Internal Categories

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December 11, 2018

A thesis submitted for the degree of
Masters by Research in Mathematics

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Contents

Abstract

In our first chapter, we will define categories, functors, and natural transformations internally to any category with pullbacks $\mathcal{E}$, and we will prove in detail that they form a **Cat**-enriched category, or 2-category $\mathbf{Cat}(\mathcal{E})$, with powers by $\mathbf{2}$ and any conical limits that $\mathcal{E}$ also has. Along the way we will describe how certain familiar notions of category theory can be made sense of internally. In Chapter Two we will explore how some properties of $\mathcal{E}$ are inherited by, or give rise to other properties in $\mathbf{Cat}(\mathcal{E})$. In Chapter Three we will investigate the extension of the assignment $\mathcal{E} \mapsto \mathbf{Cat}(\mathcal{E})$ to various 2-functors, and in particular equip one of them with various monad-like structures. One of these was remarked upon in [6], but to our knowledge the other two have not appeared elsewhere in the literature. Chapter Four will be an intermezzo on the Grothendieck Construction in preparation for Chapter Five, where we will explore factorisations of internal functors, including in particular the comprehensive factorisation.

Statement of Originality

This work has not previously been submitted for a degree or diploma at any university. To the best of my knowledge and belief, this thesis contains no material previously published or written by another person except where due reference is made in the thesis itself.

Adrian Toshar Miranda 19/10/18

Acknowledgements

This thesis would not have been possible without my wonderful support network. First and foremost to my supervisor Steve for sharing his wisdom and helping me grow as a mathematician. Not only has he taught me a great deal of mathematics this year, but he has also helped me immensely in improving my communication and critical thinking skills. I cannot overstate my appreciation for his patience and direction. Thanks also to Richard, Ross, Dom and Alex for their approachability and advice, particularly while Steve was on leave. Thanks to my fellow students for the opportunity to learn together, and in particular to Bryce for sharing his experiences as we faced similar hurdles alongside one another. Thanks to my partner Anusha, who has been an amazing source of emotional support and has helped me stay focussed on my goals. On top of this, she has also helped me typeset many of the intricate diagrams in this thesis, and I greatly appreciate it. Finally, to my family, who made sure I had a supportive environment in which to work and kept me motivated when times were tough.

Introduction

A small category consists of a set of 'arrows' with a distinguished subset of 'objects', or 'identities', and a partially defined binary operation called 'composition'. The pairs of arrows (f, g) for which composition is defined are precisely those whose 'source' and 'target' match in the following way: $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$. Furthermore, this operation is required to satisfy certain axioms pertaining to associativity and left and right unit laws. These notions are usually made precise in the language of sets and functions. However, only certain properties of sets and functions are needed in order for such notions to be meaningful. Specifically, one needs to be able to express the criteria for composability, which requires the pullback of sets. As such, the theory of categories, functors and natural transformations has models *internal to* any ambient category which has pullbacks, not just the category of sets and functions, **Set**.

The theory of internal categories was introduced by Ehresmann in the 1960s, with motivations originally coming from applications to differential geometry. It was soon realised that many familiar notions from category theory can also be made sense of internal to a category $\mathcal{E}$, often under certain relatively mild assumptions on $\mathcal{E}$. Our interests will be in categories with similar properties to **Set**, such as having exponentials, a subobject classifier, extensive coproducts, or a generating family. All but the last of these are properties of elementary toposes, while Grothendieck toposes also have generating families. In this thesis we will assume traditional category theory in **Set**, and use this to explore how category theory manifests itself internal to other such categories. We will look closely at the extension of the assignment $\mathcal{E} \mapsto \mathbf{Cat}(\mathcal{E})$ to various 2-functors, and in particular equip it with various monad-like structures. This includes a pseudocomonad structure which arises from a biadjunction, and a skew and coskew pseudomonad structure which we will see arise from a very general setting. Following this we will give an account of the Grothendieck Construction, which will allow us to view split fibrations over an internal category as categories internal to discrete fibrations over that category. In our final chapter we look at two factorisation systems on internal functors. One of these is often called the *full image factorisation*, whose left class consists of the isomorphism on objects internal functors and whose right class consists of the fully faithful internal functors. The other is often called the *comprehensive factorisation*, whose left class consists of the final internal functors and right class consists of the discrete fibrations. We will follow the two-step construction of this given in [30]. Following this will be some concluding remarks and an Appendix containing most of the diagrammatic calculations and tabulated data used in this thesis.

Conventions

In this thesis we will need to assume two universes of sets, $\mathfrak{S}$ and $\mathfrak{L}$, where all of the sets in $\mathfrak{S}$ are also sets in $\mathfrak{L}$. We will refer to members of $\mathfrak{S}$ as being *small*, and the members of $\mathfrak{L}$ as being *large*. In particular we will assume that if X and Y are small then so is the function set Y^X. A *small category* will be one whose set of objects and set of morphisms are both in $\mathfrak{S}$, while a *locally small* category will be one whose hom-sets are all in $\mathfrak{S}$ and set of objects is in $\mathfrak{L}$. We will on occasion consider categories whose set of objects is large but not necessarily small, such as categories of presheaves of small categories. Such categories will live in 2-categories, and we will need to place some restrictions on the size of categories which appear as objects in the 2-categories which we consider. We hence fix the following notational convention.

Notation 0.0.1.

1. *Let* **Set** *denote the locally small category of sets which are members of* $\mathfrak{S}$*, and arbitrary functions between them. Let* **Cat** *denote the* 2*-category of small categories, and arbitrary functors and natural transformations.*

2. *Let* **CAT** *denote the* 2*-category of locally small categories, and arbitrary functors and natural transformations between them. Thus if* $\mathcal{C}$ *is an object of* **CAT***, then all of its homs* $\mathcal{C}(X, Y)$ *are members of* $\mathfrak{S}$*, while its set of objects is a member of* $\mathfrak{L}$ *but not necessarily of* $\mathfrak{S}$*. For example* $\mathbf{Set} \in \mathbf{CAT}$*, but* $\mathbf{Set} \notin \mathbf{Cat}$*.*

3. *Let* 2-**Cat** *denote the* 2*-category of small* 2*-categories,* 2*-functors and* 2*-natural transformations.*

When we say that a map *preserves* a universal construction, we will mean that it does so up to a unique isomorphism coherent with the data of the universal construction. When this isomorphism is in fact the identity, we will say that the map *strictly preserves* the construction. Apart from the 2-categories listed above, there will also be more specialised 2-categories which we will need to consider. These will typically involve categories with some structure, and functors which preserve that structure.

Notation 0.0.2. *1. Let* **Lex** *denote the* 2*-category of small categories with finite limits, finite limit preserving functors, and arbitrary natural transformations.*

2. Let 2-**Lex** *denote the* 2*-category of small* 2*-categories with finite weighted limits in the sense of [18],* 2*-functors which preserve these, and arbitrary* 2*-natural transformations.*

3. Just as above, let **LEX** *and* 2-**LEX** *denote similar* 2*-categories to their uncapitalised versions, but with objects given by locally small categories with finite limits and finite weighted limits respectively.*

4. Let **LFP** *be the* 2*-category of locally finitely presentable categories, finitary right adjoints and arbitrary natural transformations.*

Notation 0.0.3. *We adopt the following notational convention for morphisms induced by universal properties. Unless otherwise stated, the morphism induced into a pullback P as in the following diagram by the equation $fh = gk$ will be denoted (h,k). In other words, (h,k) will denote the unique morphism satisfying $p\,(h,k) = h$ and $q\,(h,k) = k$, as in the diagram below. Similarly, the morphism induced into the composite of a sequence of composable spans by the morphisms $f_1, ...f_n$ will be denoted $(f_1, ..., f_n)$. We will be consistent with this notation, and provide diagrams where context is needed.*

$$\begin{tikzcd}
X \arrow[dr, dashed, "{(h,k)}"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "k"'] & & \\
& P \arrow[r, "p"] \arrow[d, "q"'] & A \arrow[d, "f"] \\
& B \arrow[r, "g"'] & C
\end{tikzcd}$$

1 Internal Categories, Functors and Natural Transformations

Our aim for this chapter is to introduce the concepts of internal categories, functors, and natural transformations, which will be central to the topic of this thesis. Defining and discussing examples of these will comprise the first three sections of this chapter, and this information will be collected in the form of a 2-category in Section 1.4. Finally in Section 1.5 we will introduce some further concepts, and prove some more preliminary results which will be used throughout the thesis.

1.1 Internal Categories

Let Δ denote the 'simplex category', whose objects are non-empty finite ordered sets and morphisms are order preserving functions. Identify each object in Δ with its representative ordered set $[n] := \{0, 1, 2..., n\}$ in the skeleton of Δ, and for $k \leq n$, let $\delta^n_k : [n] \to [n+1]$ denote the unique monotonic function whose image does not contain $k \in [n+1]$.

Definition 1.1.1. A category *internal* to a locally small category $\mathcal{E}$ is a diagram $\mathbb{C} : \Delta^{\text{op}} \to \mathcal{E}$ shown partially on the left below, which sends the pushout squares in Δ shown on the right below, to pullback squares in $\mathcal{E}$.

$$\begin{tikzcd}
...\mathbb{C}(2) \arrow[r, shift left=4] \arrow[r, leftarrow, shift left=2] \arrow[r] \arrow[r, leftarrow, shift right=2] \arrow[r, shift right=4] & \mathbb{C}(1) \arrow[r, shift left=2] \arrow[r, leftarrow] \arrow[r, shift right=2] & \mathbb{C}(0)
\end{tikzcd}
\qquad
\begin{tikzcd}
n+3 & n+2 \arrow[l, "\delta^{n+2}_3"'] & n+1 \arrow[l, "\delta^{n+1}_2"'] \\
n+2 \arrow[u, "\delta^{n+2}_0"] & n+1 \arrow[l, "\delta^{n+1}_2"] \arrow[u, "\delta^{n+1}_0"'] & n \arrow[l, "\delta^n_1"] \arrow[u, "\delta^n_0"']
\end{tikzcd}$$

In the literature, internal categories are usually defined as consisting of only the data shown above on the left, subject to certain conditions. The standard definition is given below, and it will be useful for intuition. These definitions are the same up to equivalence of categories.

Definition 1.1.2. Let $\mathcal{E}$ be a category with pullbacks. Then, a category $\mathbb{C} := (C_0, C_1, d_0, d_1, i, m)$ *internal* to $\mathcal{E}$ is given by the datum of a diagram
$$\begin{tikzcd} C_2 \arrow[r, "m"] & C_1 \arrow[r, shift left=2, "d_0"] \arrow[r, shift right=2, "d_1"'] & C_0 \arrow[l, "i" description] \end{tikzcd}$$
in $\mathcal{E}$, where C_0, C_1 are called the *object of objects* and *object of arrows* respectively, and the morphisms d_0, d_1, i, m are called *source*, *target*, *identity assigner* and *composition.* The *object of composable n-tuples* C_n is defined inductively for $n \geq 2$ by the pullbacks below, with the base case shown on the left:

$$\begin{tikzcd} C_2 \arrow[r, "\pi_{2,0}"] \arrow[d, "\pi_{2,1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_1 \arrow[d, "\pi_{1,1}"] \\ C_1 \arrow[r, "\pi_{1,0}"'] & C_0 \end{tikzcd} \quad \text{where } \pi_{1,0} := d_0 \text{ and } \pi_{1,1} := d_1, \text{ and} \quad \begin{tikzcd} C_n \arrow[r, "\pi_{n,0}"] \arrow[d, "\pi_{n,1}"'] \arrow[dr, phantom, "\lrcorner", very near start] & C_{n-1} \arrow[d, "\pi_{n-1,1}"] \\ C_{n-1} \arrow[r, "\pi_{n-1,0}"'] & C_{n-2} \end{tikzcd}$$

As a shorthand, the morphisms $\pi_{2,j}$ will be denoted as just π_j for $j \in \{0, 1\}$. This data is subject to axioms asserting the commutativity of certain diagrams in $\mathcal{E}$:

- Sources and targets for identities and composites:

$$\begin{tikzcd} C_0 \arrow[r, "i"] \arrow[dr, "1_{C_0}"'] & C_1 \arrow[d, "d_0"] \\ & C_0 \end{tikzcd} \quad \begin{tikzcd} C_0 \arrow[r, "i"] \arrow[dr, "1_{C_0}"'] & C_1 \arrow[d, "d_1"] \\ & C_0 \end{tikzcd} \quad \begin{tikzcd} C_2 \arrow[r, "m"] \arrow[d, "\pi_0"'] & C_1 \arrow[d, "d_0"] \\ C_1 \arrow[r, "d_0"'] & C_0 \end{tikzcd} \quad \begin{tikzcd} C_2 \arrow[r, "m"] \arrow[d, "\pi_1"'] & C_1 \arrow[d, "d_1"] \\ C_1 \arrow[r, "d_1"'] & C_0 \end{tikzcd}$$

- The associativity and left and right unit laws for composition:

$$\begin{tikzcd} C_3 \arrow[r, "m_0"] \arrow[d, "m_1"'] & C_2 \arrow[d, "m"] \\ C_2 \arrow[r, "m"'] & C_1 \end{tikzcd} \quad \begin{tikzcd} C_1 \arrow[r, "i_0"] \arrow[dr, "1_{C_1}"'] & C_2 \arrow[d, "m"] & C_1 \arrow[l, "i_1"'] \arrow[dl, "1_{C_1}"] \\ & C_1 & \end{tikzcd}$$

  Where the morphism $m_0 := (m\pi_{3,0}, \pi_1\pi_{3,1})$ is induced by the universal property of C_2 and the commutativity of the first diagram in Diagram 7.0.1, while $m_1 := (\pi_0\pi_{3,0}, m\pi_{3,1})$ is induced by the commutativity of the second diagram there, and the morphisms $i_0 := (id_0, 1_{C_1})$ and $i_1 := (1_{C_0}, id_1)$ are induced by the equations $d_0 1_C = d_0 = 1_{C_0} d_0 = d_1 i d_0$ and $d_1 1_{C_1} = d_1 = 1_{C_0} d_1 = d_0 i d_1$ respectively.

Definition 1.1.1 effectively identifies an internal category with its *nerve*, as will be discussed further in Remark 1.2.2. For our purposes, it will be preferable to Definition 1.1.2 since it determines the data of all objects of composable n-tuples up to equality rather than up to isomorphism, since they will be the values of the functor $\mathbb{C} : \Delta^{\mathrm{op}} \to \mathcal{E}$ on the objects $n \in \Delta$. Since these definitions are equivalent however, we will feel free to use the notation and notions as described in Definition 1.1.2. We describe these in more detail.

Remark 1.1.3. In Definition 1.1.2, m_0 and m_1 are thought of as taking composable triples and returning composable pairs obtained by composing the first two and the second two, respectively. For a morphism $(x, y, z) : X \to C_3$ in $\mathcal{E}$, we then see that $\pi_j m_k (x, y, z)$ is given by table in Diagram 7.0.2. In particular, $m (x, m (y, z)) = mm_1 (x, y, z) = mm_0 (x, y, z) = m (m (x, y), z)$, a relationship which we will refer to this as 'equational associativity'. Meanwhile, i_0 is thought of as taking a morphism and returning the composable pair consisting of that morphism and the identity on its domain, while i_1 returns the composable pair consisting of the morphism and the identity on its codomain.

Remark 1.1.4. From this definition, one can construct many of the familiar notions of category theory. For example,

- The *opposite* of an internal category, denoted $\mathbb{C}^{\mathrm{op}}$, by switching the roles of d_0 and d_1 and precomposing composition with the twist isomorphism $(\pi_1, \pi_0) : (\mathbb{C}^{\mathrm{op}})_2 \to C_2$.

- The *object of endomorphisms*, denoted C_{end}, is given by the domain of the equaliser $e := \text{Eq}(d_1, d_0)$. Then since $d_0 e = d_1 e$, there is an induced morphism $k : C_{\text{end}} \to C_2$ uniquely satisfying $\pi_0 e = \pi_1 e = k$. Thus we may further define C_{idemp}, the *object of idempotents*, to be the domain of the equaliser $\text{Eq}(mk, e)$.

- The kernel pairs of d_0, d_1, and m respectively give
  - C_{span}, the *object of spans*
  - C_{cospan}, the *object of cospans*
  - C_{Sq}, the *object of commutative squares.* This will feature in the construction of powers by **2** in Section 1.5.

- The pullback of $i : C_0 \to C_1$ along $m : C_2 \to C_1$ may be thought of as the 'object of composable pairs whose composite is an identity', or 'one-sided inverses' C_{inv}. To get the 'object of isomorphisms' C_{iso}, one takes the intersection (or pullback) of C_{inv} with $C_{\text{inv}}^{\text{op}}$. An internal category may be called an *internal groupoid* if the canonical map $C_{\text{iso}} \to C_1$ is invertible.

- If $\mathcal{E}$ has products and the morphism $(d_0, d_1) : C_1 \to C_0 \times C_0$ induced by the universal property of the product is a monomorphism, then we may call $\mathbb{C}$ an *internal poset.* If $\mathcal{E}$ has a terminal object, we may call $\mathbb{C}$ a *monoid* if C_0 is the terminal object.

Finally, we mention an alternative view on internal categories as monads in the bicategory $\mathbf{Span}(\mathcal{E})$, which is treated in more detail in [5]. These are spans $C_0 \xleftarrow[d_0]{} C_1 \xrightarrow{d_1} C_0$ with unit and multiplication, which correspond to identity assigners and composition respectively. Sources and targets for identities and composites say precisely that these are well defined as morphisms of spans, while the associativity and left and right unit axioms for the monad and internal category data coincide. Then an action of the internal category is a map $X \to C_0$ equipped with an algebra structure.

1.2 Internal Functors

Definition 1.2.1. Let $\mathcal{E}$ be a category with pullbacks and let $\mathbb{A}, \mathbb{B} : \Delta^{\text{op}} \to \mathcal{E}$ be categories internal to $\mathcal{E}$. An *internal functor* from $\mathbb{A}$ to $\mathbb{B}$ is a natural transformation $f : \mathbb{A} \Rightarrow \mathbb{B}$.

Remark 1.2.2. Once again, it is more common in the literature for internal functors to be defined explicitly as given by a *component on objects* $f_0 : A_0 \to B_0$ and a *component on arrows* $f_1 : A_1 \to B_1$ in $\mathcal{E}$ which satisfy the commutativity of the diagrams shown below. Here the morphism $f_2 := (f_1\pi_0, f_1\pi_1)$, is induced by the universal property of B_2 given the commutativity of the third diagram in Diagram 7.0.1. All other components of f are determined from this information by the universal property of the objects B_n in a similar way. The diagrams express f's respect for sources, targets, identities, and composition, and they all follow from the definition given above by naturality of f.

$$\begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "d_0"'] & B_1 \arrow[d, "d_0"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd} \qquad \begin{tikzcd} A_1 \arrow[r, "f_1"] \arrow[d, "d_1"'] & B_1 \arrow[d, "d_1"] \\ A_0 \arrow[r, "f_0"'] & B_0 \end{tikzcd} \qquad \begin{tikzcd} A_0 \arrow[r, "f_0"] \arrow[d, "i"'] & B_0 \arrow[d, "i"] \\ A_1 \arrow[r, "f_1"'] & B_1 \end{tikzcd} \qquad \begin{tikzcd} A_2 \arrow[r, "f_2"] \arrow[d, "m"'] & B_2 \arrow[d, "m"] \\ A_1 \arrow[r, "f_1"'] & B_1 \end{tikzcd}$$

The morphism f_2 is thought of as taking a composable pair in A and returning the composable pair given by its image under f. Given $(x, y) : X \to A_2$, the morphism f_2 composes with (x, y) to give $(f_1 x, f_1 y)$, and so the equation $f_1 m(x, y) = m(f_1 x, f_1 y)$ follows by respect for composition. Finally, note that it is clear from the definition that if $(f_0, f_1) : \mathbb{A} \to \mathbb{B}$ is an internal functor, then the same data defines an internal functor $f^{\text{op}} : \mathbb{A}^{\text{op}} \to \mathbb{B}^{\text{op}}$.

It is evident from their definition that internal categories and internal functors form a category, in fact a full subcategory of $[\Delta^{\text{op}}, \mathcal{E}]$. We write this category as $\mathbf{Cat}(\mathcal{E})_1$, using the subscript '1' to distinguish it from the 2-category which will be the subject of Section 1.4. The inclusion functor $N : \mathbf{Cat}(\mathcal{E}) \to [\Delta^{\mathrm{op}}, \mathcal{E}]$, which sends an internal category to its underlying simplicial object in $\mathcal{E}$, is often called the *nerve*. Thus the nerve may be thought of as forgetting that the objects of composable n-tuples need to be pullbacks, and a simplicial object in $\mathcal{E}$ is conversely an internal category precisely if they are. In particular, $\mathbf{Cat}(\mathcal{E})_1$ is small (resp. locally small) if $\mathcal{E}$ is small (resp. locally small), since Δ^{op} is certainly small.

The following proposition, which will be rephrased in Remark 1.2.6, will be used repeatedly throughout this thesis.

Proposition 1.2.3. *Let $f, g : \mathbb{A} \to \mathbb{B}$ be internal functors in $\mathcal{E}$ such that $f_1 = g_1$. Then $f = g$.*

Proof. Since $f_1 = g_1$, in particular $f_1 i_A = g_1 i_A$. Since f and g both preserve identities, this is equivalent to saying that $i_B f_0 = i_B g_0$. But by sources (or targets) for identities in $\mathbb{B}$, we may compose these equal morphisms in $\mathcal{E}$ with the source (or target) map of $\mathbb{B}$ to see that $f_0 = g_0$. ☐

The remark below describes a few different types of internal functors. Of particular interest are fully-faithful functors, and discrete fibrations which will both feature as the right class of orthogonal factorisation systems in $\mathbf{Cat}(\mathcal{E})$, as we will see in Sections 5.1 and 5.4. Following this remark will be a characterisation of monomorphisms in $\mathbf{Cat}(\mathcal{E})_1$.

Remark 1.2.4. One can define various familiar notions on internal functors $f : \mathbb{A} \to \mathbb{B}$.

- If the component on objects of an internal functor is an isomorphism, we may call it *isomorphism on objects.*
- If the commutative square attesting f's respect for sources or targets is in fact a pullback, we may call f a discrete opfibration or discrete fibration, respectively.
- If the square attesting f's respect for identities is a pullback, we may say that f has the property of reflecting identities.
- If the square attesting f's respect for composition is a pullback, we may call f a *discrete Conduché* functor. These may be described externally as having the special property of 'uniquely lifting factorisations'.
- Finally, f may be called *fully-faithful* if the squares attesting f's respect for sources and targets, when joined along f_1, exhibit A_1 as a limit of the zigzag $A_0 \xrightarrow{f_0} B_0 \xleftarrow[d_1]{} B_1 \xrightarrow{d_0} B_0 \xleftarrow[f_0]{} A_0$.

The component on objects of an internally fully faithful functor need not be a monomorphism, but given an internal category $\mathbb{B}$, a monomorphism $g : X_0 \to B_0$ gives rise to a fully-faithful internal functor into $\mathbb{B}$ in a canonical way, analogous to the external construction of the full-subcategory on a subset of objects. Explicitly, we may set X_1 to be the limit of a zigzag similar to the above with g in place of f_0 and then realise X_0 and X_1 to be the object of objects and object of arrows of an internal category by taking the projections as source and target maps, and inducing identities and composition maps from those of $\mathbb{B}$. Indeed, this construction does not require g to be a monomorphism. When $\mathcal{E}$ has products the condition for being fully-faithful simplifies to the condition that the first diagram in Diagram 7.0.3 is a pullback. This square always at least commutes, for any internal functor, given the commutativity of the second diagram in Diagram 7.0.3 for $j \in \{0, 1\}$. Thus we always have a morphism $f' : A_1 \to P$ into the pullback of the cospan, even if it is not itself isomorphic to A_1. We may call the internal functor *faithful* if the induced morphism f' is a monomorphism in $\mathcal{E}$.

Proposition 1.2.5. *An internal functor $f : \mathbb{A} \to \mathbb{B}$ is a monomorphism in $\mathbf{Cat}(\mathcal{E})_1$ if and only if it is faithful and f_0 is a monomorphism.*

Proof. Note that $f = (f_0, f_1)$ is a monomorphism in $\mathbf{Cat}(\mathcal{E})_1$ if and only if both f_0 and f_1 are monomorphisms in $\mathcal{E}$. Thus it suffices to show that under the assumption that f_0 is a monomorphism, f_1 is a monomorphism if and only if f is faithful. But if f_0 is a monomorphism, then so is $f_0 \times f_0$, since the diagonal functor $\mathcal{E} \to \mathcal{E} \times \mathcal{E}$ preserves limits. Hence, since monomorphisms are stable under pullback, the pullback of $f_0 \times f_0$ along $(d_0, d_1) : B_1 \to B_0 \times B_0$ is also a monomorphism. The statement then follows from the two-out-of-three property for monomorphisms. □

We now discuss some functors associated to $\mathbf{Cat}(\mathcal{E})_1$. Since $[\Delta^{\mathrm{op}}, \mathcal{E}]$ is a functor category, it has evaluation functors $\mathrm{ev}_n : [\Delta^{\mathrm{op}}, \mathcal{E}] \to \mathcal{E}$ for all $n \in \Delta^{\mathrm{op}}$, and evaluation natural transformations $\mathrm{ev}_\phi : \mathrm{ev}_n \Rightarrow \mathrm{ev}_m$ for all $\phi \in \Delta(m, n)$.

Remark 1.2.6. For all non-negative integers n, there is a functor $\mathrm{ev}_n^{\mathcal{E}} : \mathbf{Cat}(\mathcal{E})_1 \to \mathcal{E}$ defined as the nerve functor of Remark 1.2.2 followed by evaluation. Thus it acts so that $\mathbb{C} \mapsto C_n$, and $(f : \mathbb{C} \to \mathbb{D}) \mapsto (f_n : C_n \to D_n)$. There are natural transformations $d_0^{\mathcal{E}}, d_1^{\mathcal{E}} : \mathrm{ev}_1^{\mathcal{E}} \Rightarrow \mathrm{ev}_0^{\mathcal{E}}$ whose components on an internal category are given by its source and target maps respectively. Similarly, there are natural transformation $i^{\mathcal{E}} : \mathrm{ev}_0^{\mathcal{E}} \Rightarrow \mathrm{ev}_1^{\mathcal{E}} \Leftarrow \mathrm{ev}_2^{\mathcal{E}} : m^{\mathcal{E}}$ with identity assigners and composition maps as components, respectively. These can all be seen as whiskering the nerve with evaluation natural transformations. As limits in functor categories are computed pointwise, $\mathrm{ev}_2^{\mathcal{E}}$ is the pullback of $d_0^{\mathcal{E}}$ and $d_1^{\mathcal{E}}$ in the functor category $[\mathbf{Cat}(\mathcal{E})_1, \mathcal{E}]$. Then it is easy to see that the data just described combine to give a category internal to $[\mathbf{Cat}(\mathcal{E})_1, \mathcal{E}]$, as the axioms follow pointwise. We will refer to this as the *evaluation category* internal to $[\mathbf{Cat}(\mathcal{E})_1, \mathcal{E}]$, and denote it by $\underline{\mathcal{E}}$. We will often refer to $\mathrm{ev}_0^{\mathcal{E}}$, $\mathrm{ev}_1^{\mathcal{E}}$ and $\mathrm{ev}_2^{\mathcal{E}}$ as $\mathbf{Ob}_{\mathcal{E}}$, $\mathbf{Arr}_{\mathcal{E}}$ and $\mathbf{Pair}_{\mathcal{E}}$ respectively, and we will drop the subscript '$\mathcal{E}$' whenever possible. Note that Proposition 1.2.3 just says that $\mathbf{Arr}$ is faithful.

Recall that when $\mathcal{E} = \mathbf{Set}$ there is a sequence of adjunctions $\Pi_0 \dashv \mathbf{Disc} \dashv \mathbf{Ob} \dashv \mathbf{coDisc}$, where Π_0 sends a category to its set of connected components and $\mathbf{Disc}$ and $\mathbf{coDisc}$ equip a set with morphisms, giving it the structure of a discrete and codiscrete category respectively. The remarks which follow discuss sufficient conditions for a similar result to hold internally to $\mathcal{E}$. The proofs are straightforward verification of the axioms of internal categories and functors, and of the triangle identities, using the various universal properties involved. These adjunctions will feature in various places throughout this thesis.

Remark 1.2.7. Under no conditions, the functor $\mathbf{Ob}$ has a left adjoint $\mathbf{Disc} : \mathcal{E} \to \mathbf{Cat}(\mathcal{E})_1$, which sends $X \in \mathcal{E}$ to the internal category whose object of objects is X, and source, target, identity and composition maps are all 1_X. The components of the unit of this adjunction on $X \in \mathcal{E}$ are all given by identities, while the components of the counit on an internal category $\mathbb{A}$ are given by the internal functor $\epsilon_{\mathbb{A}}^{\mathcal{E}} := \left(1_{A_0}, i^A\right)$. That is, its component on objects is the identity and its component on arrows is the identity assigner.

Remark 1.2.8. If $\mathcal{E}$ has products then the functor $\mathbf{Ob}$ has a right adjoint $\mathbf{coDisc} : \mathcal{E} \to \mathbf{Cat}(\mathcal{E})_1$ which sends $X \in \mathcal{E}$ to the internal category $\mathbb{X} : \Delta^{\mathrm{op}} \to \mathcal{E}$ with $\mathbb{X}(n)$ given by the $n+1$-fold product $X \times ... \times X$. The diagonal $(1_X, 1_X) : X \to X \times X$ is the identity assigner, while source, target and composition maps are given by product projections. The counit is the identity, while the unit has its component on an internal category $\mathbb{A}$ given by the internal functor $\eta_{\mathbb{A}}^{\mathcal{E}} := (1_{A_0}, (d_0, d_1))$. That is, its component on objects is the identity and its component on arrows is induced by the universal property of the product in $\mathcal{E}$, given the data of the source and target maps of $\mathbb{A}$. The counit of the previous adjunction and the unit of this adjunction have components which are fully faithful internal functors, and an internal functor f is fully faithful if and only if the naturality square of $\eta^{\mathcal{E}}$ on f is a pullback.

Remark 1.2.9. Assume $\mathcal{E}$ has coequalisers of reflexive pairs. Then $\mathbf{Disc}$ has a left adjoint $\Pi_0 : \mathbf{Cat}(\mathcal{E}) \to \mathcal{E}$ which sends every internal category $\mathbb{A}$ to the codomain of the coequaliser q_A of its source and target, and every internal functor $(f_0, f_1) : \mathbb{A} \to \mathbb{B}$ to the morphism shown below, which is induced by the universal property of $\Pi_0\mathbb{A}$, given the serial commutativity of the square on the left:

$$\begin{tikzcd}
A_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] \arrow[d, "f_1"'] & A_0 \arrow[r, "q_A"] \arrow[d, "f_0"'] & \Pi_0 A \arrow[d, dashed, "\Pi_0(f)"] \\
B_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & B_0 \arrow[r, "q_B"'] & \Pi_0 B
\end{tikzcd}$$

The component of the unit $q : 1_{\mathbf{Cat}(\mathcal{E})} \Rightarrow \mathbf{Disc} \circ \Pi_0$ on an internal category $\mathbb{A}$ is given on objects by the coequaliser $q_{\mathbb{A}}$ above, and on arrows by the identity assigner of $\mathbb{A}$ followed by this morphism, while the component of the counit on an object $X \in \mathcal{E}$ is just given by the identity on that object. The triangle identities then hold since by the universal property of coequalisers, Π_0 maps $q_{\mathbb{A}}$ to the identity, and since coequalisers of identities are identities.

Finally, although internal categories $\mathbb{A}, \mathbb{B}$ are precisely monads in $\mathbf{Span}(\mathcal{E})$, the morphisms of these monads *may not be* internal functors. Indeed, the ones which are internal functors are precisely those for which the component spans $A_n \leftarrow X_n \rightarrow B_n$ have left legs which are identities.

1.3 Internal Natural Transformations

Definition 1.3.1. Given internal functors $(f_0, f_1), (g_0, g_1) : (A_0, A_1) \to (B_0, B_1)$, an *internal natural transformation* is a morphism $\alpha : A_0 \to B_1$ such that the following diagrams commute:

- Assignation of components:
$$\begin{tikzcd}
A_0 \arrow[r, "\alpha"] \arrow[dr, "f_0"'] & B_1 \arrow[d, "d_0"] \\
& B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
A_0 \arrow[r, "\alpha"] \arrow[dr, "g_0"'] & B_1 \arrow[d, "d_1"] \\
& B_0
\end{tikzcd}$$

- Internal naturality is given by the square on the left, where the morphisms $\alpha_0 := (\alpha d_0, g_1) : A_1 \to B_2$ and $\alpha_1 := (f_1, \alpha d_1) : A_1 \to B_2$ are respectively induced as shown in the other two diagrams, given $d_1 \alpha d_0 = g_0 d_0 = d_0 g_1$ and $d_0 \alpha d_1 = f_0 d_1 = d_1 f_1$.

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_0"] \arrow[d, "\alpha_1"'] & B_2 \arrow[d, "m"] \\
B_2 \arrow[r, "m"'] & B_1
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[rr, "d_0"] \arrow[dr, dashed, "\alpha_0"] \arrow[ddr, bend right, "g_0"'] & & A_0 \arrow[d, "\alpha"] \\
& B_2 \arrow[r, "\pi_0"] \arrow[d, "\pi_1"] & B_1 \arrow[d, "d_1"] \\
& B_1 \arrow[r, "d_0"'] & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[drr, bend left, "f_1"] \arrow[dr, dashed, "\alpha_1"] \arrow[dd, "d_1"'] & & \\
& B_2 \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] & B_1 \arrow[d, "d_1"] \\
A_0 \arrow[r, "\alpha"'] & B_1 \arrow[r, "d_0"'] & B_0
\end{tikzcd}$$

Here α_0 is thought of as taking a morphism in A and returning the composable pair in B consisting of the image of that morphism under g, together with the component of α on its domain. Similarly, α_1 is thought of as returning the composable pair consisting of the image of the morphism under f, together with the component of α on its domain. If $\alpha : A_0 \to B_1$ is an isomorphism in $\mathcal{E}$ then α has the property which may be externally described as that 'every morphism in $\mathbb{B}$ is the component of α at some object in $\mathbb{A}$'. Meanwhile, if the square giving the naturality condition is a pullback in $\mathcal{E}$, then α has the property which may be externally described as that 'every commutative square in B is the naturality square of a unique morphism in A'. We will show in Theorem 1.5.4 that these properties both hold precisely when α is the universal 2-cell of a power by $\mathbf{2}$ in $\mathbf{Cat}(\mathcal{E})$. Since all notions involved in their definition involve only finite limits, the properties of being an internal category, functor or natural transformation is both preserved by all hom-functors $\mathcal{E}(X, -) : \mathcal{E} \to \mathbf{Set}$, and jointly reflected by them.

1.4 The 2-Category $\mathbf{Cat}(\mathcal{E})$

Definition 1.4.1. Let $\begin{tikzcd} A \arrow[r, bend left=50, "f"{name=F}] \arrow[r, "g"{name=G, description}] \arrow[r, bend right=50, "h"'{name=H}] \arrow[phantom, from=F, to=G, "\Downarrow\alpha"] \arrow[phantom, from=G, to=H, "\Downarrow\beta"] & B \end{tikzcd}$ be internal natural transformations. Then define their *vertical composite* $\beta \circ \alpha$ by $A_0 \xrightarrow{(\alpha, \beta)} B_2 \xrightarrow{m} B_1$, where (α, β) is induced since $d_0 \beta = g_0 = d_1 \alpha$.

Proposition 1.4.2. *The vertical composite of internal natural transformations $\beta \circ \alpha$ as defined above is itself well-defined as an internal natural transformation.*

Proof. Considering the diagrams in Diagram 7.0.4, we see that the first two respectively show sources and targets for components of $\beta \circ \alpha$, while the third shows internal naturality. In the third diagram, the three lower quadrilaterals are associativity axioms, while the two triangles and the two upper quadrilaterals can be verified to commute by the universal property of B_2. We elaborate on the left triangle and left upper quadrilateral, as the right triangle and right upper quadrilateral are similar. Of the four diagrams in Diagram 7.0.5, the first two show the commutativity of the left triangle while the second two show the commutativity of the upper left quadrilateral. $\square$

Proposition 1.4.3. *Let $\mathbb{A}$, $\mathbb{B}$ be categories internal to $\mathcal{E}$. Then there is a category* $\mathbf{Cat}(\mathbb{A}, \mathbb{B})$ *with*

- *Objects given by internal functors from $\mathbb{A}$ to $\mathbb{B}$,*
- *Hom-sets* $\mathbf{Cat}(\mathbb{A}, \mathbb{B})(f, g)$ *given by internal natural transformations from f to g,*
- *Composition is given by the vertical composite of internal natural transformations, described above.*
- *The identity internal natural transformation 1_f of $f : \mathbb{A} \to \mathbb{B}$ is given by if_0, or equivalently by $f_1 i$.*

This category is small if $\mathcal{E}$ is locally small.

Proof. Composition can be seen to be associative by equational associativity in B. We show that identities are well-defined as internal natural transformations. Indeed, 1_f respects sources as $d_0 i f_0 = f_0$, and targets as $d_1 i f_0 = f_0$. It is natural, since in the diagram below, the lower and right triangles commute by the left and right unit laws in $\mathbb{B}$, while the upper and left triangles can be seen to commute by the universal property of B_2.

$$\begin{tikzcd}
A_1 \arrow[rr, "(1_f)_1"] \arrow[dd, "(1_f)_0"'] \arrow[dr, "f_1"] & & B_2 \arrow[dd, "m"] \\
& B_1 \arrow[ur, "i_1"'] \arrow[dl, "i_0"'] \arrow[dr, "1_{B_1}"] & \\
B_2 \arrow[rr, "m"'] & & B_1
\end{tikzcd}$$

That 1_f indeed defines a left identity for f is clear since if $\alpha : e \Rightarrow f$ is an internal natural transformation, then $1_f \alpha = m(if_0, \alpha) = m(id_0\alpha, \alpha) = m(id_0, 1_{B_1})\alpha = \alpha$. By an analogous argument, one sees that 1_f is also a right identity. This establishes that internal functors from $\mathbb{A}$ to $\mathbb{B}$ and internal natural transformations between them form a category, a required. The size conditions hold since both internal functors and internal natural transformations are determined by finite sets of morphisms of $\mathcal{E}$ subject to certain conditions. $\square$

Definition 1.4.4. The *left whiskering* and *right whiskering* pictured below are defined as βf_0 and $g_1 \alpha$ respectively.

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "f"] & \mathbb{B} \arrow[r, bend left, "g"] \arrow[r, bend right, "g'"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathbb{C}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{A} \arrow[r, bend left, "f"] \arrow[r, bend right, "f'"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbb{B} \arrow[r, "g"] & \mathbb{C}
\end{tikzcd}$$

Proposition 1.4.5. *The left and right whiskerings as defined are well-defined as internal natural transformations.*

Proof. The left whiskering respects sources as $d_0\beta f_0 = g_0 f_0 = (gf)_0$, and targets as $d_1\beta f_0 = g'_0 f_0 = (g'f)_0$. The right whiskering also respects sources as $d_0 g_1 \alpha = g_0 d_0 \alpha = g_0 f_0 = (gf)_0$, and targets as $d_1 g_1 \alpha = g_0 d_1 \alpha = g_0 f'_0 = (gf')_0$. It can then be shown by the universal property of C_2 that the whiskerings satisfy the identities $(g\alpha)_k = g_2 \alpha_k$ and $(\beta f)_k = \beta_k f_1$ for $k \in \{0, 1\}$, where for the latter identity one uses the axiom that f preserves sources and targets for $k = 0$ and $k = 1$ respectively.

The left and right whiskerings can be seen to be natural by the commutativity of the diagrams in Diagram 7.0.6. In the first of these diagrams, the triangles commute by the second of the identities above, while the quadrilateral commutes by the naturality axiom for β. Meanwhile, in the second of the diagrams, the triangles commute by the first of the identities above, and the quadrilaterals are the naturality axiom for α, and g's respect for composites. Thus the whiskerings are well-defined internal natural transformations, as required. $\square$

Theorem 1.4.6. *Let $\mathcal{E}$ be a category with pullbacks. Categories internal to $\mathcal{E}$ are the objects of a 2-category $\mathbf{Cat}(\mathcal{E})$ whose hom-categories are described in Proposition 1.4.3, and whose horizontal composition of 2-cells defined via the above whiskerings in the usual way as described in Proposition II 3.1 of [26]. If $\mathcal{E}$ is small (resp. locally small), then $\mathbf{Cat}(\mathcal{E})$ is small (resp. has small hom-categories).*

Proof. We first show that we have a uniquely defined notion of horizontal composition $\beta * \alpha : gf \Rightarrow g'f' : \mathbb{A} \to \mathbb{C}$. This requires that $(\beta f) \circ (g'\alpha) = (g\alpha) \circ (\beta f')$. To see this, consider the first diagram of Diagram 7.0.7. The quadrilateral is the naturality axiom for β, while the triangles commute by the universal property of C_2 given the commutativity of the other four diagrams of Diagram 7.0.7; the second and third for the top triangle and the fourth and fifth for the triangle on the left.

Associativity and unit laws for whiskerings are immediate from their definition as they are directly inherited from $\mathcal{E}$. The left whiskering's respect for vertical composition is also immediate, while for the right whiskering this follows since $g_1 m(\alpha, \beta) = m g_2(\alpha, \beta) = m(g_1\alpha, g_1\beta)$, by g's respect for composition and the definition of g_2. Thus internal categories, functors, and natural transformations form a 2-category, as required. The size conditions follow from the embedding of $\mathbf{Cat}(\mathcal{E})_1$ into $[\Delta^{\mathrm{op}}, \mathcal{E}]$.

$\square$

Corollary 1.4.7. *The assignment $\mathbb{A} \mapsto \mathbb{A}^{op}, (f : \mathbb{A} \to \mathbb{B}) \mapsto (f^{op} : \mathbb{A}^{op} \to \mathbb{B}^{op}), \alpha \mapsto \alpha$ constitutes an isomorphism of 2-categories $\mathbf{Cat}(\mathcal{E}) \to \mathbf{Cat}(\mathcal{E})^{co}$.*

Proof. It suffices just to check that if $\alpha : f \Rightarrow g : \mathbb{A} \to \mathbb{B}$ is an internal natural transformation, then $\alpha : g^{\mathrm{op}} \Rightarrow f^{\mathrm{op}} : \mathbb{A}^{\mathrm{op}} \to \mathbb{B}^{\mathrm{op}}$ is as well. But this is clear since switching the sources and targets of $\mathbb{B}$ swaps the axioms for sources and targets for components for α. Then the 2-natural transformations exhibiting the equivalence of 2-categories have identity components, so that it is in fact an isomorphism of 2-categories. $\square$

Remark 1.4.8. We briefly remark upon some examples of internal categories which are considerably different to the ones on which we will primarily be focussing in this thesis. In fact, the notion of internal categories, functors, and natural transformations can be made sense of internal to an ambient category $\mathcal{E}$ under weaker assumptions than the existence of all pullbacks in $\mathcal{E}$. For the data $A_2 \xrightarrow{m} A_1 \overset{\xrightarrow{d_0}}{\underset{\xrightarrow[d_1]{}}{\xleftarrow{\ i\ }}} A_0$ to be a well-defined internal category, we really just need that all pullbacks along d_0 and d_1 exist. Indeed, the original motivating example for the theory of internal categories took $\mathcal{E}$ to be the category of smooth manifolds. There the main objects of interest are internal groupoids, or *Lie groupoids*, which are formed by ensuring that their domain and codomain maps are *submersions*, a class of maps along which pullbacks do exist. One can also take $\mathcal{E}$ to be the category of abelian groups, in which case $\mathbf{Cat}(\mathcal{E})_1$ is in fact equivalent to $\mathcal{E}^{\mathbf{2}}$, as a special case of the Dold–Kan correspondence [12].

1.5 n-tuple Categories, Powers by 2, and the Double Category of Squares

In this section, we examine how the operation of sending $\mathcal{E} \mapsto \mathbf{Cat}(\mathcal{E})$ can be iterated to form what are sometimes called 'multiple categories', and what we will call 'n-tuple categories'. Following this, we describe powers by $\mathbf{2}$ in $\mathbf{Cat}(\mathcal{E})$, and prove some properties of their universal 2-cells which will be useful in later chapters. We will also see how powers of an object in any 2-category $\mathfrak{K}$ give rise to a category internal to $\mathfrak{K}$ in a well-behaved (2-functorial and limit preserving) way. The case of $\mathfrak{K} = \mathbf{Cat}(\mathcal{E})$ will be used in Section 3.4, and will be called the *double category of squares*. As well as the notion of an internal category itself, the notions of n-tuple categories and double categories of squares are also originally due to Ehresmann.

Since limits commute with one another up to isomorphism, it is clear that $\mathbf{Cat}(\mathcal{E})_1$ inherits limits from $\mathcal{E}$, with structure maps induced by universal properties. Then the functors $\mathrm{ev}_n^{\mathcal{E}} : \mathbf{Cat}(\mathcal{E})_1 \to \mathcal{E}$ of Remark 1.2.6 all strictly preserve limits. This allows the following definition.

Definition 1.5.1. 1. Recursively define n-$\mathbf{tuple\text{-}Cat}(\mathcal{E})$ as $\mathbf{Cat}(\mathcal{E})$ when $n = 1$ and $\mathbf{Cat}\big((n-1)\text{-}\mathbf{tuple\text{-}Cat}(\mathcal{E})_1\big)$ for integers $n \geq 1$. Call its objects, 1-cells and 2-cells *n- tuple categories, functors and natural transformations* in $\mathcal{E}$ respectively.

2. When $n \in \{2, 3, 4\}$ we will denote n-$\mathbf{tuple\text{-}Cat}(\mathcal{E})$ by $\mathbf{DblCat}(\mathcal{E})$, $\mathbf{TrplCat}(\mathcal{E})$ and $\mathbf{QdplCat}(\mathcal{E})$ respectively, and replace the prefix 'n-tuple' with 'double', 'triple' and 'quadruple'.

3. Viewing the data of a double category $\mathbb{A}$ like so

$$\begin{tikzcd}
A_{2,0} \arrow[d, "m"'] & & A_{2,1} \arrow[d, "m"] & & \iddots \\
A_{1,0} \arrow[rr, "i" description] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & & A_{1,1} \arrow[ll, shift right=2, "d_0"'] \arrow[ll, shift left=2, "d_1"] \arrow[d, shift right=2, "d_0"'] \arrow[d, shift left=2, "d_1"] & A_{1,2} \arrow[l, "m"] & \cdots \\
A_{0,0} \arrow[u, "i" description] \arrow[rr, "i" description] & & A_{0,1} \arrow[u, "i" description] \arrow[ll, shift right=2, "d_0"'] \arrow[ll, shift left=2, "d_1"] & A_{0,2} \arrow[l, "m"] & \cdots
\end{tikzcd}$$

each row, and each column, forms a category internal to $\mathcal{E}$. For $j \in \mathbb{N}$, respectively call the jth row and column, the jth *horizontal* and *vertical* category of $\mathbb{A}$. When $j \in \{0, 1\}$, we will call these the horizontal and vertical categories of $\{\text{objects}, \text{arrows}\}$, respectively.

4. We give a recursive definition of an n-tuple functor to be *extremal*.

   - Base case: a 1-tuple functor is extremal if it is the identity on objects.
   - A functor (f_0, f_1) internal to $(n-1)$-$\mathbf{tuple\text{-}Cat}(\mathcal{E})$ is an extremal n-tuple functor if f_0 is the identity and f_1 is an extremal $(n-1)$-tuple functor.

5. Let $\mathbf{T}_{\mathcal{E}} : \mathbf{DblCat}(\mathcal{E}) \to \mathbf{DblCat}(\mathcal{E})$ be the involutary 2-functor which swaps the vertical and horizontal categories of a double category. Call $\mathbf{T}_{\mathcal{E}}(\mathbb{A})$ the *transpose* of the double category $\mathbb{A}$, and call $\mathbb{A}$ *symmetric* if $\mathbf{T}_{\mathcal{E}}(\mathbb{A}) = \mathbb{A}$.

The language of n-tuple categories will be used in the proofs of some of the main results of Chapter Three, in particular the various monad-like structures on the assignment $\mathcal{E} \mapsto \mathbf{Cat}(\mathcal{E})_1$. We will provide some examples for the concepts in the above definition later in this section. Note that we have already seen a simple example for extremal n-tuple functors for the case when $n = 1$, such as the unit of the adjunction in Remark 1.2.8 and the counit of the adjunction in Remark 1.2.7.

More than just inheriting conical limits from $\mathcal{E}$, $\mathbf{Cat}(\mathcal{E})$ also has powers by $\mathbf{2}$. We describe the internal construction of the usual 'arrow category' in the example below, and prove its universal property in the subsequent theorem. Throughout, for $n \in \mathbb{N}\backslash\{0\}$ let $\mathbf{n} := \{0 \to ... \to (n-1)\}$ denote the path category.

Example 1.5.2. *For $\mathbb{A} \in \mathbf{Cat}(\mathcal{E})$, define the internal category $\mathbb{A}^{\mathbf{2}}$ to have object of objects A_1, and object of arrows given by the kernel pair of composition $\mathbb{A}_{Sq}$ from 1.1.4, whose projections we will refer to as p_0 and p_1. Its domain and comdomain maps are given by $d_0^{\mathbf{2}} := \pi_0 p_0$ and $d_1^{\mathbf{2}} := \pi_1 p_1$ respectively, and its identity $i^{\mathbf{2}} := (i_0, i_1)$ is induced by the universal property of the pullback. Letting $\pi_0^{\mathbf{2}}$ and $\pi_1^{\mathbf{2}}$ denote the respective pullback projections of $\left(\mathbb{A}^{\mathbf{2}}\right)_2$, set the following notation for morphisms induced into the pullback A_2.*

$$q_0 := \begin{pmatrix} (\mathbb{A}^{\mathbf{2}})_2 \xrightarrow{\pi_0^{\mathbf{2}}} \mathbb{A}_{Sq} \xrightarrow{p_0} A_2 \xrightarrow{\pi_1} A_1 \\ (\mathbb{A}^{\mathbf{2}})_2 \xrightarrow{\pi_1^{\mathbf{2}}} \mathbb{A}_{Sq} \xrightarrow{p_0} A_2 \xrightarrow{\pi_1} A_1 \end{pmatrix} \qquad q_1 := \begin{pmatrix} (\mathbb{A}^{\mathbf{2}})_2 \xrightarrow{\pi_0^{\mathbf{2}}} \mathbb{A}_{Sq} \xrightarrow{p_1} A_2 \xrightarrow{\pi_0} A_1 \\ (\mathbb{A}^{\mathbf{2}})_2 \xrightarrow{\pi_1^{\mathbf{2}}} \mathbb{A}_{Sq} \xrightarrow{p_1} A_2 \xrightarrow{\pi_0} A_1 \end{pmatrix}$$

$$r_0 := \begin{pmatrix} (\mathbb{A}^{\mathbf{2}})_2 \xrightarrow{\pi_0^{\mathbf{2}}} \mathbb{A}_{Sq} \xrightarrow{p_0} A_2 \xrightarrow{\pi_0} A_1 \\ (\mathbb{A}^{\mathbf{2}})_2 \xrightarrow{q_0} A_2 \xrightarrow{m} A_1 \end{pmatrix} \qquad r_1 := \begin{pmatrix} (\mathbb{A}^{\mathbf{2}})_2 \xrightarrow{q_1} A_2 \xrightarrow{m} A_1 \\ (\mathbb{A}^{\mathbf{2}})_2 \xrightarrow{\pi_1^{\mathbf{2}}} \mathbb{A}_{Sq} \xrightarrow{p_1} A_2 \xrightarrow{\pi_1} A_1 \end{pmatrix}$$

The morphisms q_0 and q_1 may be thought of externally as taking a composable pair of commutative squares such as the one depicted below and returning the composable pairs (a_0, b_0) and (a_1, b_1), while the morphisms r_0 and r_1 return the composable pairs corresponding to the 'two ways of traversing the boundary', (f, b_1a_1) and (b_0a_0, k) respectively. Then the composition map $m^{\mathbf{2}}$ is given by $(r_0, r_1) : (\mathbb{A}^{\mathbf{2}})_2 \to \mathbb{A}_{Sq}$.

$$\begin{tikzcd} X_0 \arrow[r, "a_0"] \arrow[d, "f"'] & Y_0 \arrow[r, "b_0"] \arrow[d, "g"] & Z_0 \arrow[d, "k"] \\ X_1 \arrow[r, "a_1"'] & Y_1 \arrow[r, "b_1"'] & Z_1 \end{tikzcd}$$

In summary, if $f : X \to (\mathbb{A}^{\mathbf{2}})_2$ is a morphism in $\mathcal{E}$ so that $\pi_j p_k \pi_l^{\mathbf{2}} f := f_{l,k,j}$ for $(j,k,l) \in \{0,1\} \times \{0,1\} \times \{0,1\}$, then $\pi_j p_k m^{\mathbf{2}} f$ is given by the first table in Diagram 7.0.8. Note that the domain and codomain maps provide the internal functors $(d_A, d_0), (c_A, d_1) : \mathbb{A}^{\mathbf{2}} \to \mathbb{A}$, and that we have an internal natural transformation $\lambda_A : d_A \Rightarrow c_A$ given by $\lambda_A := 1_{A_1} : A_1 \to A_1$. Indeed, all constructions involved are representable, so the axioms follow since they hold for $\mathcal{E} = \mathbf{Set}$.

Theorem 1.5.3. *Let $\mathcal{E}$ be a category with finite limits. Then $\mathbf{Cat}(\mathcal{E})$ has finite weighted limits.*

Proof. Recall [18] that $\mathbf{2}$ is a strong generator for $\mathbf{Cat}$, and that weighted limits may be constructed using conical limits and powers by a strong generator. It therefore suffices just to prove the universal property of the above construction. To do this, we need to give a functor F from the functor category $[\mathbf{2}, \mathbf{Cat}(\mathcal{E})(\mathbb{B}, \mathbb{A})]$ to the hom-category $\mathbf{Cat}(\mathcal{E})(\mathbb{B}, \mathbb{A}^{\mathbf{2}})$ which exhibits horizontal composition by λ_A as an isomorphism of categories natural in $\mathbb{B}$.

We define F on objects. Given $\phi : f^0 \Rightarrow f^1 : \mathbb{B} \to \mathbb{A}$ let $F\phi : \mathbb{B} \to \mathbb{A}^{\mathbf{2}}$ be the internal functor whose component on objects is given by ϕ and component on arrows is given by $(\phi_0, \phi_1) : B_1 \to \mathbb{A}_{\mathrm{Sq}}$, the morphism induced by the pullback given the naturality axiom for ϕ. Then $F\phi$ respects sources and targets automatically by definition, and can be seen to respect identities by the universal properties of $\mathbb{A}_{\mathrm{Sq}}$ and A_2 given the commutativity of the diagrams in Diagram 7.0.9 for $j \in \{0,1\}$. $F\phi$ may furthermore be seen to respect composition given the commutativity of the first two diagrams in Diagram 7.0.10 for $(j,k) \in \{0,1\} \times \{0,1\}$, where the triangular region in the second diagram commutes by the universal property of A_2, given the commutativity of the third diagram for $(k,l) \in \{0,1\} \times \{0,1\}$.

We define F on arrows. Given the commutative square Γ in $\mathbf{Cat}(\mathcal{E})(\mathbb{B}, \mathbb{A})$ as shown below, considered as a morphism from ϕ^0 to ϕ^1 in $(\mathbf{Cat}(\mathcal{E})(\mathbb{A}, \mathbb{B}))^{\mathbf{2}}$, let its image under F be induced by the universal property of $(\mathbb{A}^{\mathbf{2}})_2$ so that $\pi_k p_j F\Gamma$ is given by the second table in Diagram 7.0.8. Then sources and targets of components for $F\Gamma$ is immediate from the definition, while internal naturality follows by the commutativity of the first two diagrams in Diagram 7.0.11 for $j \in \{0,1\}$. In the second of these and when $j = 0$, the upper triangle commutes by the universal property of A_2 given the commutativity of the last two diagrams there, while all other regions can be shown to commute similarly.

For functoriality of F, let the commutative square Ω shown below be viewed as a morphism $\phi^1 \to \phi^2$ in $(\mathbf{Cat}(\mathcal{E}))^{\mathbf{2}}$.

$$\Gamma := \begin{tikzcd} f^0 \arrow[r, "\alpha^0"] \arrow[d, "\phi^0"'] & g^0 \arrow[d, "\phi^1"] \\ f^1 \arrow[r, "\alpha^1"'] & g^1 \end{tikzcd} \qquad \Omega := \begin{tikzcd} g^0 \arrow[r, "\beta^0"] \arrow[d, "\phi^1"'] & h^0 \arrow[d, "\phi^2"] \\ g^1 \arrow[r, "\beta^1"'] & h^1 \end{tikzcd}$$

Then the following calculation follows from the tables of values in Diagram 7.0.8. One can similarly use the unit laws of $\mathbb{A}$ to verify that F preserves identities, so that F is well-defined as a functor.

$$(F\Omega) \circ (F\Gamma) = m^{\mathbf{2}} \begin{pmatrix} F\Gamma \\ F\Omega \end{pmatrix} = m^{\mathbf{2}} \begin{pmatrix} (\phi^0, \alpha^1) & (\alpha^0, \phi^1) \\ (\phi^1, \beta^1) & (\beta^0, \phi^2) \end{pmatrix} = \begin{pmatrix} \phi^0 & m(\alpha^1, \beta^1) \\ m(\alpha^0, \beta^0) & \phi^2 \end{pmatrix} = F(\Omega \circ \Gamma)$$

Since $\lambda_A : A_1 \to A_1$ is just an identity morphism in $\mathcal{E}$, it is easy to see that $\lambda_A * F\Gamma = \Gamma$, and conversely for $\gamma : s \Rightarrow t : \mathbb{B} \to \mathbb{A}^{\mathbf{2}}$, we have $F(\lambda_A * \gamma) = \gamma$. Thus F is indeed an isomorphism of categories. Finally, for naturality in $\mathbb{B}$, precomposition by an internal functor $e : \mathbb{C} \to \mathbb{B}$ commutes with F since F is defined by universal properties of limits. $\square$

The following characterisation of universal 2-cells of powers by $\mathbf{2}$ in $\mathbf{Cat}(\mathcal{E})$ was promised after the definition of an internal natural transformation, and will be useful in determining when certain functors preserve powers by $\mathbf{2}$.

Theorem 1.5.4. *An internal natural transformation $\alpha : f \Rightarrow g : \mathbb{A} \to \mathbb{B}$ is the universal 2-cell exhibiting $\mathbb{A}$ as the power by $\mathbf{2}$ of $\mathbb{B}$ if and only if $\alpha : A_0 \to B_1$ is an isomorphism in $\mathcal{E}$, and the square involved in the naturality axiom for α is a pullback.*

Proof. If the internal natural transformation α is the universal 2-cell exhibiting $\mathbb{A}$ as the power by $\mathbf{2}$ of $\mathbb{B}$ then $\alpha : B_1 \to B_1$ is the identity, and the square giving its naturality axiom is just the pullback square defining $(\mathbb{B}^{\mathbf{2}})_1$. Conversely, suppose that $\alpha : A_0 \to B_1$ is an isomorphism in $\mathcal{E}$ and that the square involved in the naturality axiom for α is a pullback. These conditions respectively say that the morphisms $\alpha : A_0 \to (\mathbb{B}^{\mathbf{2}})_0$ and $u := (\alpha d_0, \alpha d_1) : A_1 \to (\mathbb{B}^{\mathbf{2}})_1 := m \times_{B_1} m$ are invertible in $\mathcal{E}$. We claim that (α, u) and (α^{-1}, u^{-1}) are the data of internal functors which give an isomorphism $\mathbb{A} \cong \mathbb{B}^{\mathbf{2}}$ in $\mathbf{Cat}(\mathcal{E})_1$. It suffices to check that these are well-defined internal functors. One can either do this directly via calculations such as those above, or use the yoneda embedding of $\mathcal{E}$ to reduce to the case when $\mathcal{E} = \mathbf{Set}$, where the proof is easier. For further details, see Propositions 3.19 and 12.1 of [6], where the second of these approaches is taken. $\square$

The remark below, and the subsequent proposition, examine the relationship between powers by $\mathbf{2}$ in $\mathbf{Cat}(\mathcal{E})$ and the various adjunctions from Remarks 1.2.7, 1.2.8, and 1.2.9. The information in the remark will be useful in Chapter Three, and the proposition will be used to prove the comprehensive factorisation in Chapter Five.

Remark 1.5.5. Let $X \in \mathcal{E}$ and $\mathbb{A} \in \mathbf{Cat}(\mathcal{E})$, and recall the adjunction $\mathbf{Disc} \dashv \mathbf{Ob}_{\mathcal{E}}$ from Remark 1.2.7. Then there are the following natural bijections:

$$\mathcal{E}(X, A_1) = \mathcal{E}\left(X, \mathbf{Ob}_{\mathcal{E}}\left(\mathbb{A}^{\mathbf{2}}\right)\right) \cong \mathbf{Cat}\left(\mathbf{Disc}_{\mathcal{E}}(X), \mathbb{A}^{\mathbf{2}}\right)_1 \cong \mathbf{Arr}_{\mathbf{Set}}\left(\mathbf{Cat}\left(\mathbf{Disc}_{\mathcal{E}}(X), \mathbb{A}\right)_1\right)$$

Thus morphisms from X to the object of arrows of an internal category $\mathbb{A}$ are in natural bijection with internal natural transformations between internal functors from the discrete category on X to $\mathbb{A}$.

Proposition 1.5.6. *Suppose $\mathcal{E}$ has coequalisers of reflexive pairs. Then the internal functor $q_{\mathbb{A}} : \mathbb{A} \to \mathbf{Disc} \circ \Pi_0(\mathbb{A})$ which was the component of the unit of the adjunction from Remark 1.2.9 is the coidentifier of the universal 2-cell of the power of $\mathbb{A}$ by $\mathbf{2}$.*

Proof. For the 1-dimensional universal property, let $f : \mathbb{A} \to \mathbb{B}$ be an internal functor so that $f.\lambda_A$ is an identity internal natural transformation. Then in particular $f_1 id_0 = f_0 id_1$, and hence since f preserves identities $i f_0 d_0 = i f_0 d_1$. By sources (or targets) for identities in $\mathbb{B}$, we may compose by either of these on the right, to get that $f_0 d_1 = f_0 d_1$. Now by the universal property of $\Pi_0(\mathbb{A})$ as a coequaliser in $\mathcal{E}$, we have an induced morphism

$u : \Pi_0(\mathbb{A}) \to B_0$, and since $\mathbf{Disc} \circ \Pi_0(\mathbb{A})$ is a discrete category, this is clearly well-defined as an internal functor. The universal property in $\mathcal{E}$ assures that it is indeed the unique internal functor commuting with f and $q_{\mathbb{A}}$.

For the 2-dimensional universal property, let $f, g : \mathbb{A} \to \mathbb{B}$ be internal functors with $f.\lambda_A$ and $g.\lambda_A$ both identities, and let $\beta : f \Rightarrow g$ be any internal natural transformation. Examining the two ways of forming the horizontal composite $\beta * \lambda$ and recalling that these must be equal, we see that $\beta d_0 = \beta d_1$ since both $f\lambda_A$ and $g\lambda_A$ are identities. Hence induce $\gamma : \Pi_0(\mathbb{A}) \to B_1$ by the universal property of the coequaliser, and define $h_0 := d_0\gamma$ and $k_0 := d_1\gamma$. It is once again clear that (h_0, ih_0) and (k_0, ik_0) constitute internal functors from $\mathbf{Disc} \circ \Pi_0(\mathbb{A})$ to $\mathbb{B}$. Furthermore, $\gamma : h \Rightarrow k$ is well-defined as an internal natural transformation. Assignation of components follows by the definitions of h_0 and k_0, while naturality follows from the left and right unit laws for $\mathbb{B}$, as in Diagram 7.0.12. Note once again that uniqueness follows from the universal property in $\mathcal{E}$. $\square$

We now give a construction of a category internal to the underlying category of any 2-category with powers $\mathfrak{K}$, and show that this construction is 2-functorial. This 2-functor will feature significantly in Section 3.4 as the component at $\mathfrak{K}$ of a pseudonatural transformation, which will itself be the unit of a biadjunction discussed there.

Theorem 1.5.7. *Let $\mathfrak{K}$ be a 2-category with powers by finite categories and denote its underlying 1-category by $\mathfrak{K}_1$. Let $A \in \mathfrak{K}$ have*

1. *$\alpha : a_0^2 \Rightarrow a_1^2 : A^{\mathbf{2}} \to A$ as the universal 2-cell of its power by $\mathbf{2}$*
2. *The diagram $a_0^3 \xrightarrow{\alpha_0} a_1^3 \xrightarrow{\alpha_1} a_2^3$ in the hom-category $\mathfrak{K}\left(A^{\mathbf{3}}, A\right)$ as the universal composable pair of 2-cells of its power by $\mathbf{3}$.*

Then

1. *The following square is a pullback in $\mathbf{Cat}(\mathcal{E})$, where $\alpha * \alpha_j' = \alpha_j$ for $j \in \{0, 1\}$.*

$$\begin{tikzcd} A^{\mathbf{3}} \arrow[r, "\alpha_0'"] \arrow[d, "\alpha_1'"'] \arrow[dr, phantom, "\lrcorner", very near start] & A^{\mathbf{2}} \arrow[d, "a_1^2"] \\ A^{\mathbf{2}} \arrow[r, "a_0^2"'] & A \end{tikzcd}$$

2. *Let $\Delta_A : A \to A^{\mathbf{2}}$ denote the diagonal morphism which satisfies $\alpha\Delta_A = 1_{1_A}$, and let $m : A^{\mathbf{3}} \to A^2$ be the unique morphism satisfying $\alpha m = \alpha_1 \circ \alpha_0$. Then the data* $\begin{tikzcd} A^{\mathbf{3}} \arrow[r, "m"] & A^{\mathbf{2}} \arrow[r, shift left=2, "a_0"] \arrow[r, shift right=2, "a_1"'] & A \arrow[l, dashed, "\Delta_A"' description] \end{tikzcd}$ *, which we collectively denote by $\eta^{\mathfrak{K}}(A)$, is a category internal to $\mathfrak{K}_1$.*
3. *The assignment $A \mapsto \eta^{\mathfrak{K}}(A)$ extends to a 2-functor $\eta^{\mathfrak{K}} : \mathfrak{K} \to \mathbf{Cat}(\mathfrak{K}_1)$ that preserves powers and any conical limits that exist in $\mathfrak{K}$.*

Proof. For part (1), first observe that the square commutes since $a_1^2\alpha_0' = a_1^3 = a_0^2\alpha_1'$. Then, for the 1-dimensional universal property, let 1-cells $x_0, x_1 : X \to A^{\mathbf{2}}$ satisfy $a_1^2x_0 = a_0^2x_1$. This commutativity condition says precisely that the codomain of $\chi_0 := \alpha.x_0$ is the domain of $\chi_1 := \alpha.x_1$. In other words, we have a composable pair in $\mathfrak{K}(X, A)$. Let $(\chi_0, \chi_1) : X \to A^{\mathbf{3}}$ be hence induced by the universal property of the power by $\mathbf{3}$. It follows from the universal property of the power by $\mathbf{2}$ that (χ_0, χ_1) is the unique 1-cell satisfying $\alpha_j'(\chi_0, \chi_1) = x_j$ for $j \in \{0, 1\}$, since $\alpha\alpha_j'(\chi_0, \chi_1) = \alpha_j(\chi_0, \chi_1) = \chi_j = \alpha.x_j$. The 2-dimensional universal property of the pullback follows from the 1-dimensional universal property, given the assumption that $\mathfrak{K}$ has powers. In fact, this argument extends by induction on n to show that $\mathfrak{K}$ has pullbacks along projections from powers by any path category $\mathbf{n}$.

For part (2), firstly note that sources and targets for identities and composition follow immediately from the definition of $\eta^{\mathfrak{K}}$ and m. Furthermore, the universal property of $A^{\mathbf{2}}$ reduces associativity and left and right unit laws of $\eta^{\mathfrak{K}}(A)$ to those of the hom-categories $\mathfrak{K}\left(A^{\mathbf{4}}, A\right)$ and $\mathfrak{K}\left(A^{\mathbf{3}}, A\right)$ respectively.

For part (3), given a 1-cell $f : A \to B$, the data $\eta^{\mathfrak{K}}(f) := \left(f, f^{\mathbf{2}}\right)$ defines an internal functor $\eta^{\mathfrak{K}}(A) \to \eta^{\mathfrak{K}}(B)$; respect for sources and targets is immediate, while respect for identities and composition follow from easy calculations similar to those above. Similarly, let $g : A \to B$ be another 1-cell and $\beta : f \Rightarrow g$ a 2-cell. Take $\eta^{\mathfrak{K}}(\beta) : A \to B^{\mathbf{2}}$ to be the 1-cell induced by the universal property of $B^{\mathbf{2}}$ given β. Then $\eta^{\mathfrak{K}}(\beta) : \eta^{\mathfrak{K}}(f) \Rightarrow \eta^{\mathfrak{K}}(g)$ is an internal natural transformation. Once again, sources and targets for components follows immediately, while internal naturality is easy to check using the universal property of the power by $\mathbf{2}$. Functoriality between hom-categories and 2-functoriality can also be checked using the 2-dimensional and 1-dimensional universal properties of $B^{\mathbf{2}}$ respectively. Finally, since all constructions involved mention only finite weighted limits, it is clear that they are preserved. □

Remark 1.5.8. When $\mathfrak{K} = \mathbf{Cat}(\mathcal{E})$, we will denote the underlying functor of the 2-functor $\eta^{\mathfrak{K}}$ above as $\delta_{\mathcal{E}}$, and we will call $\delta_{\mathcal{E}}(\mathbb{A})$ the *double category of squares in* $\mathbb{A}$, for $\mathbb{A} \in \mathbf{Cat}(\mathcal{E})$. In particular double categories of squares are *symmetric*, as defined in Definition 1.5.1. As mentioned above, the 2-functors $\eta^{\mathfrak{K}} : \mathfrak{K} \to \mathbf{Cat}(\mathfrak{K}_1)$ will feature as the components of the unit of a biadjunction in Theorem 3.4.2. Furthermore, the functors $\delta_{\mathcal{E}} : \mathbf{Cat}(\mathcal{E})_1 \to \mathbf{DblCat}(\mathcal{E})_1$ will also feature as the components of the comultiplication for a pseudocomonad structure on the assignment $\mathcal{E} \mapsto \mathbf{Cat}(\mathcal{E})_1$ in Section 3.4.

The isomorphisms up to which $\eta^{\mathfrak{K}} : \mathfrak{K} \to \mathbf{Cat}(\mathfrak{K}_1)$ preserves powers by $\mathbf{2}$ will also be of importance in Section 3.4. In the special case of $\mathfrak{K} = \mathbf{Cat}(\mathcal{E})$, these will feature as the components of the modification up to which coassociativity of the pseudocomonad holds in Theorem 3.4.4. For now, we prove naturality of certain assignments as described in the following proposition.

Proposition 1.5.9. *Let $F : \mathfrak{K} \to \mathfrak{L}$ be a 2-functor which preserves powers by $\mathbf{2}$ up to coherent isomorphisms $F\left(A^{\mathbf{2}}\right) \cong (FA)^{\mathbf{2}}$ which we denote by $\left(\eta^F_A\right)_1$. Then*

1. *The assignment $\left(\eta^F\right)_1 : A \mapsto \left(\eta^F_A\right)_1$ is 2-natural in $A \in \mathfrak{K}$.*
2. *The assignment $F \mapsto \left(\eta^F\right)_1$ is natural in F.*

Proof. Let α_X denote the universal 2-cell of X's power by $\mathbf{2}$ for X in either of $\mathfrak{K}$ or $\mathfrak{L}$, and let $f : A \to B$ be a 1-cell in $\mathfrak{K}$. Then part (1) follows from the universal property of the power by $\mathbf{2}$ in $\mathfrak{L}$ given the following calculation, wherein the second and fourth steps use 2-functoriality of F and all other steps use definitions of morphisms induced into powers by $\mathbf{2}$:

$$\begin{aligned}
\alpha_{FB}.\left(\eta^F_B\right)_1.F\left(f^{\mathbf{2}}\right) &= F\left(\alpha_B\right).F\left(f^{\mathbf{2}}\right) \\
&= F\left(\alpha_B.f^{\mathbf{2}}\right) \\
&= F\left(f.\alpha_A\right) \\
&= F(f).F\left(\alpha_A\right) \\
&= (Ff).\left(\alpha_{FA}\right).\left(\eta^F_A\right)_1 \\
&= \alpha_{FB}.(Ff)^{\mathbf{2}}.\left(\eta^F_A\right)_1
\end{aligned}$$

Let $\alpha : F \Rightarrow G$ be a 2-natural transformation. Then part (2) follows from the following calculation, where the third step uses 2-naturality of ϕ and all other steps use definitions of morphisms induced into powers by $\mathbf{2}$.

$$
\begin{aligned}
\alpha_{GA}.\left(\phi_A\right)^{\mathbf{2}}.\left(\eta_A^F\right)_1 &= \phi_A.\alpha_{FA}.\eta_A^F \\
&= \phi_A.F\left(\alpha_A\right) \\
&= G\left(\alpha_A\right).\phi_{A^{\mathbf{2}}} \\
&= \alpha_{GA}.\left(\eta_A^G\right)_1.\phi_{A^{\mathbf{2}}}
\end{aligned}
$$

□

Remark 1.5.10. A special case of Proposition 1.5.9 which will be useful later is when $\mathfrak{L} = \mathbf{Cat}\left(\mathfrak{K}_1\right)$ and F is itself the functor $\eta^{\mathfrak{K}} : \mathfrak{K} \to \mathbf{Cat}\left(\mathfrak{K}_1\right)$ of Theorem 1.5.7. Then the isomorphism $\eta_A^\eta : \eta^{\mathfrak{K}}\left(A^{\mathbf{2}}\right) \cong \left(\eta^{\mathfrak{K}}A\right)^{\mathbf{2}}$ up to which $\eta^{\mathfrak{K}} : \mathfrak{K} \to \mathbf{Cat}\left(\mathfrak{K}_1\right)$ preserves powers by $\mathbf{2}$ is the internal functor whose component on objects is the identity, and whose component on arrows the isomorphism $A^{\mathbf{2}\times\mathbf{2}} \cong \left(A^{\mathbf{2}}\right)^{\mathbf{2}}$. This is an *extremal* internal functor in the sense of Definition 1.5.1. As an even more special case, when $\mathfrak{K}$ is itself $\mathbf{Cat}\left(\mathcal{E}\right)$, the isomorphisms $\left(\delta_{\mathcal{E}}\left(\mathbb{A}\right)\right)^{\mathbf{2}} \cong \delta_{\mathcal{E}}\left(\mathbb{A}^{\mathbf{2}}\right)$ up to which $\delta_{\mathcal{E}}$ preserves powers by $\mathbf{2}$ can be described analogously as being double functors which are the identity on objects, and the unique isomorphism of internal categories $\mathbb{A}^{\mathbf{2}\times\mathbf{2}} \cong \left(\mathbb{A}^{\mathbf{2}}\right)^{\mathbf{2}}$ on arrows. This isomorphism of internal categories is again of course itself the identity on objects, and so this is an example of an extremal double functor in the sense of Definition 1.5.1. Indeed, the objects of objects of $\mathbb{A}^{\mathbf{2}\times\mathbf{2}}$ and $\left(\mathbb{A}^{\mathbf{2}}\right)^{\mathbf{2}}$ are both just $\mathbb{A}_{\mathrm{Sq}}$.

On the other hand, the objects of arrows of $\left(\mathbb{A}^{\mathbf{2}}\right)^{\mathbf{2}}$ and $\mathbb{A}^{\mathbf{2}\times\mathbf{2}}$ may both be thought of externally as 'objects of commutative cubes in $\mathbb{A}$', however they are constructed in different ways. The object of arrows of $\left(\mathbb{A}^{\mathbf{2}}\right)^{\mathbf{2}}$ is constructed as the object of 'commutative squares of commutative squares', or the kernel pair of the composition map in $\mathbb{A}^{\mathbf{2}}$. In contrast, the object of arrows of $\mathbb{A}^{\mathbf{2}\times\mathbf{2}}$ is constructed as the object of 'six commutative squares whose edges match in such a way so as to make a commutative cube'. We describe its construction. First, take the six product projections $s_1, ..., s_6 : \left(\prod_{k=1}^{6}\mathbb{A}_{\mathrm{Sq}}\right) \to \mathbb{A}_{\mathrm{Sq}}$, each corresponding to a face of a cube. Then form twelve pullbacks, each asserting the equality of a pair of edges, and finally, take the intersection of all twelve of these pullbacks.

2 Constructions $\mathbf{Cat}\left(\mathcal{E}\right)$ inherits from $\mathcal{E}$

In the last section of the previous chapter, we saw that $\mathbf{Cat}\left(\mathcal{E}\right)$ inherits conical limits from $\mathcal{E}$ since the data of an internal category mention only finite limits. We also saw that a certain 2-categorical property emerged in $\mathbf{Cat}\left(\mathcal{E}\right)$, namely that of having powers by $\mathbf{2}$. In this chapter we will look at some other properties of $\mathcal{E}$ which $\mathbf{Cat}\left(\mathcal{E}\right)$ inherits directly, namely cartesian closedness in Section 2.1 and extensivity and a natural numbers object in Section 2.2. We will also see some other 2-categorical properties emerge in $\mathbf{Cat}\left(\mathcal{E}\right)$ under certain assumptions on $\mathcal{E}$ in the final two sections of this chapter, namely copowers by $\mathbf{2}$ in Section 2.3, and a classifier for full subobjects in Section 2.4. We will the collect the results of this chapter into a list of properties $\mathbf{Cat}\left(\mathcal{E}\right)$ has when $\mathcal{E}$ is an elementary topos, and finish this chapter by remarking upon some properties that $\mathbf{Cat}\left(\mathcal{E}\right)$ does not inherit from $\mathcal{E}$.

2.1 Cartesian Closedness

Suppose that $\mathcal{E}$ is a cartesian closed category with finite limits and let $A, B \in \mathbf{Cat}\left(\mathcal{E}\right)$. We construct the exponential $\mathbb{B}^{\mathbb{A}}$. Its object of objects $\left(B^A\right)_0$ will be given by the intersection, or limit, of six equalisers into ${B_0}^{A_0} \times {B_1}^{A_1} \times {B_2}^{A_2}$. Each of the four internal functoriality axioms are encoded by one of these equalisers, while the remaining two equalisers together encode the defining property of an internal functor's component on composable pairs. The object of arrows of $\mathbb{B}^{\mathbb{A}}$ will be given by the intersection of seven equalisers into $\left(\mathbb{B}^{\mathbb{A}}\right)_0 \times \left(\mathbb{B}^{\mathbb{A}}\right)_0 \times {B_2}^{A_1} \times {B_2}^{A_1}$. Four of these equalisers encode the defining properties of the morphisms which ‘send an arrow of $\mathbb{A}$ to either side of its naturality square’, while the other three each encode an axiom for an internal natural transformation. The source, target, identity and composition maps of $\mathbb{B}^{\mathbb{A}}$ are induced by universal properties, and its universal property may be checked directly. However, it is easier to show that $\mathbf{Cat}(\mathcal{E})$ is in fact an exponential ideal of $[\Delta^{\mathrm{op}}, \mathcal{E}]$.

Theorem 2.1.1. *Let $\mathcal{A}$ be a category, let $\mathcal{E}$ be a category with finite limits and exponentials, and recall the nerve functor $N : \mathbf{Cat}(\mathcal{E})_1 \to [\Delta^{op}, \mathcal{E}]$ of Remark 1.2.2.*

1. *The functor category $[\mathcal{A}, \mathcal{E}]$ has an exponential object for $F, G : \mathcal{A} \to \mathcal{E}$ given by the functor $k \mapsto \int_n \mathcal{A}(n,k) \pitchfork Gn^{Fn}$.*
2. *The exponential G^F preserves any limits G preserves.*
3. *For $\mathbb{A} \in \mathbf{Cat}(\mathcal{E})$ and $F \in [\Delta^{op}, \mathcal{E}]$, the exponential object $(N\mathbb{A})^F : \Delta^{op} \to \mathcal{E}$ is an internal category.*
4. *The internal category $N\mathbb{A}^{N\mathbb{B}}$ has the universal property of the exponential in $\mathbf{Cat}(\mathcal{E})$.*

Proof. Part (1) follows from the following natural bijections, where $\bullet$ and $\pitchfork$ denote the **Set**-enriched copower and power, respectively:

$$\begin{aligned}
\mathcal{E}\left(X, \int_n \mathcal{A}(n,k) \pitchfork Gn^{Fn}\right) &\cong \mathcal{E}\left(\mathcal{A}(n,k) \bullet X, Gn^{Fn}\right) \\
&\cong \mathcal{E}\left(\mathcal{A}(n,k) \bullet X \times Fn, Gn\right) \\
&\cong \mathcal{E}\left((\mathcal{A}(-,k) \bullet X) \times F(-), G(-)\right) \\
&\cong [\mathcal{A},\mathcal{E}]\left(\mathcal{A}(-,k) \bullet X, G^F(-)\right) \\
&\cong \mathcal{E}\left(X, G^F(k)\right)
\end{aligned}$$

For Part (2), consider the following sequence of natural bijections, where by $P \xrightarrow{\mathcal{C}} Q$ we denote the set of morphisms from P to Q in the category $\mathcal{C}$, and by $\widehat{\mathcal{E}}$ we denote the presheaf category $[\mathcal{E}^{\mathrm{op}}, \mathbf{Set}]$. The proof follows by noticing that each natural bijection preserves the property of a morphism being invertible.

$$\begin{gathered}
G^F\left(\lim_i B_i\right) \xrightarrow{\quad \mathcal{E} \quad} \lim_i G^F(B_i) \\
\mathcal{E}\left(X, G^F\left(\lim_i B_i\right)\right) \xrightarrow{\quad \widehat{\mathcal{E}} \quad} \mathcal{E}\left(X, \lim_i G^F(B_i)\right) \\
\mathcal{E}\left(X, \int_n \mathcal{A}\left(n, \lim_i B_i\right) \pitchfork Gn^{Fn}\right) \xrightarrow{\widehat{\mathcal{E}}} \mathcal{E}\left(X, \lim_i \int_n \mathcal{A}(n, B_i) \pitchfork Gn^{Fn}\right) \\
\mathcal{E}\left(X, \int_n \lim_i \mathcal{A}(n, B_i) \pitchfork Gn^{Fn}\right) \xrightarrow{\widehat{\mathcal{E}}} \mathcal{E}\left(X, \lim_i \int_n \mathcal{A}(n, B_i) \pitchfork Gn^{Fn}\right) \\
\mathcal{E}\left(X, \lim_i \int_n \mathcal{A}(n, B_i) \pitchfork Gn^{Fn}\right) \xrightarrow{\widehat{\mathcal{E}}} \mathcal{E}\left(X, \lim_i \int_n \mathcal{A}(n, B_i) \pitchfork Gn^{Fn}\right)
\end{gathered}$$

Part (3) then follows from part (2) and the characterisation of those simplicial objects which are internal categories, namely that they preserve certain pullbacks. For part (4), the one dimensional universal property follows from Part (3). For the 2-dimensional universal property, let $\mathbf{2} \pitchfork \mathbb{C}$ denote the power by $\mathbf{2}$ in $\mathbf{Cat}(\mathcal{E})$, then since exponentiation is a right adjoint, $(\mathbf{2} \pitchfork \mathbb{C})^{\mathbb{B}}$ has at least the 1-dimensional universal property in $\mathbf{Cat}(\mathcal{E})$ of the power by $\mathbf{2}$ of the exponential $\mathbb{C}^{\mathbb{B}}$ from $\mathbf{Cat}(\mathcal{E})_1$. Then the following natural bijections show that the

2-dimensional universal property is also satisfied:

$$\begin{aligned}\{\mathbb{A}\times\mathbb{B}\Rightarrow\mathbb{C}\} &\cong \{\mathbb{A}\times\mathbb{B}\to\mathbf{2}\pitchfork\mathbb{C}\}\\ &\cong \{\mathbb{A}\to(\mathbf{2}\pitchfork\mathbb{C})^{\mathbb{B}}\}\\ &\cong \{\mathbb{A}\to\mathbf{2}\pitchfork\left(\mathbb{C}^{\mathbb{B}}\right)\}\\ &\cong \{\mathbb{A}\Rightarrow\mathbb{C}^{\mathbb{B}}\}\end{aligned}$$

□

2.2 Extensivity

Since the definition of an internal category involves pullbacks, some commutativity with pullbacks is needed for colimits in $\mathcal{E}$ to be inherited in $\mathbf{Cat}(\mathcal{E})$. The functor category $[\Delta^{\mathrm{op}},\mathcal{E}]$ has whatever colimits $\mathcal{E}$ has as they are computed pointwisely. All of these give rise to colimits in $\mathbf{Cat}(\mathcal{E})_1$ when $\mathbf{Cat}(\mathcal{E})_1$ is a reflective subcategory of $[\Delta^{\mathrm{op}},\mathcal{E}]$, which holds when $\mathcal{E}$ is locally presentable [1]. Under different assumptions, coproducts restrict to $\mathbf{Cat}(\mathcal{E})_1$ when $\mathcal{E}$ is *extensive*. Recall that extensivity means that for all $A,B\in\mathcal{E}$, the functor $\mathcal{E}/A\times\mathcal{E}/B\to\mathcal{E}/(A+B)$, which takes the coproduct, is an equivalence of categories [8].

Proposition 2.2.1. *Let $\mathcal{E}$ be an extensive category with pullbacks. Then $\mathbf{Cat}(\mathcal{E})$ has coproducts as computed in $[\Delta^{op},\mathcal{E}]$.*

Proof. Let $\mathbb{A}$ and $\mathbb{B}$ be categories internal to $\mathcal{E}$. Then the diagrams which need to be pullbacks for $\mathbb{A}+\mathbb{B}$ to be well-defined as an internal category are precisely the coproducts in $\mathcal{E}$ of the corresponding pullbacks which exhibit $\mathbb{A}$ and $\mathbb{B}$ as internal categories. But by extensivity of $\mathcal{E}$, these will be pullbacks as well. Thus $\mathbf{Cat}(\mathcal{E})_1$ has coproducts as computed in $[\Delta^{\mathrm{op}},\mathcal{E}]$. But since $\mathbf{Cat}(\mathcal{E})$ has powers as we saw in Proposition 1.5.3, the 2-dimensional universal property is also satisfied. □

In particular, the coprojections from the coproduct in $\mathbf{Cat}(\mathcal{E})$ are discrete conduche fibrations, discrete fibrations, discrete opfibrations, and reflect identities, as described in Remark 1.2.4.

Coequalisers on the other hand may not be inherited from $[\Delta^{\mathrm{op}},\mathcal{E}]$, even if coequaliser diagrams are stable under pullback. As an example, when $\mathcal{E}=\mathbf{Set}$, the two distinct functors from the terminal category $\mathbf{1}$ to the free-living morphism $\mathbf{2}$ have a coequaliser given by the free monoid on one generator, whose object of arrows is the set of natural numbers $\mathbb{N}$ rather than 2. This is because identifying the two objects of $\mathbf{2}$ produces the new composable pair, namely the unique non-identity morphism $s\in\mathbf{2}(0,1)$, which becomes composable with itself. The existence of a natural numbers object in $\mathbf{Set}$ allows us to produce this particular coequaliser in $\mathbf{Cat}(\mathbf{Set})=\mathbf{Cat}$. However, for $\mathcal{E}=\mathbf{Set}_f$, the category of finite sets, no such natural numbers object exists, and indeed neither does the coequaliser of this parallel pair in $\mathbf{Cat}(\mathbf{Set}_f)$. In general, coequalisers require the construction of a category from its presentation, which can be done in the presence of list objects [15]. We end this section by looking at a special case of list objects which are inherited in $\mathbf{Cat}(\mathcal{E})$ from $\mathcal{E}$.

Definition 2.2.2. Let $\mathcal{E}$ have finite products. A *list object* for an object $X\in\mathcal{E}$ is a diagram such as that shown below on the left, which is initial among diagrams of the form shown below on the right.

$$1\xrightarrow{\;empty\;}L(X)\xleftarrow[\;append\;]{}X\times L(X)\qquad\qquad 1\longrightarrow Y\longleftarrow X\times Y$$

When $X=1$, $L(X)$ is called a *natural numbers object.*

Corollary 2.2.3. *If $\mathcal{E}$ has finite limits and a natural numbers object, then $\mathbf{Cat}(\mathcal{E})$ has a natural numbers object.*

Proof. It is clear that as a left adjoint functor which preserves finite products, $\mathbf{Disc}:\mathcal{E}\to\mathbf{Cat}(\mathcal{E})$ preserves list objects, and so in particular gives rise to a natural numbers object in $\mathbf{Cat}(\mathcal{E})$. □

2.3 Copowers by 2

We next construct copowers by $\mathbf{2}$ in $\mathbf{Cat}(\mathcal{E})$. Recall that when $\mathcal{E} = \mathbf{Set}$, the copower by $\mathbf{2}$ of a category $\mathcal{A}$ is given by $\mathbf{2} \times \mathcal{A}$, since $\mathbf{Cat}(\mathbf{2} \times \mathcal{A}, \mathcal{B}) \cong \mathbf{Cat}(\mathbf{2}, \mathbf{Cat}(\mathcal{A}, \mathcal{B}))$ by cartesian closedness of $\mathbf{Cat}$. We therefore first construct the free-living arrow $\mathbf{2}$ internally.

Notation 2.3.1. *Let $n \in \mathbb{N}$ and $X \in \mathcal{E}$. Then by nX we denote the coproduct $\sum_{j=1}^{n} X$.*

Example 2.3.2. *Assume $\mathcal{E}$ is* lextensive *in the sense of [8], in that it is extensive and has finite limits. Denote the terminal object in $\mathcal{E}$ by 1. Then we may define the free living arrow, internal to $\mathcal{E}$, which we denote $\mathbf{2}_{\mathcal{E}}$, to have*

- *Object of objects and object of arrows given by 2 and 3 respectively, and object of n-tuples given by $n+1$.*
- *Identity assigner, source, and target maps induced as indicated in the following diagrams from the universal property of the respective coproducts.*

$$\begin{tikzcd}
1 \arrow[r, "p_1"] \arrow[d, "p_1"'] & 2 \arrow[d, dashed, "i"'] & 1 \arrow[l, "p_2"'] \arrow[d, "p_3"] \\
3 \arrow[r, equal, "\cong"] & 3 \arrow[r, equal, "\cong"] & 3
\end{tikzcd}
\qquad
\begin{tikzcd}
2 \arrow[r, "p_{1,2}"] \arrow[dr, "p_1"'] & 3 \arrow[d, dashed, "d_0"] & 1 \arrow[l, "p_3"'] \arrow[dl, "p_2"] \\
& 2 &
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "p_1"] \arrow[dr, "p_1"'] & 3 \arrow[d, dashed, "d_1"'] & 2 \arrow[l, "p_{2,3}"'] \arrow[dl, "p_2"] \\
& 2 &
\end{tikzcd}$$

- *Composition $m : 4 \to 3$ is the unique morphism making the following diagrams commute.*

$$\begin{tikzcd}
1 \arrow[r, "p_1"] \arrow[dr, "p_1"'] & 4 \arrow[d, "m"] & 1 \arrow[l, "p_4"'] \arrow[dl, "p_3"] \\
& 3 &
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "p_2"] \arrow[dr, "p_2"'] & 4 \arrow[d, "m"] & 1 \arrow[l, "p_3"'] \arrow[dl, "p_2"] \\
& 3 &
\end{tikzcd}$$

Then internal category axioms follow from extensivity and the universal property of the coproducts. Note also in particular that the following diagrams commute in $\mathcal{E}$:

$$\begin{tikzcd}
1 \arrow[r, "p_1"] \arrow[d, "p_2"'] & 3 \arrow[d, "d_1"] \\
3 \arrow[r, "d_0"'] & 2
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "p_2"] \arrow[d, "p_3"'] & 3 \arrow[d, "d_1"] \\
3 \arrow[r, "d_0"'] & 2
\end{tikzcd}$$

Remark 2.3.3. By extensivity, the data of the internal category $\mathbf{2}_{\mathcal{E}} \times \mathbb{A}$ satisfies the following properties:

- The identity assigner, source, target and composition maps are the unique morphisms satisfying the commutativity of each of the respective diagrams, for $j \in \{1, 2\}$ and $k \in \{2, 3\}$.

$$\begin{tikzcd}
A_0 \arrow[r, "p_1"] \arrow[d, "i"'] & A_0 + A_0 \arrow[d, dashed, "i"] & A_0 \arrow[l, "p_2"'] \arrow[d, "i"] \\
A_1 \arrow[r, "p_1"'] & A_1 + A_1 + A_1 & A_1 \arrow[l, "p_3"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "p_j"] \arrow[d, "d_0"'] & 3A_1 \arrow[d, dashed, "d_0"] & A_1 \arrow[l, "p_3"'] \arrow[d, "d_0"] \\
A_0 \arrow[r, "p_1"'] & 2A_0 & A_0 \arrow[l, "p_2"]
\end{tikzcd}$$

$$\begin{tikzcd}
A_1 \arrow[r, "p_1"] \arrow[d, "d_1"'] & 3A_1 \arrow[d, dashed, "d_1"] & A_1 \arrow[l, "p_k"'] \arrow[d, "d_1"] \\
A_0 \arrow[r, "p_1"'] & 2A_0 & A_0 \arrow[l, "p_2"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2 \arrow[r, "p_1"] \arrow[d, "m"'] & 4A_2 \arrow[d, dashed, "m"] & A_2 \arrow[l, "p_4"'] \arrow[d, "m"] \\
A_1 \arrow[r, "p_1"'] & 3A_1 & A_1 \arrow[l, "p_3"]
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2 \arrow[r, "p_k"] \arrow[d, "m"'] & 4A_2 \arrow[d, dashed, "m"] \\
A_1 \arrow[r, "p_2"'] & 3A_1
\end{tikzcd}$$

- The following data define internal functors $\mathbb{A} \to \mathbf{2}_{\mathcal{E}}$

$$a'_{\mathbb{A}} := \left(\begin{array}{c} A_0 \xrightarrow{\quad ! \quad} 1 \xrightarrow{\quad p_1 \quad} 2 \\ A_0 \xrightarrow{\quad ! \quad} 1 \xrightarrow{\quad p_1 \quad} 3 \end{array} \right) \qquad b'_{\mathbb{A}} := \left(\begin{array}{c} A_0 \xrightarrow{\quad ! \quad} 1 \xrightarrow{\quad p_2 \quad} 2 \\ A_0 \xrightarrow{\quad ! \quad} 1 \xrightarrow{\quad p_3 \quad} 3 \end{array} \right)$$

- The data $A_0 \xrightarrow{\quad ! \quad} 1 \xrightarrow{\quad p_2 \quad} 3$ defines an internal natural transformation $\rho'_{\mathbb{A}} : a_{\mathbb{A}} \Rightarrow b_{\mathbb{A}}$.

- Thus by the 1-dimensional universal property of $\mathbf{2}_{\mathcal{E}} \times \mathbb{A}$ as a product, there are induced internal functors $a_{\mathbb{A}} := (a'_{\mathbb{A}}, 1_{\mathbb{A}}), b_{\mathbb{A}} := (b'_{\mathbb{A}}, 1_{\mathbb{A}}) : \mathbb{A} \to \mathbf{2}_{\mathcal{E}} \times \mathbb{A}$
- Finally by the 2-dimensional universal property we have an induced internal natural transformation $\rho_{\mathbb{A}} := (\rho'_{\mathbb{A}}, 1_{1_{\mathbb{A}}}) : a_{\mathbb{A}} \Rightarrow b_{\mathbb{A}}$.

Remark 2.3.4. Recall that the category of finite sets $\mathbf{Set}_f$ is the free completion under finite coproducts of the terminal category. Furthermore, for lextensive $\mathcal{E}$, the unique coproduct preserving functor $F_{\mathcal{E}} : \mathbf{Set}_f \to \mathcal{E}$ which preserves the terminal object will also preserve all other finite limits. As will be discussed in further detail in the next chapter, any finite limit preserving functor between finite limit categories $G : \mathcal{S} \to \mathcal{E}$ gives rise to a 2-functor $\mathbf{Cat}(G) : \mathbf{Cat}(\mathcal{S}) \to \mathbf{Cat}(\mathcal{E})$, which acts componentwisely on all data. Taking $\mathcal{S} = \mathbf{Set}_f$ and applying the 2-functor $\mathbf{Cat}(F_{\mathcal{E}}) : \mathbf{Cat}(\mathbf{Set}_f) \to \mathbf{Cat}(\mathcal{E})$ to the free living arrow $\mathbf{2} \in \mathbf{Cat}(\mathbf{Set}_f)$ gives the internal category $\mathbf{2}_{\mathcal{E}}$. Furthermore, the description of the internal natural transformation $\rho_{\mathbb{A}}$ given in remark 2.3.3 may then also be described as the whiskering with the unique internal functor $\mathbb{A} \to 1$ of the internal natural transformation $\mathbf{Cat}(F_{\mathcal{E}})(\alpha)$, where α is the unique non-identity natural transformation between the two distinct functors from $\mathbf{1}$ to $\mathbf{2}$. In light of the next theorem, this is to say that $\mathbf{Cat}(F_{\mathcal{E}}) : \mathbf{Cat}(\mathbf{Set}_f) \to \mathbf{Cat}(\mathcal{E})$ preserves copowers by $\mathbf{2}$.

Theorem 2.3.5. *For $\mathbb{A} \in \mathbf{Cat}(\mathcal{E})$, the internal category $\mathbf{2}_{\mathcal{E}} \times \mathbb{A}$ has the universal property of the copower of $\mathbb{A}$ by $\mathbf{2}$ in $\mathbf{Cat}(\mathcal{E})$, with the universal 2-cell given by $\rho_{\mathbb{A}}$ as described in Remark 2.3.3.*

Proof. For $\mathbb{B} \in \mathbf{Cat}(\mathcal{E})$, we need to find an assignment both on objects and morphisms $F : \mathbf{Cat}\big(\mathbf{2}, \mathbf{Cat}(\mathcal{E})(\mathbb{A}, \mathbb{B})\big) \to \mathbf{Cat}(\mathcal{E})(\mathbf{2}_{\mathcal{E}} \times \mathbb{A}, \mathbb{B})$ which is inverse to the precomposition functor $\rho_{\mathbb{A}} * (-) : \mathbf{Cat}(\mathcal{E})(\mathbf{2}_{\mathcal{E}} \times \mathbb{A}, \mathbb{B}) \to \mathbf{Cat}\big(\mathbf{2}, \mathbf{Cat}(\mathcal{E})(\mathbb{A}, \mathbb{B})\big)$.

Suppose $\alpha : A_0 \to B_1$ defines an internal natural transformation $\alpha : f^0 \Rightarrow f^1 : \mathbb{A} \to \mathbb{B}$. Then define $(F\alpha)_0 := \left(f_0^0, f_0^1\right) : 2A_0 \to B_0$ and $(F\alpha)_1 := \left(f_1^0, \widehat{\alpha}, f_1^1\right) : 3A_1 \to B_1$ to be induced by the universal property of the coproducts, where $\widehat{\alpha}$ is the morphism $A_1 \to B_1$ given by either side of the naturality axiom for α. We show that this is well-defined as an internal functor. For $j \in \{0, 1\}$, respect for sources follows from the diagrams Diagram 7.0.13, and respect for targets follows similarly. Respect for identities follows by the first diagram in Diagram 7.0.14, while respect for composition follows by the second diagram in Diagram 7.0.14 and the two diagrams in Diagram 7.0.15. The respective triangular regions in the diagrams in Diagram 7.0.15 can be shown to commute by the universal property of B_2 using similar arguments to one another. Considering those in the first of them from left to right, their commutativity follows respectively from the pairs of diagrams in Diagram 7.0.16.

Now suppose $\beta : h \Rightarrow k : \mathbb{A} \to \mathbb{B}$ is an internal natural transformation, and let the commutative square shown below on the left be viewed as a morphism from α to β in $\mathbf{Cat}(\mathcal{E})(\mathbb{A}, \mathbb{B})^{\mathbf{2}}$. Then define the morphism $F(\gamma, \delta) : 2A_0 \to B_1$ to be induced by the universal property of the coproduct. We claim that $F(\gamma, \delta)$ is an internal natural transformation $F\alpha \Rightarrow F\beta$. Sources for components follows from the universal property of $2A_0$ given the commutativity of the diagram below on the right for $j \in \{0, 1\}$, and targets for components follows from a similar argument.

$$\begin{tikzcd} f^0 \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & f^1 \arrow[d, "\delta"] \\ g^0 \arrow[r, "\beta"'] & g^1 \end{tikzcd} \qquad \begin{tikzcd} 2A_0 \arrow[r, "F\Gamma"] & B_1 \arrow[d, "d_0"] \\ A_0 \arrow[u, "p_{1+j}"] \arrow[ur, "\beta^j"] \arrow[r, "f_0^j"'] & B_0 \end{tikzcd}$$

It is natural by the commutativity of the diagrams in Diagram 7.0.17, where similar calculations involving left and right unit laws may be used to verify the commutativity of the six upper regions of the larger diagram. Furthermore, calculations using universal properties may then be used to verify that F is an isomorphism of categories natural in $\mathbb{B}$. It is then easy to see that this assignment is indeed inverse to precomposition. $\square$

We finish this section with some applications of copowers by $\mathbf{2}$ in $\mathbf{Cat}(\mathcal{E})$, namely relating the power by $\mathbf{2}$ to the exponential by $\mathbf{2}_{\mathcal{E}}$, and seeing how $\mathbf{2}_{\mathcal{E}} \times \mathbf{Disc}_{\mathcal{E}}(-) : \mathcal{E} \to \mathbf{Cat}(\mathcal{E})$ strictly preserves generating families.

Corollary 2.3.6. *Suppose $\mathcal{E}$ has finite limits, extensive coproducts, and is cartesian closed. Then the exponential object $\mathbb{A}^{\mathbf{2}_{\mathcal{E}}}$ has the universal property of the power by $\mathbf{2}$ of $\mathbb{A}$.*

Proof. Let $\mathbb{B} \in \mathbf{Cat}(\mathcal{E})$ and observe that the following isomorphisms of categories are natural in $\mathbb{B}$: $\mathbf{Cat}(\mathcal{E})\left(\mathbb{B}, \mathbb{A}^{\mathbf{2}}\right) \cong [\mathbf{2}, \mathbf{Cat}(\mathcal{E})\left(\mathbb{B}, \mathbb{A}\right)] \cong \mathbf{Cat}(\mathcal{E})\left(\mathbf{2}_{\mathcal{E}} \times \mathbb{B}, \mathbb{A}\right) \cong \mathbf{Cat}(\mathcal{E})\left(\mathbb{B}, \mathbb{A}^{\mathbf{2}_{\mathcal{E}}}\right)$. $\square$

Corollary 2.3.7. *Recall that a family of objects $\mathcal{G}$ in a category $\mathcal{C}$ is said to be* generating *if the family of hom-functors $\mathcal{C}(X, -)$ for $X \in \mathcal{G}$ are jointly faithful. Suppose that $\mathcal{E}$ has finite limits, extensive coproducts, and a generating family of objects $\mathcal{G}$. Form the family of internal categories $\widehat{\mathcal{G}} := \{\mathbf{2}_{\mathcal{E}} \times \mathbf{Disc}(X) \,|\, X \in \mathcal{G}\}$. Then $\widehat{\mathcal{G}}$ is a generating family for $\mathbf{Cat}(\mathcal{E})_1$.*

Proof. Let $f, g : \mathbb{A} \to \mathbb{B}$ be internal functors and assume that $fh = gh$ for all internal functors $h : \mathbf{2}_{\mathcal{E}} \times \mathbf{Disc}(X) \to \mathbb{A}$ where $X \in \mathcal{G}$. To show that $\widehat{\mathcal{G}}$ is a generating family, we must show that $f_1 = g_1$ under this assumption. Denoting by α the internal natural transformation which corresponds to h via the universal property of the coproduct, $f\alpha = g\alpha$. But recall that by Remark 1.5.5, any morphism $X \to A_1$ is $\mathcal{E}$ corresponds to an internal natural transformation between internal functors from $\mathbf{Disc}(X)$ to $\mathbb{A}$ This amounts to saying that $f_1\alpha = g_1\alpha$ for all $\alpha : X \to A_1$, and hence $f_1 = g_1$ as $X \in \mathcal{G}$. $\square$

2.4 Subobject Classifiers in $\mathcal{E}$ give rise to Classifiers for Full Subobjects in $\mathbf{Cat}(\mathcal{E})$

Recall that for a category with finite limits $\mathcal{E}$, a subobject classifier $\top : 1 \to \Omega$, if it exists, is a terminal object in the category whose objects are monomorphisms in $\mathcal{E}$ and morphisms are pullback squares. Recall the adjunction $\mathbf{Ob} \dashv^1_{\eta^{\mathcal{E}}} \mathbf{coDisc}$ from Remark 1.2.8. For this section, let $\mathcal{E}$ be a category with finite limits and a subobject classifier. Call an internal functor (f_0, f_1) a *full subobject* if it is fully-faithful and f_0 is a monomorphism. This title is justified by Proposition 1.2.5.

Theorem 2.4.1. *Suppose $\mathcal{E}$ has a subobject classifier $\top : 1 \to \Omega$. Then $\mathbf{coDisc}(\top)$ is a terminal object in the category whose objects are full subobjects in $\mathbf{Cat}(\mathcal{E})$, and morphisms are pullback squares.*

Proof. Since $\mathbf{coDisc}$ is a right adjoint it preserves monomorphisms, and moreover it is clear that $\mathbf{coDisc}(\top)$ satisfies the criterion for being fully-faithful from Remark 1.2.4. Let $f : \mathbb{A} \to \mathbb{B}$ be a full subobject, let $\phi_0 : B_0 \to \Omega$ be the classifier for the monomorphism f_0, and let $\phi : \mathbb{B} \to \mathbf{coDisc}(\Omega)$ be the internal functor corresponding to ϕ_0 across the natural bijection $\mathbf{Cat}(\mathcal{E})_1(\mathbb{B}, \mathbf{coDisc}(\Omega)) \cong \mathcal{E}(\mathbf{Ob}(\mathbb{B}), \Omega)$. Thus the component on arrows of ϕ is induced by the universal property of $\Omega \times \Omega$ as a product given the morphisms $\phi_0 d_0^B$ and $\phi_0 d_1^B$. We claim that ϕ is the unique internal functor making the square below on the left a pullback in $\mathbf{Cat}(\mathcal{E})$. In fact, by adjointness it will suffice just to check that this square actually is a pullback. For the 1-dimensional universal property, let $g : \mathbb{C} \to \mathbb{B}$ be an internal functor satisfying $\phi g = \mathbf{coDisc}(\top)!$. Then define $h_0 : C_0 \to A_0$ to be the morphism induced by the universal property of A_0 as a pullback, and let h_1 be induced as in the diagram on the right below, given the commutativity of the first diagram in Diagram 7.0.25 for $j \in \{0, 1\}$.

$$
\begin{tikzcd}
\mathbb{A} \arrow[r, "!"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\mathbf{coDisc}(\top)"] \\
\mathbb{B} \arrow[r, "\phi"'] & \mathbf{coDisc}(\Omega)
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[dr, dashed, "h_1"] \arrow[drr, bend left, "g_1"] \arrow[ddr, bend right, "(h_0d_0, h_0d_1)"'] & & \\
& A_1 \arrow[r, "f_1"] \arrow[d, "(d_0, d_1)"] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \arrow[d, "(d_0, d_1)"] \\
& A_0 \times A_0 \arrow[r, "f_0 \times f_0"'] & B_0 \times B_0
\end{tikzcd}
$$

Then h respects sources and targets by the universal property of A_0 given the commutativity of the second diagram in Diagram 7.0.25, and respects identities by the universal property of A_1 as a pullback given the commutativity of the first row of diagrams in Diagram 7.0.26. Similarly, the second row of diagrams in Diagram

7.0.26 show that h respects composition, with the top triangle in the first of these commuting by the universal property of B_2 as a pullback, given the commutativity of the second of these. For uniqueness, let $h' : \mathbb{C} \to \mathbb{A}$ satisfy $fh' = g$, and hence in particular $f_j h'_j = g_j$.

Finally, for the 2-dimensional universal property, let $\alpha : g^0 \Rightarrow g^1 : \mathbb{C} \to \mathbb{B}$ be an internal natural transformation and let h^0 and h^1 denote the internal functors from $\mathbb{C}$ to $\mathbb{A}$ given by the one dimensional universal property. Consider the diagrams in Diagram 7.0.27 for $j \in \{0, 1\}$. The first shows sources and targets of components for β, and in light of the commutativity of the diagrams in Diagram 7.0.28, the second and third show internal naturality while the fourth shows uniqueness of β. This completes the proof.

□

The fullness aspect of the morphisms which $\mathbf{coDisc}(\top)$ classifies is a genuinely 2-categorical notion. It does not in contrast classify subobjects in $\mathbf{Cat}(\mathcal{E})$. Recall that a category $\mathcal{E}$ is called an *elementary topos* if it has finite limits, exponentials, and a subobject classifier. Recall also that elementary toposes can be shown to have all finite colimits and right adjoints to every pullback functor (Corollaries 2.2.9 and 2.3.4, Part A in [15]). The results of this chapter combine to give the following result.

Theorem 2.4.2. *Let $\mathcal{E}$ be an elementary topos. Then the 2-category $\mathbf{Cat}(\mathcal{E})$ has*

1. *Finite weighted limits, and coidentifiers for universal 2-cells of powers by $\mathbf{2}$*
2. *Finite Extensive Coproducts*
3. *Copowers by $\mathbf{2}$*
4. *2-categorical exponentials*
5. *A full-subobject classifier*

Furthermore, if $\mathcal{E}$ has a natural numbers object, then $\mathbf{Cat}(\mathcal{E})$ also has one.

Proof. Part (1) was Theorem 1.5.3 and Proposition 1.5.6 of the previous chapter, part (2) was Theorem 2.2.1, part (3) was Theorem 2.3.5, part (4) was Theorem 2.1.1, and part (5) was Theorem 2.4. Natural numbers objects were treated in Corollary 2.2.3. □

We conclude this chapter by remarking upon some properties that $\mathbf{Cat}(\mathcal{E})$ does not inherit from $\mathcal{E}$. We have already seen that $\mathbf{Cat}(\mathcal{E})$ need not have coequalisers even if $\mathcal{E}$ does, with a counterexample being $\mathcal{E} = \mathbf{Set}_f$ and the parallel pair being the two distinct functors from $\mathbf{1}$ to $\mathbf{2}$. Furthermore, pullback functors of $\mathcal{E}$ having right adjoints, a property known as *local cartesian closedness* which is shared by all toposes, need not give local cartesian closedness in $\mathbf{Cat}(\mathcal{E})$. Indeed, even when $\mathcal{E} = \mathbf{Set}$, certain functors have no right adjoint to their associated pullback functor. A necessary and sufficient condition for an internal functor to have a right adjoint to its pullback functor was given in [10]. This means that while $\mathbf{Cat}$ has a classifier for full subobjects from Theorem 2.4.1, it does not have a genuine subobject classifier, since we already know that it has limits and exponentials, and it is not locally cartesian closed [15].

3 The 2-Functors of the form $\mathbf{Cat}(-)$

Having described the 2-category of categories, functors, and natural transformations internal to a category with finite limits, we now examine the properties of the extension of the assignment $\mathcal{E} \mapsto \mathbf{Cat}(\mathcal{E})$ to various 2-functors. In the first section of this chapter we give a description of these 2-functors, and use this view to upgrade some of the data introduced in Chapter One to being components of pseudo (or 2)-natural transformations, or modifications. In Section 3.2 we will see how some of this data combine in a very general way to give pseudomonad-like structures on $\mathbf{Cat}(-)_1 : \mathbf{Lex} \to \mathbf{Lex}$. These are the (co)skew monads of [20]. We will recall the definitions of the various monad-like structures we use along the way. In Section 3.3 we will examine some of the properties preserved by 2-functors of the form $\mathbf{Cat}(-) : \mathbf{Lex} \to \mathfrak{K}$ where $\mathfrak{K}$ is either 2-$\mathbf{Cat}$, $\mathbf{Lex}$ or 2-$\mathbf{Lex}$. The information presented in that section will be helpful for the main result of Chapter Four, and Section 5.4. Finally, in Section 3.4 we will construct a pseudocomonad structure on $\mathbf{Cat}(-)_1 : \mathbf{Lex} \to \mathbf{Lex}$ from a biadjunction between $\mathbf{Cat}(-) : \mathbf{Lex} \to 2\text{-}\mathbf{Lex}$ and the underlying functor $\mathbf{UndLex} : 2\text{-}\mathbf{Lex} \to \mathbf{Lex}$. This biadjunction was first proven in [6], and pseudocomonadicity was also remarked upon there. In addition, we will show that the action of sending a double category to its transpose is a strict distributive law for this pseudocomonad $\mathbf{Cat}(-)_1 : \mathbf{Lex} \to \mathbf{Lex}$ over itself, and moreover a *compatible flip*. We will refer the reader to external sources for definitions of strict distributive laws [25] and compatible flips [27], but recall the definitions of other 2-categorical notions as we need.

Definition 3.0.1. 1. Recall [23] that if $F, G : \mathfrak{K} \to \mathfrak{L}$ are 2-functors between 2-categories, then a *pseudonatural transformation* consists of a function $X \mapsto \alpha_X$ from the objects of $\mathfrak{K}$ to the 1-cells of $\mathfrak{L}$, and for each $X, Y \in \mathfrak{K}$, a *constraint assigning natural isomorphism*, whose component on $f \in \mathfrak{K}(X, Y)$ is given by the isomorphism $\alpha_f : \alpha_Y.Ff \Rightarrow Gf.\alpha_X$ and is called the *pseudonaturality constraint at f*. As well as being natural in f, this data is also required to satisfy a *pseudonaturality condition* given by the equality of pastings below, for all composable pairs $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathfrak{K}$. Note that if all constraint isomorphisms α_f are in fact identities then this reduces to a 2-natural transformation.

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\alpha_X"'] & FY \arrow[r, "Fg"] \arrow[d, "\alpha_Y"'] \arrow[dl, phantom, "\swarrow \alpha_f"] & FZ \arrow[d, "\alpha_Z"] \arrow[dl, phantom, "\swarrow \alpha_g"] \\
GX \arrow[r, "Gf"'] & GY \arrow[r, "Gg"'] & GZ
\end{tikzcd}
=
\begin{tikzcd}
FX \arrow[r, "F(gf)"] \arrow[d, "\alpha_X"'] & FZ \arrow[d, "\alpha_Z"] \arrow[dl, phantom, "\swarrow \alpha_{gf}"] \\
GX \arrow[r, "G(gf)"'] & GZ
\end{tikzcd}$$

2. Recall [23] that if $\alpha, \beta : F \Rightarrow G : \mathfrak{K} \to \mathfrak{L}$ are pseudonatural transformations between 2-functors which go between 2-categories, that a *modification* $\Gamma : \alpha \Rrightarrow \beta$ is a function $X \mapsto \Gamma_X$ from the objects of $\mathfrak{K}$ to the 2-cells of $\mathfrak{L}$, subject to the axiom that the following equalities of pastings hold in $\mathfrak{L}$.

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, bend right=60, "\alpha_X"'] \arrow[d, bend left=60, "\beta_X"] \arrow[d, phantom, "\overset{\Gamma_X}{\Longrightarrow}"] & FY \arrow[d, bend left, "\beta_Y"] \arrow[dl, phantom, "\swarrow \alpha_f"] \\
GX \arrow[r, "Gf"'] & GY
\end{tikzcd}
=
\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, bend right, "\alpha_X"'] & FY \arrow[d, bend right=60, "\beta_X"'] \arrow[d, bend left=60, "\beta_Y"] \arrow[d, phantom, "\overset{\Gamma_Y}{\Longrightarrow}"] \arrow[dl, phantom, "\swarrow \beta_F"] \\
GX \arrow[r, "Gf"'] & GY
\end{tikzcd}$$

Remark 3.0.2. We also recall the following useful facts about the compositional structure of pseudonatural transformations and modifications.

- Recall [13] that given any two 2-categories $\mathfrak{K}$ and $\mathfrak{L}$ with possibly large hom-categories, there is a large 2-category $\mathbf{GRAY}(\mathfrak{K}, \mathfrak{L})$ whose objects are 2-functors from $\mathfrak{K}$ to $\mathfrak{L}$, 1-cells are pseudonatural transformations between these, and 2-cells are modifications between those. Such 2-categories will be denoted $[\mathfrak{K}, \mathfrak{L}]$.

- Recall [13] that modifications in $[\mathfrak{K}, \mathfrak{K}]$ compose pointwisely in both vertical and horizontal directions.

- Recall [13] that pseudonatural transformations $F \xrightarrow{\alpha} G \xrightarrow{\beta} H$ compose vertically by pointwise composition of their 1-cell components, and pasting together of their 2-cell components on $f \in \mathfrak{K}(X, Y)$ along Gf.

- Recall [13] that pseudonatural transformations as pictured below on the left have two possible ways of composing horizontally, given by the square shown in the middle, and that while these are not in general equal, there is always an invertible modification between them given by $X \mapsto \beta_{\alpha_X}$. The modification axiom follows from naturality of β's pseudonaturality constraint on α_X, and the pseudonaturality condition for β on each side of the square below on the right.

$$\begin{tikzcd}
\mathfrak{K} \arrow[r, bend left=60, "F"{name=A}] \arrow[r, bend right=60, "F'"'{name=B}] \arrow[from=A, to=B, Rightarrow, "\alpha"] & \mathfrak{L} \arrow[r, bend left=60, "G"{name=C}] \arrow[r, bend right=60, "G'"'{name=D}] \arrow[from=C, to=D, Rightarrow, "\beta"] & \mathfrak{M}
\end{tikzcd}
\qquad
\begin{tikzcd}
GF \arrow[rr, "G\alpha"] \arrow[dd, "\beta F"'] & & GF' \arrow[dd, "\beta F'"] \arrow[ddll, Rightarrow, "\beta_\alpha"] \\
& & \\
G'F \arrow[rr, "G'\alpha"'] & & G'F'
\end{tikzcd}$$

Notation 3.0.3. *1. Let* **Und** : 2-**Cat** → **Cat** *denote the* 2*-functor that sends a* 2*-category to its underlying category, and let* **UndLex** : 2-**Lex** → **Lex** *be defined similarly.*

2. Let [**Lex**, **Lex**] *denote the* 2*-category whose objects are* 2*-functors,* 1*-cells are pseudonatural transformations, and* 2*-cells are modifications.*

Remark 3.0.4. We recall the particular isomorphism up to which the composite of two finite limit preserving functors preserves finite limits. This will be useful in proving many of the results in this chapter. Let $\mathcal{E} \xrightarrow{F} \mathcal{S} \xrightarrow{G} \mathcal{T}$ be 1-cells in either **Lex** or **LEX**, and for a diagram $D : \mathcal{D} \to \mathcal{E}$, let $\phi \in \mathcal{S}\big(\lim(FD), F(\lim D)\big)$ and $\psi \in \mathcal{T}\big(\lim(GFD), G\lim(FD)\big)$ be the isomorphisms up to which the functors preserve the limit of the diagram. Then their composite GF preserves the limit of D up to the following isomorphism

$$\lim(GFD) \xrightarrow{\psi} G\lim(FD) \xrightarrow{G\phi} GF\lim(D)$$

Much of what we will have to say in this chapter is illuminated by the following result.

Theorem 3.0.5 (Gabriel–Ulmer Duality). *[11] Let* **LFP** *denote the* 2*-category of locally finitely presentable categories, finitary (or filtered colimit preserving) right adjoints, and natural transformations. Then the* 2*-functor* $\mathbf{LEX}(-, \mathbf{Set}) : \mathbf{Lex}^{op} \to \mathbf{CAT}$ *factors through* **LFP** *by a biequivalence* $\mathbf{Lex}^{op} \cong \mathbf{LFP}$.

3.1 The 2-Functors, and their associated Transformations and Modifications

Proposition 3.1.1. *For* $\mathcal{E} \in \mathbf{Lex}$*, the assignment* $\mathcal{E} \mapsto \mathbf{Cat}(\mathcal{E})$ *extends to a* 2*-functor* $\mathbf{Cat}(-) : \mathbf{Lex} \to 2\text{-}\mathbf{Cat}$.

Proof. As the data and axioms of an internal category mention only finite limits, given a finite limit preserving functor $F : \mathcal{E} \to \mathcal{S}$, there is a 2-functor $\mathbf{Cat}(F) : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Cat}(\mathcal{S})$ which takes the image under F of each piece of data, producing the required structure in $\mathbf{Cat}(\mathcal{S})$. Furthermore, given a 2-cell $\phi : F \Rightarrow G : \mathcal{E} \to \mathcal{S}$ in **Lex**, there is a 2-natural transformation $\mathbf{Cat}(\phi) : \mathbf{Cat}(F) \Rightarrow \mathbf{Cat}(G)$ whose component at $\mathbb{C} \in \mathbf{Cat}(\mathcal{E})$ is the internal functor $\mathbf{Cat}(F)\,\mathbb{C} \to \mathbf{Cat}(G)\,\mathbb{C}$ whose object and morphism assignment is given by the component of ϕ on C_0 and C_1 respectively; all conditions for internal functoriality follow by naturality of ϕ, as does 2-naturality of $\mathbf{Cat}(\phi)$. All of these observations combine to describe a 2-functor $\mathbf{Cat}(-) : \mathbf{Lex} \to 2\text{-}\mathbf{Cat}$; the axioms of 2-functoriality and functoriality between hom-categories are evident from the component-wise description of the data involved. □

We will often write $\mathbf{Cat}(F)\,\mathbb{C}$ and $\mathbf{Cat}(\phi)_{\mathbb{C}}$ as just $F_*\mathbb{C}$ and $\phi_{\mathbb{C}}$ respectively.

Example 3.1.2. *Given* $X \in \mathcal{E}$*, the hom-functor* $\mathcal{E}(X, -)$ *preserves finite limits and hence gives rise to a* 2*-functor* $\mathbf{Cat}(\mathcal{E}) \to \mathbf{Cat}$.

Remark 3.1.3. We remark briefly upon issues of size. As noted previously in Theorem 1.4.6, if $\mathcal{E}$ is small then so is $\mathbf{Cat}(\mathcal{E})$, and if $\mathcal{E}$ is locally small then the hom-categories of $\mathbf{Cat}(\mathcal{E})$ are small. We can therefore define a completely analogous 2-functor from **LEX** to 2-**CAT**. This would have values for locally small categories such as such as **Set**, and hence also categories of presheaves $[\mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ of small categories $\mathcal{C} \in \mathbf{Cat}$. In this light, the yoneda embedding $Y_{\mathcal{E}} : \mathcal{E} \to [\mathcal{E}^{\mathrm{op}}, \mathbf{Set}]$ of any small finite limit category is a 1-cell of **LEX**, and so another example is given by $\mathbf{Cat}(Y_{\mathcal{E}}) : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Cat}[\mathcal{E}^{\mathrm{op}}, \mathbf{Set}]$, where by $\mathbf{Cat}[\mathcal{E}^{\mathrm{op}}, \mathbf{Set}]$ we denote the 2-category $\mathbf{Cat}(\mathcal{T})$ where $\mathcal{T} := [\mathcal{E}^{\mathrm{op}}, \mathbf{Set}]$. We will return to this example after proving Theorem 3.3.2. For further information on issues of size, see [31].

Remark 3.1.4. The 2-functor $\mathbf{Cat}(-) : \mathbf{LEX} \to 2\text{-}\mathbf{CAT}$ sends representable functors $\mathcal{E}(X, -) : \mathcal{E} \to \mathbf{Set}$ to representable 2-functors $\mathbf{Cat}(\mathcal{E})(\mathbf{Disc}_{\mathcal{E}}(X), -) : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Cat}$, as is clear from the adjunction of Remark 1.2.7. Note that functors internal to $\mathcal{S} = \mathbf{Set}$ are representably fully-faithful, discrete fibrations, or discrete opfibrations if and only if they satisfy the conditions described in Remark 1.2.4. It follows that functors internal to arbitrary locally small categories with finite limits satisfy these representable notions precisely if they satisfy the conditions described in Remark 1.2.4.

In fact, the functors $\mathbf{Cat}(F)$ preserve finite weighted limits, which leads to the following proposition.

Proposition 3.1.5. *The* 2*-functor* $\mathbf{Cat}(-) : \mathbf{Lex} \to 2\text{-}\mathbf{Cat}$ *factors through the inclusion* $2\text{-}\mathbf{Lex} \to 2\text{-}\mathbf{Cat}$*, and the* 2*-functor* $\mathbf{Cat}(-)_1 := \mathbf{Und} \circ \mathbf{Cat}(-) : \mathbf{Lex} \to \mathbf{Cat}$ *factors through the inclusion* $\mathbf{Lex} \to \mathbf{Cat}$*. In other words, the functors* $\mathbf{Cat}(-) : \mathbf{Lex} \to 2\text{-}\mathbf{Cat}$ *and* $\mathbf{Cat}(-)_1 : \mathbf{Lex} \to \mathbf{Cat}$ *lift to give functors* $\mathbf{Cat}(-) : \mathbf{Lex} \to 2\text{-}\mathbf{Lex}$ *and* $\mathbf{Cat}(-)_1 : \mathbf{Lex} \to \mathbf{Lex}$*. There are also* 2*-functors* $\mathbf{LEX} \to 2\text{-}\mathbf{CAT}$ *and* $\mathbf{LEX} \to 2\text{-}\mathbf{LEX}$ *which restrict to these.*

Proof. Since we have already shown that $\mathbf{Cat}(\mathcal{E})$ has finite weighted limits, it suffices to show that for $F \in \mathbf{LEX}(\mathcal{E}, \mathcal{S})$ the 2-functor $\mathbf{Cat}(F) : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Cat}(\mathcal{S})$ preserves finite weighted limits. Conical limit preservation clearly follows from that of F, and recalling the characterising conditions for internal natural transformations which are universal 2-cells of powers by $\mathbf{2}$, as described in Theorem 1.5.4, we see that $\mathbf{Cat}(F)$ preserves these too. Finally, since hom-categories in $\mathbf{Cat}(\mathcal{E})$ have sets of objects which are no larger than the hom-sets of $\mathcal{E}$, we have the required restriction to $\mathbf{Lex}$. □

In the remark below, we describe in more detail the particular isomorphisms up to which the functors $\mathbf{Cat}(F)_1$ preserve powers by $\mathbf{2}$. These will feature as the components of the pseudonaturality constraint of a pseudonatural transformation which will be the compultimplication of the pseudocomonad on which Section 3.4 will focus.

Remark 3.1.6. As a particular case of the functors $\mathbf{Cat}(F)_1$ preserving limits, we have the isomorphisms $\delta_{F\mathbb{A}} := \left(1_{A_0}, F_{A_{\mathrm{Sq}}}\right) : (F\mathbb{A})^{\mathbf{2}} \cong F\left(\mathbb{A}^{\mathbf{2}}\right)$ in $\mathbf{Cat}(S)$, where $F_{A_{\mathrm{Sq}}} : (FA)_{\mathrm{Sq}} \cong F(A_{\mathrm{Sq}})$ are the isomorphisms up to which F preserves that pullback. It will be useful later to note that these internal functors are extremal in the sense of Definition 1.5.1. Note also that the assignment $\delta_F : \mathbb{A} \mapsto \delta_{F\mathbb{A}}$ is indeed natural in $\mathbb{A}$, and that the assignment $F \mapsto \delta_F$ is natural in F, for finite limit preserving functors F. This follows from similar calculations to those in the proof of Proposition 1.5.9.

Recall the transposition 2-functor $\mathbf{T}_{\mathcal{E}}$ defined in Definition 1.5.1, which swaps the vertical and horizontal categories in a double category. Its underlying functor will feature as the component at $\mathcal{E}$ of a strict distributive law of the pseudocomonad $\mathbf{Cat}(-)_1$ over itself, in fact a *compatible flip* in the sense of [27]. For now, we prove 2-naturality.

Proposition 3.1.7. *Let* $\mathbf{DblCat}(-) : \mathbf{Lex} \to 2\text{-}\mathbf{Lex}$ *and* $\mathbf{DblCat}(-)_1 : \mathbf{Lex} \to \mathbf{Lex}$ *be given by* $\mathbf{Cat}(-) \circ \mathbf{Cat}(-)$ *and* $\mathbf{Cat}(-)_1 \circ \mathbf{Cat}(-)_1$*, analogously to what is described in Definition 1.5.1. The assignments* $\mathcal{E} \mapsto \mathbf{T}_{\mathcal{E}} : \mathbf{DblCat}(\mathcal{E}) \to \mathbf{DblCat}(\mathcal{E})$ *and* $\mathcal{E} \mapsto T_{\mathcal{E}} := \mathbf{UndeLex}(\mathbf{T}_{\mathcal{E}}) : \mathbf{DblCat}(\mathcal{E})_1 \to \mathbf{DblCat}(\mathcal{E})_1$ *are* 2*-natural in* $\mathcal{E}$*.*

Proof. It suffices to show 2-naturality of $\mathcal{E} \mapsto \mathbf{T}_{\mathcal{E}}$, as the other will just be its whiskering with $\mathbf{UndLex}$. But it is easy to see that transposition of a double category commutes with taking the image of its data under any finite limit preserving functor. □

Remark 3.1.8. Let $n \geq 2$ and consider the involutions on n-$\mathbf{tuple}$-$\mathbf{Cat}(\mathcal{E})_1$ given by functors of the form k-$\mathbf{tuple}$-$\mathbf{Cat}\left(T_{j\text{-}\mathbf{tuple}\text{-}\mathbf{Cat}(\mathcal{E})_1}\right)_1$, where $j, k \in \mathbb{Z}/n\mathbb{Z}$ such that $j + k = n - 2$, and 0-$\mathbf{tuple}$-$\mathbf{Cat}(-)_1$ is interpreted as $1_{\mathbf{Lex}}$. We observe that these are precisely the adjacent transpositions in the group theoretic sense, which generate the symmetric group on n letters S_n as at least a subgroup of the group of automorphisms on n-$\mathbf{tuple}$-$\mathbf{Cat}(\mathcal{E})_1$. This observation will clarify some calculations in Section 3.4.

We conclude this section by examining how some of the functors associated to $\mathbf{Cat}(\mathcal{E})_1$ in Chapter One vary in $\mathcal{E} \in \mathbf{Lex}$. Proposition 3.1.11 will be used in the next section, and we will have more to say about Theorem 3.1.9 through this chapter, particularly following the proof of Theorem 3.3.2.

Theorem 3.1.9. *Let $\mathcal{E} \in \mathbf{Lex}$, let $\mathbf{Cat}(-)_1 : \mathbf{Lex} \to \mathbf{Lex}$ the the 2-functor described in Proposition 3.1.5, and recall the data of the evaluation category internal to $[\mathbf{Cat}(\mathcal{E})_1, \mathcal{E}]$ from Remark 1.2.6.*

1. *The functors $ev_n^{\mathcal{E}} : \mathbf{Cat}(\mathcal{E})_1 \to \mathcal{E}$ of Remark 1.2.6 are the components at $\mathcal{E} \in \mathbf{Lex}$ of 2-natural transformations $ev_n^{\mathcal{E}} : \mathbf{Cat}(-)_1 \Rightarrow 1_{\mathbf{Lex}}$.*

2. *The natural transformations of Remark 1.2.6 are the components at $\mathcal{E}$ of modifications.*

3. *This data defines a category internal to $[\mathbf{Lex}, \mathbf{Lex}](\mathbf{Cat}(-), \mathbf{1_{Lex}})$.*

Proof. Part (1) is clear, while the modification axiom for $d_0^{\mathcal{E}}$, $d_1^{\mathcal{E}}$, $i^{\mathcal{E}}$ and $m^{\mathcal{E}}$ follow from the definition of $\mathbf{Cat}(F)$ on objects. For example, for $m^{\mathcal{E}}$, it follows from the commutativity of the first diagram in Diagram 7.0.18. To see that this data forms an internal category it suffices to check that the pullbacks are well-defined in $[\mathbf{Lex}, \mathbf{Lex}](\mathbf{Cat}(-)_1, 1_{\mathbf{Lex}})$, since the commutativity conditions involved in the axioms of an internal category will follow pointwisely. We show that the 2-natural transformation $\mathbf{Pair}$ is the pullback of the modifications d_0 and d_1 via the projection modifications π_0 and π_1, since all other pullback conditions will follow by similar arguments. Let $\alpha : \mathbf{Cat}(-)_1 \to 1_{\mathbf{Lex}}$ be a pseudonatural transformation with component at $\mathcal{E} \in \mathbf{Lex}$ written as $\alpha^{\mathcal{E}}$ and pseudonaturality constraint written as α_F. Let $p_0, p_1 : \alpha \Rrightarrow \mathbf{Arr}$ be modifications satisfying $d_0 p_1 = d_1 p_0$. Then in particular their components on $\mathcal{E} \in \mathbf{Lex}$ satisfy this equation, and so since $\mathbf{Pair}_{\mathcal{E}}$ is the pullback of $d_0^{\mathcal{E}}$ and $d_1^{\mathcal{E}}$ in $[\mathbf{Cat}(\mathcal{E})_1, \mathcal{E}]$, we have an induced natural transformation $(p_0, p_1)^{\mathcal{E}} : \alpha^{\mathcal{E}} \Rightarrow \mathbf{Pair}_{\mathcal{E}}$. It suffices to show that the assignment $\mathcal{E} \mapsto (p_0, p_1)^{\mathcal{E}}$ is a modification, since if it is so then it will be unique by the universal property from which its components have been defined. The modification axiom requires the commutativity of the second diagram in Diagram 7.0.18, and this holds by the universal property of $(FA)_2$ as a pullback in $\mathcal{S}$ given the commutativity of the third diagram in Diagram 7.0.18 for $j \in \{0, 1\}$, where the region labelled * commutes by the modification axiom for p_j on $F \in \mathbf{Lex}(\mathcal{E}, \mathcal{S})$. □

Remark 3.1.10. Note that the category $[\mathbf{Lex}, \mathbf{Lex}](\mathbf{Cat}(-)_1, 1_{\mathbf{Lex}})$ does not have all pullbacks, but it does have those pullbacks necessary for the construction of the internal category above. Also note that had we defined internal categories in the usual way, with objects of composable n-tuples determined not up to equality but up to isomorphism, then the 2-natural transformations above would instead be only pseudonatural in $\mathcal{E}$, with pseudonaturality constraints on $F : \mathcal{E} \to \mathcal{S}$ being given by the isomorphisms up to which F preserves the relevant pullbacks.

Proposition 3.1.11. *Recall the adjunctions $\mathbf{Disc}_{\mathcal{E}} \dashv_1^{\epsilon^{\mathcal{E}}} \mathbf{Ob}_{\mathcal{E}} \dashv_{\eta^{\mathcal{E}}}^1 \mathbf{coDisc}_{\mathcal{E}}$ of Remarks 1.2.7 and 1.2.8.*

1. *The assignment $\mathcal{E} \mapsto \mathbf{Disc}_{\mathcal{E}}$ is 2-natural in $\mathcal{E}$.*

2. *The assignment $\mathcal{E} \mapsto \mathbf{coDisc}_{\mathcal{E}}$ is pseudonatural in $\mathcal{E}$, with pseudonaturality constraints on $F : \mathcal{E} \to \mathcal{S}$ being natural isomorphisms whose component on $X \in \mathcal{E}$ is the internal functor $(1_X, F_X)$, where $F_X : FX \times FX \cong F(X \times X)$ is the isomorphism up to which F preserves the product.*

3. *The assignments $\mathcal{E} \mapsto \eta^{\mathcal{E}}$ and $\mathcal{E} \mapsto \epsilon^{\mathcal{E}}$ are modifications.*

4. *This data describes a pair of adjunctions in the 2-category $[\mathbf{Lex}, \mathbf{Lex}]$.*

Proof. The proofs are similar to those of the previous theorem. Part (1) is clear from the definition. For part (2), naturality of $X \mapsto (1_X, F_X)$ follows by the universal property of the product, while pseudonaturality of $\mathcal{E} \mapsto \mathbf{coDisc}_{\mathcal{E}}$ follows from Remark 3.0.4. The modification axiom for $\mathcal{E} \mapsto \epsilon^{\mathcal{E}}$ follows from that of i, while the modification axiom for $\eta^{\mathcal{E}}$ follows from Remark 3.0.4. Finally, the triangle identities for the adjunctions in $[\mathbf{Lex}, \mathbf{Lex}]$ follow pointwisely from those of the original adjunctions. □

3.2 Skew and coSkew Pseudomonad Structures on $\mathbf{Cat}\,(-)$

In this chapter we will see that the 2-functor $\mathbf{Cat}\,(-)_1 : \mathbf{Lex} \to \mathbf{Lex}$ can be equipped with two different monad-like structures. Both of these arise from a general setting of a 2-natural transformation into an identity 2-functor having a left (respectively, right) adjoint with an identity unit (respectively, counit). We will therefore first prove the general statement and then specialise it to our example of interest. To begin, we recall the precise definitions of monad-like structures on endo-2-functors on 2-categories, which can be found in [9] and [25]. More general definitions involving pseudofunctors on bicategories also exist, but they will not be needed in this thesis. In this section we will focus on skew-pseudomonad variants of the definition below, while in Section 3.4 we will see a pseudocomonad structure on $\mathbf{Cat}\,(-)_1 : \mathbf{Lex} \to \mathbf{Lex}$. In both cases, we will have something to say about the (co)algebras for the (co)monads.

Definition 3.2.1. Let $1_{\mathfrak{K}} \xrightarrow{\eta} T \xleftarrow{\mu} T^2$ be a cospan in $\mathbf{GRAY}\big(\mathfrak{K}, \mathfrak{K}\big)$, and let τ, α and β be invertible modifications as pictured.

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] \arrow[dr, phantom, "\Downarrow\tau"] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "T\eta"] \arrow[dr, "1"'] \arrow[r, phantom, bend right, "\Downarrow\alpha"] & T^2 \arrow[d, "\mu"] \\ & T \end{tikzcd} \qquad \begin{tikzcd} T^2 \arrow[r, "\mu"] \arrow[r, phantom, bend right, "\beta\Downarrow"] & T \\ T \arrow[u, "\eta T"] \arrow[ur, "1"'] & \end{tikzcd}$$

Call the data $(T, \mu, \tau, \eta, \alpha, \beta)$ a *pseudomonad* on $\mathfrak{K}$ if the following equalities of pastings in Diagram 7.0.19 hold. Call a pseudomonad on $\mathfrak{K}^{\mathrm{op}}$ a *pseudocomonad* on $\mathfrak{K}$. If μ and η are 2-natural, and τ, α, β are identities, replace the prefix 'pseudo' with the prefix '2−'. Finally, if α is non-invertible with direction as indicated above, say the data has a *skew left unit law*, while if it has the opposite direction then say the data has a *coskew left unit law.*

Definition 3.2.2. Let $(T, \mu, \tau, \eta, \alpha, \beta)$ be a pseudomonad on $\mathfrak{K}$ as in the above definition. A *pseudoalgebra* for the pseudomonad is a 1-cell $a : TA \to A$ and two 2-cells

$$\begin{tikzcd} T^2A \arrow[r, "Ta"] \arrow[d, "\mu_A"'] \arrow[dr, phantom, "\Downarrow\alpha"] & TA \arrow[d, "a"] \\ TA \arrow[r, "a"'] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[dr, "1"'] \arrow[r, phantom, bend right, "\Downarrow\underline{\alpha}"] & TA \arrow[d, "a"] \\ & A \end{tikzcd}$$

such that the diagrams in Diagram 7.0.20 commute. If α and $\underline{\alpha}$ are identities then this is called a strict algebra. Finally, if $(T, \mu, \tau, \eta, \alpha, \beta)$ is instead a pseudo/2-comonad, then this data is called a pseudo/strict coalgebra.

We now show the general setting from which the skew and coskew pseudomonad structures on $\mathbf{Cat}\,(-)_1 :$ $\mathbf{Lex} \to \mathbf{Lex}$ arise.

Theorem 3.2.3. *Let $C : \mathfrak{K} \to \mathfrak{K}$ be a 2-functor on a 2-category, and let $\mu : C \to 1_{\mathfrak{K}}$ be a pseudonatural transformation whose pseudonaturality constraints on its own components are identities.*

1. *Suppose $\rho : 1_{\mathfrak{K}} \to C$ is right adjoint to μ in $\mathbf{GRAY}\big(\mathfrak{K}, \mathfrak{K}\big)$ with unit η and identity counit, and so that the pseudonaturality constraints of ρ on the components of μ are identities. Then C is a pseudomonad with strict associativity for a multiplication given by $C\mu$, a strict right unit law for a unit given by ρ, and a lax left unit law for ρ holding up to the modification η.*

2. *Suppose $\lambda : 1_{\mathfrak{K}} \to C$ is left adjoint to μ in $\mathbf{GRAY}\big(\mathfrak{K}, \mathfrak{K}\big)$ with counit ϵ and identity unit, and so that the pseudonaturality constraints of λ on the components of μ are identities. Then C is a pseudomonad with strict associativity for a multiplication given by $C\mu$, a strict right unit law for a unit given by λ, and a colax left unit law for λ holding up to the modification ϵ.*

3. *The components of $\mu : C \to 1_{\mathfrak{K}}$ equip their codomains with the structure of a strict algebra for both of the above monad-like structures.*

Proof. Let $\mathcal{E} \in \mathfrak{K}$, and consider the pseudonaturality constraint of μ on $\mu_{\mathcal{E}}$. This is the identity by assumption, and taking its image under C gives the strict associativity law in both cases. To see that the right unit laws hold strictly in both cases, note that for $\kappa \in \{\lambda, \rho\}$ it holds that $C\mu.C\kappa = C(\mu.\kappa) = C(1) = 1$, where the third equality holds in each case since the counit of $\mu \dashv \rho$ and the unit of $\lambda \dashv \mu$ are identities by assumption. Note that λ and ρ do indeed 'type check' as non-invertible modifications up to which the left unit laws may hold since in the following pastings, the quadrilaterals commute by the assumption on the pseudonaturality constraints of λ and ρ on the components of μ.

$$\begin{tikzcd}
C\mathcal{E} \arrow[rr, "\lambda_{C\mathcal{E}}"] \arrow[dd, equal] \arrow[dr, "\mu_{\mathcal{E}}"'] & & CC\mathcal{E} \arrow[dd, "C(\mu_{\mathcal{E}})"] \\
& \mathcal{E} \arrow[dr, "\lambda_{\mathcal{E}}"] \arrow[d, phantom, "\epsilon^{\mathcal{E}} \Swarrow"] & \\
C\mathcal{E} \arrow[rr, equal] & {} & C\mathcal{E}
\end{tikzcd}
\qquad
\begin{tikzcd}
C(\mathcal{E}) \arrow[rr, "\rho_{C(\mathcal{E})}"] \arrow[dd, equal] \arrow[dr, "\mu_{\mathcal{E}}"'] & & CC\mathcal{E} \arrow[dd, "C(\mu_{\mathcal{E}})"] \\
& \mathcal{E} \arrow[dr, "\rho_{\mathcal{E}}"] \arrow[d, phantom, "\eta^{\mathcal{E}} \Nearrow"] & \\
C\mathcal{E} \arrow[rr, equal] & {} & C\mathcal{E}
\end{tikzcd}$$

The unit axioms for the pseudomonad are just the images under C of the triangle identities for the respective adjunctions in each case. Finally, note that the multiplication law for μ as a strict algebra holds trivially, while the unit law holds in each case due to the adjunctions having identity unit or counit. Then the unit axioms for the strict algebra are just the respective triangle identities. $\square$

Applying this result to the case where $\mathfrak{K} = \mathbf{Lex}, C = \mathbf{Cat}(-)_1, \mu = \mathbf{Ob}$, and η and ϵ are from the adjunctions $\mathbf{Disc} \dashv_1^{\epsilon} \mathbf{Ob} \dashv_{\eta}^{1} \mathbf{coDisc}$, we obtain the following corollary.

Corollary 3.2.4. *Consider the* 2*-functor* $\mathbf{Cat}(-)_1 : \mathbf{Lex} \to \mathbf{Lex}$ *and the* 2*-natural transformation* $\mathbf{Cat}(\mathbf{Ob})_1$*, and recall the adjunctions in* $[\mathbf{Lex}, \mathbf{Lex}]$ *described in Proposition 3.1.11. Then* $\mathbf{Cat}(\mathbf{Ob})$ *equips* $\mathbf{Cat}(-)_1$ *with a multiplication in*

1. *A* 2*-monad on* $\mathbf{Lex}$*, with unit given by the* 2*-natural transformation* $\mathbf{Disc}$*, and a lax left unit law holding up to the modification* η*.*

2. *A pseudomonad on* $\mathbf{Lex}$*, with unit given by the pseudonatural transformation* $\mathbf{coDisc}$ *and colax left unit law holding up to the modification* ϵ*.*

Furthermore, the components of $\mathbf{Ob}$ *are strict algebras for these monad-like structures.*

3.3 Properties $\mathbf{Cat}(-)$ preserves, and the Unit Category

A consequence of the Gabriel–Ulmer Duality is that the 2-functor $\mathbf{Cat}(-)_1 : \mathbf{Lex} \to \mathbf{Cat}$ is representable, with representing object the opposite category of finitely presented models for the finite limit theory of categories; $(\mathbf{f.p.Cat})^{\mathrm{op}}$. In particular this means that $\mathbf{Cat}(-) : \mathbf{Lex} \to 2\text{-}\mathbf{Cat}$ preserves any limits which exist in $\mathbf{Lex}$. One can alternatively show by direct but lengthy calculation that it preserves products, inserters and equifiers; certain 2-categorical limits from which those 2-categorical limits known as 'PIE limits' can be constructed. These limits are inherited in $\mathbf{Lex}$ directly from $\mathbf{Cat}$ [28]. For our purposes it will be enough to know that it preserves powers and fully-faithful morphisms, which is what we show next.

Proposition 3.3.1. *The* 2*-functors* $\mathbf{Cat}(-)$*, which go from* $\mathbf{Lex}$ *to* 2-$\mathbf{Cat}$ *and* 2-$\mathbf{Lex}$ *respectively, send fully faithful functors to* 2*-fully faithful functors. The same is true for the* 2*-functors from* $\mathbf{LEX}$ *to either* 2-$\mathbf{CAT}$ *or* 2-$\mathbf{LEX}$.

Proof. Note that since 2-fully-faithful morphisms in 2-$\mathbf{LEX}$ are precisely those morphisms whose underlying 2-functors are 2-fully-faithful, it suffices to show this for $\mathbf{Cat}(-) : \mathbf{LEX} \to 2\text{-}\mathbf{CAT}$. Let $F : \mathcal{E} \to \mathcal{S}$ be a fully faithful finite limit preserving functor, and let $\mathbb{A}, \mathbb{B} \in \mathbf{Cat}(\mathcal{E})$ be internal categories. For $A, B \in \mathcal{E}$, let

$\phi_{A,B} : \mathcal{S}(FA, FB) \to \mathcal{E}(A, B)$ denote the inverse to taking the image under F, given by fully-faithfulness of F. Define the functor $\Phi_{\mathbb{A},\mathbb{B}} : \mathbf{Cat}(\mathcal{S})(F\mathbb{A}, F\mathbb{B}) \to \mathbf{Cat}(\mathcal{E})(\mathbb{A}, \mathbb{B})$ as follows:

- For an internal functor $f : F_*\mathbb{A} \to F_*\mathbb{B}$ given by the data $f := (f_0, f_1) : (FA_0, FA_1) \to (FB_0, FB_1)$, take $\Phi_{\mathbb{A},\mathbb{B}}(f)$ to have objects and arrows assignment given by $\phi_{A_j,B_j}(f_j)$ for $j \in \{0, 1\}$. This is well-defined as an internal functor since the diagrams required to commute in $\mathcal{E}$ are sent by F to commutative diagrams in $\mathcal{S}$, and F is faithful and finite limit preserving.
- For an internal natural transformation $\alpha : f \Rightarrow g : F_*\mathbb{A} \to F_*\mathbb{B}$ given by the data $\alpha \in \mathcal{S}(FA_0, FB_1)$, define $\Phi_{\mathbb{A},\mathbb{B}}(\alpha) := \phi_{A_0,B_1}(\alpha)$. Once again, this is well defined by faithfulness and finite limit preservation of F.

To complete the proof, observe that since $F_{A_j,B_k} : \mathcal{E}(A_j, B_k) \leftrightarrow \mathcal{S}(FA_j, FB_k) : \phi_{A_j,B_k}$ is a bijection, and the functors $F_{\mathbb{A},\mathbb{B}}$ and $\Phi_{\mathbb{A},\mathbb{B}}$ are entirely defined via these functions, they must constitute an isomorphism of categories. It is easy to see that the same argument applies when $\mathbf{Cat}(-)$ is restricted to $\mathbf{Lex}$. □

In particular, the yoneda embedding $Y_{\mathcal{E}} : \mathcal{E} \to [\mathcal{E}^{\mathrm{op}}, \mathbf{Set}]$ is a fully faithful functor which preserves finite limits, and hence a 1-cell in $\mathbf{LEX}$. Thus it gives rise to a 2-fully faithful 2-functor $Cat(Y_{\mathcal{E}}) : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Cat}([\mathcal{E}^{\mathrm{op}}, \mathbf{Set}])$.

Theorem 3.3.2. *The 2-functors $\mathbf{Cat}(-) : \mathbf{Lex} \to 2\text{-}\mathbf{Cat}$ and $\mathbf{Cat}(-)_1 : \mathbf{Lex} \to \mathbf{Cat}$ preserve powers by small categories, as do the 2-functors $\mathbf{Cat}(-) : \mathbf{Lex} \to 2\text{-}\mathbf{Lex}$ and $\mathbf{Cat}(-)_1 : \mathbf{Lex} \to \mathbf{Lex}$, and the analogous 2-functors with domain $\mathbf{LEX}$.*

Proof. Since the power of $\mathcal{S} \in \mathbf{LEX}$ by $\mathcal{E} \in \mathbf{Cat}$ is given by the functor category $[\mathcal{E}, \mathcal{S}]$ we need to prove that, for $\mathcal{E}$ a small category and $\mathcal{S}$ a locally small category with finite limits, there is an isomorphism of 2-categories $\mathbf{Cat}[\mathcal{E}, \mathcal{S}] \equiv 2\text{-}\mathbf{Cat}\big(\mathcal{E}, \mathbf{Cat}(\mathcal{S})\big)$. To prove this, we will construct 2-functors $\underline{\mathcal{S}}(-) : [\mathcal{E}, \mathbf{Cat}(\mathcal{S})] \to \mathbf{Cat}[\mathcal{E}, \mathcal{S}]$ and $\widehat{(-)} : \mathbf{Cat}[\mathcal{E}, \mathcal{S}] \to [\mathcal{E}, \mathbf{Cat}(\mathcal{S})]$. These 2-functors will respectively act by whiskering or composing with the evaluation category $\underline{\mathcal{S}}$, and taking images or components of data. It will be clear from their construction that these 2-functors constitute an isomorphism of 2-categories.

Given a 2-functor $F : \mathcal{E} \to \mathbf{Cat}(\mathcal{S})$, it may be whiskered and composed with the data of the evaluation category $\underline{\mathcal{S}}$ to produce a category in $[\mathcal{E}, \mathcal{S}]$, which we denote $\underline{\mathcal{S}}F$. Internal category axioms are inherited from those for $\underline{\mathcal{S}}$. Conversely, given a category $\mathbb{A} := (A_0, A_1, d_0, d_1, i, m)$ internal to $[\mathcal{E}, \mathcal{S}]$, we may define the 2-functor $\widehat{\mathbb{A}} : \mathcal{E} \to \mathbf{Cat}(\mathcal{S})$ by taking the image or component under each relevant piece of data of $\mathbb{A}$. Functoriality between hom-categories is immediate as $\mathcal{E}$ is a 1-category and hence has discrete hom-categories, while 2-functoriality of $\widehat{\mathbb{A}}$ follows from functoriality of A_0 and A_1. It is easy to see from their definition that these constructions are mutually inverse.

Given a 2-natural transformation $\phi : F \Rightarrow G : \mathcal{E} \to \mathbf{Cat}(\mathcal{S})$, whiskering it with $\mathbf{Ob}_{\mathcal{S}}$ and $\mathbf{Arr}_{\mathcal{S}}$ give an internal functor $\mathcal{S}\phi : \underline{\mathcal{S}}F \to \underline{\mathcal{S}}G$ in $[\mathcal{E}, \mathcal{S}]$. The internal functoriality axioms follow from the naturality of source, target, identity and composition of $\underline{\mathcal{S}}$. Conversely, given an internal functor $f := (f_0, f_1) : \mathbb{A} \to \mathbb{B}$ in $[\mathcal{E}, \mathcal{S}]$ and an object $X \in \mathcal{E}$, the components $(f_j)_X : A_j(X) \to B_j(X)$ for $j \in \{0, 1\}$ define an internal functor $\widehat{f_X} : \mathbb{A}X \to \mathbb{B}X$, with internal functoriality axioms inherited from those of $f : \mathbb{A} \to \mathbb{B}$. Note that 2-naturality of the assignment $X \mapsto \widehat{f_X}$ follows from naturality of f_0 and f_1. Once again, it is clear from their definition that these constructions are mutually inverse.

Given a modification $\Gamma : \phi \Rrightarrow \psi : F \Rightarrow G : \mathcal{E} \to \mathbf{Cat}(\mathcal{S})$, its component for every $X \in \mathcal{E}$ is an internal natural transformation given by the data $\Gamma^X : \mathbf{Ob}_{\mathcal{S}}(FX) \to \mathbf{Arr}_{\mathcal{S}}(GX)$. Define this to be the X component of the natural transformation $\underline{\mathcal{S}}\Gamma : \underline{\mathcal{S}}\phi \Rightarrow \underline{\mathcal{S}}\psi$. Conversely, given an internal natural transformation $\alpha : f \Rightarrow g : \mathbb{A} \to \mathbb{B}$ in $[\mathcal{E}, \mathcal{S}]$ given by the data $\alpha : A_0 \Rightarrow B_1$, we may define the modification $\widehat{\alpha} : \widehat{\phi} \Rrightarrow \widehat{\psi}$ to have component on $X \in \mathcal{E}$ given by α_X. We note then that

- The modification axiom for Γ and naturality of $\underline{\mathcal{S}}\Gamma$ coincide.

- The internal naturality axioms for Γ_X and $\underline{S}\Gamma$ coincide.
- Naturality of α and the modification axiom for $\widehat{\alpha}$ coincide.
- Internal naturality axioms for α and $\widehat{\alpha}_X$ coincide.
- These constructions are once again mutually inverse.

To finish the proof, note that both $\widehat{(-)}$ and $\underline{S}$ are functorial between hom-categories, and 2-functorial, and that this isomorphism is indeed natural in $\mathcal{E}$. □

Remark 3.3.3. Let $\mathcal{E} \in \mathbf{Lex}$ and consider in particular the isomorphism of categories $\mathbf{Cat}[\mathbf{Cat}\,(\mathcal{E})_1\,,\mathcal{E}]_1 \cong [\mathbf{Cat}\,(\mathcal{E})_1\,, \mathbf{Cat}\,(\mathcal{E})_1]$. Recall that the category of endofunctors on the right has a strict monoidal structure given by composition, and that its unit is given by the identity on $\mathbf{Cat}\,(\mathcal{E})_1$. Transporting this monoidal structure across this isomorphism of categories, one sees that the evaluation category is the unit for this transported monoidal structure on $\mathbf{Cat}[\mathbf{Cat}\,(\mathcal{E})_1\,,\mathcal{E}]_1$.

3.4 A Pseudocomonad Structure on $\mathbf{Cat}\,(-)_1$

In Section 3.2 we equipped the 2-functor $\mathbf{Cat}\,(-)_1 : \mathbf{Lex} \to \mathbf{Lex}$ with structures of pseudomonads with skew and co-skew left unit laws. In this section, we will follow the theory developed in [6] and prove a biadjunction between the 2-functor $\mathbf{UndLex} : 2\text{-}\mathbf{Lex} \to \mathbf{Lex}$ and the 2-functor $\mathbf{Cat}\,(-) : \mathbf{Lex} \to 2\text{-}\mathbf{Lex}$. We will then use this to equip $\mathbf{Cat}\,(-)_1 : \mathbf{Lex} \to \mathbf{Lex}$ with the structure of a pseudocomonad, and see that the operation of transposition on double categories is an involutary strict distributive law between this pseudocomonad and itself satisfying the Yang–Baxter equation, or a *compatible flip* in the sense of [27]. We will conclude this chapter by looking at some of its coalgebras.

In this section $\mathbf{Cat}\,(-)$ will always denote the 2-functor from $\mathbf{Lex}$ to 2-$\mathbf{Lex}$, and $\mathbf{Cat}\,(-)_1 : \mathbf{Lex} \to \mathbf{Lex}$ will always denote $\mathbf{UndLex} \circ \mathbf{Cat}\,(-)$.

The following definition is taken directly as presented in [6].

Definition 3.4.1. Let $F : \mathfrak{K} \to \mathfrak{L}$ and $G : \mathfrak{L} \to \mathfrak{K}$ be 2-functors between 2-categories, let $\epsilon : FG \Rightarrow 1_{\mathfrak{L}}$ and $\eta : 1_{\mathfrak{K}} \Rightarrow GF$ be pseudonatural transformations, and let $\theta : G\epsilon \circ \eta_G \Rrightarrow 1_G$ and $\phi : 1_F \Rrightarrow \epsilon_F \circ F\eta$ be invertible modifications. Then this data is called a *biadjunction* if both of the pastings shown in Diagram 7.0.21 are identities. In this case, G is said to be *right biadjoint* to F and F is said to be *left biadjoint* to G. The pseudonatural transformations η and ϵ are called the *unit* and *counit* respectively. If ϵ and η are 2-natural, and θ and ϕ are identity modifications, then the data is called a 2-adjunction.

Note that the modifications θ and ϕ may be thought of as mediating what would be the usual triangle identities of an ordinary adjunction. The biadjunction we will present will be simpler than the general case defined above in the following ways: the invertible modifications ϕ and θ will in fact be identities, while the counit will be $\mathbf{Ob}$, which was 2-natural, as shown in Theorem 3.1.9. We have also already encountered the data for the unit of this biadjunction in Theorem 1.5.7, which gave its components, and in Proposition 1.5.9, where we gave what we will see is the only non-identity part of its pseudonaturality constraints.

Theorem 3.4.2. *Recall the 2-functors $\eta^{\mathfrak{K}} : \mathfrak{K} \to \mathbf{Cat}\,(\mathfrak{K}_1)$ from Theorem 1.5.7, which send $A \in \mathfrak{K}$ to the category* $A^{\mathbf{3}} \xrightarrow{m} A^{\mathbf{2}} \underset{a_1}{\overset{a_0}{\rightrightarrows}} A$ (with $\Delta_A : A \to A^{\mathbf{2}}$) *internal to $\mathfrak{K}_1$, the underlying category of $\mathfrak{K}$. Recall also the natural isomorphisms $F \mapsto \eta^F$ of Remark 1.5.9, whose components on $A \in \mathfrak{K}$ are extremal internal functors in the sense of Definition 1.5.1. Recall further that their components on arrows are given by the isomorphisms up to which F preserves power of A by $\mathbf{2}$, and that extremality means that their components on objects are identities.*

1. *The assignment $\mathfrak{K} \mapsto \eta^{\mathfrak{K}}$ is pseudonatural in $\mathfrak{K}$, with pseudonaturality constraints given by the natural isomorphism η^F.*

2. *The data $(\mathbf{UndLex}, \mathbf{Cat}(-), \eta, \mathbf{Ob}, 1, 1)$ exhibits $\mathbf{Cat}(-)$ as right biadjoint to $\mathbf{UndLex}$.*

Proof. Given that the components of $\mathbf{Arr}$ are faithful, as was shown in Proposition 1.2.3, the proofs of Theorem 1.5.7 and Proposition 1.5.9 are enough to see that the required naturality conditions hold. Then pseudonaturality of $\mathfrak{K} \mapsto \eta^{\mathfrak{K}}$ follows from Remark 3.0.4. This completes the proof of part (1). For part (2), the fact that the triangle identities hold up to identity modifications is evident from extremality of the components of $\eta^{\mathfrak{K}}$, since in both triangle identities one takes either $\mathbf{Cat}(\mathbf{Ob})$ or $\mathbf{Ob}_{\mathbf{UndLex}}$, and extremality says that these will always be the identity. The coherence conditions for the biadjunction are also clear for similar reasons. ☐

Remark 3.4.3. Recall [22] that a general biadjunction as defined above in Definition 3.4.1 gives rise to a pseudocomonad as defined in Definition 3.2.1. In our particular case, the pseudocomonad structure is given by

- Taking the 2-functor $\mathbf{Cat}(-)_1$ to be the composite $\mathbf{UndLex} \circ \mathbf{Cat}$,
- Taking the counit $\mathbf{Ob}$ directly from the adjunction, and the modifications up to which the counit laws hold to be identities,
- Taking the comultiplication δ to be the whiskering $\mathbf{UndLex} \circ \eta \circ \mathbf{Cat}(-)$. Recall from Remark 1.5.8 that this has components on $\mathcal{E} \in \mathbf{Lex}$ which send a category internal to $\mathcal{E}$ to its *double category of squares*.
- Taking the modification Γ up to which coassociativity will hold to be η^{η}, the pseudonaturality constraint of η on itself. Note that this is precisely the modification mediating between the two ways of horizontally composing the pseudonatural transformation η with itself, as mentioned in Remark 3.0.2. The component natural transformations of this modification will themselves have components on $\mathbb{A} \in \mathbf{Cat}(\mathcal{E})$ given by the extremal internal triple functors whose only non-identity component will be $\left(\mathbb{A}^{\mathbf{2}}\right)^{\mathbf{2}} \cong \mathbb{A}^{\mathbf{2}\times\mathbf{2}}$.
- Noticing that the counit laws hold strictly by since the category of objects of the double category of squares of $\mathbb{A}$ is just $\mathbb{A}$ itself, and recalling from 1.5.8 that double categories of squares are symmetric.
- Either noticing that the pseudocomonad coherence axiom for comultiplication given in Definition 3.2.1 follows directly from the triangle identities, or noticing by the faithfulness of $\mathbf{Arr}$ established in Proposition 1.2.3, that it follows from the universal property of powers.

Theorem 3.4.4. *The pseudonatural transformation $\delta : \mathbf{Cat}(-)_1 \Rightarrow \mathbf{DblCat}(-)_1 := \mathbf{Cat}(\mathbf{Cat}(-)_1)_1$ and the 2-natural transformation $\mathbf{Ob} : \mathbf{Cat}(-)_1 \Rightarrow 1_{\mathbf{Lex}}$ are the comultiplication and counit respectively of a pseudocomonad structure on $\mathbf{Cat}(-)_1 : \mathbf{Lex} \to \mathbf{Lex}$, in which both counit laws hold strictly and coassociativity holds up to the modification Γ.*

Theorem 3.4.5. *The transposition 2-natural transformation $T : \mathbf{DblCat}(-)_1 \Rightarrow \mathbf{DblCat}(-)_1$ is a strict distributive law between this pseudocomonad and itself, satisfying the relevant axioms from [25]. Moreover, it is a* compatible flip *in the sense of [27].*

Proof. To see that transposition is a strict distributive law of this pseudocomonad over itself, firstly note that the counit laws are clear from symmetry of double categories in the image of δ. By the involutary nature of transposition, the comultiplication laws are logically equivalent to one another. Chasing a double category $\mathbb{A}$ around the diagram in Diagram 7.0.22 we see that its image in the bottom left and bottom right categories are $\mathbf{Cat}(\delta_{\mathcal{E}})_1 \circ T_{\mathcal{E}}(\mathbb{A})$ and $\delta_{\mathbf{Cat}(\mathcal{E})_1}(\mathbb{A})$ respectively. Recalling Remark 3.1.8, these triple categories are related to one another via sequence of transpositions and hence an element of the symmetric group on three letters, in particular one which is not its own inverse. There are two such elements in S_3, and one of them is indeed the bottom row. However, since the double categories in the image of $\delta_{\mathcal{E}}$ are symmetric, we also have that these two triple categories are themselves invariant under transposition in the first and second dimension, and second and third dimension, respectively. Thus whichever of the two elements of the symmetric group on three letters corresponds to the bottom row, it will suffice to make the diagram commute. While this is sufficient to see that the diagram above commutes, an explicit diagram chase is given in Diagram 7.0.23. We are grateful to Richard Garner for his help in its construction.

We also need to check that it satisfies the nine coherence axioms for a pseudodistributive law, of [25]. Since the pseudodistributivity and counit axioms hold strictly, the only one of these axioms which may involve non-identity isomorphisms are 4, 6, and 9. Letting $\mathbb{D}$ be a double category and using faithfulness of the components of **Arr**, it suffices to show

- For 4, that the component of $\mathbf{Cat}\left(\Gamma^{\mathcal{E}}\right)_1$ on $\mathbb{D}$ is the component of $\Gamma^{\mathbf{Cat}(\mathcal{E})_1}$ on the transpose of $\mathbb{D}$,
- For 6, that the component of $\delta_{\delta_{\mathcal{E}}}$ on $\mathbb{D}$ is its component on the transpose of $\mathbb{D}$,
- For 9, that the component of $\mathbf{Cat}\left(\Gamma^{\mathcal{E}}\right)_1$ on the transpose of $\mathbb{D}$ is the component of $\Gamma^{\mathbf{Cat}(\mathcal{E})_1}$ on $\mathbb{D}$.

These are all clear by faithfulness of **Arr**, $\mathbf{Cat}\left(-\right)_1$'s preservation of faithfulness as shown in Proposition 3.3.1, and symmetry of transposition. Finally, now that we have shown that T is indeed an involutary distributive law, the only aspect of a compatible flip as in [27] which remains to be checked is the Yang–Baxter identity for transpositions of triple categories. In light of Remark 3.1.8 this just says that the two adjacent transpositions σ_1 and σ_2 of S_3 satisfy $\sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2$, which is certainly the case. This completes the proof. □

Remark 3.4.6. We describe the conditions for a finite limit preserving functor $\Phi : \mathcal{E} \to \mathbf{Cat}\left(\mathcal{E}\right)_1$ to equip $\mathcal{E}$ with the structure of a strict coalgebra for the pseudocomonad $\mathbf{Cat}\left(-\right)_1$. For a morphism f in $\mathcal{E}$, the counit axiom for Φ says precisely that the assignment on objects of (Φf) is just f itself. Letting P denote the composite shown on the left below, the comultiplication axiom then yields the commutativity of the diagram below on the right, so that in particular $PPf = (\Phi f)_{\mathrm{Sq}}$, and $P\alpha_{\Phi X} = \alpha_{P\Phi X}$ for any natural transformation $\alpha \in \{d_0^{\mathcal{E}}, d_1^{\mathcal{E}}, i^{\mathcal{E}}, m^{\mathcal{E}}\}$ giving data for the evaluation category $\underline{\mathcal{E}}$.

$$\mathcal{E} \xrightarrow{\Phi} \mathbf{Cat}\left(\mathcal{E}\right)_1 \xrightarrow{\mathbf{Arr}_{\mathcal{E}}} \mathcal{E} \qquad \begin{tikzcd} \mathcal{E} \arrow[r, "\Phi"] \arrow[d, "P"'] & \mathbf{Cat}\left(\mathcal{E}\right)_1 \arrow[d, "(-)^{\mathcal{E}}_{\mathrm{Sq}}"] \\ \mathcal{E} \arrow[r, "P"'] & \mathcal{E} \end{tikzcd}$$

Then the 2-cell coherences for Φ as a strict coalgebra say precisely that the internal categories ΦX are such that the isomorphisms $\left((\Phi X)^{\mathbf{2}}\right)^{\mathbf{2}} \cong (\Phi X)^{\mathbf{2}\times\mathbf{2}}$, discussed in Remark 1.5.10 are equal to the isomorphisms $P\left(PPX\right) = PP\left(PX\right)$, or in other words that they are in fact identities.

Example 3.4.7. *The components of* **Disc** *on* $\mathcal{E} \in \mathbf{Lex}$ *equip* $\mathcal{E}$ *with the structure of a strict coalgebra for this pseudocomonad. The coherence axiom for counits holds trivially, and the coherence axiom for comultiplication holds since if* $\mathbb{A}$ *is discrete then* $\mathbb{A}^{\mathbf{2}^{\mathbf{2}}} = \mathbb{A}^{\mathbf{2}\times\mathbf{2}}$*, and since* **Disc** *preserves limits strictly the pseudonaturality constraint of* δ *on* **Disc** *is also the identity. Similarly, the components of* **coDisc** *on* $\mathcal{E} \in \mathbf{Lex}$ *equip* $\mathcal{E}$ *with the structure of a pseudocoalgebra for this pseudocomonad. These have a strict counit law and a comultiplication law holding up to a natural isomorphism* ϕ*. The components of* ϕ *on* $X \in \mathcal{E}$ *are given by the extremal double functors with non-identity component the unique isomorphisms between* $(X \times X) \times (X \times X)$ *and the kernel pair* $X^3 \times_{X^2} X^3$*, where we write* X^n *for the n-fold product* $X \times ... \times X$*. The counit coherence holds by extremality of* ϕ_X*, while the comultiplication coherence follows from the universal property of limits in* $\mathcal{E}$*.*

The following theorem will help us describe the coalgebras for this pseudocomonad in more detail. In particular, given a pseudocoalgebra $\Phi : \mathcal{E} \to \mathbf{Cat}\left(\mathcal{E}\right)$ there is a 2-category $\mathcal{E}_{\Phi}$ whose underlying category is $\mathcal{E}$. This 2-category will have powers by $\mathbf{2}$ for $X \in \mathcal{E}_{\Phi}$ given by PX.

Theorem 3.4.8. *For* $X, Y \in \mathcal{E}$*, define the category* $\mathcal{E}_{\Phi}\left(X, Y\right)$ *to have*

- *A set of objects given by the hom-set* $\mathcal{E}\left(X, Y\right)$*.*

- *Morphisms $\alpha : f \to g$ consisting of morphisms $\alpha \in \mathcal{E}(X, PY)$ so that $d_0\alpha = f$ and $d_1\alpha = g$.*

- *Composition $f \xrightarrow{\alpha} g \xrightarrow{\beta} h$ given by $m(\alpha, \beta)$, and identities for f given by if.*

Then

1. *The forgetful functor $\mathbf{Cat}(\mathcal{E})(\Phi X, \Phi Y) \to \mathcal{E}_\Phi(X, Y)$ is part of an isomorphism of categories.*

2. *The categories $\mathcal{E}_\Phi(X, Y)$ are hom-categories of a 2-category structure on the objects of $\mathcal{E}$.*

3. *The 2-category $\mathcal{E}_\Phi$ just described has powers by $\mathbf{2}$ given by $Y^{\mathbf{2}} = PY$, and hence all finite weighted limits. The universal 2-cells of the power by $\mathbf{2}$ of Y are given by the identity in $\mathcal{E}$ on PY.*

4. *For all $X, Y \in \mathcal{E}$ and $f, g \in \mathcal{E}(X, Y)$, let $\mathfrak{E}(X, Y)(f, g)$ be a set such that this equips $\mathcal{E}$ with a class of 2-cells giving it the structure of a 2-category with powers by $\mathbf{2}$. Recall the 2-functor η introduced in Theorem 1.5.7, which also featured as the unit of the biadjunction 3.4.2. Then $\Phi := \mathbf{UndLex}\left(\eta^{\mathfrak{E}}\right) : \mathcal{E} \to \mathbf{Cat}(\mathcal{E})_1$ is a pseudocoalgebra for the pseudocomonad $\mathbf{Cat}(-)_1$ with a strict counit law.*

Proof. For part (1), note that an object $f \in \mathcal{E}_{\Phi,X,Y}$ is sent to the internal functor Φf, while a morphism $\alpha : f \to g$ does indeed give a well-defined internal natural transformation, as can be seen by the commutativity of Diagram 7.0.24. Part (2) follows from part (1) as we may inherit the structure for horizontal composition from $\mathbf{Cat}(\mathcal{E})$. For part (3), first note that the identity on PY indeed constitutes a 2-cell from $d_0 : PY \to Y$ to $d_1 : PY \to Y$. To see that this is the universal 2-cell of a power by $\mathbf{2}$ in $\mathcal{E}_\Phi$, observe that a 2-cell $\alpha : f \Rightarrow g : X \to Y$ is also a 1-cell $\alpha : X \to PY$. Meanwhile, a commutative square of 2-cells

$$\begin{tikzcd} f \arrow[r, "\varphi"] \arrow[d, "\alpha"'] & f' \arrow[d, "\beta"] \\ g \arrow[r, "\psi"'] & g' \end{tikzcd}$$

viewed as a morphism from α to β in $[\mathbf{2}, \mathcal{E}_\Phi(X, Y)]$ induces a morphism $X \to PPY$ in $\mathcal{E}$ by the universal property of $PPY = (\Phi Y)_{\mathrm{Sq}}$, and this constitutes a 2-cell from α to α'. It follows from the universal property of the pullback that this assignment is functorial, and indeed inverse to horizontal composition by $1_P X$. That $\mathcal{E}_\Phi$ has finite weighted limits follows since $\mathcal{E} \in \mathbf{Lex}$ has conical limits, and we have just shown that it has powers by $\mathbf{2}$. For part (4), firstly note that the counit law for the pseudocoalgebra Φ is clearly strict since for $X \in \mathfrak{E}$ the object of objects of $\eta^{\mathfrak{E}}(X)$ is just X. The comultiplication law for the pseudocoalgebra Φ holds up to the isomorphism up to which powering by $\mathbf{2}$ commutes with the taking of pullbacks. The 2-cell coherences follow from universal properties. ☐

Remark 3.4.9. The identity on PY provides a choice for the universal 2-cell of the power by $\mathbf{2}$ of Y. In particular, observe that $Y \mapsto PY$ is the monad induced by powering by $\mathbf{2}$ on $\mathcal{E}_\Phi$, with unit given by i_Φ and multiplication given by the natural transformation μ_Φ which sends a commutative square to its mutual composite. Furthermore, the Kleisli category of this monad is precisely the underlying category of 0 and 2-cells of $\mathcal{E}_\Phi$, as is evident from the definition of horizontal composition in $\mathcal{E}_\Phi$.

There are also notions of pseudocoalgebra morphisms and 2-cells between them, and in this example they will be 2-functors which preserve finite weighted limits and arbitrary 2-natural transformations. We end this Chapter by quoting the result capturing this information, which is discussed in Remark 3.3.4 of [6]. There they offer a proof which uses a bicategorical version of Beck's Monadicity Theorem [22]. The Kleisli 2-category for this pseudocomonad is also described in Sections 4.3 and 4.4 of [6].

Theorem 3.4.10. *[6] The 2-category of pseudocoalgebras for the pseudocomonad $\mathbf{Cat}(-)_1$ is biequivalent to* 2-**Lex**.

4 The Grothendieck Construction

This is an expository chapter in which we will summarise the Grothendieck construction, and how it relates split opfibrations into a category, to 2-functors from that category into **Cat**. We have already seen in Theorem 3.3.2 that there is an isomorphism of 2-categories between $\mathbf{Cat}[\mathcal{E},\mathbf{Set}]$ and $[\mathcal{E},\mathbf{Cat}]$. One may recall that $[\mathcal{E},\mathbf{Set}]$ is isomorphic to the category of discrete opfibrations into $\mathcal{E}$, and dually $[\mathcal{E}^{\mathrm{op}},\mathbf{Set}]$ is isomorphic to the category of discrete fibrations into $\mathcal{E}$. Our aim in this chapter will be to show an equivalence between $\mathbf{SFib}(\mathcal{E})$ and $\mathbf{Cat}(\mathbf{DFib}(\mathcal{E})) \cong \mathbf{Cat}[\mathcal{E},\mathbf{Set}]$. As a corollary, we will obtain a result which we will need in the next chapter, which is that when $\mathbb{B} \in \mathbf{Cat}(\mathcal{E})$, there is an equivalence of 2-categories $\mathbf{Cat}(\mathbf{DFib}(\mathbb{B})) \cong \mathbf{SFib}(\mathbb{B})$. We will begin by recalling definitions. For further information on the material in this chapter, the reader should consult [7], where they also investigate the case where $\mathcal{E}$ is a 2-category or a bicategory.

Definition 4.0.1. Let $P : \mathcal{A} \to \mathcal{B}$ be a functor.

- A morphism $f : X \to Y$ in $\mathcal{A}$ is called *cartesian* with respect to P if given a morphism $g : W \to Y$ in $\mathcal{A}$ and a morphism $h : PW \to PX$ in $\mathcal{B}$ satisfying $(Pf)\,h = Pg$, there is a unique morphism $h' : W \to X$ in $\mathcal{A}$ satisfying $Ph' = h$ and $fh' = g$.

- The functor P has the structure of a *cloven fibration* if for every $X \in \mathcal{B}, Y \in \mathcal{A}$ and $f \in \mathcal{B}(X, PY)$ there is a chosen cartesian $f' \in \mathcal{A}(X', Y)$ such that $Pf' = f$. Then f' is called the *cartesian lift* of f given Y.

- A cloven fibration is *split* if
  - For any $X \xrightarrow{f} PY \xrightarrow{g} PZ$, the chosen lift of gf given Z is the composite of the chosen lift $g' : Y \to Z$ of g given Z, and the chosen lift $f : X' \to PY = Y'$ of f given Y.
  - For any $X \in \mathcal{B}$, the lift of 1_X given any object is an identity.

- If for every $f : X \to PY$ in $\mathcal{B}$ there is a unique $f' : B' \to A$ in $\mathcal{A}$ such that $Pf' = f$, then P is called a *discrete fibration*.

- A functor whose opposite is a split fibration is called a *split opfibration*, and in this case the lift will instead be referred to as *op-cartesian*.

- Let $\mathfrak{K}$ be a 2-category and let $p : A \to B$ be a 1-cell in $\mathfrak{K}$. Then p is called a cloven/split/discrete (op)fibration if
  - For every object $X \in \mathfrak{K}$, the composition functor $\mathfrak{K}(X,p) : \mathfrak{K}(X,A) \to \mathfrak{K}(X,B)$ is so.
  - These liftings are natural in X.

- For $X \in \mathfrak{K}$, let
  - $\mathbf{SFib}(X)_{\mathfrak{K}}$ be the locally full sub-2-category of $\mathfrak{K}/X$ on split fibrations and 1-cells that representably preserve chosen cartesian lifts.
  - Let $\mathbf{SoFib}(X)_{\mathfrak{K}}$ be the same but for split opfibrations and 1-cells which representably preserve chosen op-cartesian lifts.
  - Let $\mathbf{DFib}(X)_{\mathfrak{K}}$ be the locally discrete sub-2-category of $\mathfrak{K}/X$ on discrete fibrations, and arbitrary 1-cells between them.
  - Let $\mathbf{DoFib}(X)_{\mathfrak{K}}$ be the same but for discrete opfibrations.

  We will omit the subscript $\mathfrak{K}$ whenever it is clear from context.

Note that a discrete fibration is automatically a split fibration.

Example 4.0.2. *The forgetful functor from pointed sets to sets $P : \mathbf{Set}_* \to \mathbf{Set}$ which forgets the chosen element is a discrete opfibration. Given $f : P(X,x) \to Y$, the lift is given by $f : (X,x) \to (Y, f(x))$. It is easy to see that this lift is unique.*

In fact, this example is 'universal' in the sense that every discrete opfibration in **CAT** is the pullback of P along some unique functor into **Set**. This is related to the familiar construction of categories of elements of copresheaves. This chapter will describe a generalisation of this in the form of the Grothendieck Construction.

4.1 From Cat-valued 2-presheaves to Split opfibrations

Let $\mathcal{E}$ be a category and $Q : \mathcal{E} \to \mathbf{Cat}$ be a 2-functor. Define the category of elements of Q, $\widehat{Q}$, as follows:

- The objects in $\hat{Q}$ are of the form (V, v), where $V \in \mathcal{E}$ and v is an object in the category QV. We may refer to V and v as the first and second component of (V, v), respectively.
- The morphisms $(F, f) : (V, v) \to (W, w)$ in $\widehat{Q}$ consist of a morphism $F : V \to W$ in $\mathcal{E}$ and $f : (QF)\, v \to w$ is a morphism in QW. We may refer to F and f as the first and second components of (F, f), respectively.
- The morphisms $(V, v) \xrightarrow{(F,f)} (W,w) \xrightarrow{(G,g)} (X,x)$ compose to give a morphism whose first component is GF and second component is $Q\,(GF)\,v = (QG)\,(QF)\,v \xrightarrow{(QG)f} (QG)\,w \xrightarrow{g} x$, while the identity of (X, x) is given by $(1_X, 1_x)$ where 1_X is the identity of X in $\mathcal{E}$ and 1_x is the identity of x in QX.
- Associativity and left and right unit laws are inherited from $\mathcal{E}$, given 2-functoriality of Q.

There is a canonical projection $\mathrm{El}\,(Q) : \widehat{Q} \to \mathcal{E}$ which 'forgets the second component'. Note that this functor can be given the structure of a split opfibration by choosing, for any morphism $F : \mathrm{El}\,(Q)\,(V, v) \to W$ in $\mathcal{E}$, the morphism $\left(F, 1_{(QF)v}\right) : (V, v) \to (W, (QF)\,v)$. Then it is clear that $\left(F, 1_{(QF)v}\right)$ is opcartesian since whenever $(G, g) : (V, v) \to (X, x)$ is such that there exists a morphism $H : W \to X$ satisfying $HF = G$, then $(H, g) : (W, (QF)\,v) \to (X, x)$ is the unique morphism which P maps to H. Finally, the composite of chosen morphisms is itself also chosen, since the chosen morphisms are just those whose second component is an identity.

Our aim in this section will be to show how $\mathrm{El} : [\mathcal{E}, \mathbf{Cat}] \to \mathbf{SoFib}\,(\mathcal{E})$ as just described on objects extends to a 2-functor. This 2-functor will be one side of an equivalence of 2-categories which we will prove by the end of the chapter.

Given a 2-natural transformation $\sigma : Q \Rightarrow R : \mathcal{E} \to \mathbf{Cat}$, we may define the functor $\mathrm{El}\,(\sigma) : \widehat{Q} \to \widehat{R}$ in the following way:

- $(X, x) \mapsto (X, \sigma_X\,(x))$
- $\big((F, f) : (X, x) \to (Y, y)\big) \mapsto \big((F, \sigma_Y\,(f)) : (X, \sigma_X\,(x)) \to (Y, \sigma_Y\,(y))\big)$, which is well-defined as a 1-cell by 2-naturality of σ.

Functoriality of $\mathrm{El}\,(\sigma)$ follows from functoriality of σ_Y. Finally, notice that $\mathrm{El}\,(\sigma)$ commutes with the respective projection functors as it only alters the second component of any data in $\widehat{Q}$. This describes El on 1-cells.

Let $\Gamma : \sigma \Rrightarrow \tau : Q \to R : \mathcal{E} \to \mathbf{Cat}$ be a modification, with component natural transformation on $X \in \mathcal{E}$ written as $\Gamma^X : \sigma_X \to \tau_X : QX \to RX$, and its component on $x \in QX$ written as $\Gamma^X_x : \sigma_X\,(x) \to \tau_X\,(x)$. Then consider the morphism $\left(1_X, \Gamma^X_x\right) : (X, \sigma_X\,(x)) \to (X, \tau_X\,(x))$ in $\widehat{R}$. We claim that the assignment $(X, x) \mapsto \left(1_X, \Gamma^X_x\right)$ constitutes a natural transformation $\mathrm{El}\,(\Gamma) : \mathrm{El}\,(\sigma) \Rightarrow \mathrm{El}\,(\tau)$, and that this whiskers with the projection $\mathrm{El}\,(R) : \widehat{R} \to \mathcal{E}$ to give the projection $\mathrm{El}\,(Q) : \widehat{Q} \to \mathcal{E}$. Now naturality of $\mathrm{El}\,(\Gamma)$ requires that for every $(F, f) : (X, x) \to (Y, y)$ in $\mathcal{E}$, the first of the two diagrams in Diagram 7.0.31 commutes in $\widehat{R}$. On components in $\mathcal{E}$, this diagram commutes since $1_Y F = f = f 1_X$. The condition on the second component holds by the commutativity of the second diagram in Diagram 7.0.31 in RY, in which the right square is the modification axiom for Γ on F, the equalities hold by 2-naturality of σ and τ, and the left square is naturality of Γ^Y on f.

Then it is clear that this natural transformation commutes with the projections as the first component of its components are simply identities. This describes El on 2-cells. It is clear that El is well-defined as a 2-functor, since functoriality between hom-categories follows from compositional properties of natural transformations, and 2-functoriality follows from compositional properties of functors.

4.2 From Split Opfibrations to Cat-valued 2-presheaves

We now describe the 2-functor $(-)^{-1} : \mathbf{SoFib}(\mathcal{E}) \to [\mathcal{E}, \mathbf{Cat}]$ which we will show will be inverse to El. Suppose $P : \mathcal{D} \to \mathcal{E}$ is a split opfibration. Then define the 2-functor $P^{-1}(-) : \mathcal{E} \to \mathbf{Cat}$ in the following way:

- For an object $X \in \mathcal{E}$, let $P^{-1}X$ be the category whose
  - Objects are those in $\mathcal{D}$ which P maps to X,
  - Morphisms are those in $\mathcal{D}$ which P maps to 1_X,
  - Composition is as given in $\mathcal{D}$.
- For a morphism $f : X \to Y$, the functor $P_f^{-1} : P^{-1}X \to P^{-1}Y$ is defined in the following way:
  - Let $A \in P^{-1}X$, and let $f_A : A \to P_f^{-1}A$ be the chosen opcartesian 1-cell mapped to f. Then P_f^{-1} sends A to $P_f^{-1}A$.
  - Let $s : A \to B$ be a morphism in the category $P^{-1}X$, hence a 1-cell in $\mathcal{D}$ which is sent by P to 1_X. By the universal property of f_A as an opcartesian morphism for P, since $f1_X$ factors through f by 1_Y, there is an induced $P_f^{-1}(s) : P_f^{-1}A \to B'$ in $\mathcal{D}$ which is unique so that $PP_f^{-1}(s) = 1_Y$ and the square on the left in Diagram 7.0.30 commutes in the underlying category of $\mathcal{D}$. Then P_f^{-1} sends s to $P_f^{-1}(s)$.

  To see that P_f^{-1} is well defined as a functor, note that $P_f^{-1}(1_A) = (f_A, 1_Y) = 1_{Y_{A,f}}$, while for $t : B \to C$ a morphism in the category PX, the universal property of f_A as an opcartesian morphism ensures that $P_f^{-1}(t)\, P_f^{-1}(s) = P_f^{-1}(ts)$, since the diagram the right in Diagram 7.0.30 commutes.

For functoriality of this assignment, it is clear that $P1_X{}^{-1}$ fixes all objects and morphisms, and is hence the identity functor on $P^{-1}X$. Finally, let $k \in \mathcal{E}(Y, Z)$. Then by splitness of $P : \mathcal{D} \to \mathcal{E}$, chosen lifts compose to give chosen lifts, and hence their respective codomains must agree, so $P_k^{-1}P_f^{-1}$ and P_{kf}^{-1} agree on objects. For morphisms, let $s : A \to B$ be a morphism in $P^{-1}X$ as above, and consider $P_k^{-1}\big(P_f^{-1}(s)\big)$. By opcartesianness of $(kf)_A$ it suffices to note that in the first diagram in Diagram 7.0.32, the triangles commute by splitness of P, and the quadrilaterals define P_k^{-1} and P_f^{-1} on morphisms. Thus P^{-1} is well-defined as a 2-functor.

On morphisms, suppose $P : \mathcal{D} \to \mathcal{E}$ and $P' : \mathcal{D}' \to \mathcal{E}$ are split opfibrations, and let a functor $S : \mathcal{D} \to \mathcal{D}'$ satisfy $P'S = P$. Then for $X \in \mathcal{E}$ we may define the functor $(S)_X^{-1} : P^{-1}X \to P'^{-1}X$ by taking the image under S. This is well-defined by the commutativity condition on S, and is functorial by functoriality of S. We claim that the assignment $X \mapsto (S)_X^{-1}$ constitutes a natural transformation from P^{-1} to P'^{-1}. That is, for a morphism $F : X \to Y$ in $\mathcal{E}$, the second of the diagrams in Diagram 7.0.32 commutes. That this square commutes on objects is just to say that $SP_F^{-1}A = P'^{-1}_F SA$, which is indeed the case since $P = P'S$. Similarly, if we chase a morphism $t : A \to B$ around the diagram we see that the results are the same by opcartesianness of SF_A with respect to P', once again using the fact that $P = P'S$. Just as for El, it is clear that $(-)^{-1}$ is also well-defined as a 2-functor.

4.3 The Equivalence of 2-categories

We now show that El and $(-)^{-1}$ together give an equivalence of 2-categories.

Theorem 4.3.1. *The composite 2-functor $(-)^{-1}$ El is 2-naturally isomorphic to the identity 2-functor on $[\mathcal{E}, \mathbf{Cat}]$.*

Proof. For $X \in \mathcal{E}$, the category $(\mathrm{El}\,(Q))^{-1}\,(X)$ has

- Objects are (X, x) where x is an object of QX
- Morphisms are $(1_X, f) : (X, x) \to (X, x')$ where $f : x \to x'$ is a morphism in QX.

There are functors $\eta^Q_X : QX \to (\mathrm{El}\,(Q))^{-1}\,(X)$ and $\epsilon^Q_X : (\mathrm{El}\,(Q))^{-1}\,(X) \to QX$ which respectively act by 'inserting and dropping the first component'. It is then easy to see that η^Q_X and ϵ^Q_X together form an isomorphism of categories, and that both of the assignments $X \mapsto \eta^Q_X$ and $X \mapsto \epsilon^Q_X$ are 2-natural in X. We claim that the 2-transformations $\eta^Q : Q \Rightarrow (\mathrm{El}\,(Q))^{-1}$ and $\epsilon^Q : (\mathrm{El}\,(Q))^{-1} \Rightarrow Q$ are themselves the respective components at $Q \in [\mathcal{E}, \mathbf{Cat}]$ of 2-natural transformations $\eta : \mathbf{1}_{[\mathcal{E},\mathbf{Cat}]} \Rightarrow (-)^{-1}\,\mathrm{El}\,(-)$ and $\epsilon : (-)^{-1}\,\mathrm{El}\,(-) \Rightarrow \mathbf{1}_{[\mathcal{E},\mathbf{Cat}]}$. Indeed it suffices to notice that for a 2-transformation $\sigma : Q \Rightarrow R : \mathcal{E} \to \mathbf{Cat}$, the functors $(\mathrm{El}\sigma)^{-1}_X$ are constant on the first component and act just as σ_X does on the second component. Thus ϵ and η exhibit the 2-functor $(-)^{-1}\,\mathrm{El}$ as being 2-naturally isomorphic to the identity. $\square$

Theorem 4.3.2. *The composite* 2*-functor* $El\,(-)^{-1}$ *is* 2*-naturally isomorphic to the identity* 2*-functor on* $\mathbf{SoFib}\,(\mathcal{E})$.

Proof. Let $P : \mathcal{D} \to \mathcal{E}$ a split opfibration. The category $\widehat{P^{-1}}$ has

- Objects all pairs of the form (X, A) for $A \in \mathcal{D}$ and $X \in \mathcal{E}$ satisfying $PA = X$. We may write these simply as (PA, A).
- Morphisms $(F, f) : (PA, A) \to (PB, B)$ where $F : PA \to PB$ is a 1-cell in $\mathcal{E}$, and $f : P^{-1}_F A \to B$ is a morphism in $\mathcal{D}$ which P maps to 1_Y.

Then we define the functor $\epsilon'_P : \widehat{P^{-1}} \to \mathcal{D}$. On objects $(PA, A) \mapsto A$, and on morphisms $\big((F, f) : (PA, A) \to (PB, B)\big) \mapsto A \xrightarrow{F_A} P^{-1}_F A \xrightarrow{f} B$. Note that functoriality follows from splitness. We claim that the assignment $P \mapsto \epsilon_P$ defines a 2-natural transformation $\epsilon' : \mathrm{El}\,(-)^{-1} \Rightarrow \mathbf{1}_{\mathbf{LDSoF}(\mathcal{E})}$. Let $P' : \mathcal{D} \to \mathcal{E}$ be a split opfibration, and let $S : \mathcal{D} \to \mathcal{D}'$ be a functor satisfying $P'S = P$. We must then show that the following diagram commutes:

$$\begin{tikzcd}
\widehat{P^{-1}} \arrow[r, "\epsilon'_P"] \arrow[d, "\mathrm{El}(S^{-1})"'] & L \arrow[d, "S"] \\
\widehat{P'^{-1}} \arrow[r, "\epsilon'_{P'}"'] & \mathcal{D}'
\end{tikzcd}$$

Traversing this diagram in both directions we see that an object (PA, A) is sent to SA, and a morphism (F, f) is sent to $S\,(f.F_A)$.

Next, we define the 2-functor $\eta'_P : \mathcal{D} \to \widehat{P^{-1}}$ on objects as $A \mapsto (PA, A)$, and on morphisms as $\eta'_P\,(f : A \to B) = \left(Pf, v^P_f\right) : (PA, A) \to (PB, B)$, where $v^P_f : P^{-1}_{Pf} A \to B$ is the 'vertical' part of f, induced by opcartesianness of $(Pf)_A$ given 1_Y, and unique such that the following below on the left commutes in $\mathcal{D}$. Indeed, functoriality follows from that of P and the splitness condition. We claim that the assignment $P \mapsto \eta_P$ defines a 2-natural transformation $\eta' : \mathbf{1}_{\mathbf{SoFib}(\mathcal{E})} \to \mathrm{El}\,(-)^{-1}$. Let $P' : \mathcal{D} \to \mathcal{E}$ be a split opfibration, and let a functor $S : \mathcal{D} \to \mathcal{D}'$ satisfy $P'S = P$. We must then show that the diagram below on the right commutes. Traversing this diagram in both directions, we see that an object $A \in \mathcal{D}$ is sent to (PA, SA), and a morphism $f : A \to B$ in $\mathcal{D}$ is sent to $\left(Pf, v^{P'}_{Sf}\right)$.

$$\begin{tikzcd}
A \arrow[r, "(Pf)_A"] \arrow[dr, "f"'] & P^{-1}_{Pf} A \arrow[d, "v^P_f"] \\
& B
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{D} \arrow[r, "\eta'_P"] \arrow[d, "S"'] & \widehat{P^{-1}} \arrow[d, "\mathrm{El}(S^{-1})"] \\
\mathcal{D}' \arrow[r, "\eta'_{P'}"'] & \widehat{P'^{-1}}
\end{tikzcd}$$

We claim that η' and ϵ' exhibit $\mathrm{El}\,(-)^{-1}$ as being 2-naturally isomorphic to the identity on $\mathbf{SoFib}\,(\mathcal{E})$. To establish this, we must show that η'_P and ϵ'_P themselves constitute an isomorphism of 2-categories. But this is clear from their description as $\epsilon'\eta'$ and $\eta'\epsilon'$ are equal to the identity 2-functors. □

Theorem 4.3.3. *1. The 2-functors El and $(-)^{-1}$ constitute an equivalence of 2-categories between $[\mathcal{E}, \mathbf{Cat}]$ and $\mathbf{SoFib}\,(\mathcal{E})$.*

2. There is an equivalence of categories $[\mathcal{E}, \mathbf{Set}] \cong \mathbf{DoFib}\,(\mathcal{E})$.

3. For $\mathcal{S} \in \mathbf{Lex}$, $\mathbb{B} \in \mathbf{Cat}\,(\mathcal{S})$, there is an equivalence of 2-categories $\mathbf{SoFib}\,(\mathbb{B}) \cong \mathbf{Cat}\,(\mathbf{DoFib}\,(\mathbb{B}))$.

4. With this notation, there is also an equivalence of 2-categories $\mathbf{SFib}\,(\mathbb{B}) \cong \mathbf{Cat}\,(\mathbf{DFib}\,(\mathbb{B}))$.

Proof. Combining the previous two theorems yields part (1). Part (2) can be seen from the above analysis by noticing that restricting $\mathbf{SoFib}\,(\mathcal{E})$ to discrete objects corresponds to restricting $[\mathcal{E}, \mathbf{Cat}]$ to those functors which factor through $\mathbf{Set}$. Part (3) follows Theorem 3.3.2 given that the Yoneda Embedding of $\mathcal{E}$ preserves and jointly reflects finite limits, and thus reduces the proof to the case where $\mathcal{S} = \mathbf{Set}$, which was itself part (1). Part (4) is just the dual of part (3). □

5 Factorisation Systems

In this chapter we will give two orthogonal factorisation systems on $\mathbf{Cat}\,(\mathcal{E})_1$. The first of these is often called the *full image factorisation*, since its right class consists of the fully faithful functors introduced in Remark 1.2.4, which encode inclusions of full subcategories. The left class of this factorisation system consists of isomorphism-on-objects functors, which were also introduced in Remark 1.2.4. This will be treated in the first section of this chapter. The second factorisation system on $\mathbf{Cat}\,(\mathcal{E})_1$ actually comes in two varieties which are related under dualisation on the level of the 2-cells in $\mathbf{Cat}\,(\mathcal{E})$. One of these factorisation systems has as its right class the *discrete fibrations* while the other has as its right class the *discrete opfibrations*. Both of these notions were defined at the beginning of the previous chapter. These factorisation systems are called *comprehensive*, because they correspond to consistent comprehension schemes on $\mathbf{Cat}\,(\mathcal{E})_1$ in the sense of [4], though we will not investigate this aspect of them. We will treat the variant in which the right class consists of discrete fibrations in detail, and mention along the way how our considerations can be dualised to give the variant involving opfibrations. Section 5.2 will look at both classes of this factorisation system in detail and prove the required orthogonality properties. The approach we take to this factorisation closely follows that of [30]. In Section 5.3 we use 'oplax limits of 1-cells', or comma objects of the form Y/f for $f : X \to Y$, to factorise an internal functor into a right adjoint followed by a split fibration. Finally in Section 5.4, we combine several of the results we saw along the course of this thesis to further factorise the split fibration into a coidentifier followed by a discrete fibration. Composing the right adjoint and coidentifier, we will have factorised an arbitrary internal functor into a 'final functor' followed by a discrete fibration. Since many of the proofs of this chapter only require certain 2-categorical properties of $\mathbf{Cat}\,(\mathcal{E})$ we will give proofs in a more general context wherever possible.

5.1 The Full Image Factorisations of Internal Functors

In this section we will see how the full image factorisation of an internal functor into a isomorphism on objects internal functor followed by a fully faithful internal functor arises. We first prove the required orthogonality property.

Proposition 5.1.1. *Isomorphism-on-objects internal functors are left orthogonal to fully faithful internal functors.*

Proof. In the square of internal functors shown below on the left, let b be isomorphism on objects and f be fully faithful. Define $u_0 := b_0^{-1}h_0 : B_0 \to C_0$, and note that $f_0u_0 = f_0h_0b_0^{-1} = g_0b_0b_0^{-1} = g_0$. Then further define $u_1 : B_1 \to C_1$ to be the morphism induced by the pullback as shown in the middle diagram below, given the commutativity of the first diagram in the first row in Diagram 7.0.34. We claim that $u := (u_0, u_1)$ is the required internal functor shown in the diagram on the right which makes both triangles commute.

$$\begin{tikzcd} \mathbb{A} \arrow[r, "h"] \arrow[d, "b"'] & \mathbb{C} \arrow[d, "f"] \\ \mathbb{B} \arrow[r, "g"'] & \mathbb{D} \end{tikzcd} \qquad \begin{tikzcd} B_1 \arrow[d, "{(d_0,d_1)}"'] \arrow[dr, dashed, "u_1"] \arrow[drr, bend left, "g_1"] & & \\ B_0 \times B_0 \arrow[dr, bend right, "u_0 \times u_0"'] & C_1 \arrow[r, "f_1"] \arrow[d, "{(d_0,d_1)}"'] & D_1 \arrow[d, "{(d_0,d_1)}"] \\ & C_0 \times C_0 \arrow[r, "f_0 \times f_0"'] & D_0 \times D_0 \end{tikzcd} \qquad \begin{tikzcd} \mathbb{A} \arrow[r, "h"] \arrow[d, "b"'] & \mathbb{C} \arrow[d, "f"] \\ \mathbb{B} \arrow[r, "g"'] \arrow[ur, dashed, "u"] & \mathbb{D} \end{tikzcd}$$

Consider the diagrams in Diagram 7.0.33 for $j \in \{0, 1\}$. The first of these diagrams shows that u respects sources and targets, while the second and third diagrams show by the universal property of C_1 as a pullback that $u_1 b_1 = h_1$. Then, again using the universal property of C_1 as a pullback and for $j \in \{0, 1\}$, the second row of diagrams in Diagram 7.0.34 show that u respects identities, while the third row of diagrams there show that u respects composition.

Thus far we have shown that u is a well-defined internal functor making the required triangles commute. For uniqueness, suppose $v : \mathbb{B} \to \mathbb{C}$ is also an internal functor satisfying $fv = g$ and $h = vb$. Then note that in particular $v_0 b_0 = h_0 \implies v_0 b_0 b_0^{-1} = h_0 b_0^{-1} \implies v_0 = u_0$, so the assignments on objects of u and v agree. To see that their assignments on arrows also agree, we once again use the universal property of C_1. Noting that $f_1 v_1 = g_1 = f_1 u_1$, the proof is completed by the commutativity of the second diagram in the first row of Diagram 7.0.34. $\square$

We are now ready to give the construction of the full image factorisation. Let $\widehat{(-)}$ denote $\mathbf{coDisc} \circ \mathbf{Ob} : \mathbf{Cat}(\mathcal{E}) \to \mathbf{Cat}(\mathcal{E})$. Let $\mathbb{B} \in \mathbf{Cat}(\mathcal{E})$, and let $FF(\mathbb{B})$ be the full sub-2-category of $\mathbf{Cat}(\mathcal{E})/\mathbb{B}$ on fully faithful internal functors. Let $f : \mathbb{A} \to \mathbb{B}$ be an internal functor, and consider the diagram

$$\begin{tikzcd} \mathbb{A} \arrow[dr, dashed, "u_f"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "\eta^{\mathcal{E}}_{\mathbb{A}}"'] & & \\ & \mathrm{Im}(f) \arrow[r, "L(f)"] \arrow[d, "\underline{f}"] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{B} \arrow[d, "\eta^{\mathcal{E}}_{\mathbb{B}}"] \\ & \widehat{\mathbb{A}} \arrow[r, "\widehat{f}"'] & \widehat{\mathbb{B}} \end{tikzcd}$$

in which the pullback is taken in $\mathbf{Cat}(\mathcal{E})$, the boundary is the naturality square for $\eta^{\mathcal{E}}_{\mathbb{B}}$, and u_f is the internal functor induced by the commutativity of the boundary. Note that Lf is then fully-faithful, while the object assignment of u_f is an isomorphism in $\mathcal{E}$. The following corollary follows by combining this with Proposition 5.1.1.

Corollary 5.1.2. *There is an orthogonal factorisation system on* $\mathbf{Cat}(\mathcal{E})$ *whose left class consists of isomorphism on objects internal functors and right class consists of fully faithful internal functors.*

5.2 Discrete Fibrations and Final Internal Functors

The purpose of this section is to describe in some detail the two classes of internal functors which will feature in the *comprehensive factorisation*, which we will describe in Section 5.4. We take a similar route to the one in [30].

5.2.1 Discrete Fibrations

Remark 5.2.1. Recall the representable definition for discrete fibrations inside a 2-category that we gave in Definition 4.0.1. Expanding this, a 1-cell $p : C \to D$ in a 2-category $\mathfrak{K}$ is a discrete fibration if and only if for every $X \in \mathfrak{K}$ with $f : X \to C$, $g : X \to D$, $\alpha : g \Rightarrow pf$ there exists a unique 1-cell $h : X \to C$, and a unique 2-cell $\alpha' : h \Rightarrow f$ which satisfying $p\alpha' = \alpha$. The 2-cell α' is then the lift of α with respect to p. Note that in particular $pf = g$. The uniqueness clause in the definition says that if $\beta : h \Rightarrow f$ and $\beta' : h' \Rightarrow f$ satisfy $p\beta = p\beta'$, then $\beta = \beta'$; or in particular, p is representably faithful. Furthermore, taking α to be the identity, α' is an identity 2-cell if and only if $p\alpha'$ is an identity 2-cell. Finally, note that reversing the 2-cells in all of the above gives the analogous notions for discrete opfibrations.

We give some useful facts about discrete fibrations.

Proposition 5.2.2. *A functor is a discrete fibration in* **Cat** *if and only if it satisfies the condition in the remark above for* $X = 1$.

Proof. The 'only if' direction is clear. In **Cat** functors from 1 are just objects in their codomain, while natural transformations between them are just morphisms between those objects. So the condition says that if a functor $P : \mathcal{C} \to \mathcal{D}$ is a discrete fibration then for every $C \in \mathcal{C}$, $f \in \mathcal{D}(D, PC)$ there is a unique morphism f' with codomain C so that $Pf' = f$.

Assume that this condition holds for the functor P, and let $F : \mathcal{X} \to \mathcal{C}$ and $G : \mathcal{X} \to \mathcal{D}$ be functors with $\alpha : G \Rightarrow PF$ a natural transformation. Then for every $X \in \mathcal{X}$, let $\beta_X : F'X \to FX$ be the lift of $\alpha_X : GX \to PFX$, so that in particular $PX' = GX$. So, given a morphism $f \in \mathcal{X}(W, X)$, we may take the lift $F'f : F'W \to F'X$ of $Gf : GW \to GX$ as $GX = PF'X$. Then the assignment $X \mapsto F'X$ and $f \mapsto F'f$ defines a functor $F' : \mathcal{X} \to \mathcal{C}$. Indeed, given another morphism $g \in \mathcal{X}(X, Y)$ we have $P(F'g.F'f) = P\left((Gg)'.(Gf)'\right) = P(Gg)'.P(Gf)' = Gg.Gf = G(gf) = PF'(gf)$. Hence $F'(gf) = F'g.F'f$. Also, $F'1_X = (G1_X)' = 1'_{Gx} = 1_{F'x}$. So F' is a functor. Furthermore, the assignment $x \mapsto \beta_x : F'x \to Fx$ defines a natural transformation $\beta : F' \Rightarrow F$, since the image of its naturality square under P is the naturality square for α. Finally, for uniqueness, suppose that $F'' : X \to C$ is a functor and $\gamma : F'' \to F$ is a natural transformation such that $P\gamma = \alpha$. But then $P\gamma_X = \alpha_X$ for all $X \in \mathcal{X}$, and hence $\gamma_X = \beta_X$ so $\gamma = \beta$. □

Proposition 5.2.3. *Let* $f : A \to B$ *be a* 1*-cell and* $g : B \to C$ *a discrete fibration. Then* f *is a discrete fibration if and only if* gf *is a discrete fibration.*

Proof. Let $x : X \to A$ and $y : X \to C$ be 1-cells, and let $\lambda : y \Rightarrow gfx$ be a 2-cell. Since g is a discrete fibration, there exists a 1-cell $h : X \to B$ and a 2-cell $\mu : h \Rightarrow fx$ which is unique such that $g\mu = \lambda$. Hence in particular $gh = y$.

Now if f is a discrete fibration then we have a 1-cell $k : X \to A$ and a 2-cell $\nu : k \to x$ which is unique so that $f\nu = \mu$, and hence in particular $fk = h$. Thus $gf\nu = g\mu = \lambda$. Furthermore, given a 2-cell ν' with domain x so that $gf\nu' = \lambda$, by uniqueness of μ, we see that $f\nu' = \mu$ and similarly, by uniqueness of ν we have $\nu' = \nu$. So gf is a discrete fibration.

Conversely, if gf is a discrete fibration then there is a 2-cell λ' with domain x which is unique so that $gf\lambda' = \lambda$. Hence $f\lambda' = \mu$, by uniqueness of μ. Finally, suppose λ'' is a 2-cell with domain x such that $f\lambda'' = \mu$. Then $gf\lambda'' = g\mu = \lambda$, hence by uniqueness of λ' we see that $\lambda'' = \lambda'$. So f is a discrete fibration. □

Proposition 5.2.4. *Let* $p : A \to B$ *be a discrete fibration. Then* $\mathbf{DFib}(B)/p \cong \mathbf{DFib}(A)$.

Proof. An object of $\mathbf{DFib}(B)/p$ is a 1-cell $q : X \to A$ such that pq is a discrete fibration. Hence by Proposition 5.2.3, q is also a discrete fibration. Let $r : Y \to A$ be another object in $\mathbf{DFib}(B)/p$. A morphism $s \in \mathbf{DFib}(B)/p\,(r, q)$ is just a 1-cell $s : Y \to X$ such that $qs = r$. This is precisely the category $\mathbf{DFib}(A)$. □

In light of the Grothendieck Construction of the previous chapter, when $\mathfrak{K}$ is the 2-category of locally small categories this is analogous to saying that slicing over a presheaf produces a category equivalent to the presheaf category of its category of elements.

Proposition 5.2.5. *The pullback of a discrete fibration is a discrete fibration.*

Proof. Let $p : X \to Z$ be a discrete fibration, let $j : Y \to Z$ be a 1-cell and let $\pi_1 : P \to X$ and $\pi_2 : P \to Y$ exhibit P as a pullback of f and j. Let $h : A \to Y$ and $k : A \to P$ be 1-cells, with $\lambda : h \Rightarrow \pi_2 k$ a 2-cell. Then there is a 2-cell $\mu : g \Rightarrow \pi_1 k$ which uniquely satisfies $f\mu = j\lambda$, and in particular $fg = jh$.

Hence by the universal property of P, there exists a 1-cell $u : A \to P$ which is unique such that $\pi_1 u = g$ and $\pi_2 u = h$. Then by the two dimensional universal property of P, since $\mu : \pi_1 u \Rightarrow \pi_1 k$ and $\lambda : \pi_2 u \Rightarrow \pi_2 k$ satisfy $f\mu = j\lambda$, there is an induced $\lambda' : u \Rightarrow k$ which uniquely satisfies both $\pi_2\lambda' = \lambda$ and $\pi_1\lambda' = \mu$.

To show uniqueness, suppose $\lambda'' : u' \Rightarrow k$ is a 2-cell which satisfies $\pi_2\lambda'' = \lambda$. Then $f\pi_1\lambda'' = j\pi_2\lambda'' = j\lambda = f\mu$, and hence $\pi_1\lambda'' = \mu$ as f is a discrete fibration. Thus λ'' satisfies the defining properties of λ', and hence these 2-cells are equal. ☐

Remark 5.2.6. We describe the construction of comma objects from powers by $\mathbf{2}$ and pullbacks. Let $f : A \to C$ and $g : B \to C$ be 1-cells and let $\lambda : d \Rightarrow c : C^{\mathbf{2}} \to C$ be the universal 2-cell of the power by $\mathbf{2}$. The comma object f/g is then given by the iterated pullback of the zigzag from A to B, with the universal 2-cell given by whiskering with λ. This situation is depicted in the diagram below. Then C/g and f/C will comma object when f and g are identities, respectively. These are often called the *lax limit* and *oplax* limit of g and f respectively.

$$
\begin{tikzcd}
f/g \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C/g \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "g"] \\
f/C \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & C^{\mathbf{2}} \arrow[r, "c"] \arrow[d, "d"'] \arrow[dr, phantom, "\overset{\lambda}{\Nearrow}"] & C \arrow[d, "1_C"] \\
A \arrow[r, "f"'] & C \arrow[r, "1_C"'] & C
\end{tikzcd}
$$

Proposition 5.2.7. *Let $\mathfrak{K}$ be a 2-category with pullbacks and powers by $\mathbf{2}$. Let $\lambda : d \Rightarrow c : C^{\mathbf{2}} \to C$ be the universal 2-cell of the power by $\mathbf{2}$ of C, let $p : C \to D$ be a 1-cell, and let $\mu : \pi_2 \Rightarrow p\pi_1$ be the universal 2-cell of the comma object D/p. Then p is a discrete fibration if and only if the unique 1-cell $u : C^{\mathbf{2}} \to D/p$ satisfying $\mu u = p\lambda$ is an isomorphism.*

Proof. Suppose that $p : C \to D$ is a discrete fibration. Then by the universal property of D/p, the 2-cell $p\lambda$ induces a 1-cell $u : C^{\mathbf{2}} \to D/p$ which is unique so that $c = \pi_1 u$, $pd = \pi_2 u$, and $p\lambda = \mu u$. Meanwhile by the universal property of $C^{\mathbf{2}}$, μ' induces a 1-cell $v : D/p \to C^{\mathbf{2}}$ which is unique so that $\lambda v = \mu'$. Hence in particular $dv = f$, $cv = \pi_1$. But $\mu uv = p\lambda v = p\mu' = \mu$, while $\pi_1 uv = cv = \pi_1$, and $\pi_2 uv = pdv = pf = \pi_2$. Hence $uv = 1_{D/p}$ by the universal property of μ. Similarly, $p\lambda vu = p\mu' u = \mu u = p\lambda$, so $\lambda vu = \lambda$ since p is a discrete fibration. Thus $vu = 1_{C^{\mathbf{2}}}$ by the universal property of λ.

Conversely, suppose the induced $u : C^{\mathbf{2}} \to D/p$ has an inverse u^{-1}. Let $h : X \to C$, $k : X \to D$ be 1-cells and $\gamma : k \Rightarrow ph$ be a 2-cell. Then the universal property of D/p induces $q : X \to D/p$ which is unique so that $\pi_1 q = h$, $\pi_2 q = k$ and $\mu q = \gamma$. Then $\lambda u^{-1} q : du^{-1}q \Rightarrow cu^{-1}q : X \to C$ satisfies $p\lambda u^{-1} q = \mu u u^{-1} q = \mu q = \gamma$. It remains to show that this lift is unique, so let $h' : X \to C$ be a 1-cell and let $\phi : h' \Rightarrow h$ satisfy $p\phi = \gamma$, so that in particular we also have $ph' = k$. Now by the universal property of $C^{\mathbf{2}}$, ϕ induces $w : X \to C^{\mathbf{2}}$ unique so that $\phi = \lambda w$. Hence in particular, $h = cw$ and $h' = dw$. Then $\pi_1 uw = cw = h$, $\pi_2 uw = pdw = ph' = k$, and $\mu uw = p\lambda w = p\phi = \gamma$. Thus $uw = q$ since q was induced to uniquely satisfy these conditions. Hence $\lambda u^{-1} q = \lambda u^{-1} uw = \lambda w = \phi$, as required. ☐

Remark 5.2.8. All of the proofs in this subsection apply just as well to discrete opfibrations, provided that one reverses the direction of any 2-cells in sight. In particular, discrete opfibrations are also stable under pullback, a fact that we will use in Proposition 5.2.10 in the next section.

5.2.2 Final Internal Functors

In this subsection we give a description of the left class of the comprehensive factorisation system which we will prove in Section 5.4. Internal functors in this class are called *final*. For this subsection, let $\mathcal{E}$ be a category with finite limits and reflexive coequalisers, and let $\Pi_0' : \mathbf{Cat}(\mathcal{E}) \to \mathbf{LocDisc}(\mathcal{E})$ denote the transpose of the functor $\Pi_0 : \mathbf{Cat}(\mathcal{E})_1 \to \mathcal{E}$ from Remark 1.2.9 under the adjunction $\mathbf{Cat}\big(\mathbf{Cat}(\mathcal{E})_1, \mathcal{E}\big) \cong$ 2-$\mathbf{Cat}\big(\mathbf{Cat}(\mathcal{E}), \mathbf{LocDisc}(\mathcal{E})\big)$, where $\mathbf{LocDisc} : \mathbf{Cat} \to$ 2-$\mathbf{Cat}$ sends a category to itself viewed as a locally discrete 2-category, and write $\mathbf{LocDisc}(\mathcal{E})$ as just $\mathcal{E}$.

Definition 5.2.9. Call a 1-cell $f : X \to Y$ *powerful* if its pullback functor $\mathfrak{K}/Y \to \mathfrak{K}/X$ has a right adjoint. An internal functor $j : \mathbb{A} \to \mathbb{B}$ is *final* if for all powerful discrete opfibrations $q : \mathbb{X} \to \mathbb{B}$, the 2-functor $\Pi_0' : \mathbf{Cat}(\mathcal{E}) \to \mathcal{E}$ sends the pullback of j along q to an isomorphism in $\mathcal{E}$. Call the internal functor $j : \mathbb{A} \to \mathbb{B}$ *initial* if instead Π_0' sends the pullback of j along any powerful discrete fibration to an isomorphism in $\mathcal{E}$.

As mentioned previously, we will primarily be treating the case of final internal functors. However, reversing the 2-cells in all of our statements about internal final functors will give equivalent statements about internal initial functors.

Proposition 5.2.10. *A composite of final internal functors is final.*

Proof. Suppose that in the pullback diagram below, the internal functors in the bottom row are final and the internal functor q is a powerful discrete opfibration.

$$\begin{tikzcd}
\mathbb{A} \arrow[r, "f"] \arrow[d, "h"'] & \mathbb{B} \arrow[r, "g"] \arrow[d, "p"] & \mathbb{C} \arrow[d, "q"] \\
\mathbb{D} \arrow[r, "j"'] & \mathbb{E} \arrow[r, "k"'] & \mathbb{F}
\end{tikzcd}$$

Then since powerful morphisms are stable under pullback (part 3 of Corollary 2.6 in [30]), it follows from Proposition 5.2.5 by duality that p is also a discrete opfibration. But then Π_0' sends both f and g to isomorphisms. The proof is complete by 2-functoriality of Π_0'. ☐

As mentioned at the beginning of this chapter, the left class will itself be the composite of internal functors from two separate classes. In particular, they will take the form of a right adjoint followed by a coidentifier, as we will show in Section 5.3. In light of Proposition 5.2.10, it will be easier to treat these classes separately.

Proposition 5.2.11. *Coidentifiers are final, and initial.*

Proof. Let $n : A \to B$ be a coidentifier and let $p : X \to B$ be powerful. Then pullback along p is a left adjoint, so $p^*(n)$ is a coidentifier. Similarly, Π_0' is a left adjoint and so $\Pi_0'(p^*(n))$ is a coidentifier in $\mathcal{E}$. But there are no non-identity 2-cells in $\mathcal{E}$ and hence coidentifiers in $\mathcal{E}$ are isomorphisms, so n is both final and initial. ☐

We now prove two lemmas which will be needed in order to show that right adjoints are also final.

Lemma 5.2.12. *Let the first diagram below be a comma square with $q : F \to B$ a discrete opfibration, and let the second be a pullback square. Let $\lambda' : r \Rightarrow k$ be the lift of λ along q. Then there are induced* 1*-cells as depicted in the third and fourth diagrams below such that u_* is induced by $1_{qg} : qg \Rightarrow jh$, and there is an adjunction $u^* \dashv u_*$.*

$$
\begin{tikzcd}
q/j \arrow[r, "f"] \arrow[d, "r"'] & A \arrow[d, "j"] \\
F \arrow[r, "q"'] \arrow[ur, Rightarrow, "\lambda"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, "h"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "j"] \\
F \arrow[r, "q"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[dr, dashed, "u_*"] \arrow[drr, bend left, "h"] \arrow[ddr, bend right, "g"'] & & \\
& q/j \arrow[r, "f"] \arrow[d, "r"'] & A \arrow[d, "j"] \\
& F \arrow[r, "q"'] \arrow[ur, Rightarrow, "\lambda"'] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
q/j \arrow[dr, dashed, "u^*"] \arrow[drr, bend left, "f"] \arrow[ddr, bend right, "k"'] & & \\
& P \arrow[r, "h"] \arrow[d, "g"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "j"] \\
& F \arrow[r, "q"'] & B
\end{tikzcd}
$$

Proof. Note that the 2-cells λ' and 1_f satisfy the equality of pastings shown in Diagram 7.0.35, and hence by the 2-dimensional universal property of q/j we have an induced $\eta_u : 1_{q/j} \Rightarrow u_* u^*$ which is unique such that $r\eta_u = \lambda'$, and $f\eta_u = f$. Consider $\lambda' u_* : ru_* \Rightarrow ku_*$. This 2-cell whiskers with the discrete opfibration q to give λu_*, which is an identity 2-cell, and hence it is also an identity 2-cell. Thus $g = ku_*$, or $g = gu^* u_*$, and also $hu^* u_* = fu_* = h$. But since g and h are jointly monic, $u^* u_* = 1_P$. We claim that $\eta_u : 1_{q/j} \Rightarrow u_* u^*$ and $1_{1_P} : u^* u_* \Rightarrow 1_P$ are the unit and counit respectively of an adjunction $u^* \dashv u_*$. To show this we need to show that $u^* \eta_u$ and $\eta_u u_*$ are identities. For both of these, we use the universal properties defining u^* and u_* respectively.

For the first triangle identity, we firstly note that $qk\eta_u = jf\eta_u = jf = qk$, and hence $k\eta_u = k$ since q is a discrete opfibration. Now we have $u^* \eta_u = k\eta_u = k$. Also, $h\,(u^*)\,.\,(u^* \eta_u) = (hu^*)\,.\,(hu^* \eta_u) = hu^*\,(hu^* \eta_u) = f.f\eta = f$. Thus the first triangle identity follows from the universal property of u^*. For the second triangle identity, we firstly note that $q\lambda' u_* = \lambda u_* = qg$, and hence $\lambda' u_* = g$ since q is a discrete opfibration. Now, using functoriality of horizontal composition, we see $\lambda'\,(u_*)\,.\,(\eta_u u_*) = (\lambda' u_*)\,.\,(ru_*)\,.\,(r\eta_u u_*) = (\lambda' u_*)\,.g.\,(\lambda' u_*) = g$. Similarly, $(qru_*)\,.\,(qr\eta_u u_*) = (qg)\,.\,(q\lambda' u_*) = (qg)\,.\,(\lambda u_*) = qg$, and hence $r\,(u_*)\,.\,(\eta_u u_*) = g$ since q is a discrete opfibration. Finally, we have $f\,(u_*)\,.\,(\eta_u u_*) = (fu_*)\,.\,(f\eta_u u_*) = h.\,(fu_*) = h$. Thus the second triangle identity follows from the universal property of u_*, and hence we have established the adjunction as required. ☐

Lemma 5.2.13. *Consider again the comma square of Proposition 5.2.12. Instead of supposing that q is a discrete opfibration, this time instead suppose that $j_* := j$ has a left adjoint j^* with unit $\eta_j : 1_B \Rightarrow j_* j^*$ and counit $\epsilon_j : j^* j_* \Rightarrow 1_A$. Then the 1-cell $r^* : F \to q/j$ induced by $\eta_j q : q \Rightarrow j_* j^* q$ is left adjoint to r_* with the adjunction $r^* \dashv r_*$ having an identity unit.*

Proof. We first note that since $r^* : F \to q/j_*$ was induced by a universal property, it is unique such that $r_* r^* = 1_F$, $fr^* = j^* q$, and $\lambda r^* = \eta_j q$. Now consider the two 2-cells $1_{r_*} : r_* r^* r_* \Rightarrow r_*$, and $(\epsilon_j f)\,.\,(j^* \lambda) : fr^* r_* \Rightarrow r_*$. Note that $(j_* j^* \lambda)\,.\,(\eta_j q r_*) = (\eta_j j_* f)\,.\lambda$ by functoriality of horizontal composition. Thus the equality of pastings depicted below holds, and hence by the 2-dimensional universal property of q/j_*, there is an induced 2-cell $\epsilon_r : r_* r^* \Rightarrow 1_{q/j_*}$ which is unique such that $r_* \epsilon_r = r_*$ and $f\epsilon_r = (\epsilon_j f)\,.\,(j^* \lambda)$.

$$
\begin{tikzcd}
A \arrow[r, "j_*"] \arrow[rr, bend left, "1_A"] \arrow[rr, phantom, bend left=20, "\Uparrow\epsilon_j"] & B \arrow[r, "j^*"] & A \arrow[r, "j_*"] & B \\
q/j \arrow[r, "r_*"'] \arrow[u, "f"] \arrow[ur, Rightarrow, "\lambda"'] & F \arrow[r, "r^*"'] \arrow[u, "q"'] \arrow[ur, phantom, "="] & q/j \arrow[r, "r_*"'] \arrow[u, "f"] \arrow[ur, Rightarrow, "\lambda"] & F \arrow[u, "q"']
\end{tikzcd}
=
\begin{tikzcd}
& A \arrow[rr, "1_A"] \arrow[dr, "j_*"] & & A \arrow[dr, "j_*"] & \\
q/j \arrow[r, "r_*"'] \arrow[ur, "f"] & F \arrow[r, "q"'] \arrow[u, Rightarrow, "\lambda"'] & B \arrow[rr, "1_B"'] \arrow[ur, "j^*"] \arrow[u, Rightarrow, "\epsilon_j"] & & B \arrow[ul, phantom, "\Uparrow\lambda"]
\end{tikzcd}
$$

We claim that $\eta_r = 1_{1_F}$ and ϵ_r are the unit and counit of an adjunction $r^* \dashv r_*$. Indeed, the triangle identity for r_* holds immediately, while for r_* it simplifies to $(\epsilon_r r^*)\,r^*$ which is the identity. This establishes the triangle identities and completes the proof. ☐

Corollary 5.2.14. *Right adjoints are final, while left adjoints are initial.*

Proof. Consider once again the comma square for Lemmas 5.2.12 and 5.2.13, this time under both the assumption that q is a discrete opfibration and that $j_* := j$ is a right adjoint. Then the pullback projection $g = ru_*$ is the composite of two right adjoint 1-cells and is hence itself a right adjoint. Since 2-functors preserve adjoints, and adjoints in locally discrete 2-categories are just isomorphisms, g is therefore sent by Π_0' to an invertible morphism in $\mathcal{E}$, as required for j_* to be final. This completes the proof that right adjoints are final. Finally, note that the reversing of 2-cells interchanges left and right adjoints, and thus left adjoints are initial. $\square$

We now prove left orthogonality of both coidentifiers and right adjoints to discrete fibrations.

Proposition 5.2.15. *Let $\lambda : f \Rightarrow g : X \to A$ have coidentifier $n : A \to B$. Let $p : C \to D$ be a discrete fibration, and let $u : A \to C$, $v : B \to D$ satisfy $vn = pu$. Then there is a diagonal filler $w : B \to C$, which is unique such that $pw = v$ and $wn = u$.*

$$\begin{tikzcd} X \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow\lambda"] & A \arrow[r, "n"] \arrow[d, "u"'] & B \arrow[d, "v"] \arrow[dl, dashed, "w"'] \\ & C \arrow[r, "p"'] & D \end{tikzcd}$$

Proof. Note that $pu\lambda = vn\lambda$, and that this 2-cell is an identity since n coidentifies λ. Hence $u\lambda$ is also an identity since p is a discrete fibration. So by the universal property of n, we have an induced $w : B \to C$ which is unique so that $wn = u$. Then $pwn = pu = vn$, and hence $pw = v$ as n is an epimorphism, since it is a coidentifier. For uniqueness, suppose $w' : B \to C$ satisfies $pw' = v$ and $w'n = u$. Indeed, $w'n = u$ is sufficient to conclude that $w' = w$ since w was induced by the fact that $u\lambda$ is an identity. $\square$

Proposition 5.2.16. *Let $r : A \to B$ have a left adjoint $l : B \to A$ with unit $\eta : 1_B \Rightarrow rl$ and counit $\epsilon : lr \Rightarrow 1_A$. Let $p : C \to D$ be a discrete fibration, and let $u : A \to C$ and $v : B \to D$ satisfy $pu = vr$. Then there is a 1-cell $w : B \to C$ which uniquely satisfies $pw = v$ and $wr = u$.*

Proof. Since p is a discrete fibration, the 2-cell $v\eta : v \Rightarrow pul$ induces a 2-cell γ with codomain ul which uniquely satisfies $p\gamma = v\eta$. Letting $w : B \to C$ denote the domain of γ, we see that in particular $pw = v$. We claim that w is the required diagonal filler. Note that the equality of pastings depicted below holds, where the latter is just the identity 2-cell on vr by the triangle identity. Hence $wr = u$ by the discrete fibration property of p.

$$\begin{tikzcd} A \arrow[r, "1_A"] \arrow[dr, "r"'] & A \arrow[r, "u"] \arrow[d, phantom, "\Uparrow\epsilon"] & C \arrow[r, "p"] \arrow[dl, phantom, "\Uparrow\gamma"] & D \\ & B \arrow[u, "l"'] \arrow[ur, bend right, "w"'] & & \end{tikzcd} = \begin{tikzcd} A \arrow[r, "1_A"] \arrow[dr, "r"'] & A \arrow[dr, "r"] \arrow[d, phantom, "\Uparrow\epsilon"] & \\ & B \arrow[u, "l"'] \arrow[r, "1_B"'] \arrow[ur, phantom, "\Uparrow\eta"] & B \arrow[r, "v"] & D \end{tikzcd}$$

Finally, let $w' : B \to C$ satisfy $u = w'r$ and $v = pw'$. We need to show that $w' = w$. Hence consider $w'\eta : w' \Rightarrow w'rl = ul$. Then $pw'\eta = v\eta = p\gamma$. So $\gamma = w'\eta$ as p is a discrete fibration. Hence in particular their domains are equal, and so $w = w'$. $\square$

Note that under the reversing of 2-cells we have also proven that left adjoints, and coidentifiers, are left orthogonal to discrete opfibrations.

5.3 The Split Fibrations 2-monad

We are now ready to present the first stage of the comprehensive factorisation of an internal functor. As previously mentioned, since this construction is possible in any 2-category with oplax limits, we will give it at that level of generality.

Notation 5.3.1. *We adopt the convention when displaying morphisms in comma categories that the horizontally displayed arrows represent objects, and the commutative square determined by the vertical arrows represents a morphism in the comma category. Thus for example, if $F : \mathcal{A} \to \mathcal{B}$ is a functor then the following commutative square in a category $\mathcal{B}$ is a morphism from ϕ to ϕ' in the comma category $\mathcal{B}/F$.*

$$\begin{tikzcd} b \arrow[r, "\phi"] \arrow[d, "\beta"'] & Fa \arrow[d, "F\alpha"] \\ b' \arrow[r, "\phi'"'] & Fa' \end{tikzcd}$$

Proposition 5.3.2. *A* 1*-cell* $f : A \to B$ *in a* 2*-category with oplax limit of* 1*-cells as described in Remark 5.2.6 factorises as a right adjoint followed by a split fibration.*

Proof. Let $u : B/f \to A$ and $v : B/f \to B$ be comma projections, and let $\lambda : v \Rightarrow fu$ be the universal 2-cell of B/f as a comma object. Then $1_f : f \Rightarrow f1_A$ induces a 1-cell $r : A \to B/f$ which is unique such that $ur = 1_A$, $vr = f$ and $\lambda r = f$. This situation is depicted in the following diagram.

$$\begin{tikzcd}
A \arrow[r, dashed, "r"] \arrow[rr, bend left, "1_A"] \arrow[drr, bend right, "f"'] & B/f \arrow[r, "u"] \arrow[dr, "v"'] \arrow[dr, phantom, shift left=3, "\Uparrow\lambda"] & A \arrow[d, "f"] \\
 & & B
\end{tikzcd}$$

Now by the 2-dimensional universal property of B/f, since $1_u : u \Rightarrow uru$ and $\lambda : v \Rightarrow fu = vru$ satisfy $\lambda ru.\lambda = \lambda = f1_u.\lambda$, we have an induced $\eta : 1_{B/f} \Rightarrow ru$ which is unique such that $u\eta = u$ and $v\eta = \lambda$. We claim that η is the unit of an adjunction $u \dashv r$ whose counit is 1_{1_A}. The triangle identity involving the left adjoint follows as $(1_{1_A}u).(\eta r) = u$. For the triangle identity involving r, we first note that $\eta r = r$ by the universal property of B/f as $u\eta r = ur = 1_{1_A}$ and $v\eta r = \lambda r = f$. Hence $r.(\eta r) = r$. Thus we have established that $f = vr$ where r is a right adjoint.

It remains to show that v is a split fibration. Pick $X \in \mathfrak{K}$ and herein let $\underline{Y}$ denote $\mathfrak{K}(X, Y)$ for any Y in the data of the diagram above, so that we may work representably in **Cat**. We claim that given $\phi \in \underline{B}(b, fa)$ and $\beta : b' \to b$, the morphism in $\underline{B/f}$ shown below on the left is cartesian. To see this, let $g : b'' \to b'$ be a morphism in $\underline{B}$ and let the square below on the right be a morphism in $\underline{B/f}$. Both of these morphisms in $\underline{B/f}$ have source given by the top row and target given by their bottom row, as depicted.

$$(\beta, 1_a) = \begin{tikzcd}
b' \arrow[r, "\beta"] \arrow[d, "\beta"'] & b \arrow[r, "\phi"] & \underline{f}a \arrow[d, "\underline{f}(1_a)"] \\
b \arrow[rr, "\phi"'] & & \underline{f}a
\end{tikzcd}
\qquad
(\beta g, \alpha) = \begin{tikzcd}
b'' \arrow[r, "\phi'"] \arrow[d, "g"'] & \underline{f}a'' \arrow[dd, "\underline{f}(\alpha)"] \\
b' \arrow[d, "\beta"'] & \\
b \arrow[r, "\phi"'] & \underline{f}a
\end{tikzcd}$$

Then clearly $\alpha : a' \to a$ uniquely satisfies $1_a\alpha = \alpha$ and $\underline{f}\alpha\phi'' = \phi\beta g$. This shows that the projection $\underline{v} : \underline{B/f} \to \underline{B}$ is a fibration, and since the chosen cartesian lifts are sent by $\underline{u}$ to identities, they compose just as in $\underline{B}$. So the composite of chosen cartesian morphisms is a chosen cartesian morphism, and hence the fibration is split. Finally, note that all these calculations are stable under composition in $\mathfrak{K}$ by some 1-cell into X, so we have established the result required in $\mathfrak{K}$. $\square$

Theorem 5.3.3. *With notation as in the proof of Proposition 5.3.2,* $f \mapsto v$ *defines the left adjoint* L *to the forgetful* 2*-functor* $\mathbf{SFib}(B) \to \mathfrak{K}/B$*.*

Proof. We describe the 2-functor to which this assignment extends. Let $f' : A' \to B$ have the comma object B/f' with the universal 2-cell $\lambda' : v' \Rightarrow f'u'$. For a 1-cell $g \in \mathfrak{K}/B(f', f)$, or equivalently $g : A' \to A$ in $\mathfrak{K}$ such that $fg = f'$, $Lg : B/f' \to B/f$ is induced by λ' by the universal property of B/f in $\mathfrak{K}$. Thus Lg uniquely satisfies the following conditions the conditions depicted in the diagrams below.

$$\begin{tikzcd}
B/f' \arrow[r, "Lg"] \arrow[d, "u'"'] & B/f \arrow[d, "u"] \\
A' \arrow[r, "g"'] & A
\end{tikzcd}
\quad
\begin{tikzcd}
B/f' \arrow[rr, "Lg"] \arrow[dr, "v'"'] & & B/f \arrow[dl, "v"] \\
 & B &
\end{tikzcd}
\quad
\begin{tikzcd}
B/f' \arrow[r, "Lg"] & B/f \arrow[r, "u"] \arrow[dr, "v"'] \arrow[dr, phantom, shift left=3, "\Uparrow\lambda"] & A \arrow[d, "f"] \\
 & & B
\end{tikzcd}
=
\begin{tikzcd}
B/f' \arrow[r, "u'"] \arrow[dr, "v'"'] \arrow[dr, phantom, shift left=3, "\Uparrow\lambda'"] & A' \arrow[r, "g"] \arrow[d, "f'"] & A \arrow[dl, "f"] \\
 & B &
\end{tikzcd}$$

For a 2-cell $\alpha : g \Rightarrow g'$ in $\mathfrak{K}/B$, or equivalently one in $\mathfrak{K}$ such that $f\alpha = f'$, the equality of pastings in Diagram 7.0.29 holds, or equivalently the diagram of 2-cells below commutes. Hence the 2-cells $\alpha u' : uLg = gu' \Rightarrow g'u' = uLg'$ and $1_{v'} : vLg' \Rightarrow vLg$ induce the 2-cell $L\alpha : Lg \Rightarrow Lg'$ by the 2-dimensional universal property of B/f in $\mathfrak{K}$. Thus $L\alpha : Lg \Rightarrow Lg'$ is unique so that $uL\alpha = \alpha u'$ and $vL\alpha = v'$.

$$\begin{tikzcd}
vLg \arrow[r, "1"] \arrow[d, "\lambda Lg"'] & vLg' \arrow[d, "\lambda Lg'"] \\
f\alpha\,(Lg) \arrow[r, "f\alpha"'] & f\alpha\,(L'g)
\end{tikzcd}$$

To see that this assignment is functorial on hom-categories, let $\beta : g' \Rightarrow g''$ be a 2-cell in $\mathfrak{K}/B$, so that $f\beta = f'$. We see that $u\,(L\beta.L\alpha) = (uL\beta)\,.\,(uL\alpha) = (\beta u')\,.\,(\alpha u') = (\beta.\alpha)\,u$, and $v\,(L\beta.L\alpha) = (vL\beta)\,.\,(vL\alpha) = v'$. Hence $L\,(\beta.\alpha) = L\beta.L\alpha$, by the universal property of the former. Similarly, to check that the assignment strictly respects horizontal composition, we let $g \in \mathfrak{K}/B\,(f'', f')$ and check that $LgLg'$ satisfies the conditions uniquely identifying $L\,(gg')$ according to the universal property by which it is induced. Indeed, $\lambda LgLg' = \lambda' Lg' = \lambda''$, while $vLgLg' = v'Lg' = v''$ and $uLgLg' = gu'Lg' = gg'u''$. Finally noting that identities are preserved by L, we have shown that L is well defined as a 2-functor.

We claim that $r \in \mathfrak{K}/B\,(f, v)$ is the component of the unit of this adjunction at f. For 2-naturality, we show that $Lg.r' = rg$ using the universal property of B/f. Indeed, we see that $uLg.r' = gu'r' = g = urg$, $vLg.r' = v'r' = f' = fg = vrg$, and $\lambda Lg.r' = \lambda' r' = f' = fg = \lambda rg$, and hence by the universal property we have shown 2-naturality. Next, we verify the universal property of the unit. Let $h : C \to B$ be a split fibration and $k : A \to C$ be a 1-cell such that $f = hk$. Then as h is a split fibration, $\lambda : v \Rightarrow hku$ induces a chosen 2-cell $\xi : g \Rightarrow ku : B/f \to C$ such that $h\xi = \lambda$, and hence in particular $hg = v$. We claim that g is the required unique morphism of split fibrations satisfying $gr = k$ in $\mathfrak{K}/B$. To see that g preserves chosen lifts, we work representably in **Cat**. Let $(\beta, 1_a) : (\phi\beta, a) \to (\phi, a)$ be the chosen lift of $\beta : b' \to v\,(\phi, a) = b$. Then $g\,(\beta, 1_a) \in C\,(g\,(\phi\beta, a)\,, g\,(\phi, a))$ commuting with the components of $\xi : g \Rightarrow ku$ at $(\phi\beta, a)$ and (ϕ, a). But since these components are chosen, so is $g\,(\beta, 1_a)$, and hence g is a morphism of split fibrations.

Note that since ξ is chosen, so is $\xi r : gr \Rightarrow kru = k$, as can be seen when working representably in **Cat**. But $h\xi r = \lambda r = f$, so ξr is an identity 2-cell and hence $gr = k$. Now for uniqueness, suppose $g' : B/f \to C$ is a morphism of split fibrations from v to h so that $gr = k$. Working representably in **Cat**, we note that the chosen lift of $\beta \in B\,(b, b')$ will be of the form $(\beta, 1_a) \in B/f\,(\phi, \phi')$ where $\phi : b \to fa$ and $\phi' : b' \to fa$ are morphisms of B. We need to show that $g\,(\beta, 1_a) = g'\,(\beta, 1_a)$. But $hg\,(\beta, 1_x) = v\,(\beta, 1_x) = \beta$, and similarly $hg'\,(\beta, 1_x) = \beta$. Since these are both chosen lifts of β, they are equal. So $g = g'$ as required. ☐

Remark 5.3.4. Let $v : B/f \to B$ be as above and let $\lambda' : v' \Rightarrow vu'$ be the universal 2-cell exhibiting B/v as an oplax limit of v in $\mathfrak{K}$. The monad T induced by this adjunction has unit component on f given by r, and multiplication component on f given by the 1-cell $m : B/v \to B/f$ induced by the pasting below, and hence unique such that $um = uu'$, $vm = v'$, and $\lambda m = \lambda u'.\lambda'$.

$$\begin{tikzcd}
B/u \arrow[r, "u'"] \arrow[dr, "v'"'] & B/f \arrow[r, "u"] \arrow[d, "v"] \arrow[dl, phantom, "\Nearrow\lambda'"] \arrow[dr, phantom, "\Nearrow\lambda"] & A \arrow[dl, "f"] \\
& B &
\end{tikzcd}$$

Theorem 5.3.5. *The forgetful* 2*-functor* $\mathbf{SFib}\,(B) \to \mathfrak{K}/B$ *is* 2*-monadic.*

Proof. Once again, by representability, it suffices to show that this is true when $\mathfrak{K} = \mathbf{Cat}$. We begin by setting up the notation which will be used in this proof, and describing what the monad axioms say in this notation. Following this we will summarise how the structure of a T-algebra and a split fibration, and morphisms thereof, give rise to one another.

Letting v denote Tf as above, an object in B/v consists of a morphism $\psi : b' \to b$ in B, an object $a \in A$ and a morphism $\phi : b' \to fa$ in B, and will be denoted (ψ, ϕ, a). A morphism $(\psi_1, \phi_1, a_1) \to (\psi_2, \phi_2, a_2)$ in B/v consists of morphisms $\beta' : {b'}_1 \to {b'}_2$, $\beta : b_1 \to b_2$ and $\alpha : a_1 \to a_2$ so that the squares below commute, and will be denoted $(\beta', (\beta, \alpha))$.

$$
\begin{tikzcd}
b_1 \arrow[r, "\phi_1"] \arrow[d, "\beta"'] & fa_1 \arrow[d, "f\alpha"] \\
b_2 \arrow[r, "\phi_2"'] & fa_2
\end{tikzcd}
\qquad
\begin{tikzcd}
b'_1 \arrow[r, "\psi_1"] \arrow[d, "\beta'"'] & b_1 \arrow[d, "\beta"] \\
b'_2 \arrow[r, "\psi_2"'] & b_2
\end{tikzcd}
$$

Then the monad multiplication $m : B/v \to B/f$ described in Remark 5.3.4 maps the morphism $(\beta', (\beta, \alpha)) : (\psi_1, \phi_1, a_1) \to (\psi_2, \phi_2, a_2)$ to $(\beta', \alpha) : (\phi_1\psi_1, a_1) \to (\phi_2\psi_2, a_2)$. Denote the image of an object (ϕ, a) under a functor $t : B/f \to A$ as $\phi^* (a)$ and denote the image of a morphism $(\beta, \alpha) \in B/f ((\phi_1, a_1), (\phi_2, a_2))$ as $\overline{(\beta, \alpha)}$. Note then that $f\phi^* (a) = fa$. Then Tt maps the morphism $(\beta', (\beta, \alpha))$ to $\left(\beta', \overline{(\beta, \alpha)}\right) : (\psi_1, {\phi_1}^* (a)) \to (\psi_2, {\phi_2}^* (a))$. Thus the multiplication axiom for such a t exhibiting f as a T-algebra would say on the generic object $(\phi, \psi, a) \in B/v$ that $\psi^*\phi^* (a) = (\phi\psi)^* (a)$, and on the generic morphism $(\beta' (\beta, \alpha)) \in B/v ((\psi_1, \phi_1, a_1), (\psi_2, \phi_2, a_2))$ that $\overline{\left(\beta', \overline{(\beta, \alpha)}\right)} = \overline{(\beta', \alpha)}$. Meanwhile, the unit axiom for t would say on objects that ${1_{fa}}^* (a) = a$, and on morphisms that $\overline{(f\alpha, \alpha)} = \alpha$. We will proceed by showing how the structure of a split fibration and an algebra for T give rise to one another.

Assume $t : B/f \to A$ exhibits $f : A \to B$ as a T-algebra. Choose the lift of a morphism $\phi \in B (b, fa)$ to be $\overline{(\phi, 1_a)}$. It suffices to show that this morphism is cartesian for f since the lift of an identity is clearly an identity and the composite of chosen lifts is clearly also a chosen lift. For all $\alpha : a' \to a$ and $\phi : b \to fa$, the morphism $\overline{(\beta, \alpha)} : a' \to \phi^* (a)$ satisfies the commutativity of the first two triangles below. For uniqueness, suppose $\gamma : a' \to \phi^* (a)$ satisfies $f\gamma = \beta$ and the third triangle below. Then we may factorise γ as shown in the fourth triangle below, which corresponds to diagram in B shown on the right.

$$
\begin{tikzcd}
a' \arrow[r, "\overline{(\beta,\alpha)}"] \arrow[dr, "\alpha"'] & \phi^* (a) \arrow[d, "\overline{(\phi,1_a)}"] \\
& a
\end{tikzcd}
\quad
\begin{tikzcd}
fa' \arrow[r, "\beta"] \arrow[dr, "f\alpha"'] & b \arrow[d, "\phi"] \\
& fa
\end{tikzcd}
\quad
\begin{tikzcd}
a' \arrow[r, "\gamma"] \arrow[dr, "\alpha"'] & \phi^* (a) \arrow[d, "\overline{(\phi,1_a)}"] \\
& a
\end{tikzcd}
\quad
\begin{tikzcd}
a' \arrow[r, "\overline{(1_{fa'},\gamma)}"] \arrow[dr, "\gamma"'] & (f\gamma)^* (a) \arrow[d, "\overline{(\beta,1_a)}"] \\
& \phi^* (a)
\end{tikzcd}
\quad
\begin{tikzcd}
fa' \arrow[r, "1_{fa'}"] \arrow[d, "1_{fa'}"'] & fa' \arrow[d, "f\gamma"] \\
fa' \arrow[r, "f\gamma"] \arrow[d, "\beta"'] & f\phi^* (a) \arrow[d, "f(1_a)"] \\
b \arrow[r, "\phi"'] & fa
\end{tikzcd}
$$

Thus $f\overline{(1_{fa'}, \gamma)} = 1$. Hence it is enough to show uniqueness for the case where β is an identity, so herein we assume $f\gamma = 1_{fa'}$. Now $(1_{fa'}, \gamma) : (1_{fa'}, a') \to (1_{fa'}, a')$ composes with $(1_{fa'}, \alpha) : (1_{fa'}, a') \to (\phi, a')$ in B/f to give $(1_{fa'}, \alpha) : (1_{fa'}, a') \to (\phi, a)$. But functoriality of t says that the following diagram commutes

$$
\begin{tikzcd}
a' \arrow[r, "\gamma"] \arrow[dr, "\overline{(1_{fa'},\alpha)}"'] & \phi^* (a) \arrow[d, "\overline{\left(1_{fa'},\overline{(1_{fa'},\phi)}\right)}"] \\
& (f\alpha)^* (a')
\end{tikzcd}
$$

so it is enough to show that $\overline{\left(1_{fa'}, \overline{(1_{fa'}, \phi)}\right)} = 1_{\phi^*(a)}$. But this is just the multiplication axiom for t as a T-algebra, applied to the morphism $(1_b, (\phi, 1_a)) \in B/v ((1_b, \phi, a), (\phi, 1_{fa}, a))$. So f is a split fibration.

Conversely, suppose $f : A \to B$ is a split fibration, with chosen lift for $\beta : b \to fa'$ given by $\alpha : a \to a'$. Then there is a functor $t : B/f \to A$ given

- On objects by sending $\phi : b \to fa$ to the domain of its chosen lift with respect to f, as a morphism in B.
- On morphisms by sending $(\beta, \alpha) : (\phi, a) \to (\phi', a')$ to the morphism $t (\phi, a) \to t (\phi', a')$ induced by the universal property of the cartesian lift of ϕ' along f, since α composed with the cartesian lift of ϕ is also a morphism into a'.

Then $ft = v$ is clear from the definition, while the unit axiom for t as a T algebra holds since f lifts identities to identities, and the multiplication axiom for t as a T algebra can be seen to hold since chosen lifts of f compose to give a chosen lift. Furthermore, it is clear that this assignment is inverse to that described from T-algebras to split fibrations.

We now look at the 1-cells of the respective 2-categories. Let $f : A \to B$ and $f' : A' \to B$ be T-algebras, and let a functor $g : A \to A'$ be a morphism T-algebras between them. Then Tg maps an object $(\phi : b \to fa, a) \in B/f$ to $(\phi : b \to f'ga, ga)$, and a morphism (β, α) to $(\beta, g\alpha)$. Then the algebra morphism axiom for g applied to the generic cartesian morphism $(\phi, 1_a) \in B/f$ $(\phi : b \to fa, 1_{fa} : fa \to fa)$ says precisely that g preserves these and is hence itself a morphism of split fibrations.

For the converse, suppose g is a morphism of split fibrations. To establish the axiom for a morphism of split fibrations, we need to show that $t'(\beta, g\alpha) = gt(\beta, \alpha)$. Consider the commutative square in B/f shown below on the left, where $\phi : b \to fa$, $\phi' : b' \to fa$ and $\beta : b \to b'$ are morphisms in B, and $\alpha : a \to a'$ is a morphism in A. This is equivalently given by the commutative cube in B shown in the middle diagram below, and the commutative square in B shown on the right.

$$\begin{tikzcd} (\phi, a) \arrow[r, "(\beta,\alpha)"] \arrow[d, "(\phi,1_{fa})"'] & (\phi', a') \arrow[d, "(\phi',1_{fa'})"] \\ (1_{fa}, a) \arrow[r, "(f\alpha,\alpha)"'] & (1_{fa'}, a') \end{tikzcd} \qquad \begin{tikzcd} b \arrow[r, "\phi"] \arrow[d, "\beta"'] & fa \arrow[d, "f\alpha"] \\ b' \arrow[r, "\phi'"'] \arrow[d, "\phi"'] & fa' \arrow[d, "1_{fa'}"] \\ fa' \arrow[r, "1_{fa'}"'] & fa' \end{tikzcd} = \begin{tikzcd} b \arrow[r, "\phi"] \arrow[d, "\phi"'] & fa \arrow[d, "f1_a"] \\ fa \arrow[r, "1_{fa}"'] \arrow[d, "f\alpha"'] & fa \arrow[d, "\alpha"] \\ fa' \arrow[r, "1_{fa'}"'] & fa' \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "1_a"] \arrow[d, "\alpha"'] & a \arrow[d, "\alpha"] \\ a' \arrow[r, "1_{a'}"'] & a' \end{tikzcd}$$

This square is sent by t to the following square in A, where we used $\alpha = (f\alpha, \alpha)$ by the unit axiom for t.

$$\begin{tikzcd} (\phi, a) \arrow[r, "\overline{(\phi,1_{fa})}"] \arrow[d, "\overline{(\beta,\alpha)}"'] & (1_{fa}, a) \arrow[d, "\alpha"] \\ (\phi', a') \arrow[r, "\overline{(\phi',1_{fa})}"'] & (1_{fa}, a') \end{tikzcd}$$

Note that $(\phi, 1_{fa})$ and $(\phi', 1_{fa'})$ are cartesian with respect to v and v' respectively, with images under both t and t' cartesian with respect to f and f' respectively. So it follows from the universal property of cartesian morphisms that $gt(\beta, \alpha) = t'(\beta, g\alpha)$, as required.

One can easily see that 2-cells of split fibrations and 2-cells of T-algebras coincide, completing the proof. $\square$

In contrast to the inclusion of 2-categories of fully faithful internal functors into the slice categories of $\mathbf{Cat}(\mathcal{E})$, the inclusion of split fibrations into the slice categories in $\mathfrak{K}$ is not fully faithful, even when $\mathfrak{K} = \mathbf{Cat}(\mathcal{E})$. This is because 1-cells in $\mathfrak{K}$ may admit more than one split fibration structure. However, any split fibration structures they admit will be unique up to a unique isomorphism of split fibrations. Such a situation is often called *essential uniqueness*. To see this is the case for split fibrations, we prove that moreover, the split fibrations monad T is *colax idempotent*. This notion has several equivalent definitions which can be found in [19]. We recall one of these here.

Definition 5.3.6. Let $T : \mathfrak{K} \to \mathfrak{K}$ be a 2-monad with multiplication $\mu : T^2 \Rightarrow T$ and unit $\eta : 1_{\mathfrak{K}} \Rightarrow T$. Then (T, μ, η) is called *colax idempotent* if

- Every T-algebra action $x : TX \to X$ is right adjoint to $\eta_X : X \to TX$ in $\mathfrak{K}$,
- This adjunction $\eta_X \dashv x$ has an identity unit.

Note that the desired essential uniqueness of a T-algebra structure is indeed a consequence of T being colax idempotent, since adjoints are essentially unique.

Theorem 5.3.7. *The 2-monad T described in Remark 5.3.4 is colax idempotent, and in particular a 1-cell in $\mathfrak{K}$ admits at most one split fibration structure, up to unique isomorphism of split fibrations.*

Proof. We show colax idempotence of T by showing that $1_{1_A} : 1_A \Rightarrow tr = 1_A$ is the unit of an adjunction $r \dashv t$. Note that since $f : A \to B$ is a split fibration, $\lambda : v \to fu$ induces a cartesian $\alpha : t \Rightarrow u$ satisfying $f\alpha = \lambda$. So by the two dimensional universal property of B/f, there is a unique $\epsilon : tr \Rightarrow 1_{B/f}$ satisfying $v\epsilon = 1_v$ and $u\epsilon = \alpha$. Note also that αr is a chosen lift for $\lambda r = 1_v$, and hence $\alpha r = 1_{1_A}$. For the triangle identities we need to show $t\epsilon = t$, and $\epsilon r = r$. For the first of these, we use cartesianness of α. Indeed $ft\epsilon = v\epsilon = 1_v$, and $\alpha.\eta = \alpha$, so $t\epsilon = t$ since $f\alpha = \lambda$. For the second of these, we see that $v\epsilon r = vr = f$ and $u\epsilon r = \alpha r = 1_{1_A}$, so $\epsilon r = r$ follows by the 2-dimensional universal property of B/f. Then finally, since adjoints are unique up to a unique isomorphism, so must be the algebras for the 2-monad T. $\square$

Remark 5.3.8. Once again, under the reversal of 2-cells in $\mathfrak{K}$, the results of this chapter show that if $\mathfrak{K}$ has lax limits of 1-cells then any $f : A \to B$ factorises as $A \xrightarrow{l} f/B \xrightarrow{w} B$ where l is a left adjoint and w is a split opfibration, and that the forgetful functor $\mathbf{SoFib}(B) \to \mathfrak{K}/B$ is monadic, and furthermore *lax idempotent* in the sense of [19], which is the 2-cell dual of Definition 5.3.6.

5.4 Comprehensive Factorisation

Having seen how the 1-cells of a 2-category factorise into a right adjoint followed by a split fibration in the previous section, we now look at the next step of the factorisation. In $\mathbf{Cat}(\mathcal{E})$, this will factorise the split fibration into a coidentifier of the universal 2-cell of a power by $\mathbf{2}$, followed by a discrete fibration.

The first result of this section describes powers by $\mathbf{2}$ in the slice category of a 2-category with powers by $\mathbf{2}$ and pullbacks.

Proposition 5.4.1. *Let $\epsilon : e_0 \Rightarrow e_1 : E^{\mathbf{2}} \to E$ and $\beta : b_0 \to b_1 : B^{\mathbf{2}} \to B$ be universal 2-cells of the respective powers by $\mathbf{2}$ in $\mathfrak{K}$, let $p : E \to B$ be a 1-cell and let X denote the pullback in $\mathfrak{K}$ of Δ_B and $p^{\mathbf{2}}$, with projections $q : X \to B$ and $g : X \to E^{\mathbf{2}}$. Then $\epsilon g : e_0 g \Rightarrow e_1 g : X \to E$ is the universal 2-cell exhibiting $q : X \to B$ as the power by $\mathbf{2}$ of $(E, p) \in \mathfrak{K}/B$.*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & E^{\mathbf{2}} \arrow[r, bend left, "e_0"] \arrow[r, bend right, "e_1"'] \arrow[r, phantom, "\Downarrow\epsilon"] \arrow[d, "p^{\mathbf{2}}"] & E \arrow[d, "p"] \\ B \arrow[r, "\Delta_B"'] & B^{\mathbf{2}} \arrow[r, bend left, "b_0"] \arrow[r, bend right, "b_1"'] \arrow[r, phantom, "\Downarrow\beta"] & B \end{tikzcd}$$

Proof. That ϵg is well defined as a 2-cell in $\mathfrak{K}/B$ is clear since $p\epsilon g = \beta p^{\mathbf{2}} g = \beta \Delta_B q = q$. Now suppose $\gamma : h \Rightarrow k : Y \to E$ satisfies $p\gamma = s$ for some 1-cell $s : Y \to B$. To establish the universal property of ϵg as the universal 2-cell of a power by $\mathbf{2}$ in $\mathfrak{K}/B$ we must show that there is a unique 1-cell $m : Y \to X$ in $\mathfrak{K}$ such that $qm = s$ and $\gamma = \epsilon g.m$. But by the universal property of $E^{\mathbf{2}}$, γ induces a 1-cell $u : Y \to E^{\mathbf{2}}$ which is unique so that $\epsilon u = \gamma$. Now since $\beta p^{\mathbf{2}} u = p\epsilon u = p\gamma = s = \beta \Delta_B s$, $p^{\mathbf{2}} u = \Delta_B s$ by the universal property of $B^{\mathbf{2}}$. Then by the universal property of X as a pullback, there is an induced 1-cell $m : Y \to X$ which is unique so that $qm = s$ and $gm = u$, so in particular $\epsilon gm = \epsilon u = \gamma$. For uniqueness, suppose $m' : Y \to X$ satisfies $\epsilon g.m' = \gamma$ and $qm' = s$. Then $gm' = u$ by definition of u, and hence $m' = m$ by definition of m. $\square$

Since we know that $\mathbf{SFib}(\mathbb{B})$ is equivalent to $\mathbf{Cat}\big(\mathbf{DFib}(\mathbb{B})\big)$, and Proposition 1.5.6 characterised coidentifiers of universal 2-cells of powers by $\mathbf{2}$, it will suffice to know that $\mathbf{DFib}(\mathbb{B})$ has reflexive coequalisers. This is the subject of the next two propositions.

Proposition 5.4.2. *For $g \in \mathcal{E}(X,Y)$, let $g^* : \mathcal{E}/Y \to \mathcal{E}/X$ denote the pullback functor and let $\Sigma_g : \mathcal{E}/X \to \mathcal{E}/Y$ denote its left adjoint, which composes a morphism into X with g. Let* $B_2 \xrightarrow{m} B_1 \overset{\xrightarrow{d_0}}{\underset{\xrightarrow[d_1]{}}{\xleftarrow{\ i\ }}} B_0$ *be a category internal to $\mathcal{E}$. Then the composite functor* $\mathcal{E}/B_0 \xrightarrow{d_0*} \mathcal{E}/B_1 \xrightarrow{\Sigma_{d_1}} \mathcal{E}/B_0$ *is a monad T on $\mathcal{E}/B_0$.*

Proof. Let X' be the domain of the pullback of f along d_1. The unit is given on component $f : X \to B_0$ by $(1_X, if) : X \to X'$. Letting X'' denote the further pullback of Tf along d_1, the components for the monad's multiplication are similarly induced by the universal property of B_1 given the pullback projections and the composition map of $\mathbb{B}$. Then naturality of η and μ follow from the universal property of X', while the associativity and unit laws for the monad follow from those of the internal category $\mathbb{B}$. □

Note that by interchanging the roles of d_0 and d_1, we obtain an analogous result to the above for discrete opfibrations. In fact, the category of algebras for this monad is $\mathbf{DFib}(\mathbb{B})$, as shown for example in Proposition 2.2.1 of [14].

Proposition 5.4.3. *Let $\mathcal{E}$ have finite limits and pullback stable reflexive coequalisers. Then $\mathbf{DFib}(\mathbb{B})$ and $\mathbf{DoFib}(\mathbb{B})$ have finite limits and reflexive coequalisers.*

Proof. Monadic functors create limits, and also create any colimits preserved by their corresponding monad. As a left adjoint, Σ_{d_0} preserves all limits, and if coequalisers are stable under pullback in $\mathcal{E}$ then so will d_1^*. Thus since $\mathbf{DFib}(\mathbb{B})$ is the category of algebras as described, it follows that $\mathbf{DFib}(\mathbb{B})$ has finite limits and reflexive coequalisers. The case for $\mathbf{DoFib}(\mathbb{B})$ follows by duality. □

Theorem 5.4.4. *Let $\mathcal{E}$ a category with finite limits and pullback stable reflexive coequalisers. Then*

1. *The functor $\Pi_0' : \mathbf{SFib}(\mathbb{B}) \to \mathbf{DFib}(\mathbb{B})$ of Remark 1.2.9 sends the split fibration $v : \mathbb{B}/f \to B$ of Proposition 5.3.2 to a discrete fibration $p : \mathbb{C} \to \mathbb{B}$, through which it factors via the coidentifier of Remark 1.5.6.*

2. *$\mathbf{Cat}(\mathcal{E})$ has an orthogonal factorisation system whose left class consists of the final internal functors and right class consists of the discrete fibrations.*

3. *$\mathbf{Cat}(\mathcal{E})$ has an orthogonal factorisation system whose left class consists of the initial internal functors and right class consists of the discrete opfibrations.*

Proof. Taking $\mathfrak{K} = \mathbf{Cat}(\mathcal{E})$ in Proposition 5.3.2, we factorised $f : \mathbb{A} \to \mathbb{B}$ into a right adjoint $r : \mathbb{A} \to \mathbb{B}/f$ followed by a split fibration $v : \mathbb{B}/f \to B$. Since $\mathbf{SFib}(\mathbb{B})$ is monadic over $\mathbf{Cat}(\mathcal{E})/\mathbb{B}$, it inherits limits from the slice 2-category. In particular, the universal 2-cell of the power by $\mathbf{2}$ of $v : B/f \to B$ will be computed in $\mathbf{SFib}(\mathbb{B})$ just as described by Proposition 5.4.1. Now view $\mathbf{SFib}(\mathbb{B})$ as $\mathbf{Cat}(\mathbf{DFib}(\mathbb{B}))$ by part 4 of Theorem 4.3.3. By Proposition 1.5.6, the coidentifier $n : B/f \to P$ of this 2-cell will be the coequaliser of the reflexive pair corresponding to the domain and codomain maps of $v \in \mathbf{Cat}(\mathbf{DFib}(\mathbb{B}))$, which are themselves discrete fibrations. It follows that v factorises as pn where $p : P \to B$ is the discrete fibration induced by the universal property of n as a coidentifier. This proves part 1.

For part 2, note that what we have just described is a factorisation of an arbitrary internal functor into a final internal functor followed by a discrete fibration: by Proposition 5.2.11 and Corollary 5.2.14, both right adjoints and coidentifiers are final, and by Proposition 5.2.10, the composite of r and n is also final. But by Propositions 5.2.16 and 5.2.15, both of these are left orthogonal to discrete fibrations, and since we have also shown that any final internal functor factorises as a right adjoint followed by a coidentifier, we have proven the desired factorisation is orthogonal. Part 3 is an analogous conclusion to the same construction with 2-cells reversed. □

6 Concluding Remarks

In Chapter One we gave an introduction to the theory of internal categories, functors, and natural transformations. We proved that they form a 2-category with finite weighted limits, and gave examples of various familiar notions internal to a category with finite limits. These included several functors between $\mathbf{Cat}(\mathcal{E})_1$ and $\mathcal{E}$ which featured in various places along the course of this thesis. We also introduced notions of multiple categories, and a construction of a category internal to a 2-category with powers by finite categories. These featured later in Chapter Three.

In Chapter Two, we took a closer look at some other properties familiar in **Cat**, and gave an account of certain assumptions under which they generalised to $\mathbf{Cat}(\mathcal{E})$ for other finite limit categories $\mathcal{E}$. In particular, we saw that $\mathbf{Cat}(\mathcal{E})$ inherits cartesian closedness, the existence of a natural numbers object, and extensivity from $\mathcal{E}$. Meanwhile, we noted that coequalisers are *not* in general inherited by $\mathbf{Cat}(\mathcal{E})$, but they are under assumptions like the existence of list objects or local presentability of $\mathcal{E}$. We remarked that pullback functors in $\mathbf{Cat}(\mathcal{E})$ need not have right adjoints, and indeed only some do even when $\mathcal{E} = \mathbf{Set}$. We saw that copowers by $\mathbf{2}$ will exist when $\mathcal{E}$ is lextensive, and in this case $\mathbf{Cat}(\mathcal{E})_1$ inherits generating families from $\mathcal{E}$ via **Disc**. Finally, we saw that if $\mathcal{E}$ has a subobject classifier $\top : 1 \to \Omega$ then $\mathbf{Cat}(\mathcal{E})$ has a classifying full subobject, given by $\mathbf{coDisc}(\top)$.

In Chapter Three we looked at how the assignment $\mathcal{E} \mapsto \mathbf{Cat}(\mathcal{E})$ extends to various 2-functors, and saw how the various functors and natural transformations between $\mathbf{Cat}(\mathcal{E})$ and $\mathcal{E}$ varied in $\mathcal{E}$. We saw in particular that the data giving an internal category, which we collectively referred to as the evaluation category, itself forms a category internal to $[\mathbf{Lex}, \mathbf{Lex}](\mathbf{Cat}(-), 1_{\mathbf{Lex}})$, despite this category not having all pullbacks. We saw three distinct monad like structures on the 2-functor $\mathbf{Cat}(-) : \mathbf{Lex} \to \mathbf{Lex}$. Two of these were lax and colax pseudomonads structures, which arose from a very general setting, given the adjoints to **Ob**. The other was a pseudocomonad structure involving the double category of squares as the comultiplication. This arose from a biadjunction in which **Ob** also featured as the counit. We also described the pseudocoalgebras for this pseudocomonad as 2-categories with finite weighted limits. We also saw that transposition of double categories forms a compatible flip in the sense of [27].

In Chapter Four we looked at the Grothendieck Construction, from which for our purposes the main result was that the 2-category of split fibrations is equivalent to the 2-category of categories in discrete fibrations. This formed one of the main ingredients in the comprehensive factorisation in Chapter Five, where we factorised an internal functor into a right adjoint followed by a split fibration, and then further factorised the split fibration into a coidentifier followed by a discrete fibration. We also remarked upon the dual construction, factorising an internal functor into a left adjoint, followed by a coidentifier followed by a discrete opfibration.

Future work from this thesis could include investigating the lax and colax algebras for the skew and coskew pseudomonad structures on $\mathbf{Cat}(-)_1 : \mathbf{Lex} \to \mathbf{Lex}$, as well as looking at the lift of the pseudocomonad $\mathbf{Cat}(-)_1$ to its 2-categories of strict or pseudocoalgebras, using the compatible flip provided by transposition which is in particular a strict distributive law.

7 Appendix

Diagram 7.0.1.

$$
\begin{tikzcd}
C_3 \arrow[r, "\pi_{3,0}"] \arrow[d, "\pi_{3,1}"'] & C_2 \arrow[r, "m"] \arrow[d, "\pi_1"] & C_1 \arrow[dd, "d_1"] \\
C_2 \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] & C_1 \arrow[dr, "d_1"] & \\
C_1 \arrow[rr, "d_0"'] & & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_3 \arrow[r, "\pi_{3,0}"] \arrow[d, "\pi_{3,1}"'] & C_2 \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] & C_1 \arrow[dd, "d_1"] \\
C_2 \arrow[r, "\pi_0"] \arrow[d, "m"'] & C_1 \arrow[dr, "d_0"] & \\
C_1 \arrow[rr, "d_0"'] & & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2 \arrow[r, "\pi_0"] \arrow[d, "\pi_1"'] & A_1 \arrow[r, "f_1"] \arrow[d, "d_1"] & B_1 \arrow[dd, "d_1"] \\
A_1 \arrow[r, "d_0"] \arrow[d, "f_1"'] & A_0 \arrow[dr, "f_0"] & \\
B_1 \arrow[rr, "d_0"'] & & B_0
\end{tikzcd}
$$

Diagram 7.0.2.

j \ k	*0*	*1*
0	$m\,(x,y)$	x
1	z	$m\,(y,z)$

Diagram 7.0.3.

$$
\begin{tikzcd}
A_1 \arrow[r, "f_1"] \arrow[d, "{(d_0,d_1)}"'] \arrow[dr, phantom, "\lrcorner", very near start] & B_1 \arrow[d, "{(d_0,d_1)}"] \\
A_0 \times A_0 \arrow[r, "f_0 \times f_0"'] & B_0 \times B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "{(d_0,d_1)}"] \arrow[dr, "d_j"] \arrow[dd, "f_1"'] & A_0 \times A_0 \arrow[r, "f_0 \times f_0"] \arrow[d, "p_j"] & B_0 \times B_0 \arrow[dd, "p_j"] \\
 & A_0 \arrow[dr, "f_0"] & \\
B_1 \arrow[r, "{(d_0,d_1)}"'] \arrow[rr, bend left, "d_j"] & B_0 \times B_0 \arrow[r, "p_j"'] & B_0
\end{tikzcd}
$$

Diagram 7.0.4.

$$
\begin{tikzcd}
B_2 \arrow[rr, "m"] \arrow[dr, "\pi_0"] & & B_1 \arrow[dd, "d_0"] \\
 & B_1 \arrow[dr, "d_0"] & \\
A_0 \arrow[uu, "{(\alpha,\beta)}"] \arrow[ur, "\alpha"] \arrow[rr, "f_0"'] & & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
B_2 \arrow[rr, "m"] \arrow[dr, "\pi_1"] & & B_1 \arrow[dd, "d_1"] \\
 & B_1 \arrow[dr, "d_1"] & \\
A_0 \arrow[uu, "{(\alpha,\beta)}"] \arrow[ur, "\beta"] \arrow[rr, "h_0"'] & & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & & A_1 \arrow[dl, "{(f_1,\alpha d_1,\beta d_1)}"'] \arrow[d, "{(\alpha d_0,g_1,\beta d_1)}"] \arrow[dr, "{(\alpha d_0,\beta d_0,h_1)}"] \arrow[ddlll, bend right, "(\beta\circ\alpha)_1"'] \arrow[ddrrr, bend left, "(\beta\circ\alpha)_0"] & & & \\
 & & B_3 \arrow[dll, "m_1"'] \arrow[dr, "m_0"] & B_3 \arrow[dl, "m_0"'] \arrow[dr, "m_1"] & B_3 \arrow[dl, "m_1"'] \arrow[drr, "m_0"] & & \\
B_2 \arrow[drrr, bend right, "m"'] & & B_2 \arrow[dr, "m"] & & B_2 \arrow[dl, "m"'] & & B_2 \arrow[dlll, bend left, "m"] \\
 & & & B_1 & & &
\end{tikzcd}
$$

Diagram 7.0.5.

$$
\begin{tikzcd}
A_1 \arrow[r, "{(f_1,\alpha d_1,\beta d_1)}"] \arrow[d, "{(f_1,m(\alpha,\beta)d_1)}"'] \arrow[drr, "f_1"] & B_3 \arrow[r, "m_1"] & B_2 \arrow[d, "\pi_0"] \\
B_2 \arrow[rr, "\pi_0"'] & & B_1
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "{(f_1,\alpha d_1,\beta d_1)}"] \arrow[d, "{(f_1,m(\alpha,\beta)d_1)}"'] \arrow[drr, "{m(\alpha,\beta)d_1}"] & B_3 \arrow[r, "m_1"] & B_2 \arrow[d, "\pi_1"] \\
B_2 \arrow[rr, "\pi_1"'] & & B_1
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_1 \arrow[rr, "(\alpha d_0,g_1,\beta d_1)"] \arrow[dr, "(f_1,\alpha d_1)"'] \arrow[drrr, "(\alpha d_0,g_1)"] \arrow[dd, "(f_1,\alpha d_1,\beta d_1)"'] & & B_3 \arrow[rr, "m_0"] & & B_2 \arrow[dd, "\pi_0"] \\
& B_2 \arrow[drrr, "m"] & & B_2 \arrow[dr, "m"] & \\
B_3 \arrow[rr, "m_0"'] & & B_2 \arrow[rr, "\pi_0"'] & & B_1
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "(\alpha d_0,g_1,\beta d_1)"] \arrow[dr, "d_1"] \arrow[dd, "(f_1,\alpha d_1,\beta d_1)"'] & B_3 \arrow[r, "m_0"] & B_2 \arrow[dd, "\pi_1"] \\
& A_0 \arrow[dr, "\beta"] & \\
B_3 \arrow[r, "m_0"'] & B_2 \arrow[r, "\pi_1"'] & B_1
\end{tikzcd}
$$

Diagram 7.0.6.

$$
\begin{tikzcd}
A_1 \arrow[rr, "(\beta f)_1"] \arrow[dr, "f_1"] \arrow[dd, "(\beta f)_0"'] & & C_2 \arrow[dd, "m"] \\
& B_1 \arrow[ur, "\beta_1"'] \arrow[dl, "\beta_0"'] & \\
C_2 \arrow[rr, "m"'] & & C_1
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[rrr, "(g\alpha)_0"] \arrow[drr, "\alpha_0"] \arrow[dr, "\alpha_1"'] \arrow[ddd, "(g\alpha)_1"'] & & & C_2 \arrow[ddd, "m"] \\
& & B_2 \arrow[ur, "g_2"] \arrow[d, "m"] & \\
& B_2 \arrow[r, "m"'] \arrow[dl, "g_2"'] & B_1 \arrow[dr, "g_1"] & \\
C_2 \arrow[rrr, "m"'] & & & C_1
\end{tikzcd}
$$

Diagram 7.0.7.

$$
\begin{tikzcd}
A_0 \arrow[rr, "(\beta f_0, g_1'\alpha)"] \arrow[dr, "\alpha"] \arrow[dd, "(g_1\alpha, \beta f_0')"'] & & C_2 \arrow[dd, "m"] \\
& B_1 \arrow[ur, "\beta_0"'] \arrow[dl, "\beta_1"'] & \\
C_2 \arrow[rr, "m"'] & & C_1
\end{tikzcd}
\quad
\begin{tikzcd}
A_0 \arrow[rr, "(\beta f_0, {g_1}'\alpha)"] \arrow[d, "\alpha"'] \arrow[dr, "f_0"] & & C_2 \arrow[dd, "\pi_0"] \\
B_1 \arrow[r, "d_0"'] \arrow[d, "\beta_0"'] & B_0 \arrow[dr, "\beta"] & \\
C_2 \arrow[rr, "\pi_0"'] & & C_1
\end{tikzcd}
\quad
\begin{tikzcd}
A_0 \arrow[rr, "(\beta f_0, {g_1}'\alpha)"] \arrow[d, "\alpha"'] & & C_2 \arrow[dd, "\pi_1"] \\
B_1 \arrow[drr, "{g_1}'"] \arrow[d, "\beta_0"'] & & \\
C_2 \arrow[rr, "\pi_1"'] & & C_1
\end{tikzcd}
\quad
\begin{tikzcd}
A_0 \arrow[rr, "(g_1\alpha, \beta {f_0}')"] \arrow[d, "\alpha"'] & & C_2 \arrow[dd, "\pi_0"] \\
B_1 \arrow[drr, "{g_1}'"] \arrow[d, "\beta_1"'] & & \\
C_2 \arrow[rr, "\pi_0"'] & & C_1
\end{tikzcd}
\quad
\begin{tikzcd}
A_0 \arrow[rr, "(g_1\alpha, \beta {f_0}')"] \arrow[d, "\alpha"'] \arrow[dr, "{f_0}'"] & & C_2 \arrow[dd, "\pi_0"] \\
B_1 \arrow[r, "d_1"'] \arrow[d, "\beta_1"'] & B_0 \arrow[dr, "\beta"] & \\
C_2 \arrow[rr, "\pi_1"'] & & C_1
\end{tikzcd}
$$

Diagram 7.0.8.

$j \backslash k$	*0*	*1*
0	$f_{0,0,0}$	$m\left(f_{0,1,0}, f_{1,1,0}\right)$
1	$m\left(f_{0,0,1}, f_{1,0,1}\right)$	$f_{1,1,1}$

$j \backslash k$	*0*	*1*
0	ϕ^0	α^1
1	α^0	ϕ^1

Diagram 7.0.9.

$$
\begin{tikzcd}
B_0 \arrow[r, "\phi"] \arrow[dr, "f_0^k"] \arrow[d, "i"'] & A_1 \arrow[r, "(i_0,i_1)"] \arrow[d, "d_k"] & \mathbb{A}_{Sq} \arrow[d, "p_{1-k}"] \\
B_1 \arrow[drr, "f_1^k"] \arrow[d, "(\phi_0,\phi_1)"'] & A_0 \arrow[dr, "i"] & A_2 \arrow[d, "\pi_k"] \\
\mathbb{A}_{Sq} \arrow[r, "p_{1-k}"'] & A_2 \arrow[r, "\pi_x"'] & A_1
\end{tikzcd}
\qquad
\begin{tikzcd}
B_0 \arrow[r, "\phi"] \arrow[dr, equal, "1_{B_0}"] \arrow[d, "i"'] & A_1 \arrow[r, "(i_0,i_1)"] \arrow[ddr, equal, "1_{A_1}"] & \mathbb{A}_{Sq} \arrow[d, "p_j"'] \\
B_1 \arrow[r, "d_j"'] \arrow[d, "(\phi_0,\phi_1)"'] & B_0 \arrow[dr, "\phi"'] & A_2 \arrow[d, "\pi_j"] \\
\mathbb{A}_{Sq} \arrow[r, "p_j"'] & A_2 \arrow[r, "\pi_j"'] & A_1
\end{tikzcd}
$$

Diagram 7.0.10.

$$
\begin{tikzcd}
B_2 \arrow[rr, "(F\phi)_2"] \arrow[dr, "\pi_j"] \arrow[dd, "m"'] & & \left(\mathbb{A}^{\mathbf{2}}\right)_2 \arrow[rr, "m^{\mathbf{2}}"] & & \mathbb{A}_{Sq} \arrow[dl, "\pi_j^{\mathbf{2}}"'] \arrow[dd, "p_j"] \\
& B_1 \arrow[rr, "(\phi_0,\phi_1)"] \arrow[dd, "d_j"] & & \mathbb{A}_{Sq} \arrow[dd, "p_j"'] & \\
B_1 \arrow[dr, "d_j"] \arrow[dd, "(\phi_0,\phi_1)"'] & & & & A_2 \arrow[dd, "\pi_j"] \\
& B_0 \arrow[drrr, "\phi"] & & A_2 \arrow[dr, "\pi_j"] & \\
\mathbb{A}_{Sq} \arrow[rr, "p_j"'] & & A_2 \arrow[rr, "\pi_j"'] & & A_1
\end{tikzcd}
\qquad
\begin{tikzcd}
B_2 \arrow[r, "(F\phi)_2"] \arrow[d, "m"'] \arrow[dr, "f_2^k"] & \left(\mathbb{A}^{\mathbf{2}}\right)_2 \arrow[r, "m^{\mathbf{2}}"] \arrow[d, "r_{1-k}"] & \mathbb{A}_{Sq} \arrow[d, "p_{1-k}"'] \\
B_1 \arrow[d, "(\phi_0,\phi_1)"'] \arrow[drr, "f_1^k"'] & A_2 \arrow[dr, "m"] & A_2 \arrow[d, "\pi_k"'] \\
\mathbb{A}_{Sq} \arrow[r, "p_{1-k}"'] & A_2 \arrow[r, "\pi_k"'] & A_1
\end{tikzcd}
\qquad
\begin{tikzcd}
B_2 \arrow[r, "(F\phi)_2"] \arrow[d, "\pi_l"] \arrow[dd, bend right, "f_2^k"'] & \left(\mathbb{A}^{\mathbf{2}}\right)_2 \arrow[r, "r_{1-k}"] \arrow[d, "\pi_l^{\mathbf{2}}"] & A_2 \arrow[dd, bend left, "\pi_l"] \\
B_1 \arrow[r, "(F\phi)_1"] \arrow[drr, "f_1^k"] & \mathbb{A}_{Sq} \arrow[r, "p_{1-k}"] & A_2 \arrow[d, "\pi_k"'] \\
A_2 \arrow[rr, "\pi_l"'] & & A_1
\end{tikzcd}
$$

Diagram 7.0.11.

$$
\begin{tikzcd}
B_1 \arrow[rr, "(F\Gamma)_j"] \arrow[drr, "d_j"] \arrow[dd, "(F\Gamma)_{1-j}"'] \arrow[dddr, "\left(F\phi^j\right)_1"] & & \left(\mathbb{A}^{\mathbf{2}}\right)_2 \arrow[rr, "m^{\mathbf{2}}"] \arrow[dr, "\pi_j^{\mathbf{2}}"] & & \mathbb{A}_{Sq} \arrow[dd, "p_j"] \\
& & B_0 \arrow[r, "F\Gamma"] \arrow[dddrr, bend right, "\phi^j"] & \mathbb{A}_{Sq} \arrow[d, "p_j"] & \\
\left(\mathbb{A}^{\mathbf{2}}\right)_2 \arrow[dr, "\pi_j^{\mathbf{2}}"'] \arrow[dd, "m^{\mathbf{2}}"'] & & & A_2 \arrow[ddr, "\pi_j"] & A_2 \arrow[dd, "\pi_j"] \\
& \mathbb{A}_{Sq} \arrow[r, "p_j"] & A_2 \arrow[drr, "\pi_j"] & & \\
\mathbb{A}_{Sq} \arrow[rr, "p_j"'] & & A_2 \arrow[rr, "\pi_j"'] & & A_1
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r, "(F\Gamma)_0"] \arrow[d, "(F\Gamma)_1"'] \arrow[drr, "\alpha_0^j"] \arrow[ddr, "\alpha_1^j"] & \left(\mathbb{A}^{\mathbf{2}}\right)_2 \arrow[rr, "m^{\mathbf{2}}"] \arrow[dr, "r_{1-j}"] & & \mathbb{A}_{Sq} \arrow[d, "p_{1-j}"] \\
\left(\mathbb{A}^{\mathbf{2}}\right)_2 \arrow[dr, "r_{1-j}"'] \arrow[dd, "m"'] & & A_2 \arrow[ddr, "m"] & A_2 \arrow[dd, "\pi_{1-j}"] \\
& A_2 \arrow[drr, "m^{\mathbf{2}}"] & & \\
\mathbb{A}_{Sq} \arrow[r, "p_{1-j}"'] & A_2 \arrow[rr, "\pi_j"'] & & A_1
\end{tikzcd}
$$

$$
\begin{tikzcd}
B_1 \arrow[r, "(F\Gamma)_0"] \arrow[d, "d_0"] \arrow[dd, bend right, "\alpha_0^0"'] & \left(\mathbb{A}^{\mathbf{2}}\right)_2 \arrow[r, "r_1"] \arrow[d, "\pi_0^{\mathbf{2}}"] & A_2 \arrow[dd, bend left, "\pi_0"] \\
B_0 \arrow[r, "F\Gamma"] \arrow[drr, "\alpha^0"] & \mathbb{A}_{Sq} \arrow[r, "p_0"] & A_2 \arrow[d, "\pi_1"'] \\
A_2 \arrow[rr, "\pi_0"'] & & A_1
\end{tikzcd}
\qquad
\begin{tikzcd}
& \left(\mathbb{A}^{\mathbf{2}}\right)_2 \arrow[r, "r_1"] \arrow[d, "\pi_0^{\mathbf{2}}"] & A_2 \arrow[dd, bend left, "\pi_1"] \\
B_1 \arrow[ur, bend left, "(F\Gamma)_0"] \arrow[r, "F\left(\phi^1\right)"] \arrow[d, "\alpha_1^0"] \arrow[drr, "f^0"] & \mathbb{A}_{Sq} \arrow[r, "p_1"] & A_2 \arrow[d, "\pi_1"'] \\
A_2 \arrow[rr, "\pi_1"'] & & A_1
\end{tikzcd}
$$

Diagram 7.0.12.

$$
\begin{tikzcd}
\Pi_0\left(A\right) \arrow[rr, "(\gamma, id_1\gamma)"] \arrow[dr, "\gamma"] \arrow[dd, "(id_1\gamma,\gamma)"'] & & B_2 \arrow[dd, "m"] \\
& B_1 \arrow[ur, "\left(1_{B_1}, id_1\right)"] \arrow[dl, "\left(id_0, 1_{B_1}\right)"'] \arrow[dr, equal, "1_{B_1}"] & \\
B_2 \arrow[rr, "m"'] & & B_1
\end{tikzcd}
$$

Diagram 7.0.13.

$$
\begin{tikzcd}
& A_0 \arrow[rr, "p_j"] \arrow[dl, "p_j"'] \arrow[dd, "f_0^j"'] \arrow[dr, "i"'] & & 2A_0 \arrow[dr, "i"] & \\
2A_0 \arrow[dr, "(F\alpha)_0"'] & & A_1 \arrow[rr, "p_j"] \arrow[dr, "f_1^j"'] & & 3A_1 \arrow[dl, "(F\alpha)_1"] \\
& B_0 \arrow[rr, "i"'] & & B_1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
& A_1 \arrow[rr, "p_{1+2j}"] \arrow[dl, "p_{1+2j}"'] \arrow[dd, "f_1^j"'] \arrow[dr, "d_0"] & & 3A_1 \arrow[dr, "d_0"] & \\
3A_1 \arrow[dr, "(F\alpha)_1"'] & & A_0 \arrow[rr, "p_{1+j}"] \arrow[dr, "f_0^j"'] & & 2A_0 \arrow[dl, "(F\alpha)_0"] \\
& B_1 \arrow[rr, "d_0"'] & & B_0 &
\end{tikzcd}
$$

Diagram 7.0.14.

$$\begin{tikzcd}
 & A_1 \arrow[rr, "p_2"] \arrow[dl, "p_2"'] \arrow[d, "\alpha_1"'] \arrow[dr, "f_1"] \arrow[drr, "d_0"] & & 3A_1 \arrow[dr, "d_0"] & \\
3A_1 \arrow[dr, "(F\alpha)_1"'] & B_2 \arrow[r, "\pi_0"] \arrow[d, "m"'] & B_1 \arrow[drr, "d_0"'] & A_0 \arrow[r, "p_1"] \arrow[d, "f_0"] & 2A_0 \arrow[dl, "(F\alpha)_0"] \\
 & B_1 \arrow[rr, "d_0"'] & & B_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
4A_2 \arrow[d, "(F\alpha)_2"'] & A_2 \arrow[l, "p_{1+3j}"'] \arrow[r, "p_{1+3j}"] \arrow[dl, "f_2^j"] \arrow[d, "m"] & 4A_2 \arrow[d, "m"] \\
B_2 \arrow[dr, "m"'] & A_1 \arrow[r, "p_{1+2j}"] \arrow[d, "f_1^j"] & 3A_1 \arrow[dl, "(F\alpha)_1"] \\
 & B_1 &
\end{tikzcd}$$

Diagram 7.0.15.

$$\begin{tikzcd}
 & & 4A_2 \arrow[drr, "m"] & & \\
A_2 \arrow[urr, "p_2"] \arrow[rr, "m"] \arrow[dr, "\begin{pmatrix} f_1^0\pi_0 \\ f_1^0\pi_1 \\ \alpha d_1\pi_1 \end{pmatrix}"] \arrow[drrr, "\begin{pmatrix} mf_2^0 \\ \alpha d_1\pi_1 \end{pmatrix}"] \arrow[ddrr, "\begin{pmatrix} f_1^0\pi_0 \\ \widehat{\alpha}\pi_1 \end{pmatrix}"] \arrow[dd, "p_2"'] & & A_1 \arrow[rr, "p_2"] \arrow[dr, "\alpha_1"] & & 3A_1 \arrow[dd, "(F\alpha)_1"] \\
 & B_3 \arrow[rr, "m_0"'] \arrow[dr, "m_1"] & & B_2 \arrow[dr, "m"] & \\
4A_2 \arrow[rr, "(F\alpha)_2"'] & & B_2 \arrow[rr, "m"'] & & B_1
\end{tikzcd}$$

$$\begin{tikzcd}
 & & 4A_2 \arrow[drr, "m"] & & \\
A_2 \arrow[urr, "p_3"] \arrow[rr, "m"] \arrow[dr, "\begin{pmatrix} \alpha d_0\pi_0 \\ f_1^1\pi_0 \\ f_1^1\pi_1 \end{pmatrix}"] \arrow[drrr, "\begin{pmatrix} \alpha d_0\pi_0 \\ mf_2^1 \end{pmatrix}"] \arrow[ddrr, "\begin{pmatrix} \widehat{\alpha}\pi_0 \\ f_1^1\pi_1 \end{pmatrix}"] \arrow[dd, "p_3"'] & & A_1 \arrow[rr, "p_2"] \arrow[dr, "\alpha_0"] & & 3A_1 \arrow[dd, "(F\alpha)_1"] \\
 & B_3 \arrow[rr, "m_1"'] \arrow[dr, "m_0"] & & B_2 \arrow[dr, "m"] & \\
4A_2 \arrow[rr, "(F\alpha)_2"'] & & B_2 \arrow[rr, "m"'] & & B_1
\end{tikzcd}$$

Diagram 7.0.16.

$$\begin{tikzcd}
A_2 \arrow[rr, "(f_1\pi_0, f_1\pi_1, \alpha d_1\pi_1)"] \arrow[dr, "\pi_1"] \arrow[dd, "(f_1\pi_0, \widehat{\alpha}\pi_1)"'] & & B_3 \arrow[rr, "m_1"] & & B_2 \arrow[dd, "\pi_1"] \\
 & A_1 \arrow[rr, "(f_1, \alpha d_1)"] & & B_2 \arrow[dr, "m"] & \\
B_2 \arrow[rrrr, "\pi_1"'] & & & & B_1
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2 \arrow[r, "(f_1\pi_0, f_1\pi_1, \alpha d_1\pi_1)"] \arrow[dr, "\pi_0"] \arrow[dd, "(f_1\pi_0, \widehat{\alpha}\pi_1)"'] & B_3 \arrow[r, "m_1"] & B_2 \arrow[dd, "\pi_0"] \\
 & A_1 \arrow[dr, "f_1"] & \\
B_2 \arrow[rr, "\pi_0"'] & & B_1
\end{tikzcd}$$

$$\begin{tikzcd}
A_2 \arrow[rr, "(mf_2, \alpha d_1\pi_1)"] \arrow[dr, "f_2"] \arrow[dd, "(f_1\pi_0, f_1\pi_0, \alpha d_1\pi_1)"'] & & B_2 \arrow[dd, "\pi_0"] \\
 & B_2 \arrow[dr, "m"] & \\
B_3 \arrow[r, "m_0"'] & B_2 \arrow[r, "\pi_0"'] & B_1
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2 \arrow[rrrr, "(mf_2, \alpha d_1\pi_1)"] \arrow[dr, "\pi_1"] \arrow[dd, "(f_1\pi_0, f_1\pi_0, \alpha d_1\pi_1)"'] & & & & B_2 \arrow[dd, "\pi_1"] \\
 & A_1 \arrow[rr, "d_1"] & & A_0 \arrow[dr, "\alpha"] & \\
B_3 \arrow[rr, "m_0"'] & & B_2 \arrow[rr, "\pi_1"'] & & B_1
\end{tikzcd}$$

$$
\begin{tikzcd}
A_2 \arrow[rr, "m"] \arrow[dr, "f_2"] \arrow[dd, "{(mf_2,\alpha d_1\pi_1)}"'] & & A_1 \arrow[r, "\alpha_1"] \arrow[ddr, "f_1"] & B_2 \arrow[dd, "\pi_0"] \\
& B_2 \arrow[drr, "m"'] & & \\
B_2 \arrow[rrr, "\pi_0"'] & & & B_1
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2 \arrow[rr, "m"] \arrow[dr, "\pi_1"] \arrow[dd, "{(mf_2,\alpha d_1\pi_1)}"'] & & A_1 \arrow[rr, "\alpha_1"] \arrow[dr, "d_1"] & & B_2 \arrow[dd, "\pi_1"] \\
& A_1 \arrow[rr, "d_1"] & & A_0 \arrow[dr, "\alpha"] & \\
B_2 \arrow[rrrr, "\pi_1"'] & & & & B_1
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_2 \arrow[rr, "p_2"] \arrow[dr, "\pi_0"] \arrow[dd, "{(f_1\pi_0,\widehat{\alpha}\pi_1)}"'] & & 4A_2 \arrow[rr, "(F\alpha)_2"] \arrow[dr, "\pi_0"] & & B_2 \arrow[dd, "\pi_0"] \\
& A_1 \arrow[rr, "p_1"] \arrow[drrr, "f_1"'] & & 3A_1 \arrow[dr, "(F\alpha)_1"] & \\
B_2 \arrow[rrrr, "\pi_0"'] & & & & B_1
\end{tikzcd}
\qquad
\begin{tikzcd}
A_2 \arrow[rr, "p_2"] \arrow[dr, "\pi_1"] \arrow[dd, "{(f_1\pi_0,\widehat{\alpha}\pi_1)}"'] & & 4A_2 \arrow[rr, "(F\alpha)_2"] \arrow[dr, "\pi_1"] & & B_2 \arrow[dd, "\pi_1"] \\
& A_1 \arrow[rr, "p_2"] \arrow[drrr, "\widehat{\alpha}"'] & & 3A_1 \arrow[dr, "(F\alpha)_1"] & \\
B_2 \arrow[rrrr, "\pi_1"'] & & & & B_1
\end{tikzcd}
$$

Diagram 7.0.17.

$$
\begin{tikzcd}
3A_1 \arrow[dd, "(F\Gamma_1)"'] & & A_1 \arrow[ll, "p_2"'] \arrow[rr, "p_2"] \arrow[ddll, bend right, "{\left(\beta^0 d_0,\widehat{\alpha^1}\right)}"'] \arrow[ddrr, bend left, "{\left(\widehat{\alpha^0},\beta^1 d_1\right)}"] \arrow[dl, "{\left(\beta^0 d_0,\widehat{\alpha^1} id_0,g_1^1\right)}"'] \arrow[dr, "{\left(\widehat{\alpha^0} id_0,f_1^1,\beta^1 d_1\right)}"] \arrow[dd, "{\left(\widehat{\alpha^0} id_0,\beta^1 d_0,g_1^1\right)}"] & & 3A_1 \arrow[dd, "(F\Gamma_0)"] \\
& B_3 \arrow[dl, "m_1"] \arrow[dd, "m_0"] & & B_3 \arrow[dr, "m_0"] \arrow[dd, "m_1"'] & \\
B_2 \arrow[ddrr, bend right, "m"'] & & B_3 \arrow[dl, "m_0"] \arrow[dr, "m_1"'] & & B_2 \arrow[ddll, bend left, "m"] \\
& B_2 \arrow[dr, "m"'] & & B_2 \arrow[dl, "m"] & \\
& & B_1 & &
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "p_{1+2j}"] \arrow[d, "p_{1+2j}"'] \arrow[rr, bend right, "\beta_0^j"'] \arrow[dd, bend left, "\beta_1^j"] & 3A_1 \arrow[r, "(F\Gamma_0)"] & B_2 \arrow[dd, "m"] \\
3A_1 \arrow[d, "(F\Gamma_1)"'] & & \\
B_2 \arrow[rr, "m"'] & & B_1
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_2 \arrow[rr, "p_1"] \arrow[dr, "d_0"'] \arrow[dd, "\beta_0^0"'] & & 3A_1 \arrow[rr, "(F\Gamma_0)"] \arrow[dr, "d_0"] & & B_2 \arrow[dd, "\pi_0"] \\
& A_0 \arrow[rr, "p_1"] \arrow[drrr, "\beta_0^0"'] & & 2A_0 \arrow[dr, "F\Gamma"] & \\
B_2 \arrow[rrrr, "\pi_0"'] & & & & B_1
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "p_1"] \arrow[d, "\beta_0^0"'] \arrow[drr, "g_1^0"'] & 3A_1 \arrow[r, "(F\Gamma_0)"] \arrow[dr, "\left(F\alpha^1\right)_1"] & B_2 \arrow[d, "\pi_1"] \\
B_2 \arrow[rr, "\pi_1"'] & & B_1
\end{tikzcd}
$$

Diagram 7.0.18.

$$
\begin{tikzcd}
(FA)_2 \arrow[r, "m_{FA}"] \arrow[d, equal, "F_2^{\mathbb{A}}"'] & (FA)_1 \arrow[d, equal] \\
F\left(A_2\right) \arrow[r, "F(m_A)"'] & F\left(A_1\right)
\end{tikzcd}
\qquad
\begin{tikzcd}
F\left(\alpha_{\mathbb{A}}^{\mathcal{E}}\right) \arrow[r, "{F(p_0,p_1)_{\mathbb{A}}^{\mathcal{E}}}"] \arrow[d, "\alpha_{\mathcal{A}}^F"'] & F\left(A_2\right) \arrow[d, equal] \\
\alpha_{F\mathbb{A}}^{\mathcal{S}} \arrow[r, "{(p_0,p_1)_{F\mathbb{A}}^{\mathcal{S}}}"'] & (FA)_2
\end{tikzcd}
\qquad
\begin{tikzcd}
F\left(\alpha_{\mathbb{A}}^{\mathcal{E}}\right) \arrow[r, "{F(p_0,p_1)_{\mathbb{A}}^{\mathcal{E}}}"] \arrow[d, "\alpha_{\mathbb{A}}^F"'] \arrow[drr, "\left(Fp_j^{\mathcal{E}}\right)_{\mathbb{A}}"] \arrow[dr, phantom, "*"] & F\left(A_2\right) \arrow[r, equal] \arrow[dr, "F\pi_j"] & (FA)_2 \arrow[d, "\pi_j"] \\
\alpha_{F\mathbb{A}}^{\mathcal{S}} \arrow[r, "{(p_0,p_1)_{\mathbb{A}}^{\mathcal{S}}}"'] \arrow[rr, bend left, "\left(p_j^{\mathcal{S}}\right)_{\mathbb{A}}"] & (FA)_2 \arrow[r, "\pi_j"'] & FA_1
\end{tikzcd}
$$

Diagram 7.0.19.

$$\begin{tikzcd}
T^4 \arrow[r, "T^2\mu"] \arrow[d, "\mu T^2"'] \arrow[dr, "T\mu T"'] & T^3 \arrow[dr, "T\mu"] \arrow[d, phantom, "\Downarrow T\tau"] & \\
T^3 \arrow[r, phantom, "\Downarrow \tau T"] \arrow[dr, "\mu T"'] & T^3 \arrow[r, "T\mu"'] \arrow[d, "\mu T"] \arrow[dr, phantom, "\Downarrow \tau"] & T^2 \arrow[d, "\mu"] \\
& T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
=
\begin{tikzcd}
T^4 \arrow[r, "T^2\mu"] \arrow[d, "\mu T^2"'] \arrow[dr, phantom, "\Downarrow \mu_\mu"] & T^3 \arrow[d, "\mu T"'] \arrow[dr, "T\mu"] \arrow[dr, phantom, "\Downarrow \tau"] & \\
T^3 \arrow[r, "T\mu"] \arrow[dr, "\mu T"'] \arrow[dr, phantom, "\Downarrow \tau"] & T^2 \arrow[dr, "\mu"'] & T^2 \arrow[d, "\mu"] \\
& T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}$$

$$\begin{tikzcd}
T^3 \arrow[r, "1"] \arrow[dr, phantom, "\Downarrow \alpha T"] & T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"] \arrow[dr, phantom, "\Downarrow \tau"] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[u, "T\eta T"] \arrow[r, "1"'] & T^2 \arrow[r, "\mu"'] & T
\end{tikzcd}
=
\begin{tikzcd}
T^3 \arrow[r, "1"] \arrow[dr, phantom, "\Downarrow T\beta"] & T^3 \arrow[r, "T\mu"] & T^2 \arrow[d, "\mu"] \\
T^2 \arrow[u, "T\eta T"] \arrow[r, "1"'] & T^2 \arrow[ur, "1"] \arrow[r, "\mu"'] & T
\end{tikzcd}$$

Diagram 7.0.20.

$$\begin{tikzcd}
T^3A \arrow[r, "T^2a"] \arrow[d, "\mu TA"'] \arrow[dr, "T\mu A"'] & T^2A \arrow[dr, "Ta"] \arrow[d, phantom, "\Downarrow T\alpha"] & \\
T^2A \arrow[r, phantom, "\Downarrow \tau A"] \arrow[dr, "\mu A"'] & T^2A \arrow[r, "Ta"'] \arrow[d, "\mu A"] \arrow[dr, phantom, "\Downarrow \alpha"] & TA \arrow[d, "a"] \\
& TA \arrow[r, "a"'] & A
\end{tikzcd}
=
\begin{tikzcd}
T^3A \arrow[r, "T^2a"] \arrow[d, "\mu TA"'] \arrow[dr, phantom, "\Downarrow (\mu_a)^{-1}\mu A"] & T^2A \arrow[d] \arrow[dr, "Ta"] \arrow[dr, phantom, "\Downarrow \alpha"] & \\
T^2A \arrow[r, "Ta"] \arrow[dr, "\mu A"'] \arrow[dr, phantom, "\Downarrow \alpha"] & TA \arrow[dr, "a"'] & TA \arrow[d, "a"] \\
& TA \arrow[r, "a"'] & A
\end{tikzcd}$$

$$\begin{tikzcd}
T^2A \arrow[r, "1"] \arrow[dr, phantom, "\Downarrow \lambda A"] & T^2A \arrow[r, "Ta"] \arrow[d, "\mu A"] \arrow[dr, phantom, "\Downarrow \alpha"] & TA \arrow[d, "a"] \\
TA \arrow[u, "T\eta A"] \arrow[r, "1"'] & TA \arrow[r, "a"'] & A
\end{tikzcd}
=
\begin{tikzcd}
T^2A \arrow[r, "1"] \arrow[dr, phantom, "\Downarrow T\underline{\alpha}"] & T^2A \arrow[r, "Ta"] & TA \arrow[d, "a"] \\
TA \arrow[u, "T\eta A"] \arrow[r, "1"'] & TA \arrow[ur, "1"] \arrow[r, "a"'] & A
\end{tikzcd}$$

Diagram 7.0.21.

$$\begin{tikzcd}
& & FGD \arrow[dr, "\epsilon_D"] \arrow[dd, Rightarrow, "\epsilon_{\epsilon_D}"] & \\
FGD \arrow[urr, bend left, "1"] \arrow[drr, bend right, "1"'] \arrow[r, "F\eta_G D"] & FGFGD \arrow[ur, "\epsilon_{FGD}"] \arrow[dr, "FG\epsilon_D"'] \arrow[u, phantom, "\Downarrow \phi_{GD}"] \arrow[d, phantom, "\Downarrow F\theta_D"] & & D \\
& & FGD \arrow[ur, "\epsilon_D"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& & GFC \arrow[dll, bend right, "1"'] \arrow[dl, "GF\eta_C"'] \arrow[dd, Rightarrow, "\eta_{\eta_C}"'] & \\
GFC & GFGFC \arrow[l, "G\epsilon_F C"'] \arrow[u, phantom, "\Downarrow G\phi_C"] \arrow[d, phantom, "\Downarrow \theta_{GC}"] & & C \arrow[ul, "\eta_C"'] \arrow[dl, "\eta_C"] \\
& & GFC \arrow[ull, bend left, "1"] \arrow[ul, "\eta_{GFC}"] &
\end{tikzcd}$$

Diagram 7.0.22.

$$\begin{tikzcd}
\mathbf{DblCat}\left(\mathcal{E}\right)_1 \arrow[d, "\mathbf{Cat}(\delta_{\mathcal{E}})_1"'] & & \mathbf{DblCat}\left(\mathcal{E}\right)_1 \arrow[ll, "T_{\mathcal{E}}"'] \arrow[d, "\delta_{\mathbf{Cat}(\mathcal{E})_1}"] \\
\mathbf{TrplCat}\left(\mathcal{E}\right)_1 & \mathbf{TrplCat}\left(\mathcal{E}\right)_1 \arrow[l, "T_{\mathbf{Cat}(\mathcal{E})_1}"] & \mathbf{TrplCat}\left(\mathcal{E}\right)_1 \arrow[l, "\mathbf{Cat}(T_{\mathcal{E}})_1"]
\end{tikzcd}$$

Diagram 7.0.23.

$$\begin{array}{ccc}
\begin{tikzcd}
A_{1,1} \arrow[r] \arrow[d] & A_{0,1} \arrow[d] \\
A_{1,0} \arrow[r] & A_{0,0}
\end{tikzcd}
& \longmapsto &
\begin{tikzcd}
A_{1,1} \arrow[dr] \arrow[dd] & \\
& A_{1,0} \arrow[dd] \\
A_{0,1} \arrow[dr] & \\
& A_{0,0}
\end{tikzcd}
\\
\rotatebox[origin=c]{-90}{$\longmapsto$} & & \rotatebox[origin=c]{-90}{$\longmapsto$}
\\
\begin{tikzcd}
\left(\mathbb{A}^{\mathbf{2}}_{-,1}\right)_1 \arrow[rr] \arrow[dr] \arrow[dd] & & \left(\mathbb{A}^{\mathbf{2}}_{-,0}\right)_1 \arrow[dr] \arrow[dd] & \\
& A_{1,1} \arrow[rr] \arrow[dd] & & A_{0,1} \arrow[dd] \\
\left(\mathbb{A}^{\mathbf{2}}_{-,1}\right)_0 \arrow[rr] \arrow[dr] & & \left(\mathbb{A}^{\mathbf{2}}_{-,0}\right)_0 \arrow[dr] & \\
& A_{1,0} \arrow[rr] & & A_{0,0}
\end{tikzcd}
& \longmapsto &
\begin{tikzcd}
\left(\mathbb{A}^{\mathbf{2}}_{-,1}\right)_1 \arrow[rr] \arrow[dr] \arrow[dd] & & A_{1,1} \arrow[dr] \arrow[dd] & \\
& \left(\mathbb{A}^{\mathbf{2}}_{-,1}\right)_0 \arrow[rr] \arrow[dd] & & A_{1,0} \arrow[dd] \\
\left(\mathbb{A}^{\mathbf{2}}_{-,0}\right)_1 \arrow[rr] \arrow[dr] & & A_{0,1} \arrow[dr] & \\
& \left(\mathbb{A}^{\mathbf{2}}_{-,0}\right)_0 \arrow[rr] & & A_{0,0}
\end{tikzcd}
\end{array}$$

Diagram 7.0.24.

$$\begin{tikzcd}
PX \arrow[rr, "(\alpha d_0{,}g)"] \arrow[dd, "(f{,}\alpha d_1)"'] \arrow[dr, "\begin{pmatrix}\alpha d_0 & f \\ g & \alpha d_1\end{pmatrix}"'] & & (\Phi X)_2 \arrow[dd, "m"] \\
& PPX \arrow[ur, "p_0"'] \arrow[dl, "p_1"] & \\
(\Phi X)_2 \arrow[rr, "m"'] & & PX
\end{tikzcd}$$

Diagram 7.0.25.

$$\begin{tikzcd}
C_1 \arrow[rr, "g_1"] \arrow[dr, "d_j"] \arrow[dd, "(h_0d_0{,}h_0d_1)"'] & & B_1 \arrow[rr, "(d_0{,}d_1)"] \arrow[ddrr, bend left, "d_j"] & & B_0 \times B_0 \arrow[dd, "p_j"] \\
& C_0 \arrow[rr, "h_0"] \arrow[drrr, bend left, "g_0"] & & A_0 \arrow[dr, "f_0"] & \\
A_0 \times A_0 \arrow[rr, "f_0 \times f_0"'] & & B_0 \times B_0 \arrow[rr, "p_j"'] & & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[r, "h_1"] \arrow[dr, "g_1"] \arrow[dd, "d_j"'] & A_1 \arrow[r, "d_j"] \arrow[d, "f_1"] & A_0 \arrow[dd, "f_0"] \\
& B_1 \arrow[dr, "d_j"] & \\
C_0 \arrow[r, "h_0"'] \arrow[rr, bend left, "g_0"] & A_0 \arrow[r, "f_0"'] & B_0
\end{tikzcd}$$

Diagram 7.0.26.

$$\begin{tikzcd}
C_0 \arrow[r, "h_0"] \arrow[d, "i"'] \arrow[dr, "g_0"] & A_0 \arrow[r, "i"] \arrow[d, "f_0"] & A_1 \arrow[dd, "f_1"] \\
C_1 \arrow[d, "h_1"'] \arrow[drr, "g_1"] & B_0 \arrow[dr, "i"] & \\
A_1 \arrow[rr, "f_1"'] & & B_1
\end{tikzcd}
\qquad
\begin{tikzcd}
C_0 \arrow[r, "h_0"] \arrow[d, "i"'] \arrow[dr, equal] & A_0 \arrow[r, "i"] \arrow[ddr, equal] & A_1 \arrow[dd, "d_j"] \arrow[dr, "{(d_0,d_1)}"] & \\
C_1 \arrow[r, "d_j"] \arrow[d, "h_1"'] & C_0 \arrow[dr, "h_0"] & & A_0\times A_0 \arrow[dl, "p_j"] \\
A_1 \arrow[r, "{(d_0,d_1)}"'] \arrow[rr, bend left, "d_j"] & A_0\times A_0 \arrow[r, "p_j"'] & A_0 &
\end{tikzcd}$$

$$\begin{tikzcd}
C_2 \arrow[r, "h_2"] \arrow[dd, "m"'] \arrow[dr, "g_2"'] & A_2 \arrow[r, "m"] \arrow[d, "f_2"] & A_1 \arrow[dd, "f_1"] \\
 & B_2 \arrow[dr, "m"] & \\
C_1 \arrow[r, "h_1"'] \arrow[rr, bend left, "g_1"] & A_1 \arrow[r, "f_1"'] & B_1
\end{tikzcd}
\qquad
\begin{tikzcd}
C_2 \arrow[rr, "g_2"] \arrow[dr, "\pi_j"] \arrow[dd, "h_2"'] & & B_2 \arrow[dddd, "\pi_j"] \\
 & C_1 \arrow[dd, "h_1"'] \arrow[dddr, "g_1"] & \\
A_2 \arrow[dr, "\pi_j"'] \arrow[dd, "f_2"'] & & \\
 & A_1 \arrow[dr, "f_1"'] & \\
B_2 \arrow[rr, "\pi_j"'] & & B_1
\end{tikzcd}$$

$$\begin{tikzcd}
C_2 \arrow[rr, "h_2"] \arrow[dr, "\pi_j"] \arrow[dd, "m"'] & & A_2 \arrow[rr, "m"] \arrow[dr, "\pi_j"] & & A_1 \arrow[dddd, "d_j"] \arrow[ddr, "{(d_0,d_1)}"] & \\
 & C_1 \arrow[rr, "h_1"] \arrow[dd, "d_j"] & & A_1 \arrow[dddr, "d_j"'] & & \\
C_1 \arrow[dr, "d_j"'] \arrow[dd, "h_1"'] & & & & & A_0\times A_0 \arrow[ddl, "p_j"] \\
 & C_0 \arrow[drrr, "h_0"] & & & & \\
A_1 \arrow[rr, "{(d_0,d_1)}"'] & & A_0\times A_0 \arrow[rr, "p_j"'] & & A_0 &
\end{tikzcd}$$

Diagram 7.0.27.

$$\begin{tikzcd}
C_0 \arrow[rr, "h_0^j"] \arrow[d, "\beta"'] \arrow[dr, dashed, "\alpha"] \arrow[ddrr, bend left, "g_0^j"] & & A_0 \arrow[dd, "f_0"] \\
A_1 \arrow[r, "f_1"'] \arrow[d, "d_j"'] & B_1 \arrow[dr, dashed, "d_j"] & \\
A_0 \arrow[rr, "f_0"'] & & B_0
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[rr, "{\big(\beta d_0,h_1^0\big)}"] \arrow[d, "{\big(h_1^1,\beta d_1\big)}"'] & & A_2 \arrow[dl, "f_2"'] \arrow[d, "m"] \\
A_2 \arrow[r, "f_2"] \arrow[d, "m"'] & B_2 \arrow[dr, "m"] & A_1 \arrow[d, "f_1"] \\
A_1 \arrow[rr, "f_1"'] & & B_1
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[rr, "{\big(\beta d_0,h_1^1\big)}"] \arrow[d, "{\big(h_1^0,\beta d_1\big)}"'] & & A_2 \arrow[dl, "\pi_j"'] \arrow[d, "f_2"] \\
A_2 \arrow[r, "\pi_j"] \arrow[d, "f_2"'] & A_1 \arrow[dr, "f_1"] & B_2 \arrow[d, "\pi_j"] \\
B_2 \arrow[rr, "\pi_j"'] & & B_1
\end{tikzcd}$$

$$\begin{tikzcd}
C_1 \arrow[r, "{\big(h_1^0,\beta d_1\big)}"] \arrow[dd, "{\big(\beta d_0,d_1^1\big)}"'] & A_2 \arrow[r, "m"] \arrow[d, "\pi_j"] & A_1 \arrow[r, "{(d_0,d_1)}"] \arrow[ddr, "d_j"] & A_0\times A_0 \arrow[dd, "p_j"] \\
 & A_1 \arrow[ddrr, bend left, "d_j"] & & \\
A_2 \arrow[ur, "\pi_j"'] \arrow[r, "m"'] & A_1 \arrow[r, "{(d_0,d_1)}"'] \arrow[rr, bend left, "d_j"] & A_0\times A_0 \arrow[r, "p_j"'] & A_0
\end{tikzcd}$$

Diagram 7.0.28.

$$\begin{tikzcd}
C_0 \arrow[rr, "\alpha"] \arrow[d, "{\big(h_0^0,h_0^1\big)}"'] \arrow[dr, "h_0^j"] \arrow[ddrr, bend left, "g_0^j"] & & B_1 \arrow[dd, "d_j"] \arrow[dr, "{(d_0,d_1)}"] & \\
A_0\times A_0 \arrow[r, "p_j"'] \arrow[d, "f_0\times f_0"'] & A_0 \arrow[dr, "f_0"'] & & B_0\times B_0 \arrow[dl, "p_j"] \\
B_0\times B_0 \arrow[rr, "p_j"'] & & B_0 &
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[rr, "{\big(\beta d_0,d_1^1\big)}"] \arrow[dr, "d_0"] \arrow[dd, "{\big(h_1^0,\beta d_1\big)}"'] \arrow[ddrr, bend right, "h_1^0"'] & & A_2 \arrow[dd, "\pi_0"] \\
 & C_0 \arrow[dr, "\beta"] & \\
A_2 \arrow[rr, "\pi_0"'] & & A_1
\end{tikzcd}
\qquad
\begin{tikzcd}
C_1 \arrow[rr, "{\big(\beta d_0,d_1^1\big)}"] \arrow[dr, "d_1"'] \arrow[dd, "{\big(h_1^0,\beta d_1\big)}"'] \arrow[ddrr, bend left, "h_1^1"] & & A_2 \arrow[dd, "\pi_1"] \\
 & C_0 \arrow[dr, "\beta"'] & \\
A_2 \arrow[rr, "\pi_1"'] & & A_1
\end{tikzcd}$$

Diagram 7.0.29.

$$
\begin{tikzcd}
B/f' \arrow[rr, "u'"] \arrow[d, "Lg"'] \arrow[ddr, bend right=60, "v'"'] & & A' \arrow[d, "g'"] \arrow[d, bend right=60, "g"'] \arrow[d, phantom, shift right=2, "\overset{\alpha}{\Rightarrow}"] \arrow[ddl, bend left=60, "f'"] \\
B/f \arrow[rr, "u"] \arrow[dr, "v"'] & \arrow[r, phantom, "\overset{\lambda}{\Leftarrow}"] & A \arrow[dl, "f"] \\
& B &
\end{tikzcd}
=
\begin{tikzcd}
B/f \arrow[r, "u'"] \arrow[dr, "v'"'] & A' \arrow[d, "f'"] \arrow[dl, phantom, "\lambda'\Uparrow"] \\
& B
\end{tikzcd}
=
\begin{tikzcd}
B/f' \arrow[r, "Lg"] & B/f \arrow[r, "u"] \arrow[dr, "v"'] & A \arrow[d, "f"] \arrow[dl, phantom, "\lambda\Uparrow"] \\
& & B
\end{tikzcd}
$$

Diagram 7.0.30.

$$
\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "f_A"'] & B \arrow[d, "f_B"] \\
P_f^{-1}A \arrow[r, "P_f^{-1}(s)"'] & P_f^{-1}B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "f_A"'] & B \arrow[r, "t"] \arrow[d, "f_B"] & C \arrow[d, "f_C"] \\
P_f^{-1}A \arrow[r, "P_f^{-1}(s)"'] & P_f^{-1}B \arrow[r, "P_f^{-1}(t)"'] & P_f^{-1}C
\end{tikzcd}
$$

Diagram 7.0.31.

$$
\begin{tikzcd}
El(\sigma)\big((X,x)\big) \arrow[r, "El(\sigma)\left((F,f)\right)"] \arrow[d, "El\left(\Gamma_{(X,x)}\right)"'] & El(\sigma)\,(Y,y) \arrow[d, "El\left(\Gamma_{(Y,y)}\right)"] \\
El(\tau)\big((X,x)\big) \arrow[r, "El(\tau)\left((F,f)\right)"'] & El(\tau)\big((Y,y)\big)
\end{tikzcd}
\qquad
\begin{tikzcd}
\sigma_Y\,(y) \arrow[d, "\Gamma_y^Y"'] & \sigma_Y\,(QF\,(x)) \arrow[l, "\sigma_Y(f)"'] \arrow[r, equal] \arrow[d, "\Gamma_{QF(x)}^Y"] & RF\,(\sigma_X\,(x)) \arrow[d, "RF\left(\Gamma_x^X\right)"] \\
\tau_Y\,(y) & \tau_Y\,(QF\,(x)) \arrow[l, "\tau_Y(f)"] \arrow[r, equal] & RF\,(\tau_X\,(x))
\end{tikzcd}
$$

Diagram 7.0.32.

$$
\begin{tikzcd}
A \arrow[r, "s"] \arrow[d, "f_A"] \arrow[dd, bend right=60, "(kf)_A"'] & B \arrow[d, "f_B"'] \arrow[dd, bend left=60, "(kf)_B"] \\
P_f^{-1}A \arrow[r, "P_f^{-1}(s)"] \arrow[d, "k_A"] & P_f^{-1}B \arrow[d, "k_B"'] \\
P_k^{-1}P_f^{-1}A \arrow[r, "P_k^{-1}\left(P_f^{-1}(s)\right)"'] & P_k^{-1}P_f^{-1}B
\end{tikzcd}
\qquad
\begin{tikzcd}
P^{-1}X \arrow[r, "P_F^{-1}"] \arrow[d, "S_X^{-1}"'] & P^{-1}Y \arrow[d, "S_Y^{-1}"] \\
P'^{-1}X \arrow[r, "P'^{-1}_F"'] & P'^{-1}Y
\end{tikzcd}
$$

Diagram 7.0.33.

$$
\begin{tikzcd}
B_1 \arrow[rrr, "u_1"] \arrow[dr, "(d_0,d_1)"] \arrow[dd, "d_j"'] & & & C_1 \arrow[dl, "(d_0,d_1)"'] \arrow[dd, "d_j"] \\
& B_0 \times B_0 \arrow[r, "u_0 \times u_0"] \arrow[dl, "p_j"'] & C_0 \times C_0 \arrow[dr, "p_j"] & \\
B_0 \arrow[rrr, "u_0"'] & & & C_0
\end{tikzcd}
\qquad
\begin{tikzcd}
A_1 \arrow[r, "b_1"] \arrow[d, "h_1"'] & B_1 \arrow[r, "u_1"] \arrow[dr, "g_1"] & C_1 \arrow[d, "f_1"] \\
C_1 \arrow[rr, "f_1"'] & & D_1
\end{tikzcd}
$$

$$
\begin{tikzcd}
A_1 \arrow[r, "b_1"] \arrow[d, "h_1"'] \arrow[dr, "d_j"] & B_1 \arrow[r, "u_1"] \arrow[dr, "d_j"] & C_1 \arrow[r, "(d_0,d_1)"] & C_0 \times C_0 \arrow[dd, "p_j"] \\
C_1 \arrow[d, "(d_0,d_1)"'] \arrow[drrr, "d_j"] & A_0 \arrow[r, "b_0"] \arrow[drr, "h_0"] & B_0 \arrow[dr, "u_0"] & \\
C_0 \times C_0 \arrow[rrr, "p_j"'] & & & C_0
\end{tikzcd}
$$

Diagram 7.0.34.

$$
\begin{tikzcd}
B_1 \arrow[r, "g_1"] \arrow[d, "(d_0{,}d_1)"'] \arrow[dr, "d_j"] & D_1 \arrow[r, "(d_0{,}d_1)"] \arrow[dr, "d_j"] & D_0 \times D_0 \arrow[d, "p_j"] \\
B_0 \times B_0 \arrow[r, "p_j"] \arrow[d, "u_0 \times u_0"'] & B_0 \arrow[r, "g_0"] \arrow[d, "u_0"'] & D_0 \\
C_0 \times C_0 \arrow[r, "p_j"] \arrow[rr, bend right, "f_0 \times f_0"'] & C_0 \arrow[ur, "f_0"] & D_0 \times D_0 \arrow[u, "p_j"']
\end{tikzcd}
\qquad
\begin{tikzcd}
B_1 \arrow[r, "v_1"] \arrow[d, "u_1"'] \arrow[dr, "d_j"'] & C_1 \arrow[r, "(d_0{,}d_1)"] \arrow[ddr, "d_j"] & C_0 \times C_0 \arrow[dd, "p_j"] \\
C_1 \arrow[d, "(d_0{,}d_1)"'] \arrow[drr, "d_j"'] & B_0 \arrow[dr, shift left, "v_0"] \arrow[dr, shift right, "u_0"'] & \\
C_0 \times C_0 \arrow[rr, "p_j"'] & & C_0
\end{tikzcd}
$$

$$
\begin{tikzcd}
B_0 \arrow[rrr, "i"] \arrow[dr, "b_0^{-1}"] \arrow[dd, "u_0"'] & & & B_1 \arrow[dd, "u_1"] \\
& A_0 \arrow[r, "i"] \arrow[dl, "h_0"'] & A_1 \arrow[ur, "b_1"] \arrow[dr, "h_1"] & \\
C_0 \arrow[rrr, "i"] \arrow[dr, "f_0"] \arrow[dd, "i"'] & & & C_1 \arrow[dd, "f_1"] \\
& D_0 \arrow[drr, "i"] & & \\
C_1 \arrow[rrr, "f_1"'] & & & D_1
\end{tikzcd}
\qquad
\begin{tikzcd}
B_0 \arrow[r, "i"] \arrow[d, "u_0"'] \arrow[dr, equal] & B_1 \arrow[r, "u_1"] \arrow[d, "d_j"] & C_1 \arrow[dd, "d_j"'] \arrow[dr, "(d_0{,}d_1)"] & \\
C_0 \arrow[d, "i"'] \arrow[drr, equal] & B_0 \arrow[dr, "u_0"] & & C_0 \times C_0 \arrow[dl, "p_j"] \\
C_1 \arrow[r, "(d_0{,}d_1)"'] \arrow[rr, bend left, "d_j"] & C_0 \times C_0 \arrow[r, "p_j"'] & C_0 &
\end{tikzcd}
$$

$$
\begin{tikzcd}
B_2 \arrow[r, "m"] \arrow[dd, "u_2"'] \arrow[dr, "g_2"'] & B_1 \arrow[r, "u_1"] \arrow[ddr, "g_1"] & C_1 \arrow[dd, "f_1"] \\
& D_2 \arrow[dr, "m"'] & \\
C_2 \arrow[r, "m"'] \arrow[ur, "f_2"] & C_1 \arrow[r, "f_1"'] & D_1
\end{tikzcd}
\qquad
\begin{tikzcd}
B_2 \arrow[rr, "m"] \arrow[d, "u_2"'] \arrow[drr, "\pi_j"] & & B_1 \arrow[rr, "u_1"] \arrow[dr, "d_j"'] & & C_1 \arrow[dd, "d_j"'] \arrow[dr, "(d_0{,}d_1)"] & \\
C_2 \arrow[r, "\pi_j"] \arrow[d, "m"'] & C_1 \arrow[drrr, "d_j"] & B_1 \arrow[l, "u_1"'] \arrow[r, "d_j"] & B_0 \arrow[dr, "u_0"'] & & C_0 \times C_0 \arrow[dl, "p_j"] \\
C_1 \arrow[rr, "(d_0{,}d_1)"'] \arrow[rrrr, bend left, "d_j"] & & C_0 \times C_0 \arrow[rr, "p_j"'] & & C_0 &
\end{tikzcd}
$$

Diagram 7.0.35.

$$
\begin{tikzcd}
q/j \arrow[r, bend left, "r"] \arrow[r, bend right, "k"'] \arrow[r, phantom, "\Downarrow\lambda'"] \arrow[d, "u_*"'] \arrow[dr, phantom, "="] & F \arrow[r, "q"] \arrow[dr, phantom, "\Searrow\lambda"] & B \\
P \arrow[ur, "g"] \arrow[r, "u_*"'] & q/j \arrow[u, "r"] \arrow[r, "f"'] & A \arrow[u, "j"']
\end{tikzcd}
=
\begin{tikzcd}
F \arrow[r, "q"] \arrow[dr, phantom, "\Searrow\lambda"] & B \\
q/j \arrow[u, "r"] \arrow[r, "f"'] & A \arrow[u, "j"']
\end{tikzcd}
$$

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