

Canonical L_∞ -morphisms in higher Courant geometry: from twisted algebroids to higher Dirac structures

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based on joint work with A. BASU and D. FIORENZA



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- Good afternoon, and thank you for the invitation. Today I will discuss canonical morphisms between the L_∞ -algebras that arise in higher Courant and multisymplectic geometry.
- The main part is joint work with Domenico Fiorenza, now available as arXiv:2602.14702. Our goal was to answer a question posed by Marco Zambon: can his degree-one morphism from a twisted Courant algebra to the next untwisted one be constructed canonically in every degree?
- The answer is yes. The main idea is to stop comparing the higher brackets one by one and instead realize both algebras as homotopy fibers of morphisms between dg-Lie algebras. Naturality then produces the desired map.
- In the final part I will describe ongoing work with Aditya Basu and Domenico Fiorenza. There we ask whether the same viewpoint extends from graphs of closed forms to general higher Dirac structures.
- I will begin with the geometric motivation, starting from ordinary symplectic observables.

Symplectic geometry

A symplectic form $\omega \in \Omega_{\text{cl}}^2(M)$ assigns to $f \in C^\infty(M)$ a Hamiltonian vector field

$$\iota_{X_f}\omega = -df.$$

The observables form a Poisson algebra

$$(C^\infty(M), \{\cdot, \cdot\}_\omega) .$$

Higher degree

For $\sigma \in \Omega_{\text{cl}}^{r+1}(M)$, Hamiltonian data are pairs

$$(X, \alpha) \in \mathfrak{X}(M) \oplus \Omega^{r-1}(M) ,$$

$$\text{s.t. } \iota_X\sigma = -d\alpha ,$$

they assemble into an L_∞ -algebra

$$\mathfrak{Rogers}_\sigma^{(r-1)} .$$

Claim

Higher brackets encode the geometry of observables and their homotopies.

Canonical L_∞ -morphisms in higher Courant geometry

└ Introduction

└ Poisson and higher observables

Symplectic geometry

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they assemble into an L_∞ -algebra
 $\mathfrak{Rogers}_\sigma^{(r-1)}.$

Claim

Higher brackets encode the geometry of observables and their homotopies.

- Let me start from the familiar symplectic picture. A closed, non-degenerate two-form identifies the differential of a function f with a Hamiltonian vector field X_f . The Poisson bracket can then be written in terms of these vector fields and the form ω .
- The important feature is that the observables are functions and form an honest Lie algebra. The Jacobi identity holds strictly.
- If the closed form has degree $r + 1$, the Hamiltonian observable is no longer just a function. It is an $(r - 1)$ -form α , together with a vector field X satisfying $\iota_X \sigma = -d\alpha$. I keep the vector field in the notation because uniqueness may fail in the pre-plectic case.
- The binary bracket alone no longer satisfies Jacobi strictly. The defect, and all its higher coherences, are encoded by higher brackets. This is why the natural replacement of the Poisson algebra is an L_∞ -algebra, denoted here by $\mathfrak{Rogers}_\sigma^{(r-1)}$.
- The question for this talk is where this L_∞ -algebra sits and how it maps naturally to the higher Courant algebras around it.

Fix an integer $r \geq 1$. The *standard higher Courant algebroid* is

$$E^{(r-1)} = TM \oplus \wedge^{r-1} T^*M.$$

A closed form $\sigma \in \Omega_{\text{cl}}^{r+1}(M)$ twists its bracket and determines an L_∞ -algebra

$$\mathfrak{Courant}_\sigma^{(r-1)}.$$

Graphs of closed forms

The graph

$$\text{graph}(\sigma) = \{X - \iota_X \sigma \mid X \in TM\}$$

is an isotropic involutive subbundle of $E^{(r)}$.

Higher Dirac structures

General isotropic involutive subbundles

$$L \subset E^{(n)}$$

extend the graph construction and carry their own observable L_∞ -algebras.

Canonical L_∞ -morphisms in higher Courant geometry

└ Introduction

└ Higher Courant algebras as ambient structures

- Higher Courant algebroids provide the ambient geometry. In degree $r - 1$ the standard bundle is $TM \oplus \wedge^{r-1} T^*M$, and the same closed form σ twists its Dorfman bracket.
- The form σ also has a second geometric interpretation. Its graph is a subbundle of the Courant algebroid one degree higher, $E^{(r)} = TM \oplus \wedge^r T^*M$. Closedness gives involutivity, while the skew-symmetry of contraction gives isotropy.
- So the closed form appears in two roles: it twists one higher Courant bracket, and its graph defines a higher Dirac structure in the next Courant algebroid. This degree shift is exactly the shift that will occur in our canonical morphism.
- A general higher Dirac structure need not be the graph of a form. The final part of the talk asks how much of the construction survives in that wider setting.

Fix an integer $r \geq 1$. The standard higher Courant algebroid is

$$E^{(r-1)} = TM \oplus \wedge^{r-1} T^*M.$$

A closed form $\sigma \in \Omega_{cl}^{(r-1)}(M)$ twists its bracket and determines an L_∞ -algebra $\text{Courant}^{(r-1)}_\sigma$.

Graphs of closed forms

The graph

$$\text{graph}(\sigma) = \{X - \iota_X \sigma \mid X \in TM\}$$

is an isotropic involutive subbundle of $E^{(r)}$.

Higher Dirac structures

General isotropic involutive subbundles

$$L \subset E^{(r)}$$

extend the graph construction and carry their own observable L_∞ -algebras.

The starting point is the sequence

$$\mathfrak{Rogers}_{\sigma}^{(r-1)} \longrightarrow \mathfrak{Courant}_{\sigma}^{(r-1)} \longrightarrow \mathfrak{Courant}^{(r)}.$$

- ▶ The first morphism embeds multisymplectic observables into the corresponding higher Courant algebra [MZ25].
- ▶ Zambon asked whether the second morphism exists canonically in every degree [Zam10].
- ▶ The preprint [FM26] constructs it and explains both arrows through homotopy fibers of morphisms between differential graded Lie algebras.

The starting point is the sequence

$$\mathfrak{Hogez}^{(r-1)}_\sigma \longrightarrow \mathfrak{Courant}^{(r-1)}_\sigma \longrightarrow \mathfrak{Courant}^{(r)}.$$

- The first morphism embeds multisymplectic observables into the corresponding higher Courant algebra [MZ25].
- Zambon asked whether the second morphism exists canonically in every degree [Zam10].
- The preprint [FM26] constructs it and explains both arrows through homotopy fibers of morphisms between differential graded Lie algebras.

Update

The homotopy-fiber description makes the morphisms natural and suggests a path from graphs of closed forms to general higher Dirac structures.

- This sequence is the spine of the talk. On the left we have the L_∞ -algebra of Hamiltonian observables of the closed form σ . In the middle we have the σ -twisted higher Courant algebra in degree $r - 1$. On the right we have the untwisted higher Courant algebra one degree higher.
- The first arrow was constructed in joint work with Marco Zambon. Its linear term is the obvious inclusion of a Hamiltonian pair into the corresponding Courant section, but higher components are required.
- The second arrow answers Zambon's question. It is the main result of the recent preprint with Domenico Fiorenza.
- The point of the present formulation is that both arrows arise from compatible homotopy-fiber descriptions. This explains their naturality and suggests how to replace a graph $\text{graph}(\sigma)$ by a general higher Dirac structure.

1. Higher Courant L_∞ -algebras and their homotopy-fiber models.
2. The canonical morphism from the twisted degree- $(r-1)$ algebra to the untwisted degree- r algebra.
3. The multisymplectic observable morphism in the same framework.
4. Ongoing work on observable algebras of higher Dirac structures.

Canonical L_∞ -morphisms in higher Courant geometry

└ Introduction

└ Plan of the talk

- I will first recall only the pieces of L_∞ and higher Courant geometry needed for the construction. The key result of this part is the realization of Zambon's algebra as a homotopy fiber.
- I will then construct the canonical morphism. The proof has three steps: reduce to a small Cartan dg-Lie algebra, define an untwisted L_∞ -morphism there, and twist it by the closed form σ .
- After that, I will return to multisymplectic observables and show that their known morphism to the Courant algebra has the same conceptual origin.
- Finally, I will separate established results from ongoing work and explain the higher Dirac comparison problem. If time becomes short, I will compress the technical formulas in the middle and preserve this final geometric discussion.

1. Higher Courant L_∞ -algebras and their homotopy-fiber models.
2. The canonical morphism from the twisted degree- $(r-1)$ algebra to the untwisted degree- r algebra.
3. The multisymplectic observable morphism in the same framework.
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1 Introduction

2 Background

- Lie infinity algebras
- Courant algebroids
- Higher Courant Lie infinity algebras

3 Canonical morphisms via homotopy fibers

- From twisted to untwisted higher Courant algebras
- From Rogers to Courant algebras

4 Higher Dirac structures

- From graphs to higher Dirac structures

Canonical L_∞ -morphisms in higher Courant geometry

└ Background

└ Outline

- We have identified the two morphisms that organize the talk. I will now recall the higher Courant algebra and its homotopy-fiber model.
- This background section fixes the notation for the ambient dg-Lie algebras. The main construction begins immediately afterward.

■ Introduction

■ Background

- Lie infinity algebras
- Courant algebroids
- Higher Courant Lie infinity algebras

■ Canonical morphisms via homotopy fibers

- From twisted to untwisted higher Courant algebras
- From Rogers to Courant algebras

■ Higher Dirac structures

- From graphs to higher Dirac structures

Definition: Algebraic data

A graded vector space L with graded skew-symmetric maps

$$\mu_k: \wedge^k L \longrightarrow L[2 - k], \quad k \geq 1.$$

First identities

$$\mu_1^2 = 0,$$

μ_1 is a derivation of μ_2 ,

$\text{Jac}(\mu_2)$ is controlled by μ_3 .

Upshot

Equivalently, the brackets assemble into a degree-one coderivation Q_μ on $S^{\geq 1}(L[1])$ satisfying

$$Q_\mu^2 = 0.$$

Canonical L_∞ -morphisms in higher Courant geometry

└ Background

└ Lie infinity algebras

└ L_∞ -algebras

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- I will use the cohomological convention. An L_∞ -algebra is a graded vector space with a family of graded skew-symmetric brackets μ_k of degree $2 - k$.
- The first bracket μ_1 is a differential. The next identity says that it acts as a derivation of the binary bracket. The Jacobiator of μ_2 is not required to vanish: it is the boundary controlled by μ_3 , and the higher identities continue this coherence.
- Ordinary Lie algebras have only μ_2 , while dg-Lie algebras may have μ_1 and μ_2 . Thus dg-Lie algebras are strict examples of L_∞ -algebras.
- I will not need the signed shuffle identities explicitly. It is enough to remember the equivalent coalgebraic formulation: all brackets combine into one degree-one coderivation Q_μ , and every identity is contained in the single equation $Q_\mu^2 = 0$.

The standard Courant algebroid is

$$E^{(1)} = TM \oplus T^*M.$$

Structure

► anchor $\rho(X + \alpha) = X$

► symmetric pairing

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\iota_X \beta + \iota_Y \alpha)$$

► Dorfman bracket on sections

Example: Graphs of closed forms

If $\omega \in \Omega_{\text{cl}}^2(M)$, then

$$\text{graph}(\omega) = \{X - \iota_X \omega \mid X \in TM\}$$

is isotropic and involutive.

Dirac structures unify symplectic, Poisson, and constrained geometry.

Canonical L_∞ -morphisms in higher Courant geometry

└ Background

└ Courant algebroids

└ Courant algebroids and Dirac structures

The standard Courant algebroid is

$$E^{(1)} = TM \oplus T^*M.$$

Structure

- anchor $\rho(X + \alpha) = X$
- symmetric pairing

$$(X + \alpha, Y + \beta) = \frac{1}{2}(\iota_X \beta + \iota_Y \alpha)$$
- Dorfman bracket on sections

Example: Graphs of closed forms

If $\omega \in \Omega_c^2(M)$, then
 $\text{graph}(\omega) = \{X - \iota_X \omega \mid X \in TM\}$
 is isotropic and involutive.

Dirac structures unify symplectic, Poisson, and constrained geometry.

- The standard Courant algebroid is the direct sum $TM \oplus T^*M$. It has an anchor to TM , a symmetric pairing, and a Dorfman bracket. The bracket is not skew-symmetric, but its skew-symmetrization is the Courant bracket.
- A Dirac structure is, roughly, an isotropic involutive subbundle. The graph of a closed two-form is the basic example shown here. Isotropy comes from skew-symmetry of the form and involutivity is equivalent to its closedness.
- Graphs of Poisson bivectors give another fundamental family. This is why Dirac structures provide a common language for symplectic, Poisson and constrained geometries.
- For the present talk, the graph example is the important one. We will first solve the problem for graphs of closed higher forms and then ask how to remove the graph assumption.

For $r \geq 1$, set

$$E^{(r)} = TM \oplus \wedge^r T^*M.$$

Definition: σ -twisted Dorfman bracket

For $\sigma \in \Omega_{\text{cl}}^{r+2}(M)$,

$$[X + \alpha, Y + \beta]_{\sigma} = [X, Y] + \mathcal{L}_X \beta - \iota_Y d\alpha + \iota_X \iota_Y \sigma.$$

The pairing now takes values in differential forms:

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\iota_X \beta + \iota_Y \alpha) \in \Omega^{r-1}(M).$$

$r = 1$: ordinary exact Courant algebroid

Canonical L_∞ -morphisms in higher Courant geometry

└ Background

└ Courant algebroids

└ Standard higher Courant algebroids

For $r \geq 1$, set

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Definition: σ -twisted Dorfman bracketFor $\sigma \in \Omega^{r+2}_c(M)$,

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The pairing now takes values in differential forms:

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 $r = 1$: ordinary exact Courant algebroid

- The higher analogue replaces one-forms by r -forms. We write $E^{(r)} = TM \oplus \wedge^r T^*M$. For $r = 1$ this is the ordinary exact Courant algebroid.
- A closed form of degree $r + 2$ twists the Dorfman bracket by the double contraction $\iota_X \iota_Y \sigma$. The displayed formula is the only bracket formula we shall need.
- There is one essential difference from degree one: the pairing takes values in $(r - 1)$ -forms rather than functions. Consequently, the Jacobi data are distributed across several form degrees and naturally assemble into an L_∞ -algebra.
- In the main construction I will shift the index and use $E^{(r-1)}$ twisted by an $(r + 1)$ -form. I display the general formula here so that the degree bookkeeping remains visible.

Fix $\sigma \in \Omega_{\text{cl}}^{r+1}(M)$. Zambon's algebra has underlying complex

$$\mathfrak{Courant}_\sigma^{(r-1)} = \left(\Omega^0 \xrightarrow{d} \dots \xrightarrow{d} \Omega^{r-2} \xrightarrow{(0,d)} \Gamma(E^{(r-1)}) \right).$$

Low arity

- ▶ μ_1 is the de Rham differential
- ▶ μ_2 extends the skew Courant bracket

Higher arity

The brackets μ_k , $k \geq 3$, encode the failure of the Jacobi identity and the contractions of σ .

Canonical L_∞ -morphisms in higher Courant geometry

└ Background

└ Higher Courant Lie infinity algebras

└ The higher Courant L_∞ -algebra

- Zambon associates an L_∞ -algebra to this higher Courant algebroid. Its degree-zero part is the space of sections of $E^{(r-1)}$, and the negative degrees are ordinary differential forms down to functions.
- The unary bracket is the de Rham differential on the form part, with the final map $\eta \mapsto (0, d\eta)$. The binary bracket on degree-zero elements is the skew-symmetrized twisted Courant bracket.
- Higher brackets repair the failure of Jacobi and include contractions of the twisting form σ . Their explicit expressions involve universal coefficients and are not illuminating for today's argument.
- What matters is that the entire list comes from a derived-bracket construction. That observation lets us replace explicit bracket calculations by a homotopy-fiber model.

Fix $\sigma \in \Omega_{cl}^{r-1}(M)$. Zambon's algebra has underlying complex

$$\mathcal{C}ourant^{(r-1)} = \left(\Omega^0 \xrightarrow{\sigma} \Omega^1 \xrightarrow{\sigma} \Omega^2 \xrightarrow{\sigma} \dots \xrightarrow{\sigma} \Omega^{r-2} \xrightarrow{\sigma} \Gamma(E^{(r-1)}) \right).$$

Low arity

- μ_1 is the de Rham differential
- μ_2 extends the skew Courant bracket

Higher arity

The brackets μ_k , $k \geq 3$, encode the failure of the Jacobi identity and the contractions of σ .

Update

The full list of brackets follows from a derived-bracket construction.

Consider the degree- r symplectic graded manifold

$$Q_r = T^*[r]T[1]M.$$

Poisson algebra

$C^\infty(Q_r)$ carries a Poisson bracket of degree $-r$ and contains

$$C^\infty(T[1]M) \cong \Omega(M).$$

Differential

A canonical function $\mathcal{S}$ encodes d_{dR} . Closed forms σ twists the differential

$$\delta_\sigma = \{\mathcal{S} + \sigma, \cdot\}, \quad \delta_\sigma^2 = 0.$$

After shifting degrees, we obtain a dg-Lie algebra

$$\mathcal{C}_{r,\sigma} := (C^\infty(Q_r)[r], \delta_\sigma, \{\cdot, \cdot\})$$

Its negative truncation carries Getzler's derived brackets.

Canonical L_∞ -morphisms in higher Courant geometry

└ Background

└ Higher Courant Lie infinity algebras

└ The ambient dg-Lie algebra

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Its negative truncation carries Getzler's derived brackets.

- The derived brackets live inside a much simpler strict object. Take the graded symplectic manifold $Q_r = T^*[r]T[1]M$. Its functions carry a Poisson bracket of degree $-r$.
- Functions on the base $T[1]M$ are differential forms on M . The cotangent directions add the operators needed for contractions and Lie derivatives. I will use this dictionary without introducing local graded coordinates.
- The canonical Hamiltonian $\mathcal{S}$ encodes the de Rham differential. Adding the closed form σ gives the differential $\delta_\sigma = \{\mathcal{S} + \sigma, -\}$. The master equation is precisely what ensures $\delta_\sigma^2 = 0$.
- After shifting by r , the Poisson bracket has degree zero, so we obtain an ordinary dg-Lie algebra $\mathcal{C}_{r,\sigma}$. Its negative part carries the higher derived brackets that reproduce the Courant L_∞ -algebra.

Theorem: Zambon–Getzler–Bandiera

There is a natural L_∞ quasi-isomorphism

$$\mathfrak{Courant}^{(r-1)}_\sigma \simeq \mathrm{hofib}(\mathcal{C}_{r,\sigma}^{\geq 0} \hookrightarrow \mathcal{C}_{r,\sigma}).$$

1. The quotient $\mathcal{C}_{r,\sigma}/\mathcal{C}_{r,\sigma}^{\geq 0}$ is the negative truncation.
2. Getzler's derived brackets reproduce the higher Courant brackets.
3. The mapping-cone model identifies this truncation with the homotopy fiber.

Canonical L_∞ -morphisms in higher Courant geometry

└ Background

└ Higher Courant Lie infinity algebras

└ The homotopy-fiber model

Theorem: Zambon-Getzler-Bandiera

There is a natural L_∞ quasi-isomorphism

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1. The quotient $C_{r,r}/C_{r,r}^{\geq 0}$ is the negative truncation.
2. Getzler's derived brackets reproduce the higher Courant brackets.
3. The mapping-cone model identifies this truncation with the homotopy fiber.

Update

Natural maps between the ambient dg-Lie diagrams induce maps between higher Courant algebras.

- Here is the homotopical reformulation that drives the rest of the talk. Consider the inclusion of the non-negative truncation of the ambient dg-Lie algebra into the full algebra. Its homotopy fiber is naturally L_∞ quasi-isomorphic to Zambon's higher Courant algebra.
- One way to see this is to pass to the quotient, which is the negative truncation. Getzler's higher derived brackets on this quotient reproduce Zambon's brackets.
- On the other hand, the Fiorenza–Manetti suspended mapping cone models the homotopy fiber of the inclusion. Results of Pridham and Bandiera relate these two models. I have placed the mapping-cone formula in the appendix.
- The practical consequence is the last box: instead of defining a map directly between complicated Courant L_∞ -algebras, it suffices to construct a compatible map between the ambient dg-Lie diagrams. The homotopy fiber then supplies the desired L_∞ -morphism.

1 Introduction

2 Background

- Lie infinity algebras
- Courant algebroids
- Higher Courant Lie infinity algebras

3 Canonical morphisms via homotopy fibers

- From twisted to untwisted higher Courant algebras
- From Rogers to Courant algebras

4 Higher Dirac structures

- From graphs to higher Dirac structures

Canonical L_∞ -morphisms in higher Courant geometry

└ Canonical morphisms via homotopy fibers

└ Outline

- The background is now complete. In particular, the source and target Courant algebras are homotopy fibers of inclusions.
- I will now use this description to construct the canonical morphism, first in the untwisted Cartan model and then in the presence of the closed form σ .

1	Introduction
2	Background • Lie infinity algebras • Courant algebroids • Higher Courant Lie infinity algebras
3	Canonical morphisms via homotopy fibers • From twisted to untwisted higher Courant algebras • From Rogers to Courant algebras
4	Higher Dirac structures • From graphs to higher Dirac structures

Question

Does there exist a canonical L_∞ -morphism

$$\mathfrak{Courant}^{(r-1)}_\sigma \longrightarrow \mathfrak{Courant}^{(r)} \quad ?$$

Example: Zambon's morphism for $r = 1$

$$\begin{array}{ccc} 0 & \longrightarrow & \Omega^0 \oplus \mathfrak{X} \\ \downarrow & & \downarrow (d, \text{id}) \\ \Omega^0 & \xrightarrow{(d, 0)} & \Omega^1 \oplus \mathfrak{X} \end{array}$$

The vertical chain map extends to an L_∞ -morphism $\mathfrak{Courant}^{(0)}_\sigma \rightarrow \mathfrak{Courant}^{(1)}$ [Zam10].

Canonical L_∞ -morphisms in higher Courant geometry

- └ Canonical morphisms via homotopy fibers
 - └ From twisted to untwisted higher Courant algebras
 - └ Zambon's degree-one morphism

Question

Does there exist a canonical L_∞ -morphism

$$\mathcal{C}\text{ourant}^{(r-1)}_\sigma \longrightarrow \mathcal{C}\text{ourant}^{(r)} \quad ?$$

Example: Zambon's morphism for $r = 1$

$$\begin{array}{ccc} 0 & \longrightarrow & \Omega^2 \oplus \mathbb{X} \\ \downarrow & & \downarrow (d, \text{id}) \\ \Omega^2 & \xrightarrow{(d, 0)} & \Omega^2 \oplus \mathbb{X} \end{array}$$

The vertical chain map extends to an L_∞ -morphism $\mathcal{C}\text{ourant}^{(0)}_\sigma \rightarrow \mathcal{C}\text{ourant}^{(1)}$ [Zam10].

- We can now state the problem precisely. Given a closed $(r+1)$ -form σ , we want a canonical morphism from the σ -twisted Courant algebra of degree $r-1$ to the untwisted Courant algebra of degree r .
- Zambon constructed the first case, $r = 1$. The source is an ordinary Lie algebra built from vector fields with the bracket twisted by a closed two-form. The target is the Lie 2-algebra of the standard Courant algebroid.
- At the level of complexes, the vertical map differentiates the function component and keeps the vector field. Zambon showed that this linear map admits the necessary higher correction.
- He then asked whether there is an analogous canonical construction in every degree. The answer will be the theorem on the next slide, and the rest of this section explains why it exists.

On the underlying complexes, define

$$F_1(\eta) = d\eta, \quad F_1(\alpha, X) = (d\alpha, X).$$

Here η has degree below zero and $(\alpha, X) \in \Omega^{r-1} \oplus \mathfrak{X}$ has degree zero.

$$\begin{array}{ccccccc} \mathfrak{Courant}_{\sigma}^{(r-1)} & & \Omega^0 & \xrightarrow{d} & \cdots & \longrightarrow & \Omega^{r-1} \oplus \mathfrak{X} \\ & \downarrow F_1 & \downarrow d & & \downarrow d & & \downarrow (d, \text{id}) \\ & \mathfrak{Courant}^{(r)} & \Omega^1 & \xrightarrow{d} & \cdots & \longrightarrow & \Omega^r \oplus \mathfrak{X} \end{array}$$

Theorem: Fiorenza–Miti [FM26]

The chain map F_1 extends to a canonical L_{∞} -morphism $\mathfrak{Courant}_{\sigma}^{(r-1)} \longrightarrow \mathfrak{Courant}^{(r)}$.

Canonical L_∞ -morphisms in higher Courant geometry

└ Canonical morphisms via homotopy fibers

└ From twisted to untwisted higher Courant algebras

└ The canonical chain map

On the underlying complexes, define

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$$\begin{array}{ccccc} \mathcal{C}\text{ourant}^{(r-1)}_\sigma & \Omega^0 & \xrightarrow{d} & \cdots & \xrightarrow{\quad} & \Omega^{r-1} \oplus \mathbb{X} \\ \downarrow F_1 & \downarrow d & & & & \downarrow h(LM) \\ \mathcal{C}\text{ourant}^{(r)} & \Omega^1 & \xrightarrow{d} & \cdots & \xrightarrow{\quad} & \Omega^r \oplus \mathbb{X} \end{array}$$

Theorem: Fiorenza–Miti [FM26]

The chain map F_1 extends to a canonical L_∞ -morphism $\mathcal{C}\text{ourant}^{(r-1)}_\sigma \longrightarrow \mathcal{C}\text{ourant}^{(r)}$.

- The linear term is forced by the complexes. In negative degrees we apply the de Rham differential. In degree zero we differentiate the form component and leave the vector field unchanged.
- The diagram suppresses the intermediate form degrees. Horizontally we have the de Rham differential, while the rightmost vertical map is $(\alpha, X) \mapsto (d\alpha, X)$.
- The theorem says that this chain map extends canonically to a full L_∞ -morphism for every r and every closed twist σ . So the answer to Zambon's question is affirmative.
- Directly checking the L_∞ identities for Zambon's explicit brackets would be possible but opaque. We instead explain the extension by functoriality of homotopy fibers.

Recall the two models

$$\mathfrak{Courant}_\sigma^{(r-1)} \simeq \mathrm{hofib}(\mathcal{C}_{r,\sigma}^{\geq 0} \hookrightarrow \mathcal{C}_{r,\sigma}),$$

$$\mathfrak{Courant}^{(r)} \simeq \mathrm{hofib}(\mathcal{C}_{r+1}^{\geq 0} \hookrightarrow \mathcal{C}_{r+1}).$$

A homotopy-commutative square between the ambient dg-Lie models

$$\begin{array}{ccc} \mathcal{C}_{r,\sigma}^{\geq 0} & \hookrightarrow & \mathcal{C}_{r,\sigma} \\ \downarrow & & \downarrow \\ \mathcal{C}_{r+1}^{\geq 0} & \hookrightarrow & \mathcal{C}_{r+1} \end{array}$$

therefore induces the required morphism between their homotopy fibers.

Canonical L_∞ -morphisms in higher Courant geometry

- └ Canonical morphisms via homotopy fibers
 - └ From twisted to untwisted higher Courant algebras
 - └ Naturality of homotopy fibers

Naturality of homotopy fibers 15

Recall the two models

$$\mathcal{C}\text{ourant}^{(r-1)} \simeq \text{hofib}(C_{r,\sigma}^{2,0} \hookrightarrow C_{r,\sigma}),$$

$$\mathcal{C}\text{ourant}^{(r)} \simeq \text{hofib}(C_{r+1}^{2,0} \hookrightarrow C_{r+1}).$$

A homotopy-commutative square between the ambient dg-Lie models

$$\begin{array}{ccc} C_{r,\sigma}^{2,0} & \hookrightarrow & C_{r,\sigma} \\ \downarrow & & \downarrow \\ C_{r+1}^{2,0} & \hookrightarrow & C_{r+1} \end{array}$$

therefore induces the required morphism between their homotopy fibers.

Updated The problem reduces to constructing the square between the ambient models.

- Recall that both source and target have already been identified as homotopy fibers of inclusions. The source uses the twisted dg-Lie algebra $\mathcal{C}_{r,\sigma}$, while the target uses the untwisted algebra in the next degree, $\mathcal{C}_{r+1}$.
- By the universal property of the homotopy limit, a homotopy-commutative square between these inclusions induces a morphism between their homotopy fibers. This is the categorical mechanism behind the construction.
- The difficulty has now moved to the vertical arrows in this square. The full function algebras $\mathcal{C}_{r,\sigma}$ are too large to make those arrows transparent.
- We therefore look for a smaller dg-Lie subalgebra with the same negative part, hence the same derived-bracket and homotopy-fiber model. It will come entirely from Cartan calculus.

Inside $\text{Der}(\Omega)$, Cartan calculus gives the two-term dg-Lie algebra

$$\mathfrak{Cartan} = \left(\iota_{\mathfrak{X}} \xrightarrow{[\text{d}, \cdot]} \mathcal{L}_{\mathfrak{X}} \right), \quad |\iota_X| = -1, \quad |\mathcal{L}_X| = 0.$$

Definition: Cartan relations

$$\begin{aligned} [\text{d}, \iota_X] &= \mathcal{L}_X, & [\mathcal{L}_X, \iota_Y] &= \iota_{[X, Y]}, \\ [\mathcal{L}_X, \mathcal{L}_Y] &= \mathcal{L}_{[X, Y]}, & [\iota_X, \iota_Y] &= 0. \end{aligned}$$

Since $\mathfrak{Cartan}$ acts on differential forms, every shift $\Omega[r]$ is a $\mathfrak{Cartan}$ -module.

Canonical L_∞ -morphisms in higher Courant geometry

- └ Canonical morphisms via homotopy fibers
 - └ From twisted to untwisted higher Courant algebras
 - └ The Cartan dg-Lie algebra

Inside $\text{Der}(\Omega)$, Cartan calculus gives the two-term dg-Lie algebra

$$\mathfrak{Cartan} = \left(\iota_X \xrightarrow{[\cdot, \cdot]} \mathcal{L}_X \right), \quad |\iota_X| = -1, \quad |\mathcal{L}_X| = 0.$$

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Since $\mathfrak{Cartan}$ acts on differential forms, every shift $\Omega[r]$ is a $\mathfrak{Cartan}$ -module.

- Inside the derivations of the de Rham complex, contractions and Lie derivatives form a two-term dg-Lie algebra. Contraction has degree minus one, Lie derivative has degree zero, and the differential is commutator with d .
- The first Cartan identity says exactly that the differential of ι_X is $\mathcal{L}_X$. The remaining identities describe all brackets between contractions and Lie derivatives.
- These four familiar formulas are the only algebraic input required for the canonical morphism. In particular, the later L_∞ identities reduce to Cartan calculus rather than to the full higher Courant formulas.
- Since contractions and Lie derivatives act on differential forms, every shifted de Rham complex is a module over this Cartan dg-Lie algebra. This produces the small semidirect product on the next slide.

Definition: Semidirect product

$$\mathfrak{g}_r := \mathfrak{Cartan} \ltimes \Omega[r].$$

A closed form $\sigma \in \Omega_{\text{cl}}^{r+1}(M)$ is a degree-one Maurer–Cartan element of $\mathfrak{g}_r$.

The twisted differential on $\mathfrak{g}_{r,\sigma}$ is determined by

$$\begin{aligned}d_\sigma \omega &= d\omega, \\d_\sigma \iota_X &= \mathcal{L}_X + \iota_X \sigma, \\d_\sigma \mathcal{L}_X &= -\mathcal{L}_X \sigma.\end{aligned}$$

Canonical L_∞ -morphisms in higher Courant geometry

└ Canonical morphisms via homotopy fibers

└ From twisted to untwisted higher Courant algebras

└ The small dg-Lie algebra

Definition: Semidirect product

$$\mathfrak{g}_r := \mathfrak{Cartan} \ltimes \Omega[r].$$

A closed form $\sigma \in \Omega_{cl}^{r+1}(M)$ is a degree-one Maurer–Cartan element of $\mathfrak{g}_r$.The twisted differential on $\mathfrak{g}_{r,\sigma}$ is determined by

$$\begin{aligned} d_{\mathfrak{g}}\omega &= d\omega, \\ d_{\mathfrak{g}}\iota_X &= \mathcal{L}_X + \iota_X\sigma, \\ d_{\mathfrak{g}}\mathcal{L}_X &= -\mathcal{L}_X\sigma. \end{aligned}$$

- We form the semidirect product of the Cartan dg-Lie algebra with the shifted de Rham complex. The symbol $\ltimes$ records the action of contractions and Lie derivatives on forms.
- A closed $(r+1)$ -form has degree one after the shift by r . Because forms bracket trivially with one another, closedness is precisely the Maurer–Cartan equation.
- Twisting by this Maurer–Cartan element changes the differential but not the bracket. On forms it remains the de Rham differential. On a contraction it adds $\iota_X\sigma$, and on a Lie derivative it adds the corresponding derivative of σ with the displayed sign.
- This small algebra remembers exactly the Cartan operators and form terms that occur in the negative part of the large graded-symplectic model.

The inclusions $\mathfrak{g}_{r,\sigma} \hookrightarrow \mathcal{C}_{r,\sigma}$ identify their negative parts:

$$\begin{array}{ccccc} \mathfrak{g}_{r,\sigma}^{\geq 0} & \hookrightarrow & \mathfrak{g}_{r,\sigma} & \twoheadrightarrow & \mathfrak{g}_{r,\sigma}^{< 0} \\ \downarrow & & \downarrow & & \parallel \\ \mathcal{C}_{r,\sigma}^{\geq 0} & \hookrightarrow & \mathcal{C}_{r,\sigma} & \twoheadrightarrow & \mathcal{C}_{r,\sigma}^{< 0}. \end{array}$$

Theorem: Small model

$$\mathfrak{Courant}_{\sigma}^{(r-1)} \simeq \mathrm{hofib}(\mathfrak{g}_{r,\sigma}^{\geq 0} \hookrightarrow \mathfrak{g}_{r,\sigma}).$$

The same statement with $r + 1$ and zero twist models $\mathfrak{Courant}^{(r)}$.

Canonical L_∞ -morphisms in higher Courant geometry

- Canonical morphisms via homotopy fibers
 - From twisted to untwisted higher Courant algebras
 - The small homotopy-fiber model

The inclusions $\mathfrak{g}_{r,\sigma} \hookrightarrow \mathcal{C}_{r,\sigma}$ identify their negative parts:

$$\begin{array}{ccccc} \mathfrak{g}_{r,\sigma}^{\leq 0} & \hookrightarrow & \mathfrak{g}_{r,\sigma} & \hookrightarrow & \mathfrak{g}_{r,\sigma}^{\geq 0} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{C}_{r,\sigma}^{\leq 0} & \hookrightarrow & \mathcal{C}_{r,\sigma} & \hookrightarrow & \mathcal{C}_{r,\sigma}^{\geq 0} \end{array}$$

Theorem: Small model

$$\mathcal{C}\text{ourant}_\sigma^{(r-1)} \simeq \text{hofib}(\mathfrak{g}_{r,\sigma}^{\leq 0} \hookrightarrow \mathfrak{g}_{r,\sigma}).$$

The same statement with $r+1$ and zero twist models $\mathcal{C}\text{ourant}^{(r)}$.

- The inclusion of the small Cartan model into the large function algebra is not a quasi-isomorphism in every degree, and we do not need it to be. What matters is that the two algebras have the same negative part.
- The diagram makes this precise. The right vertical arrow is an equality, so the quotients by the non-negative truncations agree. Getzler's derived brackets therefore give the same higher Courant algebra.
- Equivalently, the homotopy fiber of the small inclusion is L_∞ quasi-isomorphic to $\mathcal{C}\text{ourant}_\sigma^{(r-1)}$. The untwisted algebra $\mathcal{C}\text{ourant}^{(r)}$ has the identical model after replacing r by $r+1$ and setting the twist to zero.
- From now on we can forget the large algebras $\mathcal{C}$ and work entirely with the explicit Cartan semidirect products $\mathfrak{g}$. This is the main technical simplification.

It remains to construct a homotopy-commutative square

$$\begin{array}{ccc} \mathfrak{g}_{r,\sigma}^{\geq 0} & \hookrightarrow & \mathfrak{g}_{r,\sigma} \\ \downarrow & & \downarrow \\ \mathfrak{g}_{r+1}^{\geq 0} & \hookrightarrow & \mathfrak{g}_{r+1}. \end{array}$$

Upshot

Its map on homotopy fibers will be

$$\mathsf{Courant}_{\sigma}^{(r-1)} \longrightarrow \mathsf{Courant}^{(r)}.$$

We first solve the untwisted problem, then twist the by σ .

Canonical L_∞ -morphisms in higher Courant geometry

└ Canonical morphisms via homotopy fibers

└ From twisted to untwisted higher Courant algebras

└ The reduced comparison square

It remains to construct a homotopy-commutative square

$$\begin{array}{ccc}
 \mathfrak{g}_{r,\sigma}^{>0} & \xrightarrow{\quad} & \mathfrak{g}_{r,\sigma} \\
 \downarrow & & \downarrow \\
 \mathfrak{g}_{r+1}^{>0} & \xrightarrow{\quad} & \mathfrak{g}_{r+1}
 \end{array}$$

Update

Its map on homotopy fibers will be
 $\mathfrak{Courant}_\sigma^{(r-1)} \longrightarrow \mathfrak{Courant}^{(r)}.$ We first solve the untwisted problem, then twist the by σ .

- We have reduced the theorem to this square. The horizontal arrows are simply the inclusions of the non-negative truncations into the small dg-Lie algebras.
- If we construct compatible vertical L_∞ -morphisms, naturality of the homotopy fiber gives the required map from the twisted degree $r - 1$ Courant algebra to the untwisted degree r algebra.
- There are two separate issues: increasing the shift from r to $r + 1$, and removing the Maurer–Cartan twist σ . We handle them in that order.
- First we set the twist aside and construct an untwisted morphism $\mathfrak{g}_r \rightarrow \mathfrak{g}_{r+1}$. Then we twist this morphism by σ . This is the conceptual midpoint of the proof.

Define $\Phi_1: \mathfrak{g}_r \rightarrow \mathfrak{g}_{r+1}$ by

$$\Phi_1(\omega) = d\omega, \quad \Phi_1(\iota_X) = \iota_X, \quad \Phi_1(\mathcal{L}_X) = \mathcal{L}_X.$$

This is a morphism of cochain complexes and preserves the non-negative truncations.

Failure of strictness

It is not a dg-Lie algebra morphism because

$$d \iota_X \omega \neq \iota_X d\omega.$$

The discrepancy is controlled by Cartan's identity $[d, \iota_X] = \mathcal{L}_X$.

Canonical L_∞ -morphisms in higher Courant geometry

- └ Canonical morphisms via homotopy fibers
 - └ From twisted to untwisted higher Courant algebras
 - └ The untwisted chain map

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The untwisted chain map

Define $\Phi_1: \mathfrak{g}_r \rightarrow \mathfrak{g}_{r+1}$ by

$$\Phi_1(\omega) = d\omega, \quad \Phi_1(i_X) = i_X, \quad \Phi_1(\mathcal{L}_X) = \mathcal{L}_X.$$

This is a morphism of cochain complexes and preserves the non-negative truncations.

Failure of strictness

It is not a dg-Lie algebra morphism because

$$d i_X \omega \neq i_X d\omega.$$

The discrepancy is controlled by Cartan's identity $[d, i_X] = \mathcal{L}_X$.

- The unary component is the obvious one. It differentiates the form component and acts identically on contractions and Lie derivatives. This compensates for the one-unit change in the shift.
- The map commutes with the differentials and sends the non-negative truncation of $\mathfrak{g}_r$ to that of $\mathfrak{g}_{r+1}$. It is therefore a chain map of the exact form we want.
- It is not a strict dg-Lie morphism. The only failure occurs when we bracket a contraction with a form: differentiating after contracting is not the same as contracting after differentiating.
- Cartan's identity measures the discrepancy. This is a useful failure, because it tells us exactly what the second Taylor coefficient must be: contraction of the form by the vector field.

Add one non-zero higher component:

$$\Phi_2(\iota_X, \omega) = \iota_X \omega,$$

$$\Phi_2(\omega, \iota_X) = \pm \iota_X \omega,$$

with $\Phi_2(a, b) = 0$ in all other cases.

Theorem: Untwisted morphism

$$\Phi = (\Phi_1, \Phi_2, 0, \dots): \mathfrak{g}_r \longrightarrow \mathfrak{g}_{r+1}$$

is an L_∞ -morphism and restricts to the non-negative parts.

Canonical L_∞ -morphisms in higher Courant geometry

└ Canonical morphisms via homotopy fibers

└ From twisted to untwisted higher Courant algebras

└ The quadratic correction

Add one non-zero higher component:

$$\Phi_2(\iota_X \omega) = \iota_X \omega,$$

$$\Phi_2(\omega, \iota_X) = \pm \iota_X \omega,$$

with $\Phi_2(a, b) = 0$ in all other cases.

Theorem: Untwisted morphism

$$\Phi = (\Phi_1, \Phi_2, 0, \dots): \mathfrak{g}_* \longrightarrow \mathfrak{g}_{*+1}$$

is an L_∞ -morphism and restricts to the non-negative parts.

- We add a single quadratic Taylor coefficient. It is non-zero only on a contraction and a differential form, where it returns the contraction $\iota_X \omega$. The second displayed formula is determined by graded skew-symmetry and the chosen sign convention.
- This term is a homotopy correcting the failure of Φ_1 to preserve that mixed bracket. No cubic or higher corrections are needed.
- Checking the L_∞ identities now reduces to the Cartan relations from two slides ago. Thus $\Phi = (\Phi_1, \Phi_2, 0, \dots)$ is an L_∞ -morphism.
- The quadratic term also respects the truncation. On the relevant degree-zero pairs it vanishes, so the same construction restricts to the non-negative parts.

The morphism Φ and its restriction to the non-negative parts give

$$\begin{array}{ccc} \mathfrak{g}_r^{\geq 0} & \hookrightarrow & \mathfrak{g}_r \\ \Phi|_{\geq 0} \downarrow & & \downarrow \Phi \\ \mathfrak{g}_{r+1}^{\geq 0} & \hookrightarrow & \mathfrak{g}_{r+1}. \end{array}$$

Naturality of homotopy fibers produces the untwisted comparison morphism.

Canonical L_∞ -morphisms in higher Courant geometry

- └ Canonical morphisms via homotopy fibers
 - └ From twisted to untwisted higher Courant algebras
 - └ The untwisted comparison square

- The untwisted morphism and its restriction form this strictly commutative square. The horizontal maps are again the inclusions of the non-negative truncations.
- Applying the homotopy-fiber construction already gives a canonical comparison between the untwisted Courant algebras in adjacent degrees.
- However, our source should be twisted by the closed form σ . We now reintroduce it using the standard twisting procedure for L_∞ -morphisms.
- The crucial point is that σ will twist the source differential, but its image in the target will be zero. Therefore the target remains untwisted.

The morphism Φ and its restriction to the non-negative parts give

$$\begin{array}{ccc} \mathfrak{g}^{>0} & \xrightarrow{\quad} & \mathfrak{g}_+ \\ \Phi|_{>0} \downarrow & & \downarrow \eta \\ \mathfrak{g}^{>0}_{\tau+1} & \xrightarrow{\quad} & \mathfrak{g}_{\tau+1} \end{array}$$

Updated

Naturality of homotopy fibers produces the untwisted comparison morphism.

Twist Φ by the Maurer–Cartan element σ :

$$(\Phi_\sigma)_j(x_1, \dots, x_j) = \sum_{k \geq 0} \frac{1}{k!} \Phi_{k+j}(\sigma^{\otimes k}, x_1, \dots, x_j).$$

Since Φ has only unary and binary components,

$$\Phi(\sigma) = \Phi_1(\sigma) + \frac{1}{2}\Phi_2(\sigma, \sigma) = d\sigma = 0.$$

Upshot

The source becomes $\mathfrak{g}_{r,\sigma}$, while the target remains untwisted.

Canonical L_∞ -morphisms in higher Courant geometry

└ Canonical morphisms via homotopy fibers

└ From twisted to untwisted higher Courant algebras

└ Twisting by the closed form

Twist Φ by the Maurer–Cartan element σ :

$$(\Phi_\sigma)_i(x_1, \dots, x_i) = \sum_{k \geq 0} \frac{1}{k!} \Phi_{k+i}(\sigma^{\otimes k}, x_1, \dots, x_i).$$

Since Φ has only unary and binary components,

$$\Phi(\sigma) = \Phi_1(\sigma) + \frac{1}{2}\Phi_2(\sigma, \sigma) = d\sigma = 0.$$

Upshot

The source becomes $\mathfrak{g}_{r,\sigma}$, while the target remains untwisted.

- Let me recall the twisting formula. We insert copies of the Maurer–Cartan element σ into the Taylor coefficients of Φ and sum with the usual factorial weights.
- In our case the sum is finite because Φ has only unary and binary components. The quadratic term vanishes on two forms, so $\Phi(\sigma) = \Phi_1(\sigma) = d\sigma$.
- Since σ is closed, its image is zero. Twisting therefore changes the source from $\mathfrak{g}_r$ to $\mathfrak{g}_{r,\sigma}$, while the target remains the untwisted algebra $\mathfrak{g}_{r+1}$.
- The twisted Taylor coefficients still preserve the non-negative truncation. We have therefore obtained the two vertical arrows required in the comparison square.

Twisting gives a strictly commutative square of L_∞ -morphisms:

$$\begin{array}{ccc} \mathfrak{g}_{r,\sigma}^{\geq 0} & \hookrightarrow & \mathfrak{g}_{r,\sigma} \\ \downarrow \Phi_\sigma|_{\geq 0} & & \downarrow \Phi_\sigma \\ \mathfrak{g}_{r+1}^{\geq 0} & \hookrightarrow & \mathfrak{g}_{r+1}. \end{array}$$

Theorem: Fiorenza–M. [FM26]

Passing to homotopy fibers yields the canonical morphism

$$\mathfrak{Courant}_\sigma^{(r-1)} \longrightarrow \mathfrak{Courant}^{(r)}$$

for every $r \geq 1$ and every $\sigma \in \Omega_{\text{cl}}^{r+1}(M)$.

Canonical L_∞ -morphisms in higher Courant geometry

- Canonical morphisms via homotopy fibers
 - From twisted to untwisted higher Courant algebras
 - The canonical morphism

Twisting gives a strictly commutative square of L_∞ -morphisms:

$$\begin{array}{ccc} \mathfrak{g}_{r,\sigma}^{\geq 0} & \xleftarrow{\quad} & \mathfrak{g}_{r,\sigma} \\ \downarrow \phi_{r-1,\sigma} & & \downarrow \sigma_r \\ \mathfrak{g}_{r+1}^{\geq 0} & \xleftarrow{\quad} & \mathfrak{g}_{r+1} \end{array}$$

Theorem: Fiorenza–M. [FM26]

Passing to homotopy fibers yields the canonical morphism

$$\mathfrak{Courant}_{\sigma}^{(r-1)} \longrightarrow \mathfrak{Courant}^{(r)}$$

for every $r \geq 1$ and every $\sigma \in \Omega_{n-1}^{\text{cl}}(M)$.

- We can now assemble the proof. Twisting the previous square gives a strictly commutative square of L_∞ -morphisms from the twisted small Cartan model to the untwisted model one degree higher.
- Each horizontal inclusion models a higher Courant algebra as its homotopy fiber. Naturality therefore turns the whole square into the boxed morphism $\mathfrak{Courant}_{\sigma}^{(r-1)} \rightarrow \mathfrak{Courant}^{(r)}$.
- This is the main theorem of the preprint with Domenico Fiorenza. It holds for every $r \geq 1$ and requires only that σ be closed. The construction is canonical because all ingredients are natural: de Rham differential, Cartan calculus, twisting and homotopy fiber.
- When $r = 1$, the resulting morphism agrees with Zambon's original degree-one construction. Thus the higher-dimensional answer genuinely extends the motivating example.

For $\sigma \in \Omega_{\text{cl}}^{r+1}(M)$, define

$$\text{Ham}_{\sigma}^{(r-1)} = \{(X, \alpha) \mid \iota_X \sigma = -d\alpha\}.$$

The observable algebra has underlying complex

$$\mathfrak{Rogers}_{\sigma}^{(r-1)} = \left(\Omega^0 \xrightarrow{d} \dots \xrightarrow{d} \Omega^{r-2} \xrightarrow{(0,d)} \text{Ham}_{\sigma}^{(r-1)} \right).$$

Theorem: Miti-Zambon [MZ25]

The natural inclusion of complexes extends to an L_{∞} -morphism

$$\mathfrak{Rogers}_{\sigma}^{(r-1)} \longrightarrow \mathfrak{Courant}_{\sigma}^{(r-1)}.$$

Canonical L_∞ -morphisms in higher Courant geometry

- └ Canonical morphisms via homotopy fibers
 - └ From Rogers to Courant algebras
 - └ Multisymplectic observables

For $\sigma \in \Omega_{cl}^{r-1}(M)$, define

$$\text{Ham}_\sigma^{(r-1)} = \{(X, \alpha) \mid \iota_X \sigma = -d\alpha\}.$$

The observable algebra has underlying complex

$$\mathcal{H}\text{ogers}_\sigma^{(r-1)} = \left(\Omega^0 \xrightarrow{d_0} \Omega^1 \xrightarrow{d_1} \dots \xrightarrow{d_{r-2}} \Omega^{r-1} \xrightarrow{\{d, d\}} \text{Ham}_\sigma^{(r-1)} \right).$$

Theorem: Miti-Zambon [MZ25]

The natural inclusion of complexes extends to an L_∞ -morphism

$$\mathcal{H}\text{ogers}_\sigma^{(r-1)} \longrightarrow \text{Courant}_\sigma^{(r-1)}.$$

- Let me now return to the first arrow from the introduction. For a closed $(r+1)$ -form, Hamiltonian data are pairs (X, α) satisfying $\iota_X \sigma = -d\alpha$.
- The observable complex has these Hamiltonian pairs in degree zero and lower-degree forms in negative degrees. The final differential sends a form η to the pair $(0, d\eta)$.
- The natural linear map sends (X, α) to the corresponding section $X + \alpha$ of $E^{(r-1)}$ and acts identically on the lower forms, up to the ordering convention encoded in our notation.
- Joint work with Marco Zambon shows that this inclusion of complexes extends to an L_∞ -morphism into the σ -twisted higher Courant algebra. I now explain how the same homotopy-fiber philosophy recovers this morphism.

Multicontraction with σ defines an L_∞ -morphism

$$\iota_\infty \sigma: \mathfrak{X}_{\text{ham}, \sigma} \longrightarrow A_\sigma,$$

where A_σ is the truncated de Rham complex ending in exact r -forms.

Theorem: Fiorenza–Rogers–Schreiber [FRS14]

$$\mathfrak{Rogers}_\sigma^{(r-1)} \simeq \text{hofib}(\iota_\infty \sigma).$$

The higher components are the iterated contractions

$$\iota_{X_1} \cdots \iota_{X_k} \sigma.$$

Canonical L_∞ -morphisms in higher Courant geometry

- └ Canonical morphisms via homotopy fibers
 - └ From Rogers to Courant algebras
 - └ Observables as a homotopy fiber

- Restrict attention to Hamiltonian vector fields. Repeated contraction with σ defines an L_∞ -morphism from their Lie algebra to a truncated de Rham complex A_σ .
- Its unary component sends X to $\iota_X \sigma$, which is exact by the Hamiltonian condition. The higher components are precisely the iterated contractions displayed at the bottom of the slide.
- Fiorenza, Rogers and Schreiber identify the homotopy fiber of this multicontraction morphism with the Rogers observable L_∞ -algebra. This explains geometrically where the higher brackets come from.
- We now have two homotopy-fiber descriptions: one for observables and one for the twisted Courant algebra. The desired inclusion should follow from a comparison between these two diagrams.

Multicontraction with σ defines an L_∞ -morphism

$$\iota_{\infty} \sigma : \mathfrak{X}_{\text{ham}, \sigma} \longrightarrow A_{\sigma},$$

where A_σ is the truncated de Rham complex ending in exact r -forms.

Theorem: Fiorenza-Rogers-Schreiber [FRS14]

$$\mathfrak{Rogers}^{(r-1)}_\sigma \simeq \text{hofib}(\iota_{\infty} \sigma).$$

The higher components are the iterated contractions

$$\iota_{X_1} \cdots \iota_{X_r} \sigma.$$

The maps fit into an L_∞ homotopy-commutative square:

$$\begin{array}{ccc} \mathfrak{X}_{\text{ham},\sigma} & \xrightarrow{\iota_\infty \sigma} & A_\sigma \\ \mathcal{L} \downarrow & & \downarrow \\ \mathfrak{g}_{r,\sigma}^{\geq 0} & \hookrightarrow & \mathfrak{g}_{r,\sigma}. \end{array}$$

The homotopy is the multivector version of Cartan's identity:

$$[d, \iota_{X_1 \wedge \dots \wedge X_k}] = \mathcal{L}_{X_1 \wedge \dots \wedge X_k}.$$

Canonical L_∞ -morphisms in higher Courant geometry

- Canonical morphisms via homotopy fibers
 - From Rogers to Courant algebras
 - Compatibility of the two fiber models

The maps fit into an L_∞ homotopy-commutative square:

$$\begin{array}{ccc} \tilde{X}_{\text{small},\sigma} & \xrightarrow{\text{inc},\sigma} & A_\sigma \\ \downarrow \varepsilon & & \downarrow \\ \tilde{\mathcal{H}}_{\sigma,\sigma}^{\geq 0} & \xrightarrow{\quad} & \mathcal{H}_{\sigma,\sigma} \end{array}$$

The homotopy is the multivector version of Cartan's identity:

$$[d, i_{X_1, \dots, X_k}] = \mathcal{L}_{X_1, \dots, X_k}.$$

Upshot

Passing to homotopy fibers recovers $\mathcal{H}ogers_\sigma^{(r-1)} \Rightarrow \mathcal{C}ourant_\sigma^{(r-1)}$.

- This is the comparison square. On the left, a Hamiltonian vector field acts on forms by Lie derivative, giving the map into the non-negative part of the Cartan model. On the right, the truncated de Rham target includes into the full small dg-Lie algebra.
- The square does not commute strictly at every arity. Its homotopy is the higher, multivector version of Cartan's identity shown in the middle. This is the same calculus that produced the quadratic correction earlier.
- Applying naturality of homotopy fibers yields the morphism from Rogers observables to the twisted Courant algebra. Up to the chosen models, this recovers the explicit morphism constructed with Marco Zambon.
- We can therefore view both arrows of the initial sequence through one diagrammatic principle. This unified picture motivates the move from graphs of closed forms to arbitrary higher Dirac structures.

1 Introduction

2 Background

- Lie infinity algebras
- Courant algebroids
- Higher Courant Lie infinity algebras

3 Canonical morphisms via homotopy fibers

- From twisted to untwisted higher Courant algebras
- From Rogers to Courant algebras

4 Higher Dirac structures

- From graphs to higher Dirac structures

Canonical L_∞ -morphisms in higher Courant geometry

└ Higher Dirac structures

└ Outline

- This completes the results contained in arXiv:2602.14702: the twisted-to-untwisted morphism and the reinterpretation of the observable morphism through homotopy fibers.
- I now turn to ongoing work. The question is whether the same mechanism extends from graphs of closed forms to general higher Dirac structures.

- 1 Introduction
- 2 Background
 - Lie infinity algebras
 - Courant algebroids
 - Higher Courant Lie infinity algebras
- 3 Canonical morphisms via homotopy fibers
 - From twisted to untwisted higher Courant algebras
 - From Rogers to Courant algebras
- 4 Higher Dirac structures
 - From graphs to higher Dirac structures

Let $L \subset E^{(n)} = TM \oplus \wedge^n T^*M$ be an isotropic involutive subbundle. Its Hamiltonian pairs are

$$\mathrm{Ham}_L^{(n-1)} = \{ (X, \theta) \in \mathfrak{X}(M) \oplus \Omega^{n-1}(M) \mid (X, d\theta) \in \Gamma(L) \}.$$

Zambon's brackets give an observable L_∞ -algebra on

$$\mathfrak{Zambon}_L^{(n-1)} = \left(\Omega^0 \xrightarrow{d} \dots \xrightarrow{d} \Omega^{n-2} \xrightarrow{(0,d)} \mathrm{Ham}_L^{(n-1)} \right).$$

Example: Closed forms as a special case

If $L = \mathrm{graph}(\sigma)$ for $\sigma \in \Omega_{\mathrm{cl}}^{n+1}(M)$, then

$$\mathfrak{Zambon}_{\mathrm{graph}(\sigma)}^{(n-1)} \cong \mathfrak{Rogers}_\sigma^{(n-1)}.$$

Canonical L_∞ -morphisms in higher Courant geometry

- └ Higher Dirac structures
 - └ From graphs to higher Dirac structures
 - └ Higher Dirac observables

Let $L \subset E^{(n)} = TM \oplus \wedge^n T^*M$ be an isotropic involutive subbundle. Its Hamiltonian pairs are

$$\text{Ham}_L^{(n-1)} = \{(X, \theta) \in \mathfrak{X}(M) \oplus \Omega^{n-1}(M) \mid (X, d\theta) \in \Gamma(L)\}.$$

Zambon's brackets give an observable L_∞ -algebra on

$$\text{Zambon}_L^{(n-1)} = \left(\Omega^0 \xrightarrow{d} \dots \xrightarrow{d} \Omega^{n-2} \xrightarrow{(X, d\theta)} \text{Ham}_L^{(n-1)} \right).$$

Example: Closed forms as a special case

If $L = \text{graph}(\sigma)$ for $\sigma \in \Omega_{\text{cl}}^{n-1}(M)$, then

$$\text{Zambon}_{\text{graph}(\sigma)}^{(n-1)} \simeq \mathfrak{Rogers}_\sigma^{(n-1)}.$$

- I now move from the completed results of the preprint to ongoing work. Let L be a general isotropic involutive subbundle of the standard higher Courant algebroid $E^{(n)}$.
- A Hamiltonian observable is an $(n-1)$ -form θ together with a vector field X such that the pair $(X, d\theta)$ is a section of L . We retain the pair because neither X nor θ is uniquely determined by the other for a general Dirac structure.
- Zambon's brackets turn the displayed complex into an L_∞ -algebra. In degree zero it contains Hamiltonian pairs, while the negative degrees consist of lower-degree forms, exactly as in the Rogers construction.
- If L is the graph of a closed form σ , the condition becomes $\iota_X \sigma = -d\theta$, up to the graph sign convention. We therefore recover the Rogers algebra. This special case is the consistency test for every construction that follows.

For $L = \text{graph}(\sigma)$, the observable algebra is the homotopy fiber of the multicontraction morphism

$$\iota_\infty \sigma: \mathfrak{X}_{\text{ham}, \sigma} \longrightarrow A_\sigma.$$

For a general L , the Hamiltonian condition still supplies local data

$$(X, \theta) \in \text{Ham}_L^{(n-1)}, \quad (X, d\theta) \in \Gamma(L).$$

These data suggest an L_∞ -morphism $\varphi_{L, \infty}: \mathfrak{X}_{\text{ham}, L} \rightarrow A_L$ whose higher components are built from iterated contractions of $d\theta$.

Working construction

$$\begin{array}{ccc} \mathfrak{Zambon}_L^{(n-1)} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathfrak{X}_{\text{ham}, L} & \xrightarrow{\varphi_{L, \infty}} & A_L \end{array}$$

$$\mathfrak{Zambon}_L^{(n-1)} \stackrel{?}{\simeq} \text{hofib}(\varphi_{L, \infty}).$$

Canonical L_∞ -morphisms in higher Courant geometry

- └ Higher Dirac structures
 - └ From graphs to higher Dirac structures
 - └ A candidate homotopy-fiber model

A candidate homotopy-fiber model

For $L = \text{graph}(\sigma)$, the observable algebra is the homotopy fiber of the multicontraction morphism

$$i_{\infty}\sigma: \mathbb{X}_{\text{ham},\sigma} \longrightarrow A_L.$$

For a general L , the Hamiltonian condition still supplies local data

$$(X, \theta) \in \text{Ham}_L^{(n-1)}, \quad (X, d\theta) \in \Gamma(L).$$

These data suggest an L_∞ -morphism $\varphi_{L,\infty}: \mathbb{X}_{\text{ham},L} \rightarrow A_L$ whose higher components are built from iterated contractions of $d\theta$.

Working construction

$$\begin{array}{ccc} \mathbb{Z}\text{ambon}_L^{(n-1)} & \xrightarrow{\quad} & 0 \\ \downarrow & & \downarrow \\ \mathbb{X}_{\text{ham},L} & \xrightarrow{\varphi_{L,\infty}} & A_L \end{array} \quad \mathbb{Z}\text{ambon}_L^{(n-1)} \stackrel{?}{\simeq} \text{hofib}(\varphi_{L,\infty}).$$

- For a graph $L = \text{graph}(\sigma)$, the closed form provides the multicontraction morphism directly. A general L supplies no globally distinguished form, so we must reconstruct the target and the Taylor coefficients from Hamiltonian pairs.
- Given a Hamiltonian vector field X , choose a form θ_X with $(X, d\theta_X) \in \Gamma(L)$. The candidate unary term records the class of $d\theta_X$; higher terms are built from iterated contractions of $d\theta_X$ by other Hamiltonian vector fields.
- The main subtlety is independence of the chosen Hamiltonian form. Different choices differ by a form paired with the zero vector field. We therefore quotient the top-degree target by $\text{pr}_\Omega(\Gamma(L) \cap (0, d\Omega^{n-1}))$. Isotropy of L controls the higher components.
- The question mark is intentional. We are working toward proving that the resulting L_∞ -morphism $\varphi_{L,\infty}$ has Zambon's observable algebra as its homotopy fiber. This statement does not belong to arXiv:2602.14702.

The homotopy-fiber model should produce a natural comparison morphism

$$\mathfrak{Zambon}_L^{(n-1)} \dashrightarrow \mathfrak{Courant}^{(n)}$$

and recover the closed-form morphism when $L = \text{graph}(\sigma)$.

$$\begin{array}{ccc} \mathfrak{Zambon}_{\text{graph}(\sigma)}^{(n-1)} & \xrightarrow{\cong} & \mathfrak{Rogers}_{\sigma}^{(n-1)} \\ \downarrow & & \downarrow \\ \mathfrak{Courant}^{(n)} & \xlongequal{\quad} & \mathfrak{Courant}^{(n)} \end{array}$$

Twisted higher Dirac structures

For $L' \subset E_H^{(n)}$, contractions of H obstruct the naive flat dg-Lie model. This suggests curved dg-Lie algebras and curved mapping cones as the next setting.

Canonical L_∞ -morphisms in higher Courant geometry

- └ Higher Dirac structures
 - └ From graphs to higher Dirac structures
 - └ The comparison problem

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The comparison problem

The homotopy-fiber model should produce a natural comparison morphism

$$\text{Zambon}_L^{(n-1)} \dashrightarrow \text{Courant}^{(n)}$$

and recover the closed-form morphism when $L = \text{graph}(\sigma)$.

$$\begin{array}{ccc} \text{Zambon}_{\text{graph}(\sigma)}^{(n-1)} & \xrightarrow{\cong} & \text{Rogers}_\sigma^{(n-1)} \\ \downarrow & & \downarrow \\ \text{Courant}^{(n)} & \xrightarrow{\quad} & \text{Courant}^{(n)} \end{array}$$

Twisted higher Dirac structures

For $U' \subset E_{\text{cl}}^{(0)}$, contractions of H obstruct the naive flat dg-Lie model. This suggests curved dg-Lie algebras and curved mapping cones as the next setting.

Ongoing work with A. Basu and D. Fiorenza

- Once the candidate fiber model is established, naturality should produce a comparison from the higher Dirac observable algebra to the ambient untwisted higher Courant algebra one degree higher.
- The lower square is the required compatibility test. When $L = \text{graph}(\sigma)$, Zambon's algebra identifies with the Rogers algebra, and the new comparison must reduce to the canonical morphism discussed in the previous section.
- There is a further problem when the ambient Courant bracket is itself twisted by a closed form H . Terms such as $\iota_X \iota_Y H$ obstruct the naive flat dg-Lie construction already at low arity.
- This suggests replacing flat dg-Lie algebras and ordinary mapping cones by curved versions. At present this is an exploratory direction, not a theorem. The immediate goal is first to complete the untwisted higher Dirac comparison and its compatibility with graphs.

Canonical morphisms

$$\mathfrak{Rogers}_{\sigma}^{(r-1)} \longrightarrow \mathfrak{Courant}_{\sigma}^{(r-1)} \longrightarrow \mathfrak{Courant}^{(r)}$$

Both arrows arise from natural morphisms between homotopy-fiber models.

Higher Dirac outlook

For a general higher Dirac structure L , the next comparison is

$$\mathfrak{Zambon}_L^{(n-1)} \dashrightarrow \mathfrak{Courant}^{(n)}.$$

The twisted case points toward curved dg-Lie algebras and curved mapping cones.

Thank you for your attention

Canonical L_∞ -morphisms in higher Courant geometry

└ Higher Dirac structures

└ Conclusions

- Let me summarize the two completed results. First, the canonical morphism from the twisted degree- $(r-1)$ Courant algebra to the untwisted degree- r algebra exists in every degree. Second, the previously known morphism from multisymplectic observables fits the same homotopy-fiber pattern.
- The common mechanism is more important than the individual formulas: a natural square of dg-Lie or L_∞ -morphisms induces a map between homotopy fibers.
- The higher Dirac part is the next step. For a general isotropic involutive subbundle L , we are constructing the appropriate target complex and comparison map. In the twisted setting, curvature appears naturally and points toward curved dg-Lie algebras and curved mapping cones.
- Thank you for your attention. I will be happy to discuss either the explicit formulas, the homotopy-fiber models, or the ongoing higher Dirac construction.

Canonical morphisms

$$\mathcal{H}ogets^{(r-1)}_\sigma \longrightarrow \mathcal{C}ourant^{(r-1)}_\sigma \longrightarrow \mathcal{C}ourant^{(r)}$$

Both arrows arise from natural morphisms between homotopy-fiber models.

Higher Dirac outlook

For a general higher Dirac structure L , the next comparison is

$$\mathcal{H}ombon^{(r-1)}_L \longrightarrow \mathcal{C}ourant^{(r)}.$$

The twisted case points toward curved dg-Lie algebras and curved mapping cones.

Thank you for your attention

Supplementary material

- The remaining slides contain supplementary material. I will use them only in response to questions or if more detail is useful.

For a dg-Lie morphism $\chi: \mathfrak{g} \rightarrow \mathfrak{h}$, set

$$\begin{array}{ccc} \mathrm{hofib}(\chi) & \xrightarrow{\quad} & 0 \\ \downarrow & \nearrow h & \downarrow \\ \mathfrak{g} & \xrightarrow{\chi} & \mathfrak{h}. \end{array} \quad \mathrm{hofib}(\chi) = \mathfrak{g} \times_{\mathfrak{h}}^h 0.$$

A homotopy-commutative square induces a morphism between homotopy fibers:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\chi} & \mathfrak{h} \\ F \downarrow & \nearrow & \downarrow G \\ \mathfrak{g}' & \xrightarrow{\chi'} & \mathfrak{h}' \end{array} \quad \Longrightarrow \quad \mathrm{hofib}(\chi) \longrightarrow \mathrm{hofib}(\chi').$$

Canonical L_∞ -morphisms in higher Courant geometry

└ Complementary Material

└ Homotopy fibers: definition and functoriality

For a dg-Lie morphism $\chi: \mathfrak{g} \rightarrow \mathfrak{h}$, set

$$\mathrm{hofib}(\chi) = \mathfrak{g} \times_{\mathfrak{h}}^{\chi} 0.$$

A homotopy-commutative square induces a morphism between homotopy fibers:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\chi} & \mathfrak{h} \\ \downarrow \chi & \searrow & \downarrow \chi \\ \mathfrak{g}' & \xrightarrow{\chi'} & \mathfrak{h}' \end{array} \implies \mathrm{hofib}(\chi) \longrightarrow \mathrm{hofib}(\chi').$$

Functoriality of hofib produces the L_∞ -morphisms used in the talk.

- The homotopy fiber of χ is the derived version of the ordinary fiber over zero. It is defined by replacing the pullback of $\mathfrak{g} \xrightarrow{\chi} \mathfrak{h} \leftarrow 0$ with its homotopy pullback.
- Unlike the ordinary kernel, the homotopy fiber retains the homotopy witnessing that an element maps to zero. This is essential for inclusions, whose strict kernel may vanish while their homotopy fiber is non-trivial.
- The second diagram records the property used throughout the talk. A homotopy-commutative square between two morphisms induces a morphism between their homotopy fibers.
- In the dg-Lie setting this fiber is represented by an L_∞ -algebra. The next slide gives the explicit suspended mapping-cone model.

Let $\chi: \mathfrak{g} \rightarrow \mathfrak{h}$ be a morphism of dg-Lie algebras.

Its suspended mapping cone is the complex

$$\mathrm{Cone}(\chi)[-1] = \mathfrak{g} \oplus \mathfrak{h}[-1].$$

Theorem: Fiorenza–Manetti [FM07]

This complex carries a canonical L_∞ -structure and gives a model

$$\mathrm{Cone}(\chi)[-1] \simeq \mathrm{hofib}(\chi).$$

Binary and higher brackets combine the bracket of $\mathfrak{g}$, the action on $\mathfrak{h}$, and universal Bernoulli coefficients.

Canonical L_∞ -morphisms in higher Courant geometry

└ Complementary Material

└ The mapping-cone model

Let $\chi: \mathfrak{g} \rightarrow \mathfrak{h}$ be a morphism of dg-Lie algebras.

Its suspended mapping cone is the complex

$$\text{Cone}(\chi)[-1] = \mathfrak{g} \oplus \mathfrak{h}[-1].$$

Theorem: Fiorenza–Manetti [FM07]

This complex carries a canonical L_∞ -structure and gives a model

$$\text{Cone}(\chi)[-1] \simeq \text{holib}(\chi).$$

Binary and higher brackets combine the bracket of $\mathfrak{g}$, the action on $\mathfrak{h}$, and universal Bernoulli coefficients.

- This is the explicit model behind the homotopy fibers used throughout the talk. For a dg-Lie morphism $\chi: \mathfrak{g} \rightarrow \mathfrak{h}$, the underlying complex is the suspended mapping cone $\mathfrak{g} \oplus \mathfrak{h}[-1]$.
- Fiorenza and Manetti construct canonical higher brackets on this complex. The binary and higher terms combine the bracket on $\mathfrak{g}$, the action on $\mathfrak{h}$, and universal Bernoulli coefficients.
- This L_∞ -algebra represents the homotopy fiber of χ . In the main argument, χ is the inclusion of a non-negative truncation, so the mapping cone packages the negative quotient together with the correct higher derived brackets.

In local coordinates, $T[1]M$ has

$$|x^i| = 0, \quad |\theta^i| = 1.$$

Functions are polynomial in the odd variables θ^i :

$$C^\infty(T[1]M) \cong \Gamma(\wedge T^*M) = \Omega(M).$$

The degree-one vector field

$$Q = \theta^i \frac{\partial}{\partial x^i}$$

corresponds to the de Rham differential and satisfies $[Q, Q] = 0$.

Canonical L_∞ -morphisms in higher Courant geometry

└ Complementary Material

└ Differential forms on $T[1]M$ In local coordinates, $T[1]M$ has

$$|x^i| = 0, \text{quad}|\theta^j| = 1.$$

Functions are polynomial in the odd variables θ^j :

$$C^\infty(T[1]M) \cong \Gamma(\wedge T^*M) = \Omega(M).$$

The degree-one vector field

$$Q = \theta^i \frac{\partial}{\partial x^i}$$

corresponds to the de Rham differential and satisfies $[Q, Q] = 0$.

Updator

The NQ -manifold $(T[1]M, Q)$ is the graded-geometric form of the de Rham complex.

- This slide recalls the basic graded-manifold dictionary. The fiber coordinate θ^i on $T[1]M$ has degree one, so a polynomial in the odd variables transforms exactly as a differential form.
- Consequently, the algebra of functions on $T[1]M$ is the de Rham algebra $\Omega(M)$. The vector field $Q = \theta^i \partial / \partial x^i$ acts as the exterior derivative.
- The equation $[Q, Q] = 0$ is the graded-geometric expression of $d^2 = 0$. Thus $(T[1]M, Q)$ is the NQ -manifold encoding the de Rham complex used in the ambient construction.

Let (M, σ) be pre- r -plectic and let $(X_i, \alpha_i) \in \text{Ham}_\sigma^{(r-1)}$.

Up to the standard sign convention, the higher brackets are

$$\mu_k(\alpha_1, \dots, \alpha_k) = \pm \iota_{X_k} \cdots \iota_{X_1} \sigma, \quad 2 \leq k \leq r+1.$$

In particular,

$$\begin{aligned} \mu_1 &= d, \\ \mu_2(\alpha_1, \alpha_2) &= \pm \iota_{X_2} \iota_{X_1} \sigma. \end{aligned}$$

The higher Jacobi identities follow from $d\sigma = 0$ and Cartan calculus.

Canonical L_∞ -morphisms in higher Courant geometry

└ Complementary Material

└ Higher brackets of observables

- These are the explicit Rogers brackets that were suppressed in the main talk. Their inputs in degree zero are Hamiltonian pairs (X_i, α_i) .
- Up to the chosen sign convention, the k -ary bracket is obtained by contracting the pre- r -plectic form σ with all k Hamiltonian vector fields. The arity stops at $r + 1$ for degree reasons.
- The unary bracket is the de Rham differential. The higher Jacobi identities follow from closedness of σ together with the Cartan relations. This is also why multicontraction defines the L_∞ -morphism used in the fiber model.

Let (M, σ) be pre- r -plectic and let $(X_i, \alpha_i) \in \text{Ham}_0^{(r-1)}$.

Up to the standard sign convention, the higher brackets are

$$\mu_k(\alpha_1, \dots, \alpha_k) = \pm i_{X_k} \cdots i_{X_1} \sigma, \quad 2 \leq k \leq r+1.$$

In particular,

$$\begin{aligned} \mu_1 &= d, \\ \mu_2(\alpha_1, \alpha_2) &= \pm i_{X_2} i_{X_1} \sigma. \end{aligned}$$

The higher Jacobi identities follow from $d\sigma = 0$ and Cartan calculus.

Graded manifold	Algebra of functions
$T[1]M$	$\Omega(M)$
$A[1]$ for a vector bundle A	$\Gamma(\wedge A^*)$
$\mathfrak{g}[1]$ for a Lie algebra $\mathfrak{g}$	the Chevalley–Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$
$T^*[r]T[1]M$	degree- r Poisson algebra for higher Courant geometry

A homological vector field Q turns the algebra of functions into a differential graded algebra.

Canonical L_∞ -morphisms in higher Courant geometry

└ Complementary Material

└ A graded-manifold dictionary

Graded manifold	Algebra of functions
$T[1]M$	$\Omega(M)$
$A[1]$ for a vector bundle A	$\Gamma(\wedge A^*)$
$\mathfrak{g}[1]$ for a Lie algebra $\mathfrak{g}$	the Chevalley–Eilenberg algebra $CE(\mathfrak{g})$
$T^*[r]T[1]M$	degree- r Poisson algebra for higher Courant geometry

A homological vector field Q turns the algebra of functions into a differential graded algebra.

- This table collects the graded-manifold identifications used in the talk. Shifting a vector bundle A by one turns functions into sections of the exterior algebra of A^* .
- For a Lie algebra $\mathfrak{g}$, functions on $\mathfrak{g}[1]$ form its Chevalley–Eilenberg algebra. For $T[1]M$, the same construction gives differential forms.
- Passing to the shifted cotangent bundle $T^*[r]T[1]M$ adds a degree- r symplectic form and hence a Poisson bracket of degree $-r$. This is the ambient graded Poisson algebra for the higher Courant construction.
- A homological vector field Q supplies the differential, converting each algebra of functions into a differential graded algebra.

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Canonical L_∞ -morphisms in higher Courant geometry

└ Complementary Material

└ References

└ Extended Bibliography

- These are the references used in the talk. The main new result is the joint preprint with Domenico Fiorenza, while the higher Dirac extension is ongoing work with Aditya Basu and Domenico Fiorenza.
- I also acknowledge Leonid Ryvkin for the graded-manifold material on which part of the appendix is based.