

Canonical L_∞ -morphisms in higher Courant geometry: from twisted algebroids to higher Dirac structures

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based on joint work with A. BASU and D. FIORENZA



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Geometry, Topology, and Physics (GPT) Seminar



This project has received funding from
the European Union's Horizon 2020
research and innovation programme
under grant agreement N°1010194324.

Symplectic geometry

A symplectic form $\omega \in \Omega_{\text{cl}}^2(M)$ assigns to $f \in C^\infty(M)$ a Hamiltonian vector field

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The observables form a Poisson algebra

$$(C^\infty(M), \{\cdot, \cdot\}_\omega) .$$

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Higher degree

For $\sigma \in \Omega_{\text{cl}}^{r+1}(M)$, Hamiltonian data are pairs

$$(X, \alpha) \in \mathfrak{X}(M) \oplus \Omega^{r-1}(M) ,$$

$$\text{s.t. } \iota_X\sigma = -d\alpha ,$$

they assemble into an L_∞ -algebra

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Claim

Higher brackets encode the geometry of observables and their homotopies.

Fix an integer $r \geq 1$. The *standard higher Courant algebroid* is

$$E^{(r-1)} = TM \oplus \wedge^{r-1} T^*M.$$

A closed form $\sigma \in \Omega_{\text{cl}}^{r+1}(M)$ twists its bracket and determines an L_∞ -algebra

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Graphs of closed forms

The graph

$$\text{graph}(\sigma) = \{X - \iota_X \sigma \mid X \in TM\}$$

is an isotropic involutive subbundle of $E^{(r)}$.

Higher Dirac structures

General isotropic involutive subbundles

$$L \subset E^{(n)}$$

extend the graph construction and carry their own observable L_∞ -algebras.

The starting point is the sequence

$$\mathfrak{Rogers}_{\sigma}^{(r-1)} \longrightarrow \mathfrak{Courant}_{\sigma}^{(r-1)} \longrightarrow \mathfrak{Courant}^{(r)}.$$

- The first morphism embeds multisymplectic observables into the corresponding higher Courant algebra [MZ25].

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1. Higher Courant L_∞ -algebras and their homotopy-fiber models.
2. The canonical morphism from the twisted degree- $(r-1)$ algebra to the untwisted degree- r algebra.
3. The multisymplectic observable morphism in the same framework.
4. Ongoing work on observable algebras of higher Dirac structures.

1 Introduction

2 Background

- Lie infinity algebras
- Courant algebroids
- Higher Courant Lie infinity algebras

3 Canonical morphisms via homotopy fibers

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- From graphs to higher Dirac structures

Definition: Algebraic data

A graded vector space L with graded skew-symmetric maps

$$\mu_k: \wedge^k L \longrightarrow L[2 - k], \quad k \geq 1.$$

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Upshot

Equivalently, the brackets assemble into a degree-one coderivation Q_μ on $S^{\geq 1}(L[1])$ satisfying

$$Q_\mu^2 = 0.$$

The standard Courant algebroid is

$$E^{(1)} = TM \oplus T^*M.$$

Structure

► anchor $\rho(X + \alpha) = X$

► symmetric pairing

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\iota_X \beta + \iota_Y \alpha)$$

► Dorfman bracket on sections

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Example: Graphs of closed forms

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Dirac structures unify symplectic, Poisson, and constrained geometry.

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The pairing now takes values in differential forms:

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}(\iota_X \beta + \iota_Y \alpha) \in \Omega^{r-1}(M).$$

$r = 1$: ordinary exact Courant algebroid

Fix $\sigma \in \Omega_{\text{cl}}^{r+1}(M)$. Zambon's algebra has underlying complex

$$\mathfrak{Courant}_\sigma^{(r-1)} = \left(\Omega^0 \xrightarrow{d} \cdots \xrightarrow{d} \Omega^{r-2} \xrightarrow{(0,d)} \Gamma(E^{(r-1)}) \right).$$

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Low arity

- ▶ μ_1 is the de Rham differential
- ▶ μ_2 extends the skew Courant bracket

Higher arity

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Poisson algebra

$C^\infty(Q_r)$ carries a Poisson bracket of degree $-r$ and contains

$$C^\infty(T[1]M) \cong \Omega(M).$$

Differential

A canonical function $\mathcal{S}$ encodes d_{dR} . Closed forms σ twists the differential

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After shifting degrees, we obtain a dg-Lie algebra

$$\mathcal{C}_{r,\sigma} := (C^\infty(Q_r)[r], \delta_\sigma, \{\cdot, \cdot\})$$

Its negative truncation carries Getzler's derived brackets.

Theorem: Zambon–Getzler–Bandiera

There is a natural L_∞ quasi-isomorphism

$$\mathfrak{Courant}_\sigma^{(r-1)} \simeq \mathrm{hofib}(\mathcal{C}_{r,\sigma}^{\geq 0} \hookrightarrow \mathcal{C}_{r,\sigma}) .$$

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Question

Does there exist a canonical L_∞ -morphism

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Example: Zambon's morphism for $r = 1$

$$\begin{array}{ccc} 0 & \longrightarrow & \Omega^0 \oplus \mathfrak{X} \\ \downarrow & & \downarrow (d, \text{id}) \\ \Omega^0 & \xrightarrow{(d, 0)} & \Omega^1 \oplus \mathfrak{X} \end{array}$$

The vertical chain map extends to an L_∞ -morphism $\mathfrak{Courant}^{(0)}_\sigma \rightarrow \mathfrak{Courant}^{(1)}$ [Zam10].

On the underlying complexes, define

$$F_1(\eta) = d\eta, \quad F_1(\alpha, X) = (d\alpha, X).$$

Here η has degree below zero and $(\alpha, X) \in \Omega^{r-1} \oplus \mathfrak{X}$ has degree zero.

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Theorem: Fiorenza–Miti [FM26]

The chain map F_1 extends to a canonical L_{∞} -morphism $\mathfrak{Courant}_{\sigma}^{(r-1)} \longrightarrow \mathfrak{Courant}^{(r)}$.

Recall the two models

$$\mathfrak{Courant}_{\sigma}^{(r-1)} \simeq \mathrm{hofib}(\mathcal{C}_{r,\sigma}^{\geq 0} \hookrightarrow \mathcal{C}_{r,\sigma}),$$

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A homotopy-commutative square between the ambient dg-Lie models

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Inside $\mathrm{Der}(\Omega)$, Cartan calculus gives the two-term dg-Lie algebra

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Definition: Cartan relations

$$\begin{aligned} [\text{d}, \iota_X] &= \mathcal{L}_X, & [\mathcal{L}_X, \iota_Y] &= \iota_{[X, Y]}, \\ [\mathcal{L}_X, \mathcal{L}_Y] &= \mathcal{L}_{[X, Y]}, & [\iota_X, \iota_Y] &= 0. \end{aligned}$$

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Since $\mathfrak{Cartan}$ acts on differential forms, every shift $\Omega[r]$ is a $\mathfrak{Cartan}$ -module.

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The twisted differential on $\mathfrak{g}_{r,\sigma}$ is determined by

$$\begin{aligned}d_\sigma \omega &= d\omega, \\d_\sigma \iota_X &= \mathcal{L}_X + \iota_X \sigma, \\d_\sigma \mathcal{L}_X &= -\mathcal{L}_X \sigma.\end{aligned}$$

The inclusions $\mathfrak{g}_{r,\sigma} \hookrightarrow \mathcal{C}_{r,\sigma}$ identify their negative parts:

$$\begin{array}{ccccc} \mathfrak{g}_{r,\sigma}^{\geq 0} & \hookrightarrow & \mathfrak{g}_{r,\sigma} & \twoheadrightarrow & \mathfrak{g}_{r,\sigma}^{<0} \\ \downarrow & & \downarrow & & \parallel \\ \mathcal{C}_{r,\sigma}^{\geq 0} & \hookrightarrow & \mathcal{C}_{r,\sigma} & \twoheadrightarrow & \mathcal{C}_{r,\sigma}^{<0}. \end{array}$$

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Theorem: Small model

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The same statement with $r + 1$ and zero twist models $\mathfrak{Courant}^{(r)}$.

It remains to construct a homotopy-commutative square

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Its map on homotopy fibers will be

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We first solve the untwisted problem, then twist the by σ .

Define $\Phi_1: \mathfrak{g}_r \rightarrow \mathfrak{g}_{r+1}$ by

$$\Phi_1(\omega) = d\omega, \quad \Phi_1(\iota_X) = \iota_X, \quad \Phi_1(\mathcal{L}_X) = \mathcal{L}_X.$$

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This is a morphism of cochain complexes and preserves the non-negative truncations.

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Failure of strictness

It is not a dg-Lie algebra morphism because

$$d \iota_X \omega \neq \iota_X d\omega.$$

The discrepancy is controlled by Cartan's identity $[d, \iota_X] = \mathcal{L}_X$.

Add one non-zero higher component:

$$\Phi_2(\iota_X, \omega) = \iota_X \omega,$$

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Theorem: Untwisted morphism

$$\Phi = (\Phi_1, \Phi_2, 0, \dots): \mathfrak{g}_r \longrightarrow \mathfrak{g}_{r+1}$$

is an L_∞ -morphism and restricts to the non-negative parts.

The morphism Φ and its restriction to the non-negative parts give

$$\begin{array}{ccc} \mathfrak{g}_r^{\geq 0} & \hookrightarrow & \mathfrak{g}_r \\ \Phi|_{\geq 0} \downarrow & & \downarrow \Phi \\ \mathfrak{g}_{r+1}^{\geq 0} & \hookrightarrow & \mathfrak{g}_{r+1}. \end{array}$$

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Naturality of homotopy fibers produces the untwisted comparison morphism.

Twist Φ by the Maurer–Cartan element σ :

$$(\Phi_\sigma)_j(x_1, \dots, x_j) = \sum_{k \geq 0} \frac{1}{k!} \Phi_{k+j}(\sigma^{\otimes k}, x_1, \dots, x_j).$$

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$$\Phi(\sigma) = \Phi_1(\sigma) + \frac{1}{2}\Phi_2(\sigma, \sigma) = d\sigma = 0.$$

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Upshot

The source becomes $\mathfrak{g}_{r,\sigma}$, while the target remains untwisted.

Twisting gives a strictly commutative square of L_∞ -morphisms:

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Theorem: Fiorenza–M. [FM26]

Passing to homotopy fibers yields the canonical morphism

$$\mathfrak{Courant}_\sigma^{(r-1)} \longrightarrow \mathfrak{Courant}^{(r)}$$

for every $r \geq 1$ and every $\sigma \in \Omega_{\text{cl}}^{r+1}(M)$.

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$$\text{Ham}_{\sigma}^{(r-1)} = \{(X, \alpha) \mid \iota_X \sigma = -d\alpha\}.$$

The observable algebra has underlying complex

$$\mathfrak{Rogers}_{\sigma}^{(r-1)} = \left(\Omega^0 \xrightarrow{d} \dots \xrightarrow{d} \Omega^{r-2} \xrightarrow{(0,d)} \text{Ham}_{\sigma}^{(r-1)} \right).$$

Theorem: Miti-Zambon [MZ25]

The natural inclusion of complexes extends to an L_{∞} -morphism

$$\mathfrak{Rogers}_{\sigma}^{(r-1)} \longrightarrow \mathfrak{Courant}_{\sigma}^{(r-1)}.$$

Multicontraction with σ defines an L_∞ -morphism

$$\iota_\infty \sigma: \mathfrak{X}_{\text{ham}, \sigma} \longrightarrow A_\sigma,$$

where A_σ is the truncated de Rham complex ending in exact r -forms.

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Theorem: Fiorenza–Rogers–Schreiber [FRS14]

$$\mathfrak{Rogers}_\sigma^{(r-1)} \simeq \text{hofib}(\iota_\infty \sigma).$$

The higher components are the iterated contractions

$$\iota_{X_1} \cdots \iota_{X_k} \sigma.$$

The maps fit into an L_∞ homotopy-commutative square:

$$\begin{array}{ccc} \mathfrak{X}_{\text{ham},\sigma} & \xrightarrow{\iota_\infty \sigma} & A_\sigma \\ \mathcal{L} \downarrow & & \downarrow \\ \mathfrak{g}_{r,\sigma}^{\geq 0} & \hookrightarrow & \mathfrak{g}_{r,\sigma} \end{array}$$

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The homotopy is the multivector version of Cartan's identity:

$$[d, \iota_{X_1 \wedge \dots \wedge X_k}] = \mathcal{L}_{X_1 \wedge \dots \wedge X_k}.$$

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1 Introduction

2 Background

- Lie infinity algebras
- Courant algebroids
- Higher Courant Lie infinity algebras

3 Canonical morphisms via homotopy fibers

- From twisted to untwisted higher Courant algebras
- From Rogers to Courant algebras

4 Higher Dirac structures

- From graphs to higher Dirac structures

Let $L \subset E^{(n)} = TM \oplus \wedge^n T^*M$ be an isotropic involutive subbundle. Its Hamiltonian pairs are

$$\mathrm{Ham}_L^{(n-1)} = \{(X, \theta) \in \mathfrak{X}(M) \oplus \Omega^{n-1}(M) \mid (X, d\theta) \in \Gamma(L)\}.$$

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Example: Closed forms as a special case

If $L = \mathrm{graph}(\sigma)$ for $\sigma \in \Omega_{\mathrm{cl}}^{n+1}(M)$, then

$$\mathfrak{Zambon}_{\mathrm{graph}(\sigma)}^{(n-1)} \cong \mathfrak{Rogers}_\sigma^{(n-1)}.$$

For $L = \text{graph}(\sigma)$, the observable algebra is the homotopy fiber of the multicontraction morphism

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For a general L , the Hamiltonian condition still supplies local data

$$(X, \theta) \in \text{Ham}_L^{(n-1)}, \quad (X, d\theta) \in \Gamma(L).$$

These data suggest an L_∞ -morphism $\varphi_{L, \infty}: \mathfrak{X}_{\text{ham}, L} \rightarrow A_L$ whose higher components are built from iterated contractions of $d\theta$.

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Working construction

$$\begin{array}{ccc} \mathfrak{Zambon}_L^{(n-1)} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ \mathfrak{X}_{\text{ham}, L} & \xrightarrow{\varphi_{L, \infty}} & A_L \end{array}$$

$$\mathfrak{Zambon}_L^{(n-1)} \stackrel{?}{\simeq} \text{hofib}(\varphi_{L, \infty}).$$

The homotopy-fiber model should produce a natural comparison morphism

$$\mathfrak{Zambon}_L^{(n-1)} \dashrightarrow \mathfrak{Courant}^{(n)}$$

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Twisted higher Dirac structures

For $L' \subset E_H^{(n)}$, contractions of H obstruct the naive flat dg-Lie model. This suggests curved dg-Lie algebras and curved mapping cones as the next setting.

Canonical morphisms

$$\mathcal{R}ogers_{\sigma}^{(r-1)} \longrightarrow \mathcal{C}ourant_{\sigma}^{(r-1)} \longrightarrow \mathcal{C}ourant^{(r)}$$

Both arrows arise from natural morphisms between homotopy-fiber models.

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For a general higher Dirac structure L , the next comparison is

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Thank you for your attention

Supplementary material

For a dg-Lie morphism $\chi: \mathfrak{g} \rightarrow \mathfrak{h}$, set

$$\begin{array}{ccc} \mathrm{hofib}(\chi) & \xrightarrow{\quad} & 0 \\ \downarrow & \lrcorner \nearrow h & \downarrow \\ \mathfrak{g} & \xrightarrow{\chi} & \mathfrak{h} \end{array}$$

$$\mathrm{hofib}(\chi) = \mathfrak{g} \times_{\mathfrak{h}}^h 0.$$

A homotopy-commutative square induces a morphism between homotopy fibers:

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\chi} & \mathfrak{h} \\ F \downarrow & \nearrow & \downarrow G \\ \mathfrak{g}' & \xrightarrow{\chi'} & \mathfrak{h}' \end{array} \quad \Longrightarrow \quad \mathrm{hofib}(\chi) \longrightarrow \mathrm{hofib}(\chi').$$

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Theorem: Fiorenza–Manetti [FM07]

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Binary and higher brackets combine the bracket of $\mathfrak{g}$, the action on $\mathfrak{h}$, and universal Bernoulli coefficients.

In local coordinates, $T[1]M$ has

$$|x^i| = 0, \quad |\theta^i| = 1.$$

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Up to the standard sign convention, the higher brackets are

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In particular,

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The higher Jacobi identities follow from $d\sigma = 0$ and Cartan calculus.

Graded manifold	Algebra of functions
$T[1]M$	$\Omega(M)$
$A[1]$ for a vector bundle A	$\Gamma(\wedge A^*)$
$\mathfrak{g}[1]$ for a Lie algebra $\mathfrak{g}$	the Chevalley–Eilenberg algebra $\mathrm{CE}(\mathfrak{g})$
$T^*[r]T[1]M$	degree- r Poisson algebra for higher Courant geometry

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