathbb{C}$$

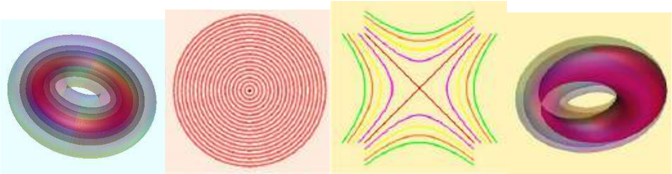
Remark: Then this geometric quantization does not see the singular points. In the example of the sphere Bohr-Sommerfeld leaves are given by integer values of height (or, equivalently) leaves which divide out the manifold in integer areas.

Action-angle coordinates with singularities

The theorem of Guillemin-Marle-Sternberg gives normal forms in a neighbourhood of fixed points of a toric action. This can be generalized to normal forms of integrable systems (not always toric) that we call non-degenerate.

Theorem (Eliasson-Miranda)

There exists symplectic Morse normal forms for integrable systems with non-degenerate singularities.



Liouville torus k_e comp. elliptic k_h hyperbolic k_f focus-focus

Description of singularities

The local model is given by $N = D^k \times \mathbb{T}^k \times D^{2(n-k)}$ and $\omega = \sum_{i=1}^{k} dp_i \wedge d\theta_i + \sum_{i=1}^{n-k} dx_i \wedge dy_i$. and the components of the moment map are:

1. Regular $f_i = p_i$ for $i = 1, ..., k$;
2. Elliptic $f_i = x_i^2 + y_i^2$ for $i = k+1, ..., k_e$;
3. Hyperbolic $f_i = x_i y_i$ for $i = k_e + 1, ..., k_e + k_h$;
4. focus-focus $f_i = x_i y_{i+1} - x_{i+1} y_i$, $f_{i+1} = x_i y_i + x_{i+1} y_{i+1}$ for $i = k_e + k_h + 2j - 1$, $j = 1, ..., k_f$.

How to compute geometric quantization in this case?

We use the approach of Cěch cohomology. Divide out the manifold in pieces and glue computations. Fix an open cover $\mathcal{A} = \{A_\alpha\}$ of M.

1. to each $(k+1)$-fold intersection of elements from the cover $\mathcal{A}$ we assign a section of $\mathcal{J}$. Let $A_{\alpha_0 \cdots \alpha_k} := A_{\alpha_0} \cap \cdots A_{\alpha_k}$,
2. A k-cochain is an assignment $f_{\alpha_0 \cdots \alpha_k} \in \mathcal{J}(A_{\alpha_0 \cdots \alpha_k})$ for each $(k+1)$-fold intersection in the cover .
3. Define a coboundary operator δ as

$$(\delta f)_{\alpha_0 \cdots \alpha_k} = \sum_{j=0}^{k} (-1)^j f_{\alpha_0 \cdots \hat{\alpha}_j \cdots \alpha_k | A_{\alpha_0 \cdots \cdots \alpha_k}}$$

4. Then the sheaf cohomology with respect to the cover is the cohomology of this complex,

$$H^k_{\mathcal{A}}(M; \mathcal{J}) = \frac{\ker \delta^k}{\mathrm{im}\, \delta^{k-1}}$$

Key point in the computation

We may choose a trivializing section of such that the potential one-form of the prequantum connection is $\Theta_0 = (xdy - ydx)$ (this uses STRONGLY the symplectic local normal forms).

Theorem

Leafwise flat sections in a neighbourhood of the singular point in the first quadrant is given by

$$a(xy)e^{\frac{i}{2}xy\ln\left|\frac{x}{y}\right|}$$

where a is a smooth complex function of one variable which is flat at the origin.

The theorem in the hyperbolic case

Theorem (Hamilton-Miranda)

The first cohomology of the neighbourhood of the figure-eight hyperbolic system has two contributions of the form $\mathbb{C}^{\mathbb{N}}$, each one corresponding to a space of Taylor series in a complex variable. It also has one $\mathbb{C}$ term for each non-singular Bohr-Sommerfeld leaf. That is,

$$\mathcal{Q}(M) = H^1(M, \mathcal{J}) \cong (\mathbb{C}^{\mathbb{N}} \oplus \mathbb{C}^{\mathbb{N}}) \oplus \bigoplus_{l \in BS_r} \mathbb{C}$$

where the sum is over the non-singular Bohr-Sommerfeld leaves.

The case of surfaces

We can use a Mayer-Vietoris argument to prove:

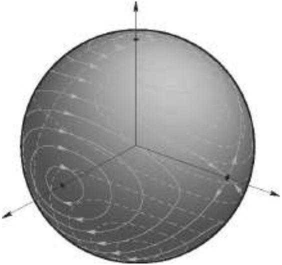
Theorem (Hamilton and Miranda)

The quantization of a compact surface endowed with an integrable system with non-degenerate singularities is given by,

$$\mathcal{Q}(M) = H^1(M; \mathcal{J}) \cong \bigoplus_{p \in \mathcal{H}} (\mathbb{C}^{\mathbb{N}} \oplus \mathbb{C}^{\mathbb{N}}) \oplus \bigoplus_{l \in BS_r} \mathbb{C} \ ,$$

where $\mathcal{H}$ is the set of hyperbolic singularities.

The case of the rigid body



Using this recipe and the quantization of this system is

$$\mathcal{Q}(M) = H^1(M; \mathcal{J}) \cong \bigoplus_{p \in \mathcal{H}} (\mathbb{C}_p^{\mathbb{N}})^2 \oplus \bigoplus_{b \in BS} \mathbb{C}_b.$$

Comparing this system with the one we get from rotations on the sphere, we obtain different results (one is infinite dimensional). Therefore this quantization depends strongly on the polarization.

Moral behind computations...in the two-dimensional singular case

1. Quantization of real polarizations systems are defined using cohomology groups with coefficients in the sheaf of flat sections.
2. In order to compute this sheaf cohomology, we can use symplectic local normal forms (action-angle coordinates with singularities).
3. These singular action-angle coordinates are defined in a neighbourhood of a non-degenerate singular point or an orbit but NOT in a neighbourhood of a singular leaf. (Eliasson, Zung, Miranda, San Vu Ngoc, Toulet...)
4. Since computations imply dealing with sheafs, symplectic local normal forms are good enough to make global computations using a Cěch approach.
5. For non-degenerate singularities on surfaces only hyperbolic singularities contribute in cohomology (with an infinite-dimensional contribution.)

What comes next? Higher-dimensional singularities...

For higher-dimensional cases, we would like to repeat the same scheme:

1. Use symplectic local normal forms to reduce the problem to a local computation.
2. Furthermore, we would like to use the fact that locally the singularities can be decomposed like a symplectic product to avoid further computations and reduce to two-dimensional blocks (for hyperbolic and elliptic singularities) and four-dimensional blocks for focus-focus singularities.
3. Key point in computations: Do we have a Künneth formula for this kind of sheafs?

Higher dimensions. The conjecture

Miranda-Solha (work in progress)

The quantization of a $2n$-dimensional symplectic manifold with singular polarization given by an integrable system with non-degenerate singularities is determined (via a Künneth-type formula) from

1. The regular Bohr-Sommerfeld leaves.
2. The hyperbolic singularities (Bohr-Sommmerfeld or not).

Focus-focus contribution

Focus-focus singularities make no contribution in quantization. In order to prove this, for this we make a strong use of a Hamiltonian S^1-action defined in a neighbourhood of a focus-focus leaf.

A Künneth formula for general Lagrangian foliations?

For general foliations, a Künneth formula is not always available but for integrable systems with non-degenerate singularities we have the following:

Theorem

Let $(M_1, \mathcal{P}_1)$ and $(M_2, \mathcal{P}_2)$ be a pair of symplectic manifolds endowed with integrable systems with non-degenerate singularities. The natural cartesian product for the foliations is Lagrangian with respect to the product symplectic structure. Let $\mathcal{J}_{12}$ be the induced sheaf of basic sections, then,

$$H^n(M_1 \times M_2, \mathcal{J}_{12}) = \bigoplus_{p+q=n} H^p(M_1, \mathcal{J}_1) \hat{\otimes} H^q(M_2, \mathcal{J}_2).$$

where ˆ stands for the completed tensor product as topological spaces.

A Künneth formula for general Lagrangian foliations?

Topological structure on these cohomological spaces

Since via the abstract De Rham theory we have identified we quantization with cohomology of a complex of smooth forms, we have an induced topological structure using the topology of the space of forms and the fact that for differential forms the differential operator d is continuous.

A sheaf digression...

These topological spaces are sometimes of infinite dimension (like in the hyperbolic case). Künneth formula for sheafs has been studied by many authors Kaup, Mümken... They use the notion of "completion" of the topological tensor product (see Grothendieck's thesis).

Theorem (Kaup)

For Fréchet sheaves $\mathcal{F}$ and $\mathcal{G}$ over compact M and N we have $H^n(M \times N, \mathcal{F} \times \mathcal{G}) = \bigoplus_{p+q=n} H^p(M, \mathcal{F}) \hat{\otimes} H^q(N, \mathcal{G}).$ *whenever,*

1. *The cohomology groups are Hausdorff.*
2. *Either $\mathcal{F}$ or $\mathcal{G}$ are nuclear.*

These conditions are fulfilled in the integrable case but not for general Lagrangian foliations (the spaces may fail to be Hausdorff). The case of general Lagrangian foliations is work in progress with Fran Presas.

Quantization and reduction

A conjecture: Quantization commutes with reduction

There is a quantization commutes with reduction principle for this model if singularities are non-degenerate.

Idea: we can use equivariant singular action-angle coordinates (Miranda-Zung, 2003) to obtain an equivariant connection 1-form and then prove that:

$$Q(M//G) = Q(M)^G$$

Next stop: Poisson manifolds...

1. Conclusion: Geometric Quantization of generic (regular or singular non-degenerate) integrable systems is determined using (eventually singular) action-angle coordinates.
2. Some integrable systems (commutative and non-commutative) like Gelfand-Ceitlin system, magnetic flow etc...are defined naturally on coadjoint orbits of the dual of a Lie algebra or, more generally, in a Poisson manifold.
3. We now want to "define" geometric quantization of these more general systems using "singular" action-angle coordinates (like collective first integrals).
4. Do we have such Poisson local normal forms theorems?

In the pipeline...

The Poisson case. (this is joint work in progress with Mark Hamilton)

- Use Vaisman approach for prequantization. Study the exact case.
- Use action-angle coordinates for Poisson structures (commutative and non-commutative ones) (Laurent-Miranda-Vanhaecke).
- Study integrable systems which are splitted.
- Study the b-case (related to Melrose's b-calculus).