# Sheaf Cohomology for Locally Compact Spaces

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These are the notes of a mini-course of five lectures given at Sheffield in the spring of 2025. The purpose of the course was to discuss sheaf cohomology of topological spaces, starting from the basic definitions and building up to sheaf cohomology with compact supports and Verdier duality. This amounts to a full six functor formalism for the derived categories of sufficiently nice topological spaces as stated in the beginning of Section 4. The notes assume a modest familiarity with homological algebra, but are otherwise self-contained, and all the statements are proved in detail. Nothing in the exposition is original work, original references are easily found on the nLab, for example.

Given the short time frame, many central topics had to be left out. A full course on the subject would at least have contained the important comparison theorems relating sheaf and Cech cohomologies, and calculations of explicit examples.

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## 1 Basic Definitions for Sheaves on a Space

For a while, $X$ will be a fixed topological space.

**1.1. (Pre)sheaves** A *presheaf* on $X$ with values in abelian groups is a functor

$$A : \mathcal{O}(X)^{\mathrm{op}} \to \underline{\mathrm{Ab}},$$

where $\underline{\mathrm{Ab}}$ is the category of abelian groups, and $\mathcal{O}(X)$ is the poset of opens in $X$ ordered by inclusion. These presheaves and natural transformations between them form a category $\mathrm{PSh}(X, \underline{\mathrm{Ab}})$. Such a presheaf $A$ is called a *sheaf* if for each family $\{U_i\}$ of open sets with union $U = \bigcup U_i$, the diagram

$$0 \to A(U) \xrightarrow{r_1} \prod_i A(U_i) \xrightarrow{r_2} \prod_{i,j} A(U_i \cap U_j) \tag{1}$$

is exact. Here $r_1(a)_i = a|U_i$ where $a \mapsto a|U_i$ is given by the presheaf structure $A(U) \to A(U_i)$. And for a family $\{a_i\} \in \prod_i A(U_i)$, $r_2(\{a_i\})_{i,j} = (a_i|U_i \cap U_j) -$

$(a_j|U_i \cap U_j)$. In other words, $A$ is a sheaf if any family $\{a_i \in A(U_i)\}$ which agree on overlaps $U_i \cap U_j$ gives a *unique* $a \in A(U)$ with $a|U_i = a_i$.

We write $\underline{\mathrm{Ab}}(X)$ for the category of sheaves of abelian groups on $X$, as a full subcategory of that of presheaves. For a sheaf $A$, one often writes $\Gamma(U, A)$ for $A(U)$, and refers to it as the group of *sections* of $A$ over $U$.

**1.2. Examples** Typical examples of sheaves are sections of a vector bundle $E \xrightarrow{\pi} X$; that is, $A(U) = \{s : U \to E | s \text{ continuous}, \pi s = \mathrm{Id}\}$. Similarly, if $X$ is a manifold, there is a sheaf $\Omega^p$ of differential $p$-forms, $\Omega^p(U) = \{p\text{-forms on } U\}$. Notice that if $X$ is *discrete*, a sheaf on $X$ is really the same as a family $\{A_x : x \in X\}$ of abelian groups.

**1.3. Remark** In much the same way, one can define sheaves on $X$ with values in other categories, as long as the category has products and equalisers to express the sheaf condition (1) as an equaliser

$$A(U) \rightarrowtail \prod A(U_i) \rightrightarrows \prod A(U_{ij}).$$

In particular, we will use the category of sheaves of $R$-modules for a commutative ring $R$, and that of bounded below cochain complexes of sheaves. The latter category will be denoted $\underline{\mathrm{Ch}}^+(X)$.

**1.4. Stalks** For a (pre)sheaf $A$, the *stalk* at a point $x$ is defined as

$$A_x = \operatorname*{colim}_{x \in U} A(U).$$

Its elements are called *germs* (of sections of $A$) *at* $x$.

**1.5. Associated Sheaf Functor** The inclusion $\underline{\mathrm{Ab}}(X) \hookrightarrow \mathrm{PSh}(X, \underline{\mathrm{Ab}})$ of the category of sheaves into that of presheaves has a left adjoint called the *associated sheaf* functor and denoted

$$\underline{a} : \mathrm{PSh}(X, \underline{\mathrm{Ab}}) \to \underline{\mathrm{Ab}}(X).$$

We will not give the explicit construction here, but just mention that $\underline{a}$ doesn't change the stalks; i.e. for a presheaf $A$ on $X$,

$$\underline{a}(A)_x = A_x. \tag{2}$$

**1.6. Exactness** The category $\underline{\mathrm{Ab}}(X)$ is an abelian category. A sequence $A \to B \to C$ is exact iff for each point $x$ the sequence $A_x \to B_x \to C_x$ of stalks is an exact sequence of abelian groups. In particular, the associated sheaf functor is exact, by (2).

**1.7. Hom and Tensor** For presheaves $A$ and $B$ on $X$, one can define $\underline{\mathrm{Hom}}$ and tensor by setting for any open $U \subseteq X$,

$$\underline{\mathrm{Hom}}(A, B)(U) = \mathrm{Hom}(A|U, B|U),$$

the group of maps of presheaves on $U$ between the restrictions of $A$ and $B$. Similarly, for a third presheaf $C$, one can define a presheaf $C \otimes A$ by

$$(C \otimes A)(U) = C(U) \otimes A(U).$$

One then has the usual mapping property,

$$\mathrm{Hom}(C \otimes A, B) = \mathrm{Hom}(C, \underline{\mathrm{Hom}}(A, B)). \tag{3}$$

If $B$ is a *sheaf* then so is $\mathrm{Hom}(A, B)$. But even if $C$ and $A$ are sheaves, $C \otimes A$ need not be. We can define a tensor product of sheaves by taking the associated sheaf of the presheaf tensor product,

$$C \otimes A := \underline{a}(C \otimes A)$$

where the left hand $\otimes$ is in $\underline{\mathrm{Ab}}(X)$ and the right-hand one is in $\mathrm{PSh}(X, \underline{\mathrm{Ab}})$. With this definition, one again has the usual mapping property (3), now for sheaves. Notice that by (2), for sheaves $C$ and $A$,

$$(C \otimes A)_x = C_x \otimes A_x$$

for stalks. In other words, although $C \otimes A$ may be a bit hard to describe explicitly, its stalks are not.

We now turn to the effect of maps between spaces on sheaves.

**1.8. Functoriality** Let $f : Y \to X$ be a map of spaces. It induces a functor $f^{-1} : \mathcal{O}(X) \to \mathcal{O}(Y)$ between posets, by composition with which we get a functor

$$f_* : \mathrm{PSh}(Y, \underline{\mathrm{Ab}}) \to \mathrm{PSh}(X, \underline{\mathrm{Ab}}), \quad f_*(B)(U) = B(f^{-1}U).$$

By general theory, this functor has a left adjoint given by Kan extension,

$$L_{f^{-1}} : \mathrm{PSh}(X, \underline{\mathrm{Ab}}) \to \mathrm{PSh}(Y, \underline{\mathrm{Ab}}).$$

The general formula for the Kan extension in this case looks like

$$L_{f^{-1}}(A)(V) = \operatorname*{colim}_{V \subseteq f^{-1}(U)} A(U), \tag{4}$$

for open sets $V \subseteq Y$ and $U \subseteq X$.

The functor $f_*$ is easily seen to map sheaves to sheaves. So we obtain an adjoint pair

$$f^* : \underline{\mathrm{Ab}}(X) \rightleftarrows \underline{\mathrm{Ab}}(Y) : f_*$$

(left adjoint on the left and on top) by composing $L_{f^{-1}}$ with the associated sheaf functor,

$$f^*(A) = \underline{a}(L_{f^{-1}}(A)).$$

The explicit formula is not so important. But the formula for the stalks

$$f^*(A)_y = A_{f(y)} \qquad (y \in Y)$$

is. This formula follows easily from (2) and (4).

Notice that for a composition $Z \xrightarrow{g} Y \xrightarrow{f} X$, we have $f_* g_* = (fg)_*$, and hence $g^* f^* = (fg)^*$.

**Examples** Some trivial but useful special cases:

- For $x : \mathrm{pt} \to X$, $x^*(A) = A_x$ is the stalk. We leave it to you to figure out what $x_*(B)$ looks like for an abelian group $B$.
- For $\gamma : X \to \mathrm{pt}$, $\gamma_*(A) = \Gamma(X, A) = A(X)$, is the group of *global sections* of the sheaf $A$. For an abelian group $B$, the sheaf $\gamma^*(B)$ is called the *constant sheaf* on $X$ corresponding to $B$. (What does it look like?)
- Let $X_{\mathrm{dis}}$ be $X$ with the discrete topology. A sheaf on $X_{\mathrm{dis}}$ is just a family of abelian groups $B = \{B_x : x \in X\}$ indexed by the points of $X$. Write $\pi : X_{\mathrm{dis}} \to X$ for the evident map. Then $\pi^*(A)_x = A_x$ for a sheaf $A$ on $X$ and any point $x$ in $X$. So $\pi_* \pi^*(A)(U) = \prod_{x \in U} A_x$. Notice that the unit

  $$A \to \pi_* \pi^*(A)$$

  of the adjunction is mono (i.e. $0 \to A \to \pi_* \pi^* A$ is exact). (It has a retraction $\pi_* \pi^*(A)_x \to A_x$ which projects to $x$.)

**1.9. Injectives** If $f : Y \to X$, then $f^* : \underline{\mathrm{Ab}}(X) \to \underline{\mathrm{Ab}}(Y)$ is exact, hence its right adjoint $f_* : \underline{\mathrm{Ab}}(Y) \to \underline{\mathrm{Ab}}(X)$ preserves injectives. It follows that the category $\underline{\mathrm{Ab}}(X)$ has *enough injectives*. Indeed, the category $\underline{\mathrm{Ab}}$ of abelian groups does, so if $A$ is a sheaf on $X$, we can embed each stalk $A_x$ into an injective, say $A_x \to I_x$. The family $I_x$ forms an injective sheaf $I$ on $X_{\mathrm{dis}}$, and we obtain an embedding of the sheaf $A$ into an injective by composing the maps

$$A \rightarrowtail \pi_* \pi^*(A) \rightarrowtail \pi_*(I)$$

where $\pi^*(A) \to I$ is the family $A_x \to I_x$ over $X_{\mathrm{dis}}$.

**1.10. Supports** Let $A$ be a sheaf on $X$, and let $a \in A(U)$. The *support* of $a$ is the set

$$\mathrm{supp}(a) = \{x \mid a_x \neq 0\} \subseteq U$$

where $a_x$ is the stalk of $a$ at $x$, i.e. the image of $a$ under $A(U) \to \lim_{x \in V} A(V)$. Notice that this is a *closed* subset of $U$!

**1.11. The Functor $f_!$** Again let $f : Y \to X$. Define for a sheaf $B$ on $Y$, and an open $U \subseteq X$

$$f_!(B)(U) = \{b \in B \mid \mathrm{supp}(b) \to U \text{ is a proper map}\}.$$

(A map between Hausdorff spaces is *proper* if it is closed with compact fibres.) Since the property of being proper is local, it follows that $f_!(B)$ is a sheaf on $X$.

Notice that there is a natural inclusion

$$f_!(B) \rightarrowtail f_*(B).$$

If $f : Y \to X$ is itself proper, then this inclusion is an identity, i.e. $f_! = f_*$. This applies in particular to the inclusion $F \hookrightarrow X$ of a *closed* subspace $F$ of $X$. If, on the other hand, $i : U \hookrightarrow X$ is the inclusion of an *open* subset, then for a sheaf $B$ on $U$, and $V \subseteq U$,

$$i_!(B)(V) = \{b \in B(V \cap U) \mid \mathrm{supp}(b) \text{ is closed in } V\}.$$

The latter condition is equivalent to the one that $b$ is the restriction of a section $\tilde{b}$ on $V$ which is zero outside $U$. With this description, it is easy to see that $i_!$ is left adjoint to $i^*$:

$$i_! : \underline{\mathrm{Ab}}(U) \rightleftarrows \underline{\mathrm{Ab}}(X) : i^* \qquad (i : U \hookrightarrow X \text{ open}).$$

**1.12. Notation** One final piece of notation: let $A$ be a sheaf on $X$, and let $Z \subseteq X$ be any subspace. Then, writing $j : Z \hookrightarrow X$ for the inclusion, one defines

$$\Gamma(Z, A) := \Gamma(Z, j^*A) = j^*(A)(Z),$$

and refers to its elements as *sections of $A$ over $Z$*.

# 2 Sheaf Cohomology

We define the cohomology groups $H^i(X, A)$ of a topological space $X$ with coefficients in the sheaf $A$, and discuss some basic properties. We will use the expression "by HA" to mean "by general facts and methods in homological algebra".

### 2.1. Definition

- Let $A$ be a sheaf of abelian groups on $X$. Since $\underline{\mathrm{Ab}}$ has enough injectives, $A$ has a resolution by injective sheaves; i.e. an exact sequence of sheaves

$$0 \to A \to I^0 \to I^1 \to \cdots$$

with each $I^p$ injective. This gives a complex $\Gamma(X, I^\bullet)$ of abelian groups, and by definition we have

$$H^p(X, A) = H^p(\Gamma(X, I^\bullet)).$$

In other words, in terms of derived functors,

$$H^p(X, A) = (R^p\Gamma)(A).$$

- Similarly, for a map $f\colon Y \to X$ and a sheaf $B$ on $Y$ with injective resolution $0 \to B \to J^\bullet$, the derived functors of $f_*$ give cohomology sheaves on $X$ via
$$R^p f_* B = H^p(f_* J^\bullet).$$

**2.2. A Bit of Homological Algebra** Let us reformulate things a bit in the language of derived categories. A morphism $A^\bullet \to B^\bullet$ in $\underline{\mathrm{Ch}}^+(X)$ is called a *quasi-isomorphism* (q.i.) if $A^\bullet_x \to B^\bullet_x$ is a quasi-ismomorphism for each $x \in X$. The derived category $D^+(X)$ is obtained from $\underline{\mathrm{Ch}}^+(X)$ by inverting the q.i.'s. It is equivalent to the category of objects $I^\bullet$ in $\underline{\mathrm{Ch}}^+(X)$ with each $I^p$ injective, and (cochain) homotopy classes of morphisms between them. Let $\mathbb{Z}[p]$ be the complex given by the constant sheaf $\mathbb{Z}$ concentrated in degree $p$. Then for $0 \to A \to I^\bullet$ as above,

$$\begin{aligned} H^p(X, A) &= [\mathbb{Z}[p], I^\bullet] \\ &= \mathrm{Hom}_{D^+(X)}(\mathbb{Z}[p], A). \end{aligned}$$

Moreover, for $f\colon Y \to X$ and $0 \to B \to J^\bullet$ as above, $Rf_*(B) = f_*(J^\bullet)$ is the total right derived functor. Here $B$ is viewed as a complex concentrated in degree 0, but the same definition applies to a general complex $B^\bullet$ in $D^+(Y)$. Notice that if $f^*$ is exact, so "doesn't need to be derived", and we obtain a derived adjunction

$$f^*\colon D^+(X) \rightleftarrows D^+(Y)\colon Rf_*$$

**2.3. First Properties** Let us go back to the concrete situation of the cohomology $H^*(X, A)$ for a sheaf $A$ on $X$.

1. Functoriality: A map $f\colon Y \to X$ induces a map

$$f^*\colon H^p(X, A) \to H^p(Y, f^*A).$$

Indeed, write $\gamma_X\colon X \to \mathrm{pt}$. Then the commutative triangle

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "\gamma_Y"'] & & X \arrow[dl, "\gamma_X"] \\ & pt & \end{tikzcd}$$

induces $R\gamma_{X*} \to R\gamma_{X*}Rf_*f^* = R\gamma_{Y*}f^*$. Or more hands-on, if $0 \to A \to I^\bullet$ and $0 \to f^*A \to J^\bullet$ are injective resolutions in $\underline{\mathrm{Ab}}(X)$ and $\underline{\mathrm{Ab}}(Y)$ respectively, then $0 \to f^*A \to f^*I^\bullet$ is a resolution so HA gives a map (which is unique up to homotopy)

$$\begin{tikzcd}
f^*A \arrow[r] \arrow[d] & f^*I^\bullet \arrow[dl, dashed] \\
J^\bullet &
\end{tikzcd}$$

hence by adjunction we get a map $I^\bullet \to f_*J^\bullet$ and hence a map of complexes

$$\Gamma(X, I^\bullet) \to \Gamma(X, f_*J^\bullet) = \Gamma(Y, J^\bullet).$$

2. The long exact sequence: A short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\underline{\mathrm{Ab}}(X)$ induces by HA a long exact sequence in cohomology,

$$\cdots \to H^i(X, A) \to H^i(X, B) \to H^i(X, C) \to H^{i+1}(X, A) \to \cdots$$

3. Mayer-Vietoris: For a sheaf $A$ on $X$ and an open subset $U \subseteq X$, let us (temporarily) write $A_U = i_*i^*A$ where $i\colon U \to X$ is the inclusion. Then

$$H^*(X, A_U) = H^*(U, i^*A) \tag{5}$$

directly from the adjunction (i.e. using the left adjoint $i_!$). Now for two open sets $U$ and $V$, the sequence

$$0 \to A_{U\cup V} \to A_U \oplus A_V \to A_{U\cap V} \to 0$$

is exact if $A$ is injective. So by HA and (5) we obtain a long exact sequence in cohomology

$$\to H^i(U \cup V, A) \to H^i(U, A) \oplus H^i(V, A) \to H^i(U \cap V, A) \to H^{i+1}(U \cup V, A) \to$$

where we have simply written $H^i(U, A)$ for $H^i(U, i^*A)$ etc.

4. Cohomology of a pair: Let $F \subseteq X$ be closed, and write $j\colon F \to X$ for the inclusion. Also let $U = X \setminus F$ be the open complement and write $i\colon U \to X$. Then the sequence

$$0 \to i_!i^*A \to A \to j_*j^*A \to 0$$

for a sheaf $A$ on $X$ is exact (this can be checked on stalks). More generally, notice that for any sheaf $B$ on $F$, the stalk $j_*(B)_x$ is zero if $x \notin F$ and $B_x$ if $x \in F$. So $j_*$ is exact and preserves injectives, hence $H^i(F, B) = H^i(X, j_*B)$. In particular, if for a sheaf $A$ on $X$ we *define*

$$H^p(X, F; A) = H^p(X, i_!i^*A)$$

we obtain a long exact sequence

$$\cdots H^i(X, F; A) \to H^i(X, A) \to H^i(F, j^*A) \to H^{i+1}(X, F, ; A) \to \cdots$$

5. Proper base change: Let $f\colon Y \to X$ be a proper map (i.e. a closed map with compact fibres). Then for any sheaf $B$ on Y and any point $x$ in $X$, there is a canonical isomorphism

$$H^p(f^{-1}(x), B) \cong R^p f_*(B)_x \tag{6}$$

where on the left we have simply written $B$ for its restriction to $f^{-1}(x)$, so for $j^*B$ where $j\colon f^{-1}(x) \to Y$ is the inclusion. We will prove this a bit later on. Let us notice that it follows that for any pullback square

$$\begin{tikzcd}
Y' \arrow[r, "q"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\
X' \arrow[r, "p"'] & X
\end{tikzcd}$$

with $f$ (and hence $f'$) proper, the canonical map

$$p^* Rf_*(B) \to Rf'_* q^*(B)$$

is a quasi-isomorphism. Indeed, this follows from (6) by computing stalks at any point $x'$ in $X$.

6. Homotopy invariance: Let $f, g\colon Y \to X$ be homotopic maps. If $A$ is a sheaf on $X$ which is constant along the homotopy, then

$$f^* = g^*\colon H^p(X, A) \to H^p(Y, f^*A) = H^*(Y, g^*A).$$

More explicitly, the condition on $A$ means that for the homotopy $H\colon Y \times [0,1] \to X$ between $f$ and $g$, the sheaf $H^*(A)$ on $Y \times [0,1]$ restricts to the constant sheaf on each segment $\{y\} \times [0,1]$. We will derive this fact as a consequnce of proper base change.

**2.4. Acyclic Resolutions** Before we turn to the proofs of the last two properties listed above, let us do something else. Let $A$ be a sheaf on $X$ and let

$$0 \to A \to B^0 \to B^1 \to \cdots$$

be a (not neccesarily injective) resolution. Suppose that $H^p(X, B^q) = 0$ for each $q \geq 0$ and each $p > 0$. Then by HA, the resolution is good enough to compute the cohomology of $A$, i.e.

$$H^p(X, A) = H^p\Gamma(X, B^\bullet).$$

One says that the $B^q$ are *acyclic* (for $\Gamma$), and that $B^\bullet$ is an acyclic resolution. Let us list some examples of acyclic sheaves (the proofs of acyclicity are given in section 2.5 below).

- Any injective sheaf is of course acyclic.

- A sheaf $B$ on $X$ is called *flabby* if for any open set $U$ of $X$ the restriction $B(X) \to B(U)$ is surjective; in other words, if every section of $B$ over an open extends to the whole space. Flabby sheaves are acyclic.

- A sheaf $B$ on $X$ is called *soft* if for any closed set $Z$ in $X$, any section of $B$ over $Z$ extends to the whole space ; i.e. if $j\colon Z \to X$ is the inclusion, $\Gamma(X, B) \to \Gamma(Z, j^*B)$ is surjective. On a paracompact space $X$, any soft sheaf is acyclic, as we will see.

- A sheaf $B$ on $X$ is called *fine* if it "admits partitions of unity". More precisely, if $B$ is a sheaf of $R$-modules, for a sheaf $R$ of rings, then if $b_i \in \Gamma(U_i, B)$ for a locally finite cover $\{U_i\}$ of $X$, there are $\rho_i \in \Gamma(U_i, R)$ with $\sum \rho_i = 1$ so that we can average the local sections $b_i$ to a global $b = \sum \rho_i b_i$. (Note that any sheaf $B$ is a sheaf of $R$-modules for a "largest" sheaf of rings $\underline{\mathrm{End}}(B) = \underline{\mathrm{Hom}}(B, B)$). A typical example is the sheaf $\Omega^p$ of $p$-forms on a manifold. On a paracompact space, every fine sheaf is acyclic. In particular for a manifold $M$, the $p$-forms give a fine resolution
$$0 \to \mathbb{R} \to \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \cdots$$
of the constant sheaf $\mathbb{R}$, showing that the cohomology of this constant sheaf can be computed as the De Rham cohomology of $M$.

There are relations between the properties of being injective, flabby, soft and fine which we now list.

1. Every injective sheaf is flabby. Indeed, if we also write $\mathbb{Z}$ for the constant sheaf on a space $X$, then $\Gamma(X, A) = \mathrm{Hom}_{\underline{\mathrm{Ab}}(X)}(\mathbb{Z}, A)$. If $U$ is open in $X$ with inclusion $i\colon U \to X$ then $i_!i^*\mathbb{Z} \to \mathbb{Z}$ is mono, so if $A$ is injective then the induced map $\mathrm{Hom}_{\underline{\mathrm{Ab}}(X)}(\mathbb{Z}, A) \to \mathrm{Hom}_{\underline{\mathrm{Ab}}(X)}(i_1 i^*\mathbb{Z}, A)$ is surjective. The conclusion follows by the adjunction between $i_!$ and $i^*$.

2. On a paracompact space, every flabby sheaf is soft. This follows from the fact that for any closed $Z$ in $X$, any section over $Z$ of a sheaf $B$ on $X$ can be extended to a neighbourhood $U$ of $X$. Or more formally, writing $i\colon Z \to X$ for the inclusion, the map
$$\operatorname*{colim}_{Z \subseteq U} \Gamma(U, B) \to \Gamma(Z, i^*B)$$
is surjective. Indeed, a section $s \in \Gamma(Z, i^*B)$ is given by a compatible family of germs $s_x \in B_x$ for $x \in Z$. Each such $s_x$ is the germ of a section $s_V \in \Gamma(V, B)$ on a neighbourhood $V$ of x. Now take a locally finite cover $\{V_i\}$ of $Z$ and corresponding such sections $s_{V_i}$. Then the set $U = \{x | (s_{V_i})_x = (s_{V_j})_x \text{ whenever } x \in V_i \cap V_j\}$ contains $Z$ by compatibility and is open by local finiteness.

3. On a paracompact space, every fine sheaf $B$ is soft. Indeed, for a section $s \in \Gamma(j^*Z, B)$ we can apply partition of unity for the cover by $X \setminus Z$ and the $V_i$ as above, and the zero-section on $X \setminus Z$ and $s_{V_i}$ on $V_i$.

**2.5. Proofs of Acyclicity** The proofs that flabby and soft sheaves are acyclic follow the same pattern, and are based on the following two properties for a short exact sequence

$$0 \to A \to B \to C \to 0$$

of sheaves on $X$.

1. If $A$ is flabby (soft) then $0 \to \Gamma A \to \Gamma B \to \Gamma C \to 0$ is exact.

2. If moreover $B$ is also flabby (soft) then so is $C$.

To see that the two properties imply acyclicity, take a flabby sheaf $B$ and an injective resolution

$$0 \to B \to I^0 \to I^1 \to \cdots$$

Let $C = \mathrm{Im}(I^0 \to I^1) = \ker(I^1 \to I^2)$. Then by 1 the sequence $0 \to \Gamma(X, B) \to \Gamma(X, I^0) \to \Gamma(X, C) \to 0$ is exact, so $H^1(X, A) = 0$. Moreover, $I^{\bullet+1}$ is an injective resolution of $C$, so $H^2(X, A) = H^1(X, C) = 0$ also since $C$ is flabby by 2. Now proceed by induction. (The argument for a soft sheaf $B$ on a paracompact space is identical).

As for the proofs of 1 and 2 above, first notice that 2 follows from 1. Indeed, if $A$ and $B$ are, say, soft, then for a closed $Z \subseteq X$ the restrictions to $Z$ are again soft and so $0 \to \Gamma(Z, A) \to \Gamma(Z, B) \to \Gamma(Z, C) \to 0$ is exact by 1. So, any section of $C$ over $Z$ can be lifted to $\Gamma(Z, B)$, then extended to all of $X$ by softness of $B$, and projected back to $C$ to get the required extension to $\Gamma(X, C)$.

It thus remains to prove 1. Here the arguments for the flabby case and the soft case are slightly different. For the flabby case, take a section $c \in \Gamma(X, C)$ and look at pairs $(U, b)$ where $U$ is open in $X$ and $b \in \Gamma(U, B)$ is a lift of $c$ on $U$. By Zorn's lemma, there is a maximal such pair $(U_0, b_0)$. If $U_0 \neq X$, take a point $x \in X \setminus U_0$. By exactness of

$$0 \to A_x \to B_x \to C_x \to 0$$

we can at least lift $c$ on $B$ to a small neighbourhood of $x$, say by a section $b_x \in \Gamma(V_x, B)$. Then $b_0$ and $b_x$ differ by a section $a \in \Gamma(V_x \cap U_0, A)$ on $V_x \cap U_0$, which can be extended to all of $V_x$ because $A$ is flabby. Now if $b - b_x = a$ on $V_x \cap U_0$ then $b_0$ on $U_0$ and $b_x + a$ on $V_x$ agree on the overlap so define a section on $U_0 \cup V_x$, contradicting maximality.

The soft case proceeds in the same spirit, but involves paracompactness. Again, take a section $c \in \Gamma(X, C)$ which we attempt to lift to $B$. We can at least do so locally, say by sections $b_i \in \Gamma(V_i, B)$ on a cover $\{V_i\}$ of $X$ indexed by a set $I$. Let $\{U_i\}$ be a locally finite refinement with $U_i \subset \bar{U}_i \subset V_i$. Now look at pairs $(S, b)$ where $S \subseteq I$ and $b$ is a lift of $c$ on $\bar{U}_S = \bigcup_{i \in S} \bar{U}_i$ (which is closed). By Zorn's lemma, there exists a maximal such pair $(S_0, b_0)$. If $S_0 \neq I$, take some $i \in I \setminus S_0$ and proceed as in the flabby case, now extending $b_0 - b_i$ from $\bar{U}_S \cap \bar{U}_i$ to $\bar{U}_i$ by softness of $A$.

**2.6. Proof of Proper Base Change** Remember we still need to prove proper base change (Section 2.3, point 5). So, let $f\colon Y \to X$ be a proper map and let $B$ be a sheaf on $Y$ with injective resolution $0 \to B \to I^\bullet$. Take a point $x \in X$ and look at the resolution

$$0 \to j^*B \to j^*I^0 \to j^*I^1 \to \cdots$$

on the fiber $f^{-1}(x)$, where $j\colon f^{-1}(x) \to Y$ is the inclusion. Now, first of all, we claim that this is a resolution by soft sheaves. Indeed, any section of $j^*(I^p)$ over a closed $K \subseteq f^{-1}(x)$ can be extended to an open neighbourhood $U$ of $K$ by the argument in Section 2.3, point 2 (now just for a *finite* cover of $K$), and next to all of $X$ since $I^n$ is flabby. So this resolution $j^*(I)$ calculates $H^*(f^{-1}(x), j^*B)$. Next, and for the same reason, any section of $I^p$ over $f^{-1}(x)$ extends to an open neighbourhood $V$ of $f^{-1}(x)$, which by properness contains a neighbourhood of the form $f^{-1}(U)$ for an open $U$ which contains $x$. In other words we have

$$\begin{aligned}\Gamma(f^{-1}(x), j^*I^p) &= \lim_{f^{-1}(x)\subseteq V} \Gamma(V, I^p)\\ &= \lim_{x\in U} \Gamma(f^{-1}(U), I^p)\\ &= f_*(I^p)_x.\end{aligned}$$

Taking cohomology then gives us that

$$H^p(f^{-1}(x), j^*B) \cong R^p f_*(B)_x$$

as required.

**2.7. Proof of Homotopy Invariance** Let $H\colon Y \times [0,1] \to X$ be a homotopy between $H_0 = f$ and $H_1 = g$, and let $A$ be a sheaf on $X$ with the property that $H^*(A)$ is constant on each slice $\{y\} \times [0,1]$. Writing $\pi$ for the projection, this means that $H^*A = \pi^*B$ for a sheaf $B$ on $Y$ (one can take $B = \pi_*H^*A$). By proper base change, we have $R^p\pi_*\pi^*(B)_y = H^p([0,1], B_y)$. This is the cohomology of $[0,1]$ with constant coefficients, which, not surprisingly, we will show vanishes for $p > 0$.

Thus, the unit $B \to R\pi_*\pi^*B$ is an isomorphism. Homotopy invariance now follows formally from this. To see this, consider the diagram

$$\begin{tikzcd} Y \arrow[r, shift left, "i_0"] \arrow[r, shift right, "i_1"'] \arrow[d, "\gamma"'] & Y \times [0,1] \arrow[r, "H"] \arrow[d, "\pi"] & X \\ pt & Y \arrow[l, "\gamma"] & \end{tikzcd}$$

It suffices, of course, to show that $i_0^* = i_1^*$ because $f^* = i_1^*H$ and $g^* = i_1^*H^*$.

Recall that the map $i_0^*$ is constructed from the unit $\eta_0\colon \mathrm{Id} \to R(i_0)_* i_0^*$ as

$$\begin{tikzcd}
R(\gamma\pi)_*\pi^*B \arrow[r, "\eta_0"] \arrow[d, "i_0^*"'] & R(\gamma\pi)_*R(i_0)_*i_0^*\pi^*B \arrow[d, "\sim"] \\
R\gamma_*B \arrow[r, "="'] & R\gamma_*R(\pi i_0)_*(\pi i_0)^*B
\end{tikzcd}$$

and similarly for $i_1^*$. Thus, taking out the left outer $R\gamma_*$ and the right outer $B$, we need to show that the diagram

$$\begin{tikzcd}
R\pi_*\pi^* \arrow[r, "\eta_0"] \arrow[d, "\eta_1"'] & R\pi_*R(i_0)_*i_0^*\pi^* \arrow[d, "\sim"] \\
R\pi_*R(i_1)_*i_1^*\pi^* \arrow[r, "\sim"'] & \mathrm{Id}
\end{tikzcd}$$

commutes. But $\mathrm{Id} \to R\pi_*\pi^*$ is an isomorphism as just mentioned, and precomposing the square with this unit evidently(*) gives a commutative diagram. This finishes the proof modulo the example below.

We briefly expand on the point (*). More explicitly, this means that the composition of units is the unit. Given adjoint pairs

$$\begin{tikzcd}
\mathbb{C} \arrow[r, shift left, "F"] & \mathbb{D} \arrow[l, shift left, "G"] \arrow[r, shift left, "F'"] & \mathbb{E} \arrow[l, shift left, "G'"]
\end{tikzcd}$$

with units $\eta\colon \mathrm{Id} \to GF$ and $\eta'\colon G'F'$, then writing $\bar{\eta}$ for the unit $\mathrm{Id} \to (G'G)(F'F)$, the diagram

$$\begin{tikzcd}
\mathrm{Id} \arrow[r, "\eta"] \arrow[dr, "\bar{\eta}"'] & GF \arrow[d, "G\eta'F"] \\
 & GG'F'F
\end{tikzcd}$$

commutes.

**2.8. Example** Let $A$ be an abelian group. Then $H^p([0,1], A) = 0$ for $p > 0$ where $A$ is viewed as a constant sheaf on $[0,1]$. There are many comparison theorems between sheaf cohomology and other types of cohomology (singular, Čech, ...) from which this follows, but unfortunately we do not have time to discuss them. Here is a more pedestrian and direct argument.

Firstly, taking a resolution $0 \to F_1 \to F_0 \to A \to 0$ by free abelian groups and notice that the cohomology commutes with sums of abelian groups (more precisely, it always commutes with finite sums and here with directed colimits by compactness of $[0,1]$) and so it suffices to consider the case where $A = \mathbb{Z}$. Consider the exact sequence

$$0 \to \mathbb{Z} \to \underline{\mathbb{R}} \to \underline{S}^1 \to 0$$

where $\underline{\mathbb{R}}$ is the sheaf of continuous real valued functions and similarly for $\underline{S}^1$. The sheaf $\underline{\mathbb{R}}$ is soft, and $\Gamma([0,1], \underline{\mathbb{R}}) \to \Gamma([0,1], \underline{S}^1)$ is surjective since $S^1$ is simply connected so $H^1([0,1], \mathbb{Z}) = 0$. To see that $H^p([0,1], \mathbb{Z}) = 0$ for $p > 1$, one shows that $\underline{S}^1$ is also soft. A sketch of this argument is that a section of $\underline{S}^1$ over a closed set $F$ in $[0,1]$ can be extended to an open $U$ containing $F$, which can be taken to be a union of open intervals with disjoint closures. Lift the section to $\underline{\mathbb{R}}$ on each of these closed intervals and extend by softness of $\underline{\mathbb{R}}$. Conclude by projecting back to $\underline{S}^1$.

# 3 Cohomology with Compact Supports

From now on, all spaces are assumed locally compact Hausdorff.

**3.1. Cohomology with Compact Support** For a sheaf $A$ on $X$ and a section $a \in A(U)$, recall the support

$$\operatorname{supp}(a) = \{x \mid a_x \neq 0\}$$

which is closed in $U$. Write

$$\Gamma_c(A) = \Gamma_c(X,\, A) = \{a \in \Gamma(X,\, A) \mid \operatorname{supp}(a) \text{ is compact}\}$$

and

$$H^n_c(X, A) = R^n\Gamma_c(X, -)(A), \quad n \geq 0$$

These are the cohomology groups with compact support of the sheaf $A$.

**3.2. $c$-Soft Sheaves** This cohomology can be calculated by any $\Gamma_c$-acyclic resolution of $A$ as before. Call a sheaf $B$ on $X$ $c$-soft if for any two compact subsets $K \subset L$ of $X$, any section of $B$ over $K$ extends to $L$; in other words, if $\Gamma(L, B) \to \Gamma(K, B)$ is surjective. For example, any flabby sheaf is $c$-soft, because a section in $\Gamma(K, B)$ can first be extended to an open neighbourhood of $K$ (as in 2.4) and then to all of $X$ by flabbiness.

The proof that $c$-soft sheaves are $\Gamma_c$-acyclic follows the same pattern as for the flabby and soft sheaves in the previous section and is a consequence of the following two properties for any short exact sequence $0 \to A \to B \to C$.

1. If $A$ is $c$-soft then we have a short exact sequence

$$0 \to \Gamma_c A \to \Gamma_c B \to \Gamma_c C \to 0.$$

2. If moreover $B$ is also $c$-soft, then so is $C$.

Property 2 follows from 1 as in 2.5. So let us prove 1.

Take a section $c \in \Gamma_c(X, C)$ with compact support $K = \mathrm{supp}(c)$. We can lift $c$ locally to $b_i \in \Gamma(U_i, B)$ for a cover of $K$ by finitely many open sets $U_1, \ldots, U_n$, which we may take to be relatively compact (recall that $X$ is assumed to be locally compact). Choose a finer cover $V_1, \ldots, V_n$ with $V_i \subset \bar{V}_i \subset U_i$ and define $\bar{V}_0 = X - \bigcup V_i$ (this is just a notation, $\bar{V}_0$ is not neccesarily the closure of an open set). Let $\tilde{b}_0 \in \Gamma(\bar{V}_0, B)$ be the zero-section. Then $\tilde{b}_0$ is a lift of $c$ over $\bar{V}_0$. We will extend $\tilde{b}_0$ to a lift $\tilde{b}_l$ over $\bar{V}_0 \cup \cdots \cup \bar{V}_l$ by induction on $l$. Suppose a lift $\tilde{b}_{l-1}$ has been found. Then $\tilde{b}_{l-1}$ and $b_l$ both lift $c$ over $\bar{V}_l \cap (\bar{V}_0 \cup \cdots \cup \bar{V}_{l-1})$, so differ by a section of $A$ there, say $\tilde{b}_{l-1} = b_l + a$ on $\bar{V}_l \cap (\bar{V}_0 \cup \cdots \cup \bar{V}_{l-1})$. By $c$-softness of $A$, we can now extend $a$ to $\bar{V}_l$. Then $\tilde{b}_{l-1}$ and $b_l + a$ together define a section $\tilde{b}_l$ over $\bar{V}_0 \cup \cdots \cup \bar{V}_l$. When we reach $l = n$, this gives a lift of $c$ in $\Gamma_c(B)$.

**3.3. Change of Base** Consider a map $f : Y \to X$ and recall the functor $f_! : \underline{\mathrm{Ab}}(Y) \to \underline{\mathrm{Ab}}(X)$. This is a subfunctor of $f_*$ (hence preserves monos) and coincides with $f_*$ if $f$ is proper. A basic property of $f_!$ is the following.

**Lemma 3.1.** *The functor $f_!$ preserves c-softness.*

*Proof.* Let $B$ be a $c$-soft sheaf on $Y$ and let $b \in \Gamma(K, f_! B)$ where $K$ is a compact subset of $X$. We can extend $b$ to a section (again called) $b \in \Gamma(U, f_! B)$ on a neighbourhood $U$ of $K$ (cf 2.4). So $b \in \Gamma(f^{-1}(U), B)$ with $\mathrm{supp}(b) \to U$ proper. Let $V$ be a smaller relatively compact neighbourhood of $K$ with $K \subseteq V \subseteq \bar{V} \subseteq U$. Then $S = \mathrm{supp}(b) \cap f^{-1}(\bar{V})$ is compact. Choose a relatively compact neighbourhood $W$ of $S$ and consider the section of $B$ which is $b$ on $\bar{W} \cap f^{-1}(\bar{V})$ and zero on $\bar{W} - W$. Since $B$ is $c$-soft, we can extend this section to a section $\tilde{b}$ on all of $\bar{W}$. This section is zero on the boundary, so extends to a section, again called $\tilde{b}$ on all of $Y$ which is zero outside $W$. This last section extends $b$. Indeed, it agrees with $b$ on $\bar{W} \cap f^{-1}(\bar{V})$, while $b$ and $\tilde{b}$ are both zero outside $\bar{W}$. This proves the lemma. ☐

Next, consider for a sheaf $B$ on $Y$ and a point $x$ in $X$ the canonical map

$$f_!(B)_x \to \Gamma_c(f^{-1}(x), B) \tag{7}$$

This map is always injective. Indeed, suppose $b_x \in f_!(B)_x$ is represented by a section $b \in \Gamma(f^{-1}(U), B)$ with $\mathrm{supp}(b) \to U$ proper. Suppose $b|_{f^{-1}(x)} = 0$. Let $V$ be a relatively compact neighbourhood of $x$ with $V \subseteq \bar{V} \subseteq U$. Then $\mathrm{supp}(b) \cap f^{-1}(\bar{V})$ is compact and disjoint from $f^{-1}(x)$. So there is a relatively compact neighbourhood $W \supseteq \mathrm{supp}(b) \cap f^{-1}(\bar{V})$ with $f^{-1}(x) \cap \bar{W} = \varnothing$. Then $b = 0$ on $f^{-1}(V - f(\bar{W}))$, hence represents 0 in $f_!(B)_x$.

This map 7 is surjective if $B$ is $c$-soft. To see this, take $b \in \Gamma_c(f^{-1}(x), B)$ and let $V$ be a relatively compact neighbourhood in $Y$ of $\mathrm{supp}(b) \subseteq f^{-1}(x)$. Then the section of $B$ on the compact set $(f^{-1}(x) \cap \bar{V}) \cup (\bar{V} - V)$ which is $b$ on $f^{-1}(x) \cap \bar{V}$ and zero on $\bar{V} - V$ extends to all of $\bar{V}$ by $c$-softness of $B$, and next to all of $Y$ by zero outside $V$. This gives a global section $\tilde{b} \in \Gamma(Y, B)$ with support in $\bar{V}$ and agreeing with $b$ on $f^{-1}(x)$. Let us record this as follows.

**Lemma 3.2.** *For a map $f : Y \to X$ and a $c$-soft sheaf $B$ on $Y$, the canonical map $f_!(B)_x \to \Gamma_c(f^{-1}(x), B)$ is an isomorphism.*

It follows from the previous lemma that $c$-soft sheaves on $Y$ are $f_!$-acyclic, because their restrictions to $f^{-1}(x)$ are evidently again $c$-soft. More generally, it follows that if $\varphi : B^\bullet \to C^\bullet$ is a quasi-isomorphism between complexes of $c$-soft sheaves, then $f_!(B^\bullet) \to f_!(C^\bullet)$ is again a quasi-isomorphism. An elementary way to see this is to use the mapping cone $C(\varphi)$ defined by $C(\varphi)^n = B^{n+1} \oplus C^n$ with differential $d(b, c) = (-db, dc - \varphi b)$. A map $\varphi : B^\bullet \to C^\bullet$ is a quasi-isomorphism if and only if $C(\varphi)$ is acyclic (i.e. $C(\varphi)$ is quasi-isomorphic to zero) and $f_!$ evidently commutes with the construction of the mapping cone. We now deduce the following property:

**Proposition 3.3.** *Let $f : Y \to X$ and let $A$ be an arbitrary sheaf on $Y$. Then the canonical map*

$$R^p f_!(A)_x \to H^p_c(f^{-1}(x), A), \qquad (p \geq 0)$$

*is an isomorphism.*

*Proof.* If $0 \to A \to B^0 \to B^1 \to \cdots$ is a $c$-soft resolution, then $R^p f_!(A)_x$ is the cohomology of $f_!(B^\bullet)_x$, while $H^p_c(f^{-1}(x), A)$ is that of $\Gamma_c(f^{-1}(x), B^\bullet)$. These two agree by the lemma. $\square$

**Corollary 3.4.** *For any pullback square*

$$\begin{tikzcd} Y' \arrow[r, "q"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "p"] & X \end{tikzcd}$$

*the canonical map $p^* R^n f_! \to R^n f'_! q^*$ is an isomorphism.*

*Proof.* This follows from the previous proposition, by inspecting for a sheaf $A$ on $Y$ the map $p^* R^n f_! A \to R^n f'_! q^* A$ on stalks. $\square$

**3.4. Functoriality** Recall that a map $f : Y \to X$ induces for any sheaf $A$ on $X$ a map $f^* = H^*(X, A) \to H^*(Y, f^* A)$. The construction of this map $f^*$ depended on the unit $\mathrm{Id} \to Rf_* f^*$, and does not work for compactly supported cohomology. However, if $f$ is proper, then $f_! = f_*$ and the same construction now gives a map $f^* : H^*_c(X, A) \to H^*_c(Y, f^* A)$. So compactly supported cohomology is contravariant along proper maps.

For the same reason, the argument from lecture 2 for homotopy invariance goes through only if the relevant homotopy is itself a proper map. Explicitly, if $f, g : Y \rightrightarrows X$ are proper maps which are homotopic by a proper homotopy $H : Y \times [0, 1] \to X$, then $f^* = g^* : H^*_c(X, A) \to H^*_c(Y, f^* A) = H^*(Y, g^* A)$ for any sheaf $A$ on $X$ which is constant along $H$.

**3.5. Some Exact Sequences**

(a) As for ordinary cohomology, a short exact sequence of sheaves

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

induces a long exact sequence in compactly supported cohomology:

$$\cdots \longrightarrow H_c^n(X,A) \longrightarrow H_c^n(X,B) \longrightarrow H_c^n(X,C) \longrightarrow H_c^{n+1}(X,A) \longrightarrow \cdots$$

(b) For an open inclusion $i : U \hookrightarrow X$ and its closed complement $j : Z \hookrightarrow X$, the sequence

$$0 \longrightarrow i_! i^* A \longrightarrow A \longrightarrow j_* j^* A \longrightarrow 0$$

for a sheaf $A$ on $X$ is exact (as in 2.3 above). Since $j_* = j_!$ for a closed embedding, this gives a long exact sequence

$$\cdots \longrightarrow H_c^n(U,A) \longrightarrow H_c^n(X,A) \longrightarrow H_c^n(Z,A) \longrightarrow H_c^{n+1}(U,A) \longrightarrow \cdots$$

(where we have also just written $A$ for its restriction to $U$ and $Z$ respectively). In particular, $A$ is $c$-soft if and only if $H_c^1(U,A) = 0$ for each open $U$.

**Example** Consider $\mathbb{R}^d \subset S^d$ with closed complement the point $p$ at infinity. We already proved in lecture 2 that cohomology with constant coefficients is homotopy invariant. So by the usual induction using Mayer-Vietoris we find that $H^n(S^d,\mathbb{Z}) = \mathbb{Z}$ for $n = 0$, $d$ and zero otherwise. The long exact sequence above (for $U = \mathbb{R}^d$, $X = S^d$, $Z = \{p\}$) now gives that $H_c^n(\mathbb{R}^d,\mathbb{Z}) = \mathbb{Z}$ for $n = d$ and zero otherwise. We will come back to this example when discussing cohomological dimension.

(c) **Mayer-Vietoris for opens** For an open subset $U \subseteq X$ with embedding $i : U \to X$, let us (temporarily) write

$$A_{(U)} = i_! i^* A$$

If $A$ is injective or $c$-soft, so is $i^* A$, hence $A_{(U)}$ is $c$-soft. In particular, $H_c^n(X, A_{(U)}) = H_c^n(U, i^* A)$. The stalk of $A_{(U)}$ at a point $x$ is $A_x$ if $x \in U$ and is zero otherwise. If $U \subseteq U'$ there is an evident map $A_{(U)} \to A_{(U')}$. Now, if $V$ is another open set, we have an exact sequence

$$0 \longrightarrow A_{(U\cap V)} \longrightarrow A_{(U)} \oplus A_{(V)} \longrightarrow A_{(U\cup V)} \longrightarrow 0$$

as one easily checks. Thus, again writing $A$ also for its restrictions to open sets, we obtain a long exact Mayer-Vietoris sequence

$$\cdots \to H_c^n(U\cap V, A) \to H_c^n(U,A) \oplus H_c^n(V,A) \to H_c^n(U \cup V, A) \to$$
$$\to H_c^{n+1}(U\cap V, A) \to \cdots$$

(d) **Mayer-Vietoris for closed sets** Now suppose $F \subseteq X$ is a closed set with embedding $j : F \hookrightarrow X$. Then clearly if $A$ is a $c$-soft sheaf on $X$, its restriction $j^*(A)$ on $F$ is again $c$-soft. Moreover, $j_* = j_!$ and the unit $A \to j_*j^*A$ induces a restriction map $H_c(X, A) \to H_c(F, j^*A) = H_c(X, j_!j^*A)$. Let us write $A_F = j_!j^*A$. Now if $G$ is another closed set, the sequence

$$0 \longrightarrow A_{F\cup G} \longrightarrow A_F \oplus A_G \longrightarrow A_{F\cap G} \longrightarrow 0$$

is exact, so we obtain a long exact Mayer-Vietoris sequence

$$\cdots \to H_c^n(F \cup G, A) \to H_c^n(F, A) \oplus H_c^n(G, A) \to H_c^n(F \cap G, A) \to \\ \to H_c^{n+1}(F \cup G, A) \to \cdots$$

(**Remark:** Note the difference with the one for open sets!)

# 4 The Derived Category and the Functor $f^!$

**4.1. A Review in Terms of Derived Categories** Let us summarize what we have seen so far, and rephrase it in terms of derived categories. Let $D^+(X)$ be the derived category of bounded below cochain complexes of sheaves of abelian groups. If $R$ is a (commutative) ring, we write $D_R^+(X)$ for the analogous derived category of bounded below complexes of sheaves of $R$-modules. These categories are obtained from the categories $\underline{\mathrm{Ch}}^+(X)$ and $\underline{\mathrm{Ch}}_R^+(X)$ by inverting the quasi-isomorphisms. Recall that a map $A^\bullet \to B^\bullet$ is a quasi-isomorphism if and only if the map of stalks $A_x^\bullet \to B_x^\bullet$ is so, for every point $x$ in $X$. If $f : Y \to X$ is a continuous map, there are adjoint functors

$$f^* : \underline{\mathrm{Ch}}^+(X) \rightleftarrows \underline{\mathrm{Ch}}^+(Y) : f_*$$

The functor $f^*$ is exact and preserves quasi-isomorphisms. (Remember $f^*(A^\bullet)_x = A^\bullet_{f(x)}$.) The functor $f_*$ is only left exact and needs to be derived. One obtains a well-defined functor $Rf_* : D^+(Y) \to D^+(X)$ by setting $Rf_*(A^\bullet) = f_*(I^\bullet)$ where $A^\bullet \to I^\bullet$ is a quasi-isomorphism into a complex consisting of injective sheaves. (It suffices to take flabby sheaves, or soft ones if the spaces are paracompact.) Thus, we obtain adjoint functors

$$f^* : D^+(x) \rightleftarrows D^+(X) : f_* \tag{8}$$

where we have now written $f_*$ instead of $Rf_*$, for exhibiting it as a right adjoint between derived categories, it is evident that the right derived functor is meant. This is a customary abuse of notation.

The category $D^+(X)$ inherits an internal hom from $\underline{\mathrm{Ch}}^+(X)$ sometimes written $R\underline{\mathrm{Hom}}(B^\bullet, A^\bullet)$ for emphasis, and explicitly calculated as $\underline{\mathrm{Hom}}(B^\bullet, I^\bullet)$ for an injective resolution $A^\bullet \xrightarrow{\sim} I^\bullet$. The functor $R\underline{\mathrm{Hom}}(B^\bullet, -)$ has a left adjoint given by deriving the tensor product, sometimes written

$$C^\bullet \mapsto C^\bullet \otimes^{\mathbb{L}} B^\bullet.$$

When working with $R$-modules, this can be explicitly calculated by taking a resolution of $C^\bullet$ (or of $B^\bullet$) which is flat. Let us work over a field $k$ for simplicity, so that we can simply take the tensor product as before. This then gives the usual adjunction formula

$$\mathrm{Hom}_{D_k^+(X)}(C^\bullet \otimes B^\bullet, A^\bullet) \simeq \mathrm{Hom}_{D_k^+(X)}(C^\bullet, R\underline{\mathrm{Hom}}(B^\bullet, A^\bullet)).$$

Next, for a continuous map $f : Y \to X$ between locally compact spaces, we defined a functor

$$f_! : \underline{\mathrm{Ch}}^+(Y) \to \underline{\mathrm{Ch}}^+(X).$$

We have seen that as a consequence of Theorem 3.2 , it preserves quasi-isomorphisms between complexes of c-soft sheaves. It thus gives a well-defined functor at the level of derived categories, again denoted

$$f_! : D^+(Y) \to D^+(X)$$

(although some people might prefer to denote it more explicitly as $Lf_!$). It is calculated by taking c-soft resolutions. As per Theorem 3.4, this functor satisfies the change-of-base formula stating that for any pullback square

$$\begin{tikzcd} Y' \arrow[r, "q"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ X' \arrow[r, "p"] & X \end{tikzcd}$$

the canonical map $p^* f_! \to f'_! q^*$ is an isomorphism at the level of derived categories, as in the diagram

$$\begin{tikzcd} D^+(Y') \arrow[d, "f'_!"] & D^+(Y) \arrow[l, "q^*"'] \arrow[d, "f_!"] \arrow[dl, phantom, "\Leftarrow \simeq"] \\ D^+(X') & D^+(X) \arrow[l, "p^*"] \end{tikzcd}.$$

When working over a field, one can also check directly that the *projection formula* holds, stating that for $A^\bullet$ in $D_k^+(X)$ and $B^\bullet$ in $D_k^+(Y)$ the map

$$f_!(B^\bullet \otimes f^* A^\bullet) \xrightarrow{\sim} f_!(B^\bullet) \otimes A^\bullet$$

is an isomorphism in $D_k^+(X)$. (Look at the stalks at a point $x$ in $X$, where $A_x^\bullet$ is a complex of vector spaces.) The same formula still holds over a ring $R$ instead of the field $k$ if one takes the derived tensor. Notice also that $f^*$ preserves the tensor, ie is a strong monoidal functor.

So much for our review. We conclude this minicourse by constructing a right adjoint $f^!$ to the functor $f_!$, as in

$$f_! : D_k^+(X) \rightleftarrows D_k^+(Y) : f^! \tag{9}$$

when working over a field, and under the condition that $Y$ has *finite cohomological dimension*. Let us begin by reviewing this last condition.

**4.2. Cohomological Dimension** A locally compact space $Y$ is said to have cohomological dimension $\leq d$ if $H^n_c(Y, A) = 0$ for *any* sheaf A and any $n > d$. The cohomological dimension $\dim(Y)$ of $Y$ is the smallest such $d$. We will see later that this agrees with the usual dimension for manifolds. For now, let us only record the following lemma.

**Lemma 4.1.** *If $Y$ has cohomological dimension $\leq d$, then any sheaf $B$ on $Y$ has a c-soft resolution of length at most $d$, denoted*

$$0 \to B \to C^0 \to \cdots \to C^d \to 0$$

*Proof.* Let $0 \to B \to C^\bullet$ be any c-soft resolution, and let $D = \ker(C^d \to C^{d+1})$ then $0 \to B \to C^0 \to \cdots \to C^{d-1} \to D \to 0$ is a resolution, and it suffices to prove that $D$ is c-soft. For this we use the criterion, discussed in 3.5 (c), that $H^1_c(U, D)$ has to vanish for any open $U \subseteq Y$. It is indeed the case, since by "general HA" (a long exact sequence argument) $H^n_c(U, D) \simeq H^{n+d}_c(U, B)$ for every $n$. In particular $H^1_c(U, D) = H^{d+1}_c(U, B) = H^{d+1}_c(Y, i_! B)$ for the inclusion $i : U \hookrightarrow Y$, which is zero by assumption on $Y$. □

**Example.** For a manifold $M$ of dimension $d$ and for $k = \mathbb{R}$, the sequence $0 \to A \to A \otimes \Omega^0 \to \cdots \to A \otimes \Omega^d \to 0$ is a typical such resolution, where $\Omega^i$ is the sheaf of differential $i$-forms. (The sheaf $A \otimes \Omega^i$ is fine, hence c-soft.)

Now let us go back to the problem of constructing a right adjoint $f^!$ as above. It would be enough to have a functor $f^! : \underline{\mathrm{Ch}}^+_k(X) \to \underline{\mathrm{Ch}}^+_k(Y)$ at the level of complexes which satisfied the mapping property

$$\mathrm{Hom}_{\underline{\mathrm{Ch}}^+_k(X)}(f_! B^\bullet, A^\bullet) \simeq \mathrm{Hom}_{\underline{\mathrm{Ch}}^+_k(Y)}(B^\bullet, f^! A)$$

at least for a complex of injective sheaves $A^\bullet$ on $X$, and a complex of c-soft sheaves $B^\bullet$ on $Y$, or a c-soft resolution of an arbitrary complex $B^\bullet$ on $Y$. Now for any sheaf $C$ on $Y$ and any open set $V \subseteq Y$ with inclusion $i : V \hookrightarrow Y$ we have

$$C(V) = \mathrm{Hom}(\underline{k}_{(V)}, C)$$

Where $\underline{k}$ is the constant sheaf on $Y$ corresponding to the field $k$ and $\underline{k}_{(V)} = i_! i^*(\underline{k})$. So this suggests defining

$$f^!(A^\bullet)(V) = \underline{\mathrm{Hom}}(f_! B^\bullet_{(V)}, A^\bullet)$$

where $B^\bullet_{(V)}$ is some c-soft resolution of $\underline{k}_{(V)}$. There are a few things to check to see whether this definition works. The first obstacle is that more generally for any two bounded below complexes $M^\bullet$ and $N^\bullet$, the $\underline{\mathrm{Hom}}$-complex is graded as

$$\underline{\mathrm{Hom}}(M^\bullet, N^\bullet)^p = \prod_i \underline{\mathrm{Hom}}(M^i, N^{p+i})$$

and this is in general bounded below only if $M^\bullet$ is bounded above. So we would need a *bounded* c-soft resolution of $\underline{k}_{(V)}$. This is where the finite cohomological dimension of $Y$ comes in. Explicitly, if $Y$ is of dimension $d$, let us fix a c-soft resolution

$$\underline{k} \to \Omega^0 \cdots \to \Omega^d \to 0$$

of the constant sheaf $\underline{k}$ on $Y$. (The notation is chosen to suggest the analogy with differential forms.) Then for any open $V \subseteq Y$,

$$\underline{k}_{(V)} \to \Omega^0_{(V)} \to \cdots \to \Omega^d_{(V)} \to 0$$

is again a c-soft resolution. So we now *define*

$$f^!(A^\bullet)(V) = \underline{\mathrm{Hom}}(f_!\Omega^\bullet_{(V)}, A^\bullet)$$

for this specific resolution. Notice that $f^!(A^\bullet)$ naturally has the structure of a *presheaf* on $X$, because if $V \subseteq W$ then there is an inclusion $\Omega^\bullet_{(V)} \hookrightarrow \Omega^\bullet_{(W)}$ hence we can define $f^!(A^\bullet)(W) \to f^!(A^\bullet)(V)$ by precomposition with $f_!\Omega^\bullet_{(V)} \to f_!\Omega^\bullet_{(W)}$. Let us check that this presheaf is in fact a sheaf. This means that we have to check for an arbitrary open cover $V = \bigcup V_i$ that

$$f^!(A^\bullet)(V) \to \prod_i f^!(A^\bullet)(V_i) \rightrightarrows \prod_{i,j} f^!(A^\bullet)(V_{ij})$$

is an equalizer. Or equivalently that

$$\bigoplus_{i,j} f_!\Omega^\bullet_{(V_{ij})} \rightrightarrows \bigoplus_i f_!\Omega^\bullet_{(V_i)} \to f_!\Omega^\bullet_{(V)}$$

is a coequalizer. This can be checked for the stalks at a point $x$ which look like

$$\bigoplus_{i,j} \Gamma_c(f^{-1}(x), \Omega^\bullet_{(V_{ij})}) \rightrightarrows \bigoplus_i \Gamma_c(f^{-1}(x), \Omega^\bullet_{(V_i)}) \to \Gamma_c(f^{-1}(x), \Omega^\bullet_{(V)}) \qquad (10)$$

Now for a *directed* cover $V = \bigcup V_i$, the sheaf $\Omega^\bullet$ is the colimit of the sheaves $\Omega^\bullet_{(V_i)}$, hence

$$\Gamma_c(f^{-1}(x), \Omega_{(V)}) = \varinjlim \Gamma_c(f^{-1}(x), \Omega_{(V_i)})$$

which is another way of expressing that 10 is a coequalizer. And the case of a *finite* cover reduces (by induction) to the case of two opens. So we have to check for two opens $V$ and $W$ that, writing $V^x$ for $f^{-1}(x) \cap V$ and $W^x$ for $f^{-1}(x) \cap W$, the sequence

$$\Gamma_c(V^x \cap W^x, \Omega^\bullet) \to \Gamma_c(V^x, \Omega^\bullet) \oplus \Gamma_c(W^x, \Omega^\bullet) \to \Gamma_c(V^x \cup W^x, \Omega^\bullet) \to 0$$

is exact. This follows from the Mayer-Vietoris sequence for opens, as the next term $H^1_c(V^x \cap W^x, \Omega^\bullet)$ vanishes by c-softness of $\Omega^\bullet$. So we have proved

**Lemma 4.2.** *For any complex $A^\bullet$ on $X$, the presheaf $f^!(A)$ defined by*

$$f^!(A^\bullet)(V) = \underline{\mathrm{Hom}}(f_!\Omega^\bullet_{(V)}, A^\bullet)$$

*(where $V$ ranges over open subsets of $Y$) is a sheaf with values in bounded below complexes, ie an object of $\underline{\mathrm{Ch}}^+(Y)$.*

We still need to check that this is a well-defined functor on the derived category, and that it satisfies the adjunction property. The first property is easily taken care of by restricting the functor to complexes of injective sheaves $A^\bullet$, because weak equivalences between injectives are homotopy equivalences. (If you know this for bounded below complexes from HA but are unsure about sheaves, notice that injective sheaves are also injective as presheaves and check this for each open $V$ separately.) So now it remains to check that the adjunction property holds, ie that for *any* complex of c-soft sheaves $B^\bullet$ on $Y$ and any complex $A^\bullet$ of injectives on $X$, we have a bijective correspondence

$$\mathrm{Hom}_{D_k^+(X)}(f_! B^\bullet, A^\bullet) \simeq \mathrm{Hom}_{D_k^+(Y)}(B^\bullet, f^! A^\bullet) \tag{11}$$

between maps in the derived categories $D_k^+(X)$ and $D_k^+(Y)$. We will prove something stronger: there is a natural isomorphism of cochain complexes

$$\underline{\mathrm{Hom}}(f_!(B^\bullet \otimes \Omega^\bullet), A^\bullet) \xrightarrow[\simeq]{\theta} \underline{\mathrm{Hom}}(B^\bullet, f^! A^\bullet) \tag{12}$$

for any $A^\bullet$ in $\underline{\mathrm{Ch}}^+(X)$ and any $B^\bullet$ in $\underline{\mathrm{Ch}}^+(Y)$.

Let us first construct the map $\theta$. Working degree by degree for notational convenience, ie for sheaves $A$ and $B$ of vector spaces, suppose we are given a map $\phi : f_!(B \otimes \Omega^p) \to A$ of sheaves on $X$. We want to construct a map

$$\theta(\phi) : B \to f^!(A)^{-p}$$

of sheaves on $Y$. Let $V \subseteq Y$ be open and $b \in B(V)$. Then $\theta(\phi)_V(b)$ is to be a map

$$\theta(\phi)_V(b) : f_!(\Omega^p_{(V)}) \to A$$

of sheaves on $X$. So let $U \subseteq X$ be open, and let $\omega \in f_!(\Omega^p_{(V)})(U)$. So $\omega \in \Gamma(f^{-1}(U), \Omega^p_{(V)})$ is a section in $\Gamma(f^{-1}(U) \cap V, \Omega^p)$ with two properties :

- $\mathrm{supp}(\omega) \to U$ is a proper map
- $\omega$ can be extended to a section $\omega \in \Gamma(f^{-1}(U), \Omega^p)$ which vanishes outside $V$.

We can now define

$$\theta(\phi)_V(b)_U(\omega) = \phi_U(b \otimes \omega) \in A(U) \tag{13}$$

(This makes sense as $b \otimes \omega \in \Gamma(U, f_!(B \otimes \Omega^p))$ because $\omega \in \Gamma(f^{-1}(U), \Omega^p)$ has support inside $V$ so $b \otimes \omega$ is defined on all of $f^{-1}(U)$ although $b$ is only defined on $V$.) It is now straightforward (but a bit laborious) to check that this definition 13 gives a well-defined map 12 of complexes, natural in $A^\bullet$ and $B^\bullet$.

Next, let us notice that if we can prove that $\theta$ is a quasi-isomorphism for $A^\bullet$ injective and $B^\bullet$ c-soft then the desired isomorphism 11 follows, because $B^\bullet \xrightarrow{\sim} B^\bullet \otimes \Omega^\bullet$ hence $f_!(B^\bullet) \xrightarrow{\sim} f_!(B^\bullet \otimes \Omega^\bullet)$ because these sheaves $B^\bullet$ and $B^\bullet \otimes \Omega^\bullet$ are c-soft. For the latter, we in fact have more generally

**Lemma 4.3.** *Let $B$ and $C$ be sheaves on the space $Y$. If one of them is c-soft then so if $B \otimes C$.*

*Proof.* Let us say $C$ is c-soft. For the proof of the Lemma, we will use a resolution of $B$ that will again be of use later on. Recall that for an open $V \subseteq Y$ there is a bijective correspondence between maps $\underline{k}_{(V)} \to B$ and sections $b \in B(V)$. So there is an epimorphism of sheaves (and even of presheaves)

$$\bigoplus_{b,V} \underline{k}_{(V)} \twoheadrightarrow B$$

where the sum ranges over all such $V \subseteq Y$ and $b \in B(V)$. The kernel of this map is a sheaf of the same type, namely a sum of sheaves $\underline{k}_{(U)}$ ranging over pairs of sections $b \in B(V)$ and $b' \in B(V')$ and agreeing on $U \subseteq V \cap V'$. So we have an exact sequence

$$0 \to P_1 \to P_0 \to B \to 0$$

Where $P_0$ and $P_1$ are of the form $\bigoplus \underline{k}_{(V_j)}$ for a family of opens $V_j$. Tensoring with $C$ gives an exact sequence

$$0 \to P_1 \otimes C \to P_0 \otimes C \to B \otimes C \to 0$$

and by property 2 in 3.2 it suffices to observe that $P_1 \otimes C$ and $P_0 \otimes C$ are c-soft. This is indeed the case since each $\underline{k}_{(V)} \otimes C = i_! i^*(\underline{k}) \otimes C = i_!(i^*(\underline{k}) \otimes i^*(C)) = i_! i^*(C)$ is c-soft (where $i : V \hookrightarrow X$ denotes the inclusion). This proves the lemma. $\square$

Using the same resolution $0 \to P_1 \to P_0 \to B \to 0$, we can now easily deduce the fact that 12 is an isomorphism. Consider for single sheaves $A$ on $X$ and $B$ on $Y$ the diagram

$$\begin{array}{ccccccc}
0 \longrightarrow & \underline{\mathrm{Hom}}(f_!(B \otimes \Omega), A) & \longrightarrow & \underline{\mathrm{Hom}}(f_!(P_0 \otimes \Omega), A) & \longrightarrow & \underline{\mathrm{Hom}}(f_!(P_1 \otimes \Omega), A) \\
& \downarrow \theta & & \downarrow \theta & & \downarrow \theta \\
0 \longrightarrow & \underline{\mathrm{Hom}}(B, f^! A) & \longrightarrow & \underline{\mathrm{Hom}}(P_0, f^! A) & \longrightarrow & \underline{\mathrm{Hom}}(P_1, f^! A)
\end{array}$$

Where the rows are exact (the top row is because $f_!$ preserves exact sequences of c-soft sheaves). The two right-hand instances of $\theta$ in the diagram are isomorphisms as indicated because for a single open set $V$, the map

$$\underline{\mathrm{Hom}}(f_!(\underline{k}_{(V)} \otimes \Omega), A) \to \underline{\mathrm{Hom}}(k_{(V)}, f^! A)$$

is an isomorphism by definition of $f^!(A)$. It follows that the left-hand map $\theta$ is an isomorphism as well. The case of complexes $B^\bullet$ and $A^\bullet$ follows by naturality.

**Remark.** Notice that is follows from the isomorphism 12 (for single sheaves) that $f^!(A)$ is injective whenever $A$ is.

Let us summarize what we have proved

**Theorem 4.4.** *Let $f : Y \to X$ be a map between locally compact Hausdorff spaces, and assume that $Y$ is of finite cohomological dimension. Then the functor $f_!$ has a right adjoint $f^!$ as in*

$$f_! : D_k^+(Y) \rightleftarrows D_k^+(X) : f^!$$

*explicitly defined for an injective complex $A^\bullet$ on $X$ by*

$$f^!(A^\bullet)(V) = \underline{\mathrm{Hom}}(f_!\Omega^\bullet_{(V)}, A^\bullet)$$

*where $0 \to \underline{k} \to \Omega^0 \to \cdots \to \Omega^d \to 0$ is a c-soft resolution of the constant sheaf $\underline{k}$ on $Y$.*

We conclude this section by considering the special case where $X$ is a single point, so $f_! = \Gamma_c(Y, -)$. Then $D_k^+(X)$ is simply the derived category of bounded below cochain complexes of vector spaces (over the field $k$), and every vector space is injective. Moreover every vector space is a sum of copies of $k$ itself, so $f^!$ is determined by the sheaf $f^!(\underline{k})$ given by

$$\begin{aligned} f^!(\underline{k})^p(V) &= \underline{\mathrm{Hom}}(\Gamma_c(Y, \Omega^{-p}_{(V)}), \underline{k}) \\ &= \Gamma_c(V, \Omega^{-p})^\vee, \end{aligned}$$

the linear dual of $\Gamma_c(V, \Omega^{-p})$. This complex of sheaves

$$f^!(\underline{k})^{-d} \to f^!(\underline{k})^{-d+1} \to \cdots \to f^!(\underline{k})^0 \to 0 \tag{14}$$

is a complex of sheaves on $Y$ (where $d = \dim(Y)$) called the *dualizing complex.* Let

$$\mathcal{O}_Y = \ker(f^!(\underline{k})^{-d} \to f^!(\underline{k})^{-d+1})$$

This defines a sheaf on $Y$ called the orientation sheaf (cf. the case where $Y$ is a manifold below).

**Corollary 4.5** (Poincaré duality)**.** *Let $Y$ be a locally compact space of cohomological dimension $d$, and suppose the complex 14 is exact. Then there is a natural isomorphism*

$$H^p(Y, \mathcal{O}_Y) = H^{d-p}(Y, \underline{k})^\vee$$

*Proof.* The complex 14 shifted to the right by $d$ is an injective resolution of $\mathcal{O}_Y$, by hypothesis. Recall the shift convention $A^\bullet[p]^i = A^{i-p}$ and writing $\underline{\mathrm{Hom}}$ for the group of morphisms in the relevant derived category, $D_k^+(Y)$ and $D_k^+(pt)$, respectively. We have

$$\begin{aligned} H^p(Y, \mathcal{O}_Y) &= \underline{\mathrm{Hom}}(\underline{k}[p], f^!(\underline{k})[d]) \\ &= \underline{\mathrm{Hom}}(\underline{k}[p-d], f^!(\underline{k})) \\ &= \underline{\mathrm{Hom}}(f_!\underline{k}[p-d], \underline{k}) \\ &= \underline{\mathrm{Hom}}(\Gamma_c(Y, \Omega^\bullet)[p-d], \underline{k}) \\ &= H_c^{d-p}(Y, \underline{k})^\vee. \end{aligned}$$

□

**Example.** Let $Y$ be a manifold of (usual) dimension $d$. Let us first check that it is also of cohomological dimension $d$. We already saw in 3.5 (b) that $H_c^p(\mathbb{R}^d, \underline{k}) = 0$ unless $p = d$ for which it is $k$, so for an open chart $i : U \hookrightarrow Y$ we have $k = H_c^d(U, \underline{k}) = H_c^d(Y, \underline{k}_{(U)})$, showing that the cohomological dimension of $Y$ is at least $d$. On the other hand, the cohomological dimension of $Y$ is *local* in the sense that if a space $Y$ is covered by open sets $U$ of cohomological dimension $\leq d$, then $Y$ also has dimension $\leq d$. Indeed for a sheaf $B$ on $Y$ take a resolution

$$0 \to P_1 \to P_0 \to B \to 0$$

as in the proof of Theorem 4.3, where $P_i$ is a sum of sheaves of the form $k_{(U)}$. Then the long exact sequence gives that $H^p(Y, B) = 0$ for $p > d$ since $H^p(Y, k_{(U)}) = H^p(U, \underline{k}) = 0$ for $p > d$ by assumption.

This shows in particular that the dualizing complex has no cohomology in degrees other than $-d$. (The cohomology in degree $-p$ of the stalk $f^!(\underline{k})_x$ is $\varinjlim_{x \in U} H_c^p(U, \underline{k})^\vee$, which is zero unless $p = d$.) So the assumption of the corollary is satisfied.