INFINITE-DIMENSIONAL $\mathbb{Z}_2$ SUPERMANIFOLDS

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ABSTRACT

In this paper the theory of finite-dimensional supermanifolds of Berezin, Leites and Kostant is generalized in two directions.

First, we introduce infinite-dimensional supermanifolds "locally isomorphic" to arbitrary Banach (or, more generally, locally convex) supermanifolds. This is achieved by considering supermanifolds as functors (equipped with some additional structure) from the category of finite-dimensional Grassman superalgebras into the category of the corresponding smooth manifolds (Banach or locally convex). For example, flag supermanifolds of Banach Superspaces as well as unitary supergroups of Hilbert Superspaces are constructed.

Second, we define "generalized" supermanifolds, graded by Abelian groups \( \mathbb{Z}_2 \), instead of the group \( \mathbb{Z}_2 \) of supermanifolds. The corresponding superfields, describing, potentially, particles with more general statistics than Bose and Fermi, generally speaking, turn out to have an infinite number of components.

0. Introduction

0.1. One of the purposes of this paper is to extend the theory of finite-dimensional supermanifolds of Berezin-Leites-Kostant\(^{1-5}\) so as to include infinite-dimensional supermanifolds "locally isomorphic", in a sense, to arbitrary Banach (or, more generally, locally convex) supermanifolds.

The other purpose is to construct \( \mathbb{Z}_2 \)-graded supermanifolds, related in the same manner to \( \mathbb{Z}_2 \)-graded superalgebras as ordinary supermanifolds are related to ordinary (i.e. \( \mathbb{Z}_2 \)-graded) superalgebras. In particular, we want to have the correspondence ("super" Lie functor):

\[
\mathbb{Z}_2 \text{-Lie supergroups} \xrightarrow{\text{Slic}} \mathbb{Z}_2 \text{-graded Lie superalgebras}.
\]

0.2. An evident obstacle we are faced with trying to define infinite-dimensional supermanifold is that the language of topological spaces with sheaves of superalgebras of superfields on them is inadequate for infinite dimensions.

Hence, to extend the theory of Berezin-Leites-Kostant we are, simultaneously, to reformulate it.

This aim is achieved here by considering, say, Banach supermanifolds (the category of which is denoted further as \( \text{Man} \)) as functors from the category of finite-dimensional real Grassman superalgebras into the category of smooth Banach manifolds, equipped with some additional structure, whereas supersmooth morphisms of Banach supermanifolds are defined as functor morphisms preserving this structure.

The corresponding structure admits a simple characterization in terms of linear algebra and topology in the functor category \( \text{Man} \). Namely, Banach supermanifolds could be defined locally (as Banach superregions) as open subfunctors of some "linear supermanifolds" constructed out of Banach superspaces (Sects. 2 and 3), whereas supersmooth morphisms of Banach superregions are just
those functor morphisms, whose "weak derivative" morphism is a family of linear morphisms (in the sense of Sect.1.3). Globally-supermanifolds could be defined as functors of the category \text{Man}^{Gr}, equipped with some super-smooth atlas of Banach superspaces.

The arising neglecting functor \text{Man} \to \text{Man}^{Gr} could be interpreted as a "geometrization" of the Yoneda point functor for supermanifolds composed with the functor of "restriction to finite-dimensional superpoints" (see Sect. 8.1).

0.1. The category \text{Man} of Banach manifolds embeds in the category \text{Man}^{Gr} of Banach supermanifolds through a generalization of Beresin's "Grassmann analytic continuation" \cite{22} (see Sect. 4.2); the natural isomorphism of the subcategory of locally finite-dimensional supermanifolds with the category of supermanifolds of Beresin-Leites-Kostant is established in Sect.4.7.

0.4. Sections 5 to 7 of the present paper are devoted to a development of the theory of Banach supermanifolds along much standard lines as vector bundles (Sect.5), inverse function theorems and related topics (Sect.6), and Lie supergroups (Sect.7). From the author's point of view, the main result here, shining some additional light on the nature and metaphysics of supermanifolds, is Th.4.4.1 and its Coroll.4.4.2 stating that Banach superalgebras (of any given type) "are" algebras (of the same type) in the category \text{Man}^{Gr} and vice versa.

Among the variety of possible examples of Banach supermanifolds and Banach Lie supergroups there were chosen such as flag supermanifolds (Sect.4.6) and unitary supergroups of Hilbert superspaces (Sect.7.2).

0.5. In Sect.8 is constructed, for any Banach supermanifold \mathcal{M} its supergroup of superdiffemorphisms \text{Sm}(\mathcal{M}) (which is not, generally speaking, a Lie supergroup); besides, for any vector bundle \mathcal{E} in the category \text{Man}^{Gr} there is defined the functor \hat{\mathcal{E}} \circ \mathcal{E} in the category \text{Man}^{Gr}; under the assumption, will play an important role in the theory of infinite-dimensional representations of Lie supergroups due to the fact that actions of Lie supergroups on the vector bundle \mathcal{E} induce linear actions on the functor \hat{\mathcal{E}} \circ \mathcal{E}.

0.6. In Sect.9 the definition of the category \text{Man}^{Gr} is iterated to produce the category \text{Man}^{Gr}^{*} of "super"manifolds, which could be defined as functors of the functor category \text{Man}^{Gr^{k}} (equipped with some additional structure) or, equivalently, as functors of the functor category \text{Man}^{Gr^{k}}. The main result here is Th.9.2.1 stating that the category of \mathbb{Z}_{\geq 0}^{k} graded Banach superalgebras of any type is equivalent to the category of "ordinary" algebras of the same type in the category \text{Man}^{Gr}. Besides, a big part of elementary differential geometry (inverse function theorems, Lie theory, etc.) generalizes literally to the case of the category \text{Man}^{Gr}.

0.7. To conclude, note that one can extend as well the theory of locally convex and tame Fréchet smooth manifolds \cite{19} defining locally convex (resp. tame Fréchet) supermanifolds. The counterpart of Nash-Moser inverse function theorem is valid for tame Fréchet supermanifolds as well. This generalization will be considered in a separate publication.

Notations and conventions. Throughout the paper \text{Set}, \text{Top}, \text{Man}, \text{Thm} denote the category of sets, topological spaces, smooth Banach manifolds and smooth real Banach vector bundles, respectively; \text{Gr} will denote the subcategory of the category of real Grassmann finite-dimensional superalgebras, containing exactly one Grassmann superalgebra \Lambda^1 with 1 odd generator (in particular, \Lambda^2 - \mathcal{R}).

The category of \mathcal{D}-valued functors defined on the category \mathcal{C} is denoted \mathcal{D}^{\mathcal{C}}; the class of objects of the category \mathcal{D} is denoted here as \text{Obj}(\mathcal{D}).

If a category \mathcal{D} is a category with a terminal object, the latter will be denoted as \text{pt} and all morphisms of the type \text{pt} \to X will be called points of \text{pt}.

All vector spaces and superspaces are over the field \mathcal{X} = \mathbb{R} or \mathcal{X} = \mathcal{K}.

For \ell \in \mathbb{Z} denote \mathbb{1} the element \mathbb{1}(mod2) \in \mathbb{Z}_2^\ell; (-1)^{\mathbb{1}} = 1 and (-1)^{\mathbb{1}} = -1.

The variables \Lambda, \Lambda', etc. will run the set of objects of the category \text{Gr} (the sole exception being Sect. 9, where, it is permitted for \Lambda to run over "generalized" Grassmann superalgebras).
1. Linear Algebra in Categories

This section deals with such things as algebras, superalgebras, polylinear morphisms, etc., in categories with finite products.

Throughout the section \( \mathcal{D} \) will be some fixed category with finite products.

The most compact way to define an algebraic structure of some type \( T \) on an object \( X \) belonging to \( \mathcal{D} \) is to use the Yoneda embedding \( \mathcal{D} \to \mathcal{D}^\mathcal{D} \) (see, e.g., Ref. [6]). In what follows, \( \mathcal{D} \) will be identified with its image in the functor category \( \text{Set}^{\mathcal{D}^\mathcal{D}} \). The fact that \( \text{Hom} \) commutes with products permits one to define a structure of type \( T \) on an object \( X \) pointwise, reducing it to the case \( \mathcal{D} = \text{Set} \) (see §1 of Ref. [6]).

1.1. Rings in categories. For example, an object \( R \) of \( \mathcal{D} \) together with morphisms \( R \times R \to R \), \( R \times R \to R \) and \( p : R \to R \), where \( p \) is the terminal object in \( \mathcal{D} \), is said to be the (commutative) ring with unity in the category \( \mathcal{D} \) if for every object \( Y \) of the category \( \mathcal{D} \) the triple \( (R(Y), \cdot_Y, +_Y) \) is a (commutative) ring and \( e_Y(p) \) is the unity of this ring. Recall that we have identified the object \( R \) with the functor \( \text{Hom}(R) \to \text{Hom} \), whereas morphisms \( +, \cdot \) with the corresponding functor morphisms \( + : (e_Y, +_Y, 0_Y) \to (e_Y, +_Y, 0_Y) \) and \( \cdot : (e_Y, \cdot_Y, 1_Y) \to (e_Y, \cdot_Y, 1_Y) \), respectively.

To the end of section 1 R will be some fixed commutative ring with unity in the category \( \mathcal{D} \).

1.2. \( R \)-Modules. An object \( V \) of \( \mathcal{D} \) together with a morphism \( R \times V \to V \) is called an \( R \)-module if for every \( Y \in \mathcal{D} \) the pair \( (V(Y), \cdot_Y) \) is an \( R(Y) \)-module (all modules over commutative rings with unity in the category \( \text{Set} \) are supposed to be unitary). Given two \( R \)-modules \( V \) and \( V' \), a functor morphism \( f : V \to V' \) is morphism of \( R \)-modules if for every \( Y \in \mathcal{D} \) the map \( f_Y : V(Y) \to V'(Y) \) is morphism of \( R(Y) \)-modules.

The category \( \text{Mod}_R(\mathcal{D}) \) of \( R \)-modules of the category \( \mathcal{D} \) is an additive category; it has, in particular, direct sums and zero object.

The category \( \text{Mod}_R(\text{Top}) \) coincides, evidently, with the category of topological vector spaces over the field \( K \), whereas the category \( \text{Mod}_R(\text{Man}) \) is the category of Banach spaces over \( K \).

1.3. Polynan-linear morphisms. Let \( V_1, \ldots, V_n, V \) be \( R \)-modules and \( Z \) be some object of \( \mathcal{D} \). A morphism \( f : Z \times V_1 \times \cdots \times V_n \to V \) will be called a \( Z \)-family of \( R \)-\( n \)-linear morphisms if for every \( Y \in \mathcal{D} \) the map \( f_Y : Z(Y) \times V_1(Y) \times \cdots \times V_n(Y) \to V(Y) \) is \( Z(Y) \)-family of \( R(Y) \)-\( n \)-linear maps, i.e., if for every \( z \in Z(Y) \) the partial map \( f_Y(z, \cdot, \cdots, \cdot) \) sending \( V_1(Y) \times \cdots \times V_n(Y) \to V(Y) \) into \( V(Y) \) is \( R(Y) \)-\( n \)-linear. The set \( \text{Lin}_R(Z(V_1, \ldots, V_n); V) \) of \( Z \)-families of \( R \)-\( n \)-linear morphisms of \( V_1 \times \cdots \times V_n \) into \( V \) is canonically equipped with the structure of Abelian group (\( \left< f + g \right> = f + g \)).

In particular, a morphism \( f : V_1 \times \cdots \times V_n \to V \) is called \( R \)-\( n \)-linear if it is a \( p \)-family of \( R \)-\( n \)-linear morphisms for the terminal object \( p \).

The corresponding Abelian group of \( R \)-\( n \)-linear morphisms will be denoted as \( \text{Lin}_R(V_1, \ldots, V_n; V) \) or, simply, as \( \text{Lin}_R(V_1, \ldots, V_n) \).

Note that the correspondence \( f \mapsto g \) on \( (V_1, x) \), where \( x \) belongs to \( L_R(V_1, \ldots, V_n) \) and \( g : \times V \to V \) is the \( R \)-module structure of \( V \), defines the natural isomorphism

\[
\text{Lin}_R(V_1, \ldots, V_n, V) \cong \text{Lin}_R(R(V_1, \ldots, V_n) V)
\]

of Abelian groups.

1.4. \( R \)-Algebras. 1.4.1. An \( R \)-module \( A \) together with an \( R \)-bilinear morphism \( A \times A \to A \) is called an \( R \)-algebra; \( R \)-algebra \( A \) is said to be (anti)commutative, resp. associative, resp. Lie, resp. Jordan algebra, if for every \( Y \in \mathcal{D} \) the \( R(Y) \)-algebra \( (A(Y), \cdot_Y) \) is (anti)-commutative, resp. associative, etc.

If \( A \) is associative (resp. Lie) \( R \)-algebra, then a pair \( (A, A \times A \to A) \) is called a left \( A \)-module if \( V \) is an \( R \)-module and \( \rho : R \times V \to V \) is an \( R \)-bilinear morphism such that for every \( Y \in \mathcal{D} \) the pair \( (V(Y), \cdot_Y) \) is the left \( A(Y) \)-module. Morphisms of \( R \)-algebras and of left modules are defined in an obvious way.

1.4.2. We leave it to the reader to define the general notion of \( R \)-algebras of type \( T \) as a sequence \( V_1, \ldots, V_n \) of \( R \)-modules ("ground objects") equipped with a sequence \( f_1, \ldots, f_n \) of \( R \)-polynar-linear morphisms defined on them ("ground operations"), satisfying some set
of "laws" of the type \( g = 0 \), where \( g \) is an \( R \)-polylinear morphism constructed in a finite number of steps from the ground operations by means of compositions like \( h \circ (h_1, x \ldots, x, h_m) \) with \( R \)-linear operators \( h, h_1, \ldots, h_m \), addition of \( R \)-polylinear morphisms, as well as compositions of \( R \)-polylinear morphisms with canonical isomorphisms of the type
\[
\bigotimes \frac{k}{k} V^* \otimes k \quad \text{and} \quad (k \otimes k) \otimes k \quad \text{arising from the commutativity and associativity of products; the number \( n \) of ground objects, the "spectrum" of ground operations as well as "laws" - all depending on the type \( T \). Morphisms of algebras of type \( T \) can be defined as families of \( R \)-linear morphisms sending every ground object of one algebra into the corresponding ground object of another and commuting with every ground operation.}

The category of \( R \)-algebras of the type \( T \) in the category \( \mathcal{D} \) will be denoted as \( \mathcal{T}_R(\mathcal{D}) \).

1.4.3. Example. Let the type \( T \) be "left modules over Lie algebras". Then there are two ground objects \( A \) and \( V \), two ground operations \( \otimes A \rightarrow A \) and \( \otimes V \rightarrow V \), and three "laws": three-linear Jacobi identity and bilinear anticommutativity law for \( \otimes \), as well as three-linear identity stating that \( V \) is left \( A \)-module. The Jacobi identity, for example, can be written, up to canonical isomorphisms of associativity of products, as \( \sum \alpha \cdot (\alpha \cdot \beta) = 0 \), where the sum runs over "even" permutation isomorphisms \( A \otimes A \otimes A \rightarrow A \otimes A \otimes A \), arising from the commutativity of products.

1.4.4. Remark. Another, more invariant and consistent (but more involved at the same time), way to define \( R \)-algebras of type \( T \) is, following some ideas of Lowen /7/, (see also §18 of Ref. /6/), to define "type" \( T \) as an additive strict monoidal category with some additional structures, whereas \( R \)-algebras of type \( T \) in the category \( \mathcal{D} \) to define as functors (preserving all of the structures involved) from the category \( T \) into the "category of \( R \)-polylinear morphisms" of the category \( \mathcal{D} \). We assume here more naive point of view on "universal polylinear algebra" in categories, hoping to present constructions in a more complete version of this work.

For the reader unsatisfied of "do-it-yourself" prescriptions in the "definition" of \( R \)-algebras of the type \( T \) the author should note that for all practical purposes of the present work the variable \( T \) of type could be assumed to run over the following finite set: "modules", "algebras", "commutative (resp. associative, resp. Lie) algebras" and "modules over associative (or Lie) algebras".

1.5. Internal functors of polylinear morphisms. Let \( n \) be a natural number. The functor \( \mathcal{L}_R^n: (\mathcal{M}_R(\mathcal{D}))^n \rightarrow \mathcal{M}_R(\mathcal{D}) \) such that there exists the functor isomorphism
\[
\mathcal{L}_R^n(W, \mathcal{X}_R^n(V_1, \ldots, V_n; V)) \rightarrow \mathcal{L}_R^{n+1}(W, V_1, \ldots, V_n; V),
\] (1.5.1)
will be called internal \( \mathcal{L}_R^n \)-functor. Of course, the functors \( \mathcal{L}_R^n \) not necessarily exist (excepting the trivial case \( n = 0 \), when \( \mathcal{L}_R^0 = \mathcal{I}d \)).

Let for a given \( n \) the functor \( \mathcal{L}_R^n \) exist. Setting in the Eq. (1.5.1), \( W = \bigotimes (V_1, \ldots, V_n; V) \) define the \( R \)-linear \( n+1 \)-linear evaluation morphisms
\[
ev_n: \bigotimes (V_1, \ldots, V_n; V) \times V_1 \ldots \times V_n \rightarrow V
\] (1.5.2)
as follows: \( \mathcal{L}_R^n(W, \bigotimes (V_1, \ldots, V_n; V)) \). The Yoneda lemma (see, e.g., Ref. /10/) implies that for every \( f \in \mathcal{L}_R^n(W, \bigotimes (V_1, \ldots, V_n; V)) \) the identity
\[
\mathcal{L}_R^n(f) = \bigotimes (f \circ \mathcal{I}d_{V_1} \ldots \mathcal{I}d_{V_n})
\] (1.5.3)
holds.

The r.h.s. of (1.5.3) is defined when \( f \) is an arbitrary morphism with codomain \( \mathcal{L}_R^n(W, \bigotimes (V_1, \ldots, V_n; V)) \) generating thus (due to the \( R \)-polylinearity of \( \mathcal{L}_R^n \)) the morphisms
\[
\mathcal{L}_R^n(W_1, \ldots, W_n; \bigotimes (V_1, \ldots, V_n; V)) \rightarrow \mathcal{L}_R^{n+1}(W_1, \ldots, V_n; V)
\] (1.5.4)
and
\[
\mathcal{J}(z, \bigotimes (V_1, \ldots, V_n; V)) \rightarrow \mathcal{L}_R^n(z; V_1, \ldots, V_n; V)
\] (1.5.5)
which, evidently, are natural on all of the arguments.

The functor \( \mathcal{L}_R^n \) will be called algebraically coherent if the functor morphisms \( \mathcal{L}_R^n \) are isomorphisms for all \( n \); it will be called coherent if, in addition, \( \mathcal{L}_R^n \) is an isomorphism. The category \( \mathcal{D} \) will be said to have (algebraically) coherent \( \mathcal{L}_R^n \)-functors.
if for every \( n \in \mathbb{N} \) there exists a functor \( \mathcal{L}_R^n \) which is (algebraically) coherent.

If \( \mathcal{D} \) has algebraically coherent \( \mathcal{L}_R \)-functors, one can easily construct the functor isomorphisms
\[
\mathcal{L}_R^n(W_1, \ldots, W_n) \cong \mathcal{L}_R^n(W_1, \ldots, W_n; \mathcal{V}) \quad (1.5.6)
\]
"internalizing" the isomorphisms (1.5.4).

Moreover, one can define in this case the \( R \)-bilinear internal composition morphism
\[
\text{comp}: \mathcal{L}_R(V, W) \times \mathcal{L}_R(V', W') \longrightarrow \mathcal{L}_R(V \times V', W)
\]
(1.5.7) as the inverse image by \( \mathcal{L}_R(V, W) \) of the morphism
\[
\mathcal{L}_R(V, W) \times \mathcal{L}_R(V, W) \cong \mathcal{L}_R(V, W) \times \mathcal{L}_R(V, W) \longrightarrow \mathcal{L}_R(V \times V, W).
\]
(1.5.8)

Taking in (1.5.7) \( V = V' = W \) one can verify that the multiplication \( \text{comp}: \mathcal{L}_R(V, W)^2 \longrightarrow \mathcal{L}_R(V, W) \) turns \( \mathcal{L}_R(V, W) \) into an associative algebra with unity
\[
\mathcal{L}_R(V, W) \cong \mathcal{L}_R(V, W) \quad (1.5.9)
\]
defined as the image of \( \text{Id}_V \) by the isomorphism
\[
\mathcal{L}_R(V, W) \cong \mathcal{L}_R(R, V) \cong \mathcal{L}_R(R, \mathcal{L}_R(V, W)).
\]
(1.5.10)

The reader could verify that the existence of algebraically coherent \( \mathcal{L}_R \)-functors implies the existence of algebraically coherent \( \mathcal{L}_R^n \)-functors for every \( n \in \mathbb{N} \).

Let now \( \mathcal{D} \) have coherent \( \mathcal{L}_R \)-functors. Taking \( Z = p \) (the terminal object in \( \mathcal{D} \)) in (1.5.4) we obtain the canonical isomorphism which permits us to identify the Abelian group of points of \( \mathcal{L}_R^n(V_1, \ldots, V_n) \) with the Abelian group \( \mathcal{L}_R^n(V_1, \ldots, V_n) \).

Let us reinterpret the functor morphism \( \mathcal{L}_R^n \) defined by (1.5.4) in terms of \( R \)-modules in the functor category \( \mathcal{D} = \text{Set} \mathcal{D} \). Equip the set \( \mathcal{L}_R^n(Z; V_1, \ldots, V_n) \) of all \( Z \)-families of \( R \)-polylinear morphisms of \( V_1 \times \ldots \times V_n \) into \( V \) with the structure of an \( R \)-module, defining the multiplication of a morphism \( \varphi: Z \times V_1 \times \ldots \times V_n \longrightarrow V \) belonging to \( \mathcal{L}_R^n(Z; V_1, \ldots, V_n) \) on some morphism \( \psi: Z \longrightarrow R \) in \( \mathcal{D} \) by means of Yoneda embedding \( \mathcal{D} \hookrightarrow \mathcal{D} \) as follows:
\[
(rf)(z, v_1, \ldots, v_n) = r f(z, v_1, \ldots, v_n), \quad z \in Z, v_i \in V_i(Y).
\]
(1.5.11)

Then the functor \( \mathcal{L}_R^n(V_1, \ldots, V_n; \mathcal{V}) \) in the functor category \( \mathcal{D} \) defined by the equation
\[
\mathcal{L}_R^n(V_1, \ldots, V_n; \mathcal{V})(z) := \mathcal{L}_R^n(Z; V_1, \ldots, V_n) \quad (1.5.12)
\]
turns actually into an \( R \)-module in the functor category \( \mathcal{D} \). If there exists in \( \mathcal{D} \) the functor \( \mathcal{L}_R^n \), the morphism \( \rho^n \) defined by (1.5.4) turn out to be \( R(Z) \)-linear, producing together (when \( Z \) runs in \( \mathcal{D} \)) some morphism
\[
\mathcal{L}_R^n(V_1, \ldots, V_n; \mathcal{V}) \xrightarrow{\rho^n} \mathcal{L}_R^n(V_1, \ldots, V_n; \mathcal{V})
\]
(1.5.13)
of \( R \)-modules in \( \mathcal{D} \).

The existence of coherent \( \mathcal{L}_R \)-functors in \( \mathcal{D} \) implies, hence, that \( R \)-modules \( \mathcal{L}_R^n(V_1, \ldots, V_n; \mathcal{V}) \) are representable. The inverse is not true as shows Example 3 below.

**Examples.**

1) In the category \( \text{Mat} \) of smooth Banach manifolds there exist coherent \( \mathcal{L}_R \)-functors: \( \mathcal{L}_R^n(V_1, \ldots, V_n) \) is \( \mathcal{L}_R^n(V_1, \ldots, V_n) \) equipped with the topology of uniform convergence on bounded sets of \( \mathcal{V} \).

2) In the category of Banach manifolds of class \( C^\infty \) (i.e. continuous) there exist algebraically coherent \( \mathcal{L}_R \)-functors, defined as in Example 1 above, which are not coherent.

3) For every functor category \( \mathcal{C} = \text{Set} \mathcal{D} \) and every \( R \)-modules \( V_1, \ldots, V_n; \mathcal{V} \) in \( \mathcal{D} \) the functor \( \mathcal{L}_R^n(V_1, \ldots, V_n; \mathcal{V}) \) in \( \mathcal{C} = \text{Set} \mathcal{C} \), defined by (1.5.12), is representable by the functor in \( \mathcal{C} \) obtained by restriction of the argument \( Z \) in (1.5.12) to the subcategory \( \mathcal{C} \) of \( \mathcal{C} \). Nevertheless, the category \( \mathcal{C} \) gives an example of a topos with no internal \( \mathcal{L}_R \)-functors, excepting the trivial \( \mathcal{L}_R \), if one takes, say, \( R \) to be the constant ring \( R(\Lambda) = R \) (the category \( \mathcal{C} \) is defined in Sect. 3.4 below).

### 1.6 Tensor product

The category \( \mathcal{D} \) will be said to have tensor products over \( R \) if for every \( R \)-modules \( V_1, \ldots, V_n; \mathcal{V} \), there exists an \( R \)-module \( V_1 \otimes \ldots \otimes V_n; \mathcal{V} \) and a natural isomorphism
\[
\mathcal{L}_R(V_1, \ldots, V_n, \mathcal{V}) \cong \mathcal{L}_R^0(V_1, \ldots, V_n, \mathcal{V})
\]
(1.6.1)

In close analogy with construction of functor isomorphisms \( \mathcal{L}_R^n \) of the preceding section we can define the functor morphisms
The category \( \mathcal{D} \) will be said to have coherent tensor products over the ring \( R \) if all of the morphisms (1.6.2) are isomorphisms.

If \( \mathcal{D} \) has coherent tensor products over \( R \), then canonical isomorphisms (1.3.1) generate natural (on \( V \)) isomorphisms
\[
R \otimes_R V \xrightarrow{\lambda_V} V \xrightarrow{\mu_V} V \otimes_R R
\]
(1.6.3)
and the tensor product can (and will) be chosen in such a way that
\[
\lambda_V = \rho_V = \text{Id}_V
\]
for every \( R \)-module \( V \).

**Examples:**
1) The category \( \text{Man} \) has coherent tensor products over the field \( K \) (completion of the algebraic tensor product w.r.t. the projective topology [9]).

2) For every category \( \mathcal{C} \), the corresponding category \( \text{Set}^\mathcal{C} \) of set-valued functors has coherent tensor products which can be defined pointwise:
\[
(V \otimes^\mathcal{C} V')(X) = V(X) \otimes_R V'(X)
\]
(1.6.4)

Note that Yoneda imbedding \( \mathcal{C} \to \text{Set}^\mathcal{C} \) does not commute, in general, with tensor products (counterexample: \( \mathcal{C} = \text{Man} \) and \( R = \mathbb{R} \)).

1.7. \( R \)-Supermodules: An \( R \)-module \( V \) in the category \( \mathcal{D} \) together with a fixed direct sum decomposition
\[
V = gV + \nu V
\]
(1.7.1)
will be called an \( R \)-supermodule. The submodule \( gV \) (resp. \( \nu V \)) is even (resp. odd) submodule of \( V \). Morphisms of \( R \)-supermodules are defined as morphisms of underlying \( R \)-modules commuting with the corresponding direct sum decompositions.

The category of \( R \)-supermodules in \( \mathcal{D} \) will be denoted \( \mathcal{S} \text{Mod}_R(\mathcal{D}) \).

Let \( V_1, \ldots, V_n \) be \( R \)-supermodules. The fact that the functor \( i^R(\mathbb{Z}; \ldots) \) commutes with (finite) direct sums permits one canonically equip \( i^R(\mathbb{Z}; V_1, \ldots, V_n) \) with the structure of \( R(\mathbb{Z}) \)-supermodule as follows:
\[
i^R(\mathbb{Z}; V_1, \ldots, V_n) = \bigoplus_{d \in \mathbb{Z}} i^R(\mathbb{Z}; dV_1, \ldots, dV_n) \quad (i^d + e = K)
\]
(1.7.2)
In particular, \( V(\mathbb{Z}) = i^R(\mathbb{Z}; V) \) is an \( R \)-supermodule if \( V \) is.

Algebraically coherent \( L_R \)-functors and/or coherent tensor products, if they exist, commute with direct sums which permits one to define the structure of \( R \)-supermodules on \( L^R(V_1, \ldots, V_n; V) \) and \( V_1 \otimes_R \cdots \otimes_R V_n \) by means of direct sum decompositions similar to (1.7.2).

Noting that the set of morphisms of an \( R \)-supermodule \( V \) into an \( R \)-supermodule \( W \) is naturally isomorphic to \( gL(V, W) \) and that the natural isomorphisms (1.5.1) and (1.6.1) are actually morphisms of \( R(\mathbb{Z}) \)-supermodules, one can see that the \( R \)-supermodule \( V_1 \otimes_R \cdots \otimes_R V_n \) (resp. \( L^R(V_1, \ldots, V_n; V) \)) represents (resp. corepresents) the corresponding functor of even polylinear morphisms. In particular, the canonical evaluation (and internal composition morphisms defined by eqs. (1.5.2) and (1.5.7), respectively, are even polylinear morphisms of \( R \)-supermodules.

1.8. Change of parity functor. Define the functor
\[
\Gamma: \mathcal{S} \text{Mod}_R(\mathcal{D}) \to \mathcal{S} \text{Mod}_R(\mathcal{D})
\]
(1.8.1)
called the change of parity functor as follows:
\[
\Gamma(V) = \Gamma + \nu V : \quad \Gamma(f) = f
\]
(1.8.2)
The fact that every (even) \( R \)-polylinear morphism \( f: V_1 \times \cdots \times V_n \to W \) is at the same time an (even) \( R \)-polylinear morphism \( f: W \to R \times \cdots \times R \) permits one to construct the natural isomorphism
\[
L^R(V_1, \ldots, V_n; V) \cong L^R(\nu V_1, \ldots, \nu V_n; V)
\]
(1.8.3)
using the isomorphisms (1.5.1).

1.9. \( R \)-Superalgebras: Let \( V_1, \ldots, V_n \) be \( R \)-supermodules and \( \mathcal{G}_n \) be the permutation group of the set \( \{1, \ldots, n\} \). On the union
\[
\bigcup_{\sigma \in \mathcal{G}_n} L^R(V_{\sigma(1)}, \ldots, V_{\sigma(n)}; V)
\]
the "graded" right action of the permutation group \( \mathcal{G}_n \) can be defined in such a way that for the generator \( \sigma_i := (i, i+1) \) belonging to \( \mathcal{G}_n \) this action is determined by the equations
\[
\begin{align*}
(i, i+1)f^\mathcal{G}_n(V_1, \ldots, V_n) &= \sum_{i \leq j < i+1} (-1)^{i+j} f^\mathcal{G}_n(V_{\sigma^{-1}(1)}, \ldots, V_{\sigma^{-1}(i)}, V_{i+1}, \ldots, V_{\sigma^{-1}(i+1)}, \ldots, V_{\sigma^{-1}(n)})
\end{align*}
\]
(1.9.1)
Here again \( f = \sum f^\mathcal{G}_n \) is identified with the corresponding natural transformation of functors through Yoneda imbedding, \( V_1 \) is arbitrary elements of \( R(\mathbb{Y}) \)-supermodule \( V_1(\mathbb{Y}) \), whereas \( \nu V_1 \) (resp. \( V_1 \)) denotes the even (resp. odd) part of the element \( V_1 \) (i.e. \( V_1 = V_1^0 + V_1^1 \)).
An $R$-linear morphism $\sigma$ belonging to an $R(p)$-supermodule $I_R^{{\mathcal{D}}(V;V')} := I^R_R(V_1, \ldots, V_p; V')$ will be called supersymmetric if it is invariant w.r.t. the action of $S_R$ defined above. Denote $\text{Sym}_R^{{\mathcal{D}}}(V;V')$ the set of all supersymmetric morphisms of $V^p$ into $V'$; it is in fact an $R(p)$-subsupermodule of $I_R^{{\mathcal{D}}(V;V')}$.

Replacing now in the definition of $R$-algebras of type $T$ ground objects by $R$-supermodules, ground operations by even polylinear morphisms, and replacing in every "law" $g \circ \sigma$ every composition $g \circ \sigma$ of an $R$-polylinear morphism $\sigma$ with canonical isomorphism $\sigma$ of commutativity of products by its "$\mathbb{Z}_2$-graded" counterpart $\sigma_\mathbb{Z}_2$, defined by eq.(1.9.1), we will arrive to the definition of $R$-superalgebras of the type $T$ in the category $\mathcal{D}$; the corresponding category will be denoted as $S_T^R(\mathcal{D})$.

1.10. $R$-superalgebras. To define $\mathbb{Z}_2$-graded $R$-superalgebras we have simply to replace the direct sum decomposition (1.7.1) by the decomposition $V = \bigoplus_{n \in \mathbb{Z}_2} V_n$. Given $\mathbb{Z}_2$-graded $R$-supermodules $V_1, \ldots, V_n$, the module $I^\mathbb{Z}_2_R(V_1, \ldots, V_n; V)$, $\bigwedge^R_R(V_1, \ldots, V_n; V)$ and $\mathfrak{L}_{\mathbb{Z}_2} V_n$ could be canonically turned into $\mathbb{Z}_2$-graded supermodules just as it was made in subsect. 1.7 for the case $k = 1$. In order to define $\mathbb{Z}_2$-graded superalgebras of some type $T$ we need as well to introduce the $\mathbb{Z}_2$-graded action of permutation groups on $R$-polylinear morphisms.

This is done by setting in the counterpart of eq.(1.9.1) for the case of arbitrary $k$, the indices $E$ and $E'$ to belong to $\mathbb{Z}_2^n$, and defining the factor $(-1)^{E' \cdot E}$ for $E = (E_1, \ldots, E_n)$ and $E' = (E'_1, \ldots, E'_n)$ as $(-1)^{E' \cdot E} = \prod_{i=1}^n (-1)^{E' \cdot E_i}$.

2. Superrepresentable Modules in Functor Categories.

In this section some classes of "vector spaces" in the functor categories $\mathcal{Set}^R$, $\mathcal{Top}^R$ and $\mathcal{Man}^R$ are introduced, playing the crucial part in the theory of supermanifolds. All the definitions and results are given here; for the case of the category $\mathcal{Set}^R$, but could be literally applied to the categories $\mathcal{Top}^R$ and $\mathcal{Man}^R$ as well.

Define in the functor category $\mathcal{Set}^R$ the functor $\mathcal{R}$ as follows: $\mathcal{R}(\Lambda) := \mathfrak{P}(\Lambda) = \mathfrak{P}(\Lambda)$, where $\mathfrak{P}(\Lambda)$ is a morphism of Grassmann superalgebras and $\lambda \in \Lambda$.

The ring structures on each $\mathfrak{P}(\Lambda)$ generate the structure of a commutative ring with unity on $\mathcal{R}$.

Let now $V$ be some real vector superspace ($\mathcal{R}$-superspace in $\mathcal{Set}$). Define the functor $\mathcal{V}$ as follows:

$$\overline{\mathcal{V}}(\Lambda) := \mathfrak{P}(\Lambda) \otimes \mathfrak{P}(\Lambda) \rightarrow \mathfrak{V}(\Lambda), \quad \overline{\mathcal{V}}(\Lambda) := \mathfrak{P}(\Lambda) \otimes \mathfrak{P}(\Lambda)$$

for $\mathfrak{P}(\Lambda) \rightarrow \Lambda'$. The canonical $\Lambda$-module structures on each $\mathfrak{V}(\Lambda)$ turn the functor $\mathfrak{V}$ into an $\mathcal{R}$-module.

At last, let $f : V_1 \otimes \ldots \otimes V_n \rightarrow V$ be an even $R$-linear map of vector superspaces. Define the functor morphism $\overline{f} : \overline{\mathcal{V}}(\Lambda) \otimes \ldots \otimes \overline{\mathcal{V}}(\Lambda) \rightarrow \overline{\mathfrak{V}}(\Lambda)$ such that every component $\overline{\mathcal{V}}(\Lambda) : V_1(\Lambda) \otimes \ldots \otimes V_n(\Lambda) \rightarrow \overline{\mathfrak{V}}(\Lambda)$ of $\overline{f}$ is the $\Lambda$-linear map uniquely determined by the equations

$$\overline{f}(\lambda_1 \otimes \ldots \otimes \lambda_n) = \lambda_1 \otimes \ldots \otimes \lambda_n$$

for every $\lambda_i \otimes v_i \in \overline{\mathcal{V}}(\Lambda)$.

The functor morphism $\overline{f}$ is $\mathcal{R}$-linear.

If $V$ is a complex vector superspace then $\overline{f}$ turns out to be a $\mathcal{C}$-module, where the ring $\mathfrak{C}(\Lambda) = \mathfrak{P}(\Lambda) \otimes \mathfrak{J}(\Lambda)$ is the complexification of the ring $\mathfrak{P}(\Lambda)$; if $f$ is $\mathcal{C}$-linear then $\overline{f}$ is $\mathcal{R}$-linear.

The main properties of the correspondence $V \mapsto \overline{\mathcal{V}}$ and $f \mapsto \overline{f}$ are summarized in the following proposition.
Proposition 2.1. Let $V_1,\ldots,V_n,V$ be vector superspaces over the field $K$. Then

a) The map $\varphi_L : L_X(V_1,\ldots,V_n;V) \rightarrow L_X(V_1,\ldots,V_n;V)$ is an isomorphism of $K$-modules (taking into account that the set $\overline{X}(p)$ of points of $\overline{X}$ coincides with $X$);

b) If $A_1,\ldots,A_n$ are $K$-polynomial maps such that for every $i$ the codomain of $A_i$ is $V_i$ then for every $f \in L_X(V_1,\ldots,V_n;V)$ the identity $\varphi_L(A_1,\ldots,A_n) f = \varphi_L(A_1 f,\ldots,A_n f)$ holds;

c) If $f \in L_X(V_1,\ldots,V_n;V)$ and $A$ belongs to the permutation group $S_n$ then the identity $\varphi_L(A) f = A f \circ A$ holds, where $A f$ is the "graded" action of $A$ on $f$ and $A f$ is the "ordinary" composition (in $Set^{GR}$) of $f$ with permutation isomorphism $A$.

Corollary 2.2. The correspondence $V \mapsto \overline{V}$ and $f \mapsto \overline{f}$ defines full and faithful functor $Mod_X(Set) \rightarrow Mod_X(Set^{GR})$ respecting finite direct sums; more generally, it generates, for every type $V$ of polynormal algebraic structure, the full and faithful functor $\overline{V} : Set \rightarrow K^X(Set^{GR})$.

An $\overline{K}$-module (or, more generally, $\overline{K}$-algebra of some type $V$) $\overline{V}$ in the functor category $Set^{GR}$ will be called superrepresentable if it is isomorphic to $\overline{V}$ for some $\overline{K}$-supermodule (resp. $\overline{K}$-superalgebra of the type $V$) $\overline{V}$ in $Set$. In $Set^{GR}$ there exist, of course, $\overline{K}$-modules which are not superrepresentable. For example, if $V$ is some vector superspace over $\overline{K}$, then the $\overline{K}$-module $\overline{V}^{nil}$, defined as $\overline{V}^{nil}(A) = \overline{V}(\Lambda^{nil}(A))$, where $\Lambda^{nil}$ is the ideal of nilpotents of the Grassmann algebra $\Lambda$, is not superrepresentable. Note that for every $\Lambda$ the identity $\overline{V}(\Lambda) = \overline{V} \oplus \overline{V}^{nil}(\Lambda)$ holds.

In conclusion, we note that a $\overline{K}$-module in $Top^{GR}$ or in $Man^{GR}$ is superrepresentable exactly if it is superrepresentable considered as a $\overline{K}$-module in $Set^{GR}$.

3. Riemann superregions

3.1. Topology on the functor category $Top^{GR}$. Let $\mathcal{F}$ and $\mathcal{F}'$ be some functors in $Top^{GR}$. The functor $\mathcal{F}'$ is called a subfunctor of the functor $\mathcal{F}$ if for every $\Lambda \in GR$ the topological space $\mathcal{F}'(\Lambda)$ is the topological subspace of the topological space $\mathcal{F}(\Lambda)$ and, besides, if the family $\{\mathcal{F}'(\Lambda)\mid \Lambda \in GR\}$ forms a functor morphism (denoted further as $\mathcal{F}' \subseteq \mathcal{F}$); the functor $\mathcal{F}'$ of the functor $\mathcal{F}$ is called open if every $\mathcal{F}'(\Lambda)$ is open in $\mathcal{F}(\Lambda)$.

Given two open subfunctors $\mathcal{F}'$ and $\mathcal{F}''$ of the functor $\mathcal{F}$ one can define the open subfunctor $\mathcal{F}' \cap \mathcal{F}''$ of $\mathcal{F}$ pointwise as follows: $(\mathcal{F}' \cap \mathcal{F}'')(\Lambda) := \mathcal{F}'(\Lambda) \cap \mathcal{F}''(\Lambda)$. Similarly, given a family $\{\mathcal{F}_\Lambda\}$ of open subfunctors of $\mathcal{F}$ one can define the open subfunctor $\bigcup \mathcal{F}_\Lambda$ of $\mathcal{F}$ by the equalities $(\bigcup \mathcal{F}_\Lambda)(\Lambda) = \bigcup (\mathcal{F}_\Lambda)(\Lambda)$. Besides, the initial functor $\mathcal{F}$ is an open subfunctor of every functor $\mathcal{F}$ in $Top^{GR}$.

The topologies just defined on functors in $Top^{GR}$ incorporate themselves to produce some Grothendieck pretopology (see, e.g., Ref. [16]) on the functor category $Top^{GR}$.

Namely, call a functor morphism open if it can be represented as $\mathcal{F}' \subseteq \mathcal{F}$, where $\mathcal{F}' \subseteq \mathcal{F}$ is an isomorphism and $\mathcal{F}$ is an open subfunctor of $\mathcal{F}$. A family $\{\mathcal{F}_\Lambda\}$ of functor morphisms will be called an open covering of the functor $\mathcal{F}$ if each $\mathcal{F}_\Lambda$ is an open morphism and, besides, if for every $\Lambda \in GR$ the family $\{\mathcal{F}_\Lambda(\mathcal{F}(\Lambda))\}$ is an open covering (in the usual sense) of the topological space $\mathcal{F}(\Lambda)$. It is elementary to verify that the class of open coverings defined here really is a (Grothendieck) pretopology on the category $Top^{GR}$.

Note that the obvious neglecting functor $Man^{GR} \rightarrow Top^{GR}$ induces some pretopology on the category $Man^{GR}$: a family $\{\mathcal{F}_\Lambda\}$ of morphisms in $Man^{GR}$ is an open covering of the functor $\mathcal{F}$ iff it is an open covering of $\mathcal{F}$ considered as the family of morphisms in $Top^{GR}$.

Throughout the rest of the paper the categories $Top^{GR}$ and $Man^{GR}$ are supposed to be equipped with the pretopologies defined above.
To give an example of open subfunctors consider an arbitrary functor $F$ in $\text{Top}^{\mathcal{O}}$. Let $U$ be an open subset in $\mathcal{Z} = \mathcal{Z}^{\mathcal{G}}$ (the base of the functor $\mathcal{Z}$) and for every $\Lambda$ let $\mathcal{E}_{\Lambda} : \Lambda \to \mathbb{K}$ be the only morphism of Grassman superalgebra. The family $\{ \mathcal{H}_{\Lambda} \}_{\Lambda \in \mathcal{G}}$ defines an open subfunctor of the functor $\mathcal{F}$ which will be denoted as $\mathcal{F}_{U}$. It turns out that if the functor $\mathcal{F}$ is locally isomorphic to locally convex superrepresentable modules then all its open subfunctors are of this type.

In more details, a $\mathbb{K}$-module $\mathcal{V}$ in $\text{Top}^{\mathcal{O}}$ will be called locally convex (resp. Banach, resp. Fréchet, etc.) $\mathbb{K}$-module if for every $\Lambda$ the topological vector space is locally convex (resp. Banach, resp. Fréchet, etc.). An open subfunctor of a superrepresentable locally convex (resp. Banach, etc.) $\mathbb{K}$-module will be called locally convex (resp. Banach, etc.) superrepresentation, real or complex depending on whether $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. The functor $\mathcal{F}$ will be said to be locally isomorphic to locally convex superrepresentations if there exists an open covering $\{ \mathcal{U}_{\alpha} \to \mathcal{F} \}$ of $\mathcal{F}$ such that each $\mathcal{U}_{\alpha}$ is a locally convex superrepresentation.

Proposition 3.1.1. If a functor $\mathcal{F}$ in $\text{Top}^{\mathcal{O}}$ is locally isomorphic to locally convex superrepresentations, then every open subfunctor $\mathcal{F}'$ of the functor $\mathcal{F}$ coincides with the functor $\mathcal{H}_{\mathcal{F}}$, where $\mathcal{F} = \mathcal{F}^{\mathcal{G}}$ is the base of the functor $\mathcal{F}$.

3.2. Supersmooth morphisms of Banach superregions. In what follows, $\mathcal{V}$, $\mathcal{V}'$, $\mathcal{W}$, etc. will denote (Banach) superrepresentable modules.

Given two real (Banach) superregions $\mathcal{V}_{U}$ and $\mathcal{V}'_{U}$, the functor morphism $\mathcal{F} : \mathcal{V}_{U} \to \mathcal{V}'_{U}$, will be called supersmooth if for every $\Lambda$ the map $\mathcal{F}_{\Lambda} : \mathcal{V}_{U}(\Lambda) \to \mathcal{V}'_{U}(\Lambda)$ is smooth and, besides, for every $u \in \mathcal{V}_{U}(\Lambda)$ the derivative map $D_{\Lambda}^{\mathcal{F}}(u) : \mathcal{V}(\Lambda) \to \mathcal{V}'(\Lambda)$ is $\Lambda$-linear.

The latter condition is equivalent, in turn, to the following one: the "weak superderivative" morphism $D^{\mathcal{F}} : \mathcal{V}_{U} \times \mathcal{V} \to \mathcal{V}'$, defined by equalities $(D^{\mathcal{F}})(u,v) = D_{\Lambda}^{\mathcal{F}}(u).v$ for $u \in \mathcal{V}_{U}(\Lambda)$ and $v \in \mathcal{V}(\Lambda)$, is a $\mathcal{H}_{\mathcal{U}}$-family of $\mathbb{K}$-linear morphisms.

It is evident that a composition of supersmooth morphisms is again supersmooth, hence $\mathcal{B}$-superregions and supersmooth morphisms between them define a category, which will be called the category of supersmooth Banach superregions and denoted $\mathcal{S}_{\text{Reg}}$.

Given a $\mathcal{B}$-superregion $\mathcal{U} = \mathcal{V}_{U}$, every open subfunctor $\mathcal{U}'$ of $\mathcal{U}$ (equal according to Prop.3.1.1 to some superregion $\mathcal{V}_{U'}$, with $U'$ being open in $U$) will be called an open subsuperregion of $\mathcal{U}$. The inclusion morphism $\mathcal{U}' \subseteq \mathcal{U}$ is, obviously, supersmooth. Hence, one can define the pretopology on the category $\mathcal{S}_{\text{Reg}}$, induced by that on the category $\mathcal{C}_{\text{Reg}}$ along the obvious neglecting functor

$$\mathcal{S}_{\text{Reg}} \to \mathcal{C}_{\text{Reg}}$$

(3.2.1) The category $\mathcal{S}_{\text{Reg}}$ will be assumed further to be equipped with this induced pretopology.

Remark. It is quite evident how one can define the category of real superanalytic superregions. As to the complex analytic case, there arise two evident possibilities, namely, using complex Banach superregions in the functor category $\text{Top}^{\mathcal{O}}$, or using instead from the very beginning the category $\mathcal{C}_{\text{Reg}}$ of complex finite-dimensional Grassman superalgebra and copying preceding constructions for the functor category $\text{Top}^{\mathcal{O}}$ on the place of $\text{Top}^{\mathcal{O}}$. Nevertheless, it follows from Prop.2.1 for $\mathbb{K} = \mathbb{C}$, that the two arising categories of complex superanalytic superregions are equivalent.

We will restrict ourselves in this paper only with the super-smooth-case, but most of the results of this work (if not all) are valid, with obvious changes, for the $\mathbb{K}$-superanalytic case as well.

3.3. The structure of supersmooth morphisms. Here is given some characterisation of supersmooth morphisms which, being rather technical, turns out to be, nevertheless, a very useful tool in various proofs and constructions.
Let \( \phi: \overline{V}_U \rightarrow \overline{V}'_U \) be a natural transformation of B-superrregions. The family \( \{ f_i : i \in \mathbb{N} \} \) will be called skeleton of \( f \) if the following conditions are satisfied:

1) \( f_0 = f_{\phi}; U \rightarrow U' \) and \( f_i: U \rightarrow \mathcal{L}(\overline{V}_U) \) for \( i > 1 \) are smooth maps such that for every \( u \in U \) the \( \mathbb{R}\)-linear map \( f_i(u) \) is supersymmetric in the sense of Sect.1.9; here \( \overline{V} \) is considered as purely odd Banach superspace;

2) for every Grassman superalgebra \( A \) and every \( u \in U \),

\[
\lambda \in \overline{V}_{null}(A), \quad \lambda \in \overline{V}_{null}(A)
\]

holds in the latter expression \( u + \lambda + \lambda \) is considered as an element of the B-region \( \overline{V}_U(A) \) in accord with the canonical decompositions (2.4) and

\[
\overline{V}_{null}(A) = \overline{V}_{null}(A) \oplus \overline{V}_{null}(A),
\]

(3.3.1)

\[
\overline{V}_{null}(A) = \overline{V}_{null}(A)
\]

(3.3.2)

\[
\overline{V}_{null}(A) = \overline{V}_{null}(A)
\]

(3.3.3)

\[
\rho_{\phi}(u) \text{ is identified with an element of } \mathcal{L}(\overline{V}_U(A)) \text{ via the canonical isomorphism of the type (1.5,4), and the } \mathcal{R}\text{-polylinear morphism } \overline{V}_{null}(A) \text{ in } \text{Top}^\mathcal{Q} \text{ is defined by eq. (2.3), which shows that the sum in (3.3.1) is actually finite (for the Grassman algebra } A \text{ with } i \text{ odd generators only terms with } 2k+m \leq i \text{ could be non-zero).}

\]

Proposition 3.3.1. a) A skeleton of \( f \) exists if it uniquely determines. Every family \( \{ f_i : i \in \mathbb{N} \} \) of smooth maps such that \( f_0: U \rightarrow U' \) and \( f_i: U \rightarrow \mathcal{L}(\overline{V}_U) \) for \( i > 1 \) is the skeleton of some functor morphism \( f \).

Now, Prop.2.1 permits one to prove the following important result.

Theorem 3.3.2. The following conditions on the functor morphism \( f: \overline{V}_U \rightarrow \overline{V}'_U \) of Banach superregions in \( \text{Top}^\mathcal{Q} \) are equivalent:

1) \( f \) is supersmooth;

2) Each component \( f_A \) of \( f \) is smooth and the derivative \( D_{\overline{V}}(x): \overline{V}(A) \rightarrow \overline{V}'(A) \) is linear for every \( x \) of the form \( x = \mathcal{V}(\lambda)_i(u) \),

where \( \mathcal{V} : \mathcal{R} \rightarrow \mathcal{A} \) is the initial morphism of Grassman superalgebras and \( u \in U \);

3) \( f \) has a skeleton.

For a number of applications it is necessary to know the expression of the skeleton \( \{ (g \circ f)_i \} \) of the composition \( (g \circ f) \) of supersmooth morphisms \( g \) and \( f \) in terms of skeletons of \( g \) and \( f \).

A bit of combinatorics produces the following result.

Proposition 3.3.3. Let \( \phi: \overline{V}_U \rightarrow \overline{V}'_U \) and \( g: \overline{V}'_U \rightarrow \overline{V}''_U \) be supersmooth morphisms of B-superrregions with the skeletons \( \{ f_i \} \) and \( \{ g_i \} \), respectively. Then the skeleton \( \{ (g \circ f)_i \} \) of the composition \( g \circ f \) is determined by the expression

\[
(g \circ f)_i(u) = \sum \frac{1}{i!} \overline{V}_{null}(A) \overline{V}_{null}(A) \overline{V}_{null}(A)
\]

(3.3.4)

\[
= \sum \frac{1}{i!} \overline{V}_{null}(A) \overline{V}_{null}(A) \overline{V}_{null}(A)
\]

(3.3.5)

\[
\text{for every } u \in U \text{, where the sum runs over } 1, m, \text{ and odd } \beta, \text{ even } \alpha \text{ and odd } \beta, \text{ such that } \sum \alpha \beta = i \text{ on } \text{and } S \text{ is the projection operator of supersymmetrization of } \mathcal{R}\text{-linear maps defined as}
\]

\[
S: h = \frac{1}{n!} \sum \frac{1}{i!} \overline{V}_{null}(A) \overline{V}_{null}(A) \overline{V}_{null}(A)
\]

(3.3.6)

the action of \( S \) being defined by eq. (1.91).

In what follows, the supersmooth morphism \( f \) of B-superrregions will be identified either with the family \( \{ f_A \} \) of its components or with its skeleton \( \{ f_i \} \), depending on the circumstances.

Remarks. 1) The skeleton \( \{ f_i \} \) of a supersmooth morphism \( f = \{ f_i \} \) codes itself the naturality properties of the family \( \{ f_i \} \). Permitting one to give a direct (i.e. doing without functors) description of B-superrregions and supermanifolds (=objects of the category \( \text{Man}^\mathcal{Q} \) equipped with some structure. From this point of view Theorem 3.3.2 gives an explicit description of the neglecting functor

\[
N': \text{Reg}^\mathcal{Q} \rightarrow \text{Man}^\mathcal{Q} \rightarrow \text{Man}^\mathcal{Q},
\]

(3.3.7)

\[
\text{where } c(\mathcal{V}) = \{ (f_A)_i \} \text{ and } c(\mathcal{V}) = \{ (f_A)_i \}.
\]

One can prolong the "path of conceptual simplification" (3.3.5)
adding to its end one functor more, namely, the functor \( \text{Man}^N \rightarrow \text{Man} \)
\[ \mathcal{N} \rightarrow \bigcup_{j \in \mathcal{N}} \mathcal{A}_j \] sending \( \{ \mathcal{N}_i \}_{i \in \mathcal{N}} \) into \( \bigcup_{i \in \mathcal{N}} \mathcal{A}_i \). This permits one to visualize supermanifolds and their morphisms as ordinary manifolds and smooth maps between them writing \( \mathcal{N} = \bigcup_{i \in \mathcal{N}} \mathcal{A}_i \) and \( \mathcal{N} = \bigcup_{i \in \mathcal{N}} \mathcal{A}_i \). Note that the functor \( \bigcup \) does not commute with products which implies, e.g., that Lie supergroups (= groups of the category \( \text{SReg} \)) are not groups at all (considered in Set).

2) Theorem (3.3.2) together with Prop.(3.3.3) permits one as well to give an equivalent "abstract" definition of supermanifolds doing without both functors and Grassman algebras. Namely, one can define Banach superregions as pairs \( (U, x, Y) \), where \( U \) is an open region in a banach space \( Y \) and \( x \) is some Banach space; morphisms here are "abstract skeletons" \( \{ x_i \}_{i \in \mathcal{N}} \) and the composition of morphisms is then to be defined by eq.(3.3.3).

This was just this definition that was used in the author's work\(^9\) devoted to an extension to Banach supermanifolds of results of M.Batchelor\(^{10}\) and V.Palamodov\(^{11}\) on the structure of finite-dimensional supermanifolds.

3.4. The categories \( \text{SReg}^\mathcal{N}(m) \). Besides the category \( \text{SReg} \) of supermanifolds one can as well construct a family of categories, "approximating" in a sense the category \( \text{SReg} \).

Namely, denote as \( \mathcal{G}^\mathcal{N}(m) \) the full subcategory of the category \( \mathcal{G}^\mathcal{N} \), consisting of all Grassman algebras with not more than \( m \) odd generators. In the functor category \( \text{Top}^\mathcal{G} \) one can define the ring \( \mathcal{G}(m) \), superrepresentable \( \mathcal{G}(m) \), modules, topology and superregions in close analogy with the preceding case, with obvious changes. For example, the skeleton of supersmooth morphism \( f \) is now a family \( \{ f_i \}_{i \in \mathcal{N}} \), satisfying the corresponding conditions. The corresponding category of supersmooth superregions will be denoted \( \text{SReg}^\mathcal{N}(m) \), whereas objects of it will be called \( m \)-out superregions or, simply, \( m \)-superregions. We will write sometimes \( \text{SReg}^\mathcal{N}(m) \) instead of \( \text{SReg}^\mathcal{N} \) in order to unify notations.

Note that the counterparts of all of the results of the present section, in particular, Theorem 3.3.2 remain valid for the category \( \text{SReg}^\mathcal{N}(m) \) with arbitrary \( m \), though p. a) of Prop.(2.1) fails: the map \( f \rightarrow \overline{f} \) is bijective only for \( n \)-linear morphisms with \( n \leq m \).

There exist obvious functors \( \mathfrak{R}^\mathcal{N}(m) : \text{SReg}^\mathcal{N}(m) \rightarrow \mathfrak{R}^\mathcal{N}(m) \) for \( n \leq m \leq n \infty \), induced by inclusion functor \( \mathfrak{G}^\mathcal{N}(m) : \text{SReg}^\mathcal{N}(m) \rightarrow \mathfrak{G}^\mathcal{N}(m) \) (on skeleton's language \( \mathfrak{R}^\mathcal{N}(m) = \{ \mathfrak{R}_i^\mathcal{N}(m) \}_{i \in \mathcal{N}} \). Obviously, the category \( \text{SReg}^\mathcal{N}(m) \) is naturally equivalent to (and will be identified with) the category \( \text{SReg} \) of smooth Banach regions, whereas Theorem 3.3.2 and Prop. 3.3.3 imply that the category \( \text{SReg}^\mathcal{N}(m) \) is naturally equivalent to the category \( \text{VMan}_m \) of smooth trivial vector bundles over Banach regions (to an \( m \)-superregion \( \overline{V}^\mathcal{N}(m) \) corresponds the vector bundle \( U \times V^\mathcal{N}(m) \rightarrow \overline{V}^\mathcal{N} \). The same Theorem 3.3.2 and Prop. 3.3.3 imply, moreover, the existence of functors \( \mathcal{G}^\mathcal{N} : \text{Reg}^\mathcal{N} \rightarrow \text{Reg}^\mathcal{N}(m) \) and \( \mathcal{G}^\mathcal{N} : \text{Reg}^\mathcal{N} \rightarrow \text{Reg}^\mathcal{N}(m) \) on skeleton's language, \( \mathcal{G}^\mathcal{N} \) being faithful, whereas \( \mathcal{G}^\mathcal{N} \) being full and faithful and left adjoint to the functor \( \mathcal{G}^\mathcal{N} \). In particular, the following result is valid.

Proposition 3.4.1. The category \( \text{Reg} \) of smooth regions in Banach spaces could be identified (through the functor \( \mathcal{G}^\mathcal{N} \)) with full subcategory of \( \text{SReg}^\mathcal{N}(m) \) and for every \( m \)-superregion \( \overline{V}^\mathcal{N}(m) \) there exists the canonical monomorphism \( \mathcal{G}^\mathcal{N} : \overline{V}^\mathcal{N}(m) \rightarrow \overline{V}^\mathcal{N} \) being the component of the natural transformation \( \mathcal{G}^\mathcal{N} : \text{Reg} \rightarrow \text{Reg}^\mathcal{N} \) defined by the adjunction.

One can observe that the correspondence sending a smooth map \( f : U \rightarrow U' \) of \( \mathcal{N} \)-regions to the smooth map \( \mathcal{G}^\mathcal{N}(f) \) defined by the "Tagel expansion" (3.3.1) with the skeleton \( \{ f, \mathcal{G}^\mathcal{N}(f), \ldots \} \), is just the infinite-dimensional counterpart of Berezin's "Grassman analytic continuation"\(^{2/}\).

Note in conclusion that \( m \)-supermanifolds with finite \( m \) (glued out of \( m \)-superregions) play an important part in construction of
invariants of Banach supermanifolds being the counterparts (on the functor's language) of "a-th infinitesimal neighbourhoods" of supermanifolds exploited in Palamodov's work.


4.1. The definition of the category $\text{SMan}$. We can define now Banach supermanifolds by means of atlases on functors of the category $\text{Man}^{\text{Gr}}$.

Let $F$ be a functor in $\text{Man}^{\text{Gr}}$. An open covering $\mathcal{U} = \{U_i \to F\}$ of the functor $F$ is called a (super)smooth atlas on $F$ if every $U_i$ is a B. superregion and every pullback

$$
\begin{array}{ccc}
U_i & \xrightarrow{h} & U_B \\
\downarrow & & \downarrow \\
U & \xrightarrow{f} & F
\end{array}
$$

(4.1.1)

can be chosen in such a way, that $U_B$ is a B. superregion and the pullback projections be supersmooth. Two atlases $\mathcal{A}$ and $\mathcal{M}$ on $F$ are said to be equivalent if $\mathcal{A} \cup \mathcal{M}$ is an atlas as well; it is an equivalence relation on the class of atlases on $F$.

A Banach supermanifold is a functor $\mathcal{M}$ in $\text{Man}^{\text{Gr}}$ together with an equivalence class of atlases on it; elements of every atlas of the corresponding equivalence class will be called charts of the supermanifold $\mathcal{M}$. We will not distinguish in notations between a supermanifold and its underlying functor.

Let $\mathcal{M}$ and $\mathcal{M}'$ be B. supermanifolds. A functor morphism $\mathcal{M} \to \mathcal{M}'$ will be said to be supersmooth if for every charts $U \to \mathcal{M}$ and $U' \to \mathcal{M}'$ of $\mathcal{M}$ and $\mathcal{M}'$, respectively, the pullback

$$
\begin{array}{ccc}
U \times_{\mathcal{M}} \mathcal{M}' & \xrightarrow{h} & U' \\
\downarrow & & \downarrow \\
U & \xrightarrow{f} & \mathcal{M}'
\end{array}
$$

(4.1.2)

could be chosen in such a way, that $U \times_{\mathcal{M}} \mathcal{M}'$ be a B. superregion and the pullback projections $h$ and $f$ be supersmooth.

Composition of two supersmooth morphisms is again supersmooth, which permits one to define correctly the category $\text{SMan}$ of Banach supermanifolds. The set of morphisms of a supermanifold $\mathcal{M}$ into the supermanifold $\mathcal{M}'$ will be denoted $\text{SMan}(\mathcal{M}, \mathcal{M}')$.

Let $\mathcal{M}$ be a B. supermanifold and let $\mathcal{M}'$ be an open subfunctor of $\mathcal{M}$. There exists on the functor $\mathcal{M}$ the only structure of the supermanifold such that the inclusion $\mathcal{M}\hookrightarrow \mathcal{M}'$ is supersmooth morphism. The functor $\mathcal{M}'$ equipped with this structure will be called an open subsupersmanifold of $\mathcal{M}$. Note that in accord with Prop. 3.1.1 every open subsupersmanifold $\mathcal{M}'$ of the supermanifold $\mathcal{M}$ is of the form $\mathcal{M}' = \mathcal{M}|_U$ for some open subset $U$ of the base manifold $\mathcal{M} := \mathcal{M}(\mathcal{R})$ of $\mathcal{M}$. Inclusions of open supermanifolds generate in a standard way (cf. Sect.3.1) some pretopology on the category $\text{SMan}$. This pretopology is induced by the canonical pretopology on the category $\text{Man}^{\text{Gr}}$ along the neglecting functor

$$
\text{SMan} \xrightarrow{\Omega} \text{Man}^{\text{Gr}},
$$

(4.1.3)

continuing the functor (3.2.1) and denoted by the same letter (the category $\text{SReg}$ is, of course, assumed to be imbedded into $\text{SMan}$ by means of attaching to a superregion $\mathcal{U}$ the trivial atlas $\text{Id}_\mathcal{U}$). The category $\text{SMan}$ will be assumed to be equipped with the pretopology just defined.

Remark. In the definition of supermanifolds one can use as well the neglecting functor $\text{SReg} \xrightarrow{\Omega} \text{Set}^{\text{Gr}}$ instead of the functor (3.2A). The definition of atlases on $\text{Set}$-valued functors and supersmooth morphisms follows closely that given above for $\text{Man}$-valued functors, with some changes caused by the fact that the pretopology on $\text{SReg}$ is not induced by that on $\text{Set}^{\text{Gr}}$ (where open coverings are defined as such families $\{U_i \xrightarrow{U_i} \mathcal{R}\}$ that for every $\mathcal{U}$ the family $\{U_i \xrightarrow{U_i} \mathcal{R}\}$ is an epi family of monos). This changes are the following ones: we are to demand that pullback projections $\mathcal{M}$ and $\mathcal{M}'$ in the pullback (4.1.1) as well as projection $\mathcal{M}$ in the pullback (4.1.2) are open, considered as morphisms of $\text{SReg}$. We will arrive as a result to the category $\text{SMan}$ of supermanifolds as $\text{Set}$-valued functors on $\text{Gr}$ with atlases on them.

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It turns out that the functor $\text{SMan} \to \text{SMan}'$, generated by the
neglecting functor $\text{Man}^{0}\text{r} \to \text{Set}$, is not only a natural equiva-
ience of categories but even an isomorphism of them, permitting one to
identify this two categories.

In practice, both categories $\text{SMan}$ and $\text{SMan}'$ will be used,
depending on circumstances: whereas general definitions look simpler
than "modulo manifolds", some concrete supermanifolds (e.g., Grass-
mannians) arise naturally first as $\text{Set}$-valued functors.

4.2. The categories $\text{SMan}(m)$. One can define the categories
$\text{SMan}(m)$ of $m$-supermanifolds starting from the categories $\text{SMarg}$ and
repeating almost literally the definitions of the preceding subsection. Besides,
each category $\text{SMan}(m)$ will be equipped with the
pretopology induced on the category $\text{SMan}^{0\text{r}(m)}$ along the
neglecting functor $\text{SMan}(m) \to \text{SMan}^{0\text{r}(m)}$.

If $\mathcal{D}$ and $\mathcal{D}'$ are categories with pretopologies on them, the
functor $\mathcal{D} \to \mathcal{D}'$ is called continuous if it respects open coverings
and pullbacks of open (belonging to some open covering) morphisms.
Example, the neglecting functor $\text{Man}(m)$ as well as the functors
$\mathcal{N}_m$, $\mathcal{L}_m$ and $\mathcal{L}_m'$, defined in Sect.3.4 are continuous.

Proposition 4.2.1. a) The category $\text{SMan}(m)$ of $m$-supermanifolds
is continuously naturally equivalent to the category $\text{Man}$ of $m$-
manifolds; b) the category $\text{SMan}(1)$ of $1$-supermanifolds is naturally
equivalent to the category $\text{VBan}$ of smooth Banach real vector bundles
(continuously, if one equips $\text{VBan}$ with the pretopology generated by
open inclusions of vector bundles); c) the functors $\mathcal{N}_m$, $\mathcal{L}_m$ and
$\mathcal{L}_m'$ have continuous extensions (denoted here by the same letters)
$\mathcal{N}_m'$: $\text{SMan}^{0\text{r}} \to \text{SMan}^{0\text{r}}$, $\mathcal{L}_m'$: $\text{Man} \to \text{SMan}^{0\text{r}}$, $\mathcal{L}_m'$: $\text{VBan} \to \text{SMan}^{0\text{r}}$ (4.2.1)
such that $\mathcal{L}_m'$ is full and faithful and left adjoint to $\mathcal{N}_m'$, whereas
$\mathcal{L}_m'$ is faithful functor such that $\mathcal{L}_m'$ is

In particular, for every supermanifold $\mathcal{M}$ there exists the
canonical monomorphism $\mathcal{L}_m' (\mathcal{M}) \to \mathcal{M}$.

4.3. Products of supermanifolds. Let $\mathcal{M}$ and $\mathcal{M}'$ be superman-


4.4. Linear algebra in the category of supermanifolds. Let $f: x_1 \otimes \ldots \otimes x_n \to \mathcal{V}$ be an even $K$-linear map of $B$-superspaces.

4.4.2. The functor $\text{SMan}(m) \to \text{SMan}^{0\text{r}}$ is in a natural equivalence of categories.

Corollary 4.4.2. For every $\text{polynomial type } T$ of algebraic struc-
ture the functor (4.4.2) establishes a natural equivalence of the
category of $K$-superalgebras of the type $T$ in $\text{Man}$ with the category of
$K$-algebras of the same type $T$ in $\text{SMan}$.

Corollary 4.4.3. In the category $\text{SMan}$ of supermanifolds there exist coherent tensor products over $\mathbb{F}$ as well as coherent internal
$\mathcal{L}_m$-functors (see Sects. 1.5 and 1.6 for the definition).
We choose the functors $\mathcal{L}_K$ and $\mathcal{O}_F$ in such a way that for every B-superspace $V_1, \ldots, V_n$, the identities
\[
\mathcal{L}_K(V_1, \ldots, V_n; V) = \mathcal{L}_K(V_1, \ldots, V_n; V) \quad \text{and} \quad \mathcal{O}_F(V_1, \ldots, V_n; V) = V_1 \otimes \cdots \otimes V_n
\] hold.

Note that, generally speaking, the dinatural (on $\mathcal{L}_K$) morphism
\[
\mathcal{L}_K(V; V)(A) \to \mathcal{L}_K(V(A); V(V)(A))
\] defined by equations
\[
\phi_A^{(\lambda \otimes f)}(\lambda \circ f) = \lambda \phi_{V(\lambda)}(f(v))
\] for every $\lambda \in \mathcal{L}_K$ and $\phi_{V(A)}(f(v))$, is not an isomorphism.

Similarly, $(V_1 \otimes \cdots \otimes V_n)(A)$ is not isomorphic, in general, to $V_1(A) \otimes \cdots \otimes V_n(A)$.

An important role plays as well the image of the change of parity functor $\Pi$ along the natural isomorphism (4.4.2). It will be denoted $\bar{\Pi} : \text{Mod}_E(S\text{Man}) \to \text{Mod}_E(S\text{Man})$ and will be chosen in such a way that
\[
\bar{\Pi}(V) = \Pi(V) \quad \text{and} \quad \bar{\Pi}(F) = \Pi(F).
\]

At last, choose and fix, for every type $T$ of polylinear algebraic structure some functor
\[
S : T : S\text{Man} \to S\text{Man}
\] quasi-inverse to the functor (4.4.1). It seems that there is no canonical choice for this "supersization" functor $S$.

4.5. Linear algebra in $\text{SMan}$. The counterpart of Th.4.4.1 is valid as well for the category $\text{SMan}$, if $1 \leq m < \infty$, but the Corollaries 4.4.2 and 4.4.3 fail to be true for this case. Nevertheless, if a polylinear type $T$ of algebraic structure is such that all its ground operations and laws are not more than $m$-linear then the category of $K$-superalgebras of the type $T$ in $\text{Man}$ is equivalent to the category of $K$-algebras of the same type in $\text{SMan}$. For example:

Proposition 4.5.1. Let $m \geq 3$. Then the category of Banach Lie superalgebras (resp. modules over Lie superalgebra) over $K$ is naturally equivalent to the category of Lie algebras (resp. modules over Lie algebras) over $K$ in the category $\text{SMan}$. Theorem 4.6. Example: Grassmanians and flag supermanifolds. Here is constructed the supermanifold $\mathcal{F}_n(V)$ of flags of any given length $n$ for any $K$-module $V$ in the category $\text{SMan}$ (the complex case could be treated quite similarly). The definition of $\mathcal{F}_n(V)$ considered as a set-valued functor is, essentially, that given by Yu.Manin[12] in the context of the algebraic supergeometry for the finite-dimensional case. As to the supersmooth structure on $\mathcal{F}_n(V)$, here is used a superized and "analytically continued on $n$" version of "coordinate free" atlases for ordinary Grassmanians (see, e.g., Ref.[13]), what makes things look a bit more transparent.

A Banach $\Lambda$-supersuperalgebra $S$ will be called free if it is isomorphic to a $\Lambda$-supersuperalgebra $\Lambda \otimes E$ for some real B-superspace $E$. A Banach $\Lambda$-supersuperalgebra $E'$ of $E$ will be called direct if there exists a Banach $\Lambda$-supersuperalgebra $E''$ such that $E' \cong \Lambda \otimes E''$.

Proposition 4.6.1. Let $V$ be a real B-superspace and $E$ be a free direct $\Lambda$-supersuperalgebra of $\Lambda \otimes E$. Then for every $\text{S\text{Man}}$-structure $\varphi : \Lambda \to \Lambda'$ of Grassmanian superalgebra the $\Lambda$-subsuperalgebra $\mathcal{F}_n(V)$ of $\Lambda \otimes E$, generated by the real subsuperalgebra $1(V)$ of $\Lambda \otimes E$, is free and direct.

This implies that one can correctly define, for a given $K$-module $V$ and any natural number $n$, the functor $\mathcal{F}_n(V)$ in $\text{Set}^{\mathcal{C}}$ such that $\mathcal{F}_n(V)(A)$ is the set of all sequences $E_1, \ldots, E_n$ of $\Lambda$-supersuperalgebras with $E_i$ being a free direct $\Lambda$-supersuperalgebra of $E_{i+1}$ for every $1 \leq i < n$.

Define now in a canonical way the supersmooth structure on the functor $\mathcal{F}_n(V)$.

Consider first the case of a Grassmanian $\mathcal{F}_1(V)$, where $V$ is some B-superspace.

Let $E'$ and $E''$ be $\Lambda$-supersuperalgebras. Denoting the set (and, actually the $\pi_1$-sequence of morphisms of $E'$ into $E''$ as $\text{Hom}_E(E', E'')$, define the morphism of $\alpha$-modules (setting $E' = \Lambda \otimes V'$ and $E'' = \Lambda \otimes V''$).
If \( V \) is an arbitrary \( \mathbb{K} \)-module, then there exists due to

\[ \text{Theorem 4.4.1} \]

\[ \text{an isomorphism } \overline{V} \longrightarrow V \] for some real \( B \)-superspace \( V \); it induces, obviously, an isomorphism \( \overline{F}_n(V) \longrightarrow \overline{F}_n(\overline{V}) \) for every natural \( n \). Define a supersmooth structure on \( \overline{F}_n(V) \) as the image of the supersmooth structure on \( \overline{F}_n(\overline{V}) \) defined above. This structure does not depend, in fact, on the choice of an isomorphism \( J \).

\[ \text{4.7. Connection with Berezin-Lotts-Keurentjes theory.} \]

Define an \( \mathbb{K} \)-superalgebra \( \mathcal{R} \) in the category \( \mathcal{S\text{-Sup}} \) as the functor

\[ \mathcal{R}(\Lambda) := \Lambda, \quad \mathcal{R}(\varphi) : V \longrightarrow \Lambda' \]

with an \( \mathbb{K} \)-superalgebra structure on it generated by \( \mathbb{K} \)-superalgebra structures on every \( \Lambda \) when \( \Lambda \) runs in \( \mathcal{S} \). The reader could verify that \( \mathcal{R} \) is considered as an \( \mathbb{K} \)-algebra in \( \mathcal{S\text{-Sup}} \) is isomorphic to the \( \mathbb{K} \)-algebra \( \mathcal{R} \), where the real superalgebra \( \mathcal{R} \) coincides as an \( \mathbb{R} \)-algebra with \( \mathcal{R} \), but is not trivial as a superspace: \( \mathcal{R} \) is an \( \mathcal{R} \).

The \( \mathbb{K} \)-superalgebra \( \mathcal{R} \) is commutative. It plays the role of coordinate ring for supermanifolds.

Let \( \mathcal{M} \) be a supermanifold. In accord with Sect.1 the set \( SC^0(\mathcal{M}) := SC^0(\mathcal{M}, \mathcal{R}) \) is canonically equipped with the structure of commutative superalgebra over \( SC^0(\mathcal{M}, \mathcal{R}) \). Moreover, it is evident that for every \( \alpha \in \mathcal{R} \) the functor morphism \( \mathcal{R} : \mathcal{M} \longrightarrow \mathcal{R} \), defined by equations \( \mathcal{R}(\alpha)(x) := \alpha \cdot x \), is supersmooth. The corresponding imbedding \( \mathcal{M} \hookrightarrow SC^0(\mathcal{M}, \mathcal{R}) \) equips, canonically, the set \( SC^0(\mathcal{M}) \) with the structure of an \( \mathcal{R} \)-superalgebra.

Elements of the superalgebra \( SC^0(\mathcal{M}) \) will be called superfields on the supermanifold \( \mathcal{M} \).

Example. Let \( U \subseteq \mathbb{R}^{2n+1} \) be a finite-dimensional superregion.

Let \( x_i \in \mathbb{R}^{2n+1} \) and \( \theta_j \in \mathbb{R}^{2n+1} \) for \( i = 1, \ldots, n \) and \( j = 1, \ldots, m \) where \( \theta_j \) (resp. \( \theta_{2j-1} \)) are projections of \( \mathbb{R}^{2n+1} \) onto even (resp. odd) coordinates. Then

\[ SC^0(\mathcal{M}) \cong C^0(\Sigma_1, \ldots, \Sigma_m) \otimes \Lambda(\theta_1, \ldots, \theta_m), \]

where \( C^0(\Sigma_1, \ldots, \Sigma_m) \) and \( \Lambda(\theta_1, \ldots, \theta_m) \) is the Grassman
superalgebra with generators $\theta_1, \ldots, \theta_n$.

Let $\mathfrak{M}$ be a B. supermanifold. The correspondence $U \mapsto \mathcal{O}^{\mathfrak{f}}(\mathfrak{M})_U$, where $U$ runs over all open subsets in the base manifold $\mathfrak{M}$ of $\mathfrak{M}$, defines a sheaf of $\mathfrak{R}$-superalgebras on $\mathfrak{M}$. Denote the corresponding sheaved space $\text{Sh}(\mathfrak{M})$. Every morphism $f: \mathfrak{M} \to \mathfrak{M}'$ of B. supermanifolds generates, in an evident manner, some morphism $\text{Sh}(f): \text{Sh}(\mathfrak{M}) \to \text{Sh}(\mathfrak{M}')$ of spaces sheaved with $\mathfrak{R}$-superalgebras. This defines the functor $\text{Sh}$ from the category $\mathfrak{SMAN}$ to the category of topological spaces sheaved with $\mathfrak{R}$-superalgebras.

A Banach supermanifold $\mathfrak{M}$ will be called locally finite-dimensional if there exists an atlas $\{ V_i | U_i \to \mathfrak{M} \}$ such that every Banach supercurve $V_i$ is finite-dimensional. Let $\mathfrak{SMAN}_{\text{fin}}$ be the full subcategory of $\mathfrak{SMAN}$ whose objects are just locally finite-dimensional supermanifolds.

**Proposition 4.7.1.** The functor $\text{Sh}$ establishes an equivalence of the category $\mathfrak{SMAN}_{\text{fin}}$ with the category of supermanifolds of Berezin-Leites-Kostant.

4.8. **BLK-supermanifolds as variable $\Lambda$-supermanifolds.** Here are clarified some relations between supermanifolds of Berezin-Leites-Kostant (BLK-supermanifolds, for brevity) and various types of "supermanifolds over finite-dimensional Grassman algebras"/14-16/.

In what follows, the category $\mathfrak{BLK-SMan}$ of BLK-supermanifolds will be identified with the category $\mathfrak{SMAN}_{\text{fin}}$ through the functor $\text{Sh}$ defined in Sect.4.4.

Denote, for any $\Lambda \in \mathfrak{Bx}$, the category of $\mathfrak{G}^\Lambda$-supermanifolds over $\Lambda$ of Alice Rogers/15/ as $\Lambda^\Lambda \mathfrak{Man}$; the category of $\mathfrak{H}$-supermanifolds/14-15/ as $\Lambda^\Lambda \mathfrak{Man}$; the category of supermanifolds over $\Lambda$ in the sense of Ref./16/ as $\Lambda^\Lambda \mathfrak{JMan}$.

One has the inclusions of categories

$$\Lambda^\Lambda \mathfrak{Man} \subset \Lambda^\Lambda \mathfrak{Man} \subset \Lambda^\Lambda \mathfrak{JMan}.$$  

Note, that the category $\Lambda^\Lambda \mathfrak{Man}$ does not coincide, generally speaking, with the category $\Lambda^\Lambda \mathfrak{JMan}$ (for example, $\mathfrak{G}^\Lambda$-linearity of derivative maps imposes no restrictions at all in the case of $\Lambda = \Lambda^\Lambda$).

One can see immediately from the definition of the Jadczyk-Pilch supersmoothness (equality smoothness + $\Lambda^\Lambda$-linearity of derivatives), that "evaluation at point $\Lambda$" ($\mathfrak{M} \mapsto \mathfrak{M}(\Lambda)$, $x \mapsto x_\Lambda$) defines for every $\Lambda \in \mathfrak{Bx}$ some functor

$$\mathfrak{BLK-SMan} \longrightarrow \Lambda^\Lambda \mathfrak{JMan}.$$ 

The functor $\Lambda_\Lambda$ is, for any $\Lambda$, neither full nor faithful.

When $\Lambda$ runs in $\mathfrak{Bx}$ we obtain, hence, for every BLK-supermanifold $\mathfrak{M}$ (resp. for every morphism $f$ of supermanifolds) some "object section" $\Lambda \mapsto \mathfrak{M}(\Lambda)$ (resp. some "morphism section" $f \mapsto f_\Lambda$) of the "bundle" $\coprod_{\Lambda \in \mathfrak{Bx}} \Lambda^\Lambda \mathfrak{JMan} \longrightarrow \mathfrak{Bx}$, which permits us to consider BLK-supermanifolds as "variable" Jadczyk-Pilch supermanifolds depending on a discrete parameter $\Lambda$.

Now one can formulate the relation between the category of BLK-supermanifolds and the supermanifolds of Jadczyk-Pilch in the following tautological motto: BLK-supermanifolds (and their morphisms) are just those sections of the "bundle" $\coprod_{\Lambda \in \mathfrak{Bx}} \Lambda^\Lambda \mathfrak{JMan} \longrightarrow \mathfrak{Bx}$, which are "analytic" (in functorial) on the discrete parameter $\Lambda$.

Moreover, Theorem 3.3.2 implies that every functorial on $\Lambda$ section of the bundle $\coprod_{\Lambda \in \mathfrak{Bx}} \Lambda^\Lambda \mathfrak{JMan} \longrightarrow \mathfrak{Bx}$ belongs, in fact, to the subbundle $\coprod_{\Lambda \in \mathfrak{Bx}} \Lambda^\Lambda \mathfrak{Man} \longrightarrow \mathfrak{Bx}$.

Note that the category $\Lambda^\Lambda \mathfrak{JMan}$ contains the category $\mathfrak{SMAN}_\Lambda$ of M. Batchelor/14/ as a full subcategory and is equivalent to our category $\mathfrak{SMAN}_{\text{fin}}(\Lambda)$ of locally finite-dimensional $\Lambda$-supermanifolds. In particular, for every $L \supset \Lambda$ there is defined a "projection" functor $\Pi_L^\Lambda: \Lambda^\Lambda \mathfrak{JMan} \longrightarrow \Lambda^\Lambda \mathfrak{Man}$ (see Prop.4.2.1), and BLK-supermanifolds can be characterized in terms of projective limits as

$$\mathfrak{BLK-SMan} \approx \text{Proj lim} \left[ \Lambda^\Lambda \mathfrak{Man}_L \right],$$

in addition to M. Batchelor's characterization of them as inductive limits of her categories $\mathfrak{SMAN}_L$. 
To conclude with, the author hopes the reader could see now, that pretentious declarations of A. Rogers stating that her definition of $O^n$-supermanifolds "embraces" that of [MLX]-supermanifolds \(^1/2\), is exactly as reasonable as, say, the statement that the "definition of complex numbers embraces that of analytic functions".

5. Vector Bundles in the Category of Supermanifolds.

5.1. The definition. The triple $(\mathcal{M}, \mathcal{V}, \overline{\mathcal{V}})$ will be called a trivial real vector bundle in the category $\text{Sman}$ or, simply, trivial s-vector bundle, if $\mathcal{M}$ is a supermanifold (called base of a given s-vector bundle), $\mathcal{V}$ is an $\mathcal{R}$-module and $\overline{\mathcal{V}} : \mathcal{M} \times \mathcal{V} \rightarrow \mathcal{M}$ is a canonical projection. We will often write simply $\mathcal{M} \times \mathcal{V}$ instead of $(\mathcal{M} \times \mathcal{V}, \mathcal{M}, \overline{\mathcal{V}})$. Morphism of a trivial s-vector bundle $\mathcal{M} \times \mathcal{V}$ into a trivial s-vector bundle $\mathcal{M}' \times \mathcal{V}'$ is a pair $(f: \mathcal{M} \rightarrow \mathcal{M}', \epsilon: \mathcal{M} \rightarrow \mathcal{M}')$ such that $f^* \epsilon = g: \mathcal{M} \rightarrow \mathcal{M}'$ and $\epsilon^* f = \zeta: \mathcal{M} \rightarrow \mathcal{M}'$ is an $\mathcal{M}$-family of $\mathcal{R}$-linear morphisms (see Section 1.3 for the definition).

Open s-vector subbundles and the corresponding pretopology on the category of trivial s-vector bundles are defined in an evident way; besides, one has an obvious neglecting functor sending trivial s-vector bundles into the functor category $\text{Sman}^{\text{Gr}}$. This permits one to define the category $\text{Sman}$ of (Banach) s-vector super-bundles by means of atlases on functors in $\text{Sman}^{\text{Gr}}$ just in the same way as we have defined supermanifolds, with obvious changes (for the abstract theory of gluing, atlases, etc. see the authors paper \(^1/2\)). In particular, there is defined the canonical neglecting functor

$$\text{Sman} \rightarrow \text{Sman}^{\text{Gr}}$$  \hspace{1cm} (5.1.1)

Besides, there is defined the functor $\text{Sman} \rightarrow \text{Sman}$ sending any s-vector bundle $\mathcal{E} \rightarrow \mathcal{M}$ to its base $\mathcal{M}$.

Note that due to Corollary 4.4.3 s-vector bundles could be constructed by means of cocycles, i.e. families of morphisms of supermanifolds of the type $(\mathcal{A} \rightarrow \mathcal{M})_x$ where $(\mathcal{A} \rightarrow \mathcal{M})_x$ is some open covering of a supermanifold $\mathcal{M}$ by open subsupermanifolds and the family $(\theta)$ satisfies cocycle conditions

$$\theta_{\beta} \theta_{\alpha} = 1, \quad \theta_{\beta} \theta_{\gamma} \theta_{\alpha} = 1.$$  \hspace{1cm} (5.1.2)

The products in the l.h.s. of eqs.\(^1/2\) are defined just because $\mathcal{E} \rightarrow \mathcal{M}$ is an $\mathcal{R}$-algebra (see Sect. 1.1); in fact, the corresponding "functions" $\theta_{\beta}$ "take values" in the Lie supergroup (= group in the category $\text{Sman}$) $\mathcal{D}^s \mathcal{V}$ defined in Sect. 7.1 below.

5.2. Inverse images. Let $\mathcal{E} \rightarrow \mathcal{M}$ be a s-vector bundle with a base $\mathcal{M}$ and let $f: \mathcal{M} \rightarrow \mathcal{M}'$ be some morphism of supermanifolds. Define in the category $\text{Sman}^{\text{Gr}}$ a functor $f^* \mathcal{E}$ pointwise as $(f^* \mathcal{E})(\Lambda) = f^* \mathcal{E}(\Lambda)$. The functor $f^* \mathcal{E}$ can be canonically equipped with the structure of a vector bundle in such a way that it becomes an inverse image of a vector bundle $\mathcal{E}$ along the morphism $f$ with all usual properties of inverse images. The bundle $f^* \mathcal{E} \rightarrow \mathcal{M}'$ is, as a matter of fact, the pullback projection of the pullback of $\mathcal{E} \rightarrow \mathcal{M}$ along $f$.

In particular, if $p: \mathcal{M} \rightarrow \mathcal{M}'$ is some point of $\mathcal{M}$, define the fiber $\mathcal{E}_x$ of the vector bundle $\mathcal{E}$ at point $x$ as follows: $\mathcal{E}_x = x^* \mathcal{E}$; the fiber $\mathcal{E}_x$ is canonically equipped with the structure of $\mathcal{R}$-module. If, besides, $f: \mathcal{M} \rightarrow \mathcal{M}'$ is some morphism of s-vector bundles, then there is defined, due to the properties of inverse images, the canonical morphism $f_*: \mathcal{E}_x \rightarrow \mathcal{E}'_{p(x)}$.

5.3. The tangent functor and superderivative morphisms. Define, for every supermanifold $\mathcal{M}$, the functor $\mathcal{T}_{\mathcal{M}}$ in $\text{Sman}^{\text{Gr}}$ pointwise as follows $\mathcal{T}_{\mathcal{M}}(\Lambda) = \Lambda[\mathcal{M}]$. If $f: \mathcal{M} \rightarrow \mathcal{M}'$ is morphisms of supermanifolds, define the functor morphism $\mathcal{T}_f: \mathcal{T}_{\mathcal{M}} \rightarrow \mathcal{T}_{\mathcal{M}'}$ as $(\mathcal{T}_f)_x = \mathcal{T}_x$. This determines some functor $\text{Sman} \rightarrow \text{Sman}^{\text{Gr}}$ which actually lifts to the functor (denoted here by the same letter $\mathcal{T}$)

$$\text{Sman} \rightarrow \text{Sman}$$  \hspace{1cm} (5.3.1)

along the neglecting functor (5.1.1). The functor (5.3.1) will be called the tangent functor.

If $p: \mathcal{M} \rightarrow \mathcal{M}'$ is a point of $\mathcal{M}$ and $f: \mathcal{M} \rightarrow \mathcal{M}'$ is some morphism of supermanifolds, we will write $\mathcal{T}_x \mathcal{M}$ instead of $(\mathcal{T}_x \mathcal{M})_x$ and $\mathcal{T}_x$ instead of $(\mathcal{T}_x)_x$.

Given a B-superspace $\mathcal{V}_y$, one can identify the tangent bundle $\mathcal{T}(\mathcal{V}_y)$ with the trivial s-vector bundle $\mathcal{V}_y \times \mathcal{V}$; if $f: \mathcal{V}_y \rightarrow \mathcal{V}_z$ is a supermorphism of B-superspaces, then the morphism $f_*: \mathcal{T}_y \rightarrow \mathcal{T}_z$ is just the weak superderivative morphism $\mathcal{D}^s \mathcal{T}$ defined in Sect. 3.2 as $(\mathcal{D}^s \mathcal{T})(u, v) = \mathcal{D}^s \mathcal{T}(u, v)$. 

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In accord with Corollary 4.4.3 there exists the only morphism $D\varepsilon: V/\varepsilon \rightarrow L_{\mathcal{R}}(V;V')$ (the derivative morphism of $f$) such that $D\varepsilon = ev(D\varepsilon \times i_{\varepsilon})$.

Supervector bundles possess many of the properties of ordinary supervector bundles. It is left to the reader to formulate, say, the chain rule (using morphism comp. of Sect.4.5) reflecting the functoriality of $T^\mathcal{R}$.

5.4. Vector bundles in the category $\text{Sh}^n([m])$. One can define the category $\text{Sh}^n([m])$ of $m$-supervector bundles on $m$-supermanifolds with finite $m$, repeating literally the corresponding definitions of the case $m=0$; one can define as well the tangent functor $\text{Sh}^n([m]) \xrightarrow{T} \text{Sh}^n([m])$.

Note, nevertheless, that, generally speaking, vector bundles in $\text{Sh}^n([m])$ (with finite $m\neq 0$) could not be constructed by means of cocycles; besides, the supervector map $D\varepsilon$ (contrary to the tangent map $T\varepsilon$) for a morphism $\varepsilon$ in $\text{Sh}^n([m])$ with finite $m\neq 0$, could be uniquely determined only as morphism in $\text{Sh}^n([m-1])$.

5.5. Vector functors. The definition and the main properties of vector functors for the category $\text{Sh}^n([m])$ is similar to that for the "non-supervariable" case (see, e.g., Ref.[69]).

In particular, for given supervector bundles $E$ and $E'$ over one and the same base $\mathcal{M}$ one can define supervector bundles $E \oplus E'$ and $L(E,E')$ in such a way that locally (i.e., for supervector bundles $(\mathcal{M};V) \oplus (\mathcal{M};V') = \mathcal{M} \times (V \oplus V')$ and $L(\mathcal{M};V,V') = \mathcal{M} \times L_{\mathcal{R}}(V,V')$, where $L_{\mathcal{R}}$ denotes the trivial supervector bundle $\mathcal{M} \rightarrow \mathcal{M}$.

One should note that whereas the functors of evaluation at point $\lambda \in \mathcal{M}$ commute with $E \oplus E'$ (i.e., $(E\oplus E')(\lambda) = E(\lambda) \oplus E'(\lambda)$), this is not the case for $L(E,E')$.

5.6. Change of parity functor for $\mathcal{M}$-supervector bundles. An important role in the theory of supervector bundles plays the natural extension $\overline{\text{Ext}}: \text{Sh}^n([m]) \rightarrow \text{Sh}^n([m])$ of the change of parity functor $\overline{\text{Ext}}: \text{Sh}^n([m]) \rightarrow \text{Sh}^n([m])$ defined in Sect.4.4.

To define the functor $\overline{\text{Ext}}$ for trivial supervector bundles note first of all that the natural isomorphism

$$L_{\mathcal{R}}(V;V') \cong L_{\mathcal{R}}(\mathcal{M};V;V')$$

extends, as a consequence of Corollary 4.4.3, to the natural isomorphism

$$L_{\mathcal{R}}(\mathcal{M};V;V') \cong L_{\mathcal{R}}(\mathcal{M};V;V'),$$

which sends an $\mathcal{M}$-family of $\mathcal{R}$-linear morphisms $f: \mathcal{M} \rightarrow V'$ to an $\mathcal{M}$-family of $\mathcal{R}$-linear morphisms.

\[\overline{\text{Ext}}(V;V') \cong L_{\mathcal{R}}(\mathcal{M};V;V')\]  

where $\overline{\text{Ext}}$ is defined in Sect.4.5 (see eq. (1.5.5)).

If now $\mathcal{M} \times \mathcal{V}$ is a trivial supervector bundle, put $\overline{\text{Ext}}(\mathcal{M} \times \mathcal{V}) = \mathcal{M} \times \overline{\text{Ext}}(\mathcal{V})$.

Besides, one can define the action of $\overline{\text{Ext}}$ on morphisms of trivial supervector bundles using the isomorphism (5.6.2) and the fact that the set of all morphisms $f_{\lambda}$ of a trivial supervector bundle $\mathcal{M} \times \mathcal{V}$ into a trivial supervector bundle $\mathcal{M} \times \mathcal{V}'$ over a fixed morphism $\lambda: \mathcal{M} \rightarrow \mathcal{M}'$ of bases, is in evident one-to-one correspondence with the set $L_{\mathcal{R}}(\mathcal{M};\mathcal{V};\mathcal{V}')$.

In fact, the action $\overline{\text{Ext}}$ thus defined, is an extension of the functor $\overline{\text{Ext}}$ of Sect.4.4 to the category of trivial supervector bundles; this extension is, obviously, continuous functor which permits one to construct automatically the desired functor

$$\overline{\text{Ext}}: \text{Sh}^n([m]) \rightarrow \text{Sh}^n([m])$$

by means of "completion of functors by continuity" procedure, described in Ref.[11] (5.6.4).

5.7. The functor $\text{Ext}_{\mathcal{R}}$. Let $\mathcal{V}$ be an $\mathcal{R}$-module. Define an $\mathcal{R}$-module $\overline{\mathcal{V}}$ as

$$\overline{\mathcal{V}} = \mathcal{R} \otimes \mathcal{V},$$

where $\mathcal{R}$ is the "coordinate ring" defined in Sect.4.7. If, further, $f: \mathcal{V} \rightarrow \mathcal{V}'$ is a morphism of $\mathcal{R}$-modules, define the morphism $f_{\lambda}: \overline{\mathcal{V}} \rightarrow \overline{\mathcal{V}}'$ as $f_{\lambda} = \lambda \cdot f$.

The correspondence $\mathcal{V} \rightarrow \overline{\mathcal{V}}$ and $f_{\lambda} \rightarrow f_{\lambda}$ is, in fact, a functor, and there exists an evident functor isomorphism

$$\overline{\mathcal{V}} = \mathcal{V} \otimes \overline{\mathcal{V}}.$$

Besides, if $\overline{\mathcal{V}}$ is some isomorphism of $\mathcal{R}$-modules, it generates for every $\lambda$ an isomorphism

$$\lambda \otimes \mathcal{V} = \overline{\mathcal{V}}(\lambda) \otimes \mathcal{V}(\lambda),$$

permitting one to equip $\overline{\mathcal{V}}$ with the structure of an $\mathcal{R}$-module. This structure does not depend, in fact, on the choice of an isomorphism $I$, and for every morphism $f$ of $\mathcal{R}$-modules the morphism $f_{\lambda}$ turns out to be a morphism of $\mathcal{R}$-modules.

We have defined thus the functor $\text{Ext}_{\mathcal{R}}$ as the functor from the category of $\mathcal{R}$-modules to the category of $\mathcal{R}$-modules.

The functor $\text{Ext}_{\mathcal{R}}$ is, in fact, a covariant supervector functor, so that one can extend it to the whole category of $\mathcal{R}$-modules. Bearing in mind the canonical isomorphism (5.7.2), one can as well simply define $\overline{\mathcal{V}}$ as
for every s-vector bundle $\mathcal{E}$.

Moreover, for every s-vector bundle $\mathcal{E}$ the s-vector bundle \( \mathcal{E}_R \) can be canonically equipped with the structure of a bundle of \( R \)-modules (to define the latter, replace $R$ by $\mathcal{E}$ in the definition of s-vector bundles). The details are left to the reader.

### 5.8. Extended sections of a vector bundle. Let $\mathcal{E} \xrightarrow{\pi} \mathcal{M}$ be an s-vector bundle. Morphism $s : \mathcal{M} \to \mathcal{E}$ of supermanifolds is called a section of s-vector bundle $\mathcal{E}$ if $\pi \circ s = 1_{\mathcal{M}}$. Denote $\Gamma(\mathcal{E})$ the set of sections of $\mathcal{E}$. Sections of $\mathcal{E}_R$ will be called extended sections of $\mathcal{E}$ and we will write $\Gamma_R(\mathcal{E})$ instead of $\Gamma(\mathcal{E}_R)$.

Extended sections of the tangent bundle $T\mathcal{M}$ of a supermanifold $\mathcal{M}$ will be called vector fields on $\mathcal{M}$; extended sections of $T^*\mathcal{M}$ will be called differential 1-forms (co-vector fields) on $\mathcal{M}$.

Let $\mathcal{M} \times \mathcal{V} \to \mathcal{M}$ be a trivial s-vector bundle and $s \in \Gamma(\mathcal{M} \times \mathcal{V})$ be its section. One can see that $s = (1_{\mathcal{M}}, s')$, where $s'$ is the principal part of the section $s$. The correspondence $s \mapsto s'$ determines a bijection

\[
\Gamma(\mathcal{M} \times \mathcal{V}) \to \mathcal{C}(\mathcal{M}, \mathcal{V}).
\]

(5.8.1)

If, besides, $\mathcal{V}$ is an $R$-module, then the bijection (5.8.1) permits one to equip the set of sections $\Gamma(\mathcal{M} \times \mathcal{V})$ of the bundle of $R$-modules $\mathcal{M} \times \mathcal{V}$ with the structure of an $\mathcal{S}(\mathcal{M})$-module.

More generally, if $\mathcal{E} \xrightarrow{\pi} \mathcal{M}$ is an arbitrary bundle of $R$-modules, the set $\Gamma(\mathcal{E})$ of sections of $\mathcal{E}$ can be naturally equipped with the structure of an $\mathcal{S}(\mathcal{M})$-module in such a way that for every atlas $\{\mathcal{E}_a \xrightarrow{\pi_a} \mathcal{M}\}$ (with $\mathcal{E}_a$ being trivial bundles of $R$-modules) all induced maps $\Gamma(\mathcal{E}) \xrightarrow{\pi_a} \Gamma(\mathcal{E}_a)$ are morphisms of modules over commutative associative $\mathcal{E}$-superalgebras with unity.

### 5.9. Differential of a superfield. If $\mathcal{V}_U$ is a $\mathcal{M}$-superfield, one can identify its tangent bundle $T(\mathcal{V}_U)$ with the trivial bundle $\mathcal{V}_U \times \mathcal{V}$ and cotangent bundle $T^*(\mathcal{V}_U)$ with the trivial s-vector bundle $\mathcal{V}_U \times \mathcal{E}_R(\mathcal{V}, R)$. Let now $f : \mathcal{V}_U \to \mathcal{M}$ be a superfield on $\mathcal{V}_U$.

Noting that the evident natural isomorphism $\mathcal{L}_R(\mathcal{V} \times \mathcal{V}^*) \cong \mathcal{L}_R(\mathcal{V} \times \mathcal{V})$ of $R$-modules generetates the natural isomorphism

\[
\mathcal{L}_R(\mathcal{V}, \mathcal{V}^*) \cong \mathcal{L}_R(\mathcal{V}, \mathcal{V})
\]

(5.9.1)

of $R$-modules, define the differential $df$ of the superfield $f$ as a covector field on $\mathcal{V}_U$ with principal part (df)\_\| defined as a composition

\[
\mathcal{V}_U \xrightarrow{\partial = \pi^*} \mathcal{V}(\mathcal{V}, \mathcal{V}) \xrightarrow{\mathcal{L}_R(\mathcal{V}, R)} \mathcal{L}_R(\mathcal{V}, R) \cong \mathcal{L}_R(\mathcal{V}, \mathcal{V})
\]

(5.9.2)

Let now $\mathcal{M}$ be an arbitrary supermanifold and $A = \{U_a \xrightarrow{i_a} \mathcal{M}\}$ be an atlas of $\mathcal{M}$. The atlas $A$ generates, in an evident way, some atlas $\{U_a \times \mathcal{V}_U \xrightarrow{i_a} \mathcal{V}(\mathcal{V}, \mathcal{V})\}$ on $T^*(\mathcal{V}_U)$. Let $f : \mathcal{M} \to \mathcal{M}$ be a superfield on $\mathcal{M}$. Then there exists the only covector field $df$ on $\mathcal{M}$ such that $(df)_{|\mathcal{V}_U} = i^*_a(df)_{|\mathcal{V}_U}$ for every $\alpha$. The covector field $df$ thus defined does not depend on the choice of an atlas $A$.

### 5.10. Action of vector fields on superfields and the Lie bracket. Let first $\mathcal{V}_U$ be a $\mathcal{M}$-superfield, $f : \mathcal{V}_U \to \mathcal{M}$ be a superfield on $\mathcal{V}_U$ and $\mathcal{F}$ be some vector field on $\mathcal{V}_U$ with principal part $\mathcal{F}_0 : \mathcal{V}_U \to \mathcal{V}$. Define the superfield $\mathcal{F}f$ on $\mathcal{V}_U$ as a composition

\[
\mathcal{F} \circ f : \mathcal{V}_U \xrightarrow{(df)_0} \mathcal{V}_0 \xrightarrow{\mathcal{F}_0} \mathcal{V} \to \mathcal{V}_U \cong \mathcal{V}.
\]

(5.10.1)

Let now $\mathcal{M}$ be an arbitrary supermanifold, $\mathcal{F}$ be a superfield on $\mathcal{M}$ and $\mathcal{F}$ be a vector field on $\mathcal{M}$. Then there exists the only superfield $\mathcal{F}f$ on $\mathcal{M}$ such that for every chart $U \to \mathcal{M}$ on $\mathcal{M}$ the identity $(\mathcal{F}f)_0 = \mathcal{F}_0 = \mathcal{F}_U$ holds.

For every vector field $\mathcal{F}$ the map $f \to \mathcal{F}f$ is a supervariant derivation of the $\mathcal{S}(\mathcal{M})$-module $\mathcal{S}(\mathcal{M})$.

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two vector fields on $\mathcal{M}$, then there exists the only vector field $\mathcal{F}_1 \circ \mathcal{F}_2$ on $\mathcal{M}$ such that for every superfield $f$ on $\mathcal{M}$ the identity

\[
[\mathcal{F}_1, \mathcal{F}_2]f = \mathcal{F}_1f \circ \mathcal{F}_2f - \sum_{\alpha} (-1)^{|\alpha|} f^*_\alpha \sum_{\beta} (\mathcal{F}_1^\alpha \epsilon \mathcal{F}_2^\beta - \mathcal{F}_2^\beta \epsilon \mathcal{F}_1^\alpha)
\]

(5.10.2)

holds. The real superalgebra $\mathcal{S}(\mathcal{M})$ of vector fields on $\mathcal{M}$, equipped with the operation $[\cdot, \cdot]$, is a real Lie superalgebra.

### 6. Immersions, Submersions, Subsupermanifolds, etc.

#### 6.1. Definitions. We will call every morphism of supermanifolds of the form

\[
\mathcal{M} \xrightarrow{\mathcal{M} \xrightarrow{i} \mathcal{M} \xrightarrow{\mathcal{M} \to \mathcal{M}'} \mathcal{M}'}
\]

(resp. of the form $\mathcal{M} \xrightarrow{i} \mathcal{M} \xrightarrow{\mathcal{M} \to \mathcal{M}'} \mathcal{M}$) a standard embedding (resp. a standard projection).

A morphism $\mathcal{M} \xrightarrow{i} \mathcal{M} \xrightarrow{\mathcal{M} \to \mathcal{M}'} \mathcal{M}$ of supermanifolds will be called an immersion (resp. a submersion, resp. a local isomorphism) if there exists a family of pullbacks

\[
\mathcal{U}_a \xrightarrow{i_a} \mathcal{U}_a' \xrightarrow{\iota_a} \mathcal{M}'
\]

(6.1.1)
such that \( \{ U_\alpha \}_{\alpha \in \mathcal{A}} \) is an open covering of \( \mathcal{M} \), every \( f' \) is an open morphism and every \( f \) is a standard embedding (resp. standard projection, resp. an isomorphism); morphisms \( f \) will be called an embedding if there exists a family of pullbacks

\[
\begin{align*}
\xi[f(U_i)] & \overset{f^*} \rightarrow U_i' \\
U_i & \overset{f} \rightarrow U_i' \\
(\alpha & \in \mathcal{A})
\end{align*}
\]

(6.1.2)

such that \( \{ U_i' \}_{i \in \mathcal{I}} \) is an open covering of \( \mathcal{M}' \) and every \( f_i \) is a standard embedding. At last, a supermanifold \( \mathcal{M} \) will be called a sub-supermanifold of a supermanifold \( \mathcal{M}' \), if \( \mathcal{M} \) is a set-valued subfunctor of the functor \( \mathcal{M}' \) and if, besides, the inclusion morphism \( \mathcal{M} \subset \mathcal{M}' \) is an embedding (which includes that it is smooth).

6.2. Morphisms criteria module supermanifolds.

Proposition 6.2.1. a) If some morphism \( f \) of supermanifolds is an immersion (resp. submersion, resp. local isomorphism, resp. embedding) then for every \( \lambda \) the morphism \( f_{\lambda} \) of \( \lambda \)-manifolds is an immersion (resp. submersion, resp. local isomorphism, resp. embedding).

b) If some morphism \( f \) of supermanifolds is such that the morphism \( f_{\lambda} \) of \( \lambda \)-manifolds is an immersion (resp. submersion, resp. local isomorphism, resp. embedding) then \( f \) is an immersion (resp. submersion, resp. local isomorphism, resp. embedding).

Corollary 6.2.2. A morphism \( f: \mathcal{M} \rightarrow \mathcal{M}' \) of supermanifolds is an isomorphism iff the morphism \( f^*_{\lambda}(f^*_\lambda(\mathcal{M})) \rightarrow f^*_{\lambda}(\mathcal{M}') \) of vector bundles (see Sect.4.2) is an isomorphism.

6.3. Differential criteria for morphisms. Let \( h: \mathcal{M} \rightarrow \mathcal{M} \) be a morphism of supermanifolds. An open sub-supermanifold \( U \) of \( \mathcal{M} \) will be called an open neighborhood of the morphism \( h \) if \( h \) lifts to \( \mathcal{M} \) along the inclusion morphism \( U \subset \mathcal{M} \). A morphism \( f: \mathcal{M} \rightarrow \mathcal{M}' \) is said to be an immersion (resp. submersion, resp. local isomorphism) in some neighborhood of the morphism \( h \) if there exists an open neighborhood \( U \subset \mathcal{M} \) of \( h \) such that the morphism \( f(U) \rightarrow f(U) \) is an immersion (resp. submersion, resp. local isomorphism).

Proposition 6.3.1 (Inverse function theorem). A morphism \( f: \mathcal{M} \rightarrow \mathcal{M}' \) of supermanifolds is a local isomorphism in some neighborhood of a point \( p \) if the morphism \( f_{\lambda} \mathcal{M} \rightarrow f_{\lambda} \mathcal{M}' \) is an isomorphism of \( \overline{E} \)-modules.

To formulate the corresponding criterions for immersions and submersions we need a notion of direct morphisms of modules. Let \( \overline{V} \) be an \( \overline{E} \)-module and \( \overline{V}' \) be some submodule. The submodule \( \overline{V}' \) is called direct if there exists an \( \overline{E} \)-module \( \overline{V} \) and an isomorphism \( \overline{V}' \cong \overline{V} \) of \( \overline{E} \)-modules. More generally, a morphism \( g: \overline{V}' \rightarrow \overline{V} \) of \( \overline{E} \)-modules is called direct if it is isomorphic (as an object of the category of \( \overline{E} \)-modules over \( \overline{V} \)) to the inclusion of some direct submodule of \( \overline{V} \).

Proposition 6.3.2. Let \( f: \mathcal{M} \rightarrow \mathcal{M}' \) of supermanifolds be an immersion in some neighborhood of a point \( p \) iff the morphism \( f_*^\mathcal{M} \rightarrow f_*^\mathcal{M}' \) is direct if it is a submersion in some neighborhood of \( x \) iff \( f_*^\mathcal{M} \rightarrow f_*^\mathcal{M}' \) is an epimorphism and Ker\( f_*^\mathcal{M} \rightarrow f_*^\mathcal{M}' \) is a direct submodule of the \( \overline{E} \)-module \( f_*^\mathcal{M} \).

In conclusion of this section we will formulate some useful criteria permitting one to see whether some smooth subfunctor \( \mathcal{N} \) of a supermanifold \( \mathcal{M} \) is a sub-supermanifold of \( \mathcal{M} \) (a functor \( \mathcal{N} \) in \( \mathbf{Man} \)) is called a smooth subfunctor of the functor \( \mathcal{M} \) in \( \mathbf{Man} \) if for every \( \lambda \) the manifold \( \mathcal{M}(\lambda) \) is a submanifold of the manifold \( \mathcal{M}(\lambda) \) and the set of inclusions \( \{ \mathcal{N}(\lambda) \subset \mathcal{M}(\lambda) \} \) \( \lambda \in \mathcal{G} \) is a functor morphism).

Theorem 6.3.3. Let \( \mathcal{N} \) be a smooth subfunctor of a supermanifold \( \mathcal{M} \). If for every point \( x \) of \( \mathcal{M} \), the functor \( \mathcal{N}_x \) is a superrepresentable submodule of the \( \overline{E} \)-module \( \mathcal{M}_x \), then there exists on the functor \( \mathcal{N} \) the structure of a sub-supermanifold of \( \mathcal{M} \) (here, of course, the tangent bundle \( \overline{T}_{\mathcal{N}} \mathcal{N} \) and its "fiber" \( \mathcal{N}_x \)) for a functor \( \mathcal{N} \) in \( \mathbf{Man} \) are defined pointwise).

7. Lie Supergrupoids.

7.1. Definition and examples. A group (object) in the category \( \mathbf{Man} \) will be called a Lie supergroup.

Proposition 1. Let \( \mathcal{A} \) be an associative \( \overline{E} \)-algebra with unity in \( \mathbf{Man} \). Define \( \mathcal{A}^\ast = \mathcal{A}[\mathcal{A}]^\ast \), where \( \mathcal{A}^\ast \) is the Lie group of invertible elements of the \( \overline{E} \)-algebra \( \mathcal{A} \). Then for every \( \lambda \) the manifold \( \mathcal{A}^\ast(\Lambda) \) is a Lie group; the Lie group structures on all \( \mathcal{A}^\ast(\Lambda) \) generate the structure of a Lie supergroup on \( \mathcal{A}^\ast \). In particular, if \( \mathcal{V} \) is an \( \overline{E} \)-module in \( \mathbf{Man} \), then \( \mathcal{L}^\mathcal{V}(\mathcal{V},\mathcal{V})^\ast \) is an associative \( \overline{E} \)-algebra with unity in \( \mathbf{Man} \) (see Sect.1.5 and Coroll.4.4.3). The Lie supergroup \( \mathcal{L}^\mathcal{V}(\mathcal{V},\mathcal{V})^\ast \) will be denoted \( \mathcal{L}^\mathcal{V}(\mathcal{V})^\ast \).
Let \( G \) be a Lie supergroup and let \( \mathcal{K} \) be a submanifold of \( G \) such that for every \( \Lambda \) the manifold \( \mathcal{K}(\Lambda) \) is a subgroup of \( G(\Lambda) \) (and, hence, a Lie subgroup of the Lie group \( G(\Lambda) \)). The structures of Lie groups on \( \Lambda \) produce, when \( \Lambda \) runs in \( G_{(\Lambda)} \), the structure of a Lie supergroup on \( \mathcal{K} \); the supermanifold \( \mathcal{K} \) equipped with this structure of a Lie supergroup is called a Lie sub-

A variety of examples of Lie subgroups one can obtain considering involutions in associative algebras with unity in the category \( \text{Man} \). Let \( \mathcal{K} \) be an associative \( \mathbb{K} \)-algebra with unity in \( \text{Man} \). An \( \mathbb{K} \)-linear morphism \( I : \mathcal{K} \rightarrow \mathcal{K} \) is called an involution on the algebra \( \mathcal{K} \) if \( I^2 = 1 \) and, besides, if \( I \) is an antiautomorphism of the algebra \( \mathcal{K} \). Involution \( I \) is an involution on \( \mathcal{K} \) if \( I^2 = 1 \) and, besides, if \( I \) is an antiautomorphism of the algebra \( \mathcal{K} \), i.e., for every \( \Lambda \) and every \( a, b \in \mathcal{K}(\Lambda) \) the identity \( I(a \cdot b) = I(b) \cdot I(a) \) holds.

**Proposition 7.1.1.** Let \( I : \mathcal{K} \rightarrow \mathcal{K} \) be an involution in an associative \( \mathbb{K} \)-algebra \( \mathcal{K} \) with unity in \( \text{Man} \). Define for every \( \Lambda \) the subset \( \mathcal{K}_I(\Lambda) \) of \( \mathcal{K}(\Lambda) \) as the set of all \( a \in \mathcal{K}(\Lambda) \) such that \( I(a) = a \). Then the family \( \{ \mathcal{K}_I(\Lambda) \} \) generates a subfunctor \( \mathcal{K}_I \) in \( \text{Man} \). The functor \( \mathcal{K}_I \) is a subfunctor of \( \mathcal{K}^* \) and, moreover, a Lie supermanifold of the Lie supergroup \( \mathcal{K}^* \).

**Example 2:** Hilbert superspaces and unitary Lie supergroups. Let \( V \) be a complex Banach space super and \( \langle \cdot, \cdot \rangle : V^* \times V \rightarrow \mathbb{F} \) be an even non-degenerate superhermitian form on \( V \), i.e., \( \langle \cdot, \cdot \rangle \) is \( F \)-sesquilinear on the first argument, \( \mathbb{F} \)-linear on the second argument and, besides, the identity

\[
\langle x, y \rangle = \sum_{i \in \mathbb{F}} (-1)^{|i|} \langle x^i, y \rangle
\]

(7.1.1)

holds. Note that the non-degeneracy of \( \langle \cdot, \cdot \rangle \) implies, together with condition (7.1.1) that \( V \) has the topology of a Hilbert space.

Define the map \( \mathbb{T} : \mathcal{L}(V \times V) \rightarrow \mathcal{L}(V \times V) \) (superhermitian conjugation) as follows:

\[
\langle x^\Lambda y, z \rangle = \sum_{i, j \in \mathbb{F}} (-1)^{|i|} \langle x^{i \Lambda} y, z \rangle
\]

(7.1.2)

(we write, respecting traditions, \( x^\Lambda \) instead of \( x^\Lambda(\Lambda) \)).

The morphism \( \mathbb{T} : \mathcal{L}(V \rightarrow V) \rightarrow \mathcal{L}(V \rightarrow V) \) is an involution in the algebra \( \mathcal{L}(V \rightarrow V) \). The Lie supermanifold \( \mathcal{H} \) of the Lie group \( \mathcal{G}(V \rightarrow V) \) will be called a pseudounitary group of the of the complex Banach super space \( V \), associated with the superhermitian form \( \langle \cdot, \cdot \rangle \); it will be denoted \( \mathcal{H}(V, \langle \cdot, \cdot \rangle) \).

Note that the forms \( \langle \cdot, \cdot \rangle \) and \( \langle \cdot, \cdot \rangle \) are Hermitian forms on \( V \) and \( V \), respectively. If this two forms are definite then the pair \( (V, \langle \cdot, \cdot \rangle) \) (or, simply, \( V \)) will be called a Hilbert super space, whereas the corresponding Lie group supergroup \( \mathcal{G}(V, \langle \cdot, \cdot \rangle) \) will be called the unitary supergroup of the Hilbert super space \( V \).

### 7.2 Lie theory

**7.2.1 Lie theory.** Let \( \mathcal{G} \) be a Lie supergroup and \( \mathcal{P} \rightarrow \mathcal{G} \) be its unity.

For every \( \Lambda \in \mathcal{G} \) the space \( \mathcal{T} \mathcal{G}(\mathcal{G}(\Lambda)) = \mathcal{T}_e \mathcal{G}(\Lambda) \) is at the same time the Lie algebra \( \mathcal{L}(\mathcal{G}(\Lambda)) \) of the Lie group \( \mathcal{G}(\Lambda) \). The structures of Lie algebras on \( \mathcal{L}(\mathcal{G}(\Lambda)) \) generate, when \( \Lambda \) runs in \( \mathcal{G} \), the structure of a Lie algebra in the category \( \text{Man} \) on the fiber \( \mathcal{T}_0 \mathcal{G} \) of the tangent bundle \( \mathcal{T}_\mathcal{G} \) of the Lie supergroup \( \mathcal{G} \). The \( \mathbb{F} \)-module \( \mathcal{T}_\mathcal{G} \) equipped with this structure will be called the Lie algebra (in the category \( \text{Man} \)) of the Lie supergroup \( \mathcal{G} \) and will be denoted \( \mathcal{L}(\mathcal{G}) \).

In fact, the function \( \mathbb{L} \) continues, in an evident way, as some functor from the category of Lie supergroups to the category of Lie algebras of the category \( \text{Man} \) of \( \mathbb{L}(\mathcal{G}) \) of Lie functions; composing the Lie functor \( \mathbb{L} \) with the supermanification functor \( S \) of Sect.4.4 (see (4.4.7)) we will obtain the functor \( S: \mathbb{L}(\mathcal{G}) \).

The Lie superalgebra \( \mathcal{S}(\mathcal{G}) \) will be called the Lie superalgebra of the Lie super-

### 7.3 Exponential morphisms.

**7.3.1 Exponential morphisms.** Define now for every Lie supergroup \( \mathcal{G} \) the exponential morphism

\[
\exp : \mathcal{L}(\mathcal{G}) \rightarrow \mathcal{G}
\]

pointwise: \( \exp(\Lambda) = \exp(\mathcal{G}(\Lambda)) \). It is a functor morphism due to functoriality properties of exponential maps in the ordinary Lie theory.

**Proposition 7.3.1.** The exponential morphism \( \exp \) is super smooth; it is a local isomorphism at any open neighborhood of the point zero \( \mathcal{G}(0) \rightarrow \mathcal{L}(\mathcal{G}) \).

### 7.4 The structure of Lie supergroups

**7.4.1 The structure of Lie supergroups.** Let \( \mathcal{G} \) be a Lie supergroup. For every \( \Lambda \) let \( \mathcal{H}(\Lambda) \) be the kernel of the morphism \( \mathcal{G}(\Lambda) \rightarrow \mathcal{G}(\mathcal{G}(\Lambda)) \) of Lie groups, where \( \mathbb{F}: \Lambda \rightarrow \mathbb{K} \) in the terminal morphism of superspaces. It is evident, that the Lie group \( \mathcal{G}(\Lambda) \) is a semidirect product...
\[ G(\Lambda) \simeq \mathfrak{g}(\Lambda) \] (7.4.1)

Besides, \( X_\Lambda(\Lambda) \) is a nilpotent Lie group.

Consider now the canonical \( \mathcal{K} \)-module decomposition \( L(\mathcal{G}) \equiv L(\mathcal{G}) \oplus \mathfrak{g}(\mathcal{G}) \), where \( \mathfrak{g}(\mathcal{G}) \cong L(\mathcal{G}) \) is the "ordinary" Lie algebra and \( \mathfrak{g}(\mathcal{G}) \) is the superpoint corresponding to the odd part of the Lie superalgebra \( SL(\mathcal{G}) \). Due to Prop. 7.1.1, the exponential morphism \( \exp \) isomorphically maps the superpoint \( \mathfrak{g}(\mathcal{G}) \) onto some superpoint \( \mathfrak{g}(\mathcal{G}) \cong \mathfrak{g}(\mathcal{G}) \) (in fact, \( \mathfrak{g}(\mathcal{G}) \cong X_\Lambda(\Lambda) \) for every \( \Lambda \)).

Let for every \( \Lambda \) the map \( \iota_\Lambda : \mathcal{G}(\mathcal{G})(\Lambda) \times \mathfrak{g}(\mathcal{G})(\Lambda) \to \mathcal{G}(\mathcal{G})(\Lambda) \) is the restriction of the multiplication in the group \( \mathcal{G}(\mathcal{G})(\Lambda) \), i.e., \( \iota_\Lambda(\mathbf{g}, x) = g \times x \) (the functor \( \mathcal{E}_\Lambda^\iota \) of Grassmann analytical continuation is defined in Sect. 4.2).

**Proposition 7.4.1.** The family \( \{ \iota_\Lambda \}_{\Lambda \in \mathcal{K}} \) determines some supergroup isomorphism

\[ \iota_\Lambda : \mathcal{G}(\mathcal{G})(\Lambda) \times \mathfrak{g}(\mathcal{G})(\Lambda) \to \mathcal{G}(\mathcal{G})(\Lambda) \] (7.4.2)

of supermanifolds. In particular, every Lie supergroup in a simple supermanifold.

**7.5. Inverse Lie theory modulo manifolds.**

**Proposition 7.5.1.** Let \( \mathcal{G} \) be a Lie superalgebra and \( \mathcal{G} \) be a Lie group such that \( l(\mathcal{G}) = \mathfrak{g} \).

Let, further, there exists a linear smooth action of the Lie group \( \mathcal{G} \) on the Banach space \( \mathfrak{g} \) such that the corresponding infinitesimal action of the Lie algebra \( \mathfrak{g} \) on the space \( \mathfrak{g} \) coincides with the adjoint action (determined by the Lie bracket in \( \mathfrak{g} \)). Then there exists the only (up to an isomorphism) Lie supergroup \( \mathcal{G} \) such that its Lie superalgebra \( SL(\mathcal{G}) \) coincides with \( \mathfrak{g} \) and \( \mathcal{G} \) coincides with \( \mathcal{G} \).

**7.6. Linear representations of Lie supergroups.** Let \( V \) be a \( \mathcal{K} \)-module and \( \mathcal{G} \) be a Lie supergroup. An action \( \rho : \mathcal{G} \times V \to V \) of \( \mathcal{G} \) on \( V \) is called a \( \mathcal{K} \)-linear representation of \( \mathcal{G} \) (or a \( \mathcal{G} \)-module over \( \mathcal{K} \)) if \( \rho \) is a family of \( \mathcal{K} \)-linear morphisms. As a trivial consequence of Corollary 4.4.3 one obtains that the canonical representation of the Lie supergroup \( \mathcal{G} \) on \( V \) (i.e., the restriction of the evaluation morphism \( ev \) into \( \mathcal{G}(\mathcal{G}) \)) is universal among all linear actions of Lie supergroups on the \( \mathcal{K} \)-module \( V \); in particular, \( \mathcal{K} \)-linear representations of a Lie supergroup \( \mathcal{G} \) are in a bijective correspondence with the set of all morphisms of \( \mathcal{G} \) into \( \mathcal{G}(\mathcal{G})(\Lambda) \).

**7.7. Groups in \( \text{Bun}[\mathcal{G}] \).** Groups in the category \( \text{Bun}[\mathcal{G}] \) will be called \( \mathcal{G} \)-Lie supergroups. The following proposition permits one to reduce Lie supergroups and their representations to \( \mathcal{G} \)-Lie supergroups and their representations in \( \mathcal{G} \).

**Proposition 7.7.1.** Let \( \mathcal{G} \). The functor \( \mathcal{F}_\mathcal{G} \) (defined in Sect. 4.2) generates an equivalence of the category of Lie supergroups with the category of \( \mathcal{G} \)-Lie supergroups: for a given Lie supergroup \( \mathcal{G} \) the category of linear representations of \( \mathcal{G} \) is equivalent to the category of linear representations of the \( \mathcal{G} \)-Lie supergroup \( \mathcal{F}_\mathcal{G}(\mathcal{G}) \).

**7.8. Factor-supergroups of Lie supergroups.** Let \( \mathcal{G} \) be a Lie supergroup and \( \mathcal{H} \) be some its Lie subgroup. Define the functor \( \mathcal{G}/\mathcal{H} \) pointwise as follows: \( \mathcal{G}/\mathcal{H}(\Lambda) \equiv \mathcal{G}(\Lambda)/\mathcal{H}(\Lambda) \); canonical projections \( \pi : \mathcal{G}(\Lambda) \to \mathcal{G}(\Lambda)/\mathcal{H}(\Lambda) \) define the functor morphism \( \pi \). Then \( \mathcal{G}/\mathcal{H} \) is a Lie supergroup w.r.t. multiplication \( \mathcal{G}/\mathcal{H} \times \mathcal{G}/\mathcal{H} \to \mathcal{G}/\mathcal{H} \) defined pointwise.

This Lie supergroup will be called the factor-supergroup of \( \mathcal{G} \) over \( \mathcal{H} \).

**8. Supergroups of Superdiffeomorphisms.**

In this section will be constructed supergroups of superdiffeomorphisms of supermanifolds being the counterparts of groups of diffeomorphisms in a standard theory of manifolds. This supergroups exist as group objects in the functor category \( \text{Set}^{\mathcal{K}} \). The latter topos seems to play the same role in the theory of supermanifolds as the topos \( \text{Set} \) plays in the theory of manifolds: it is the "environment" for various types of objects which arise naturally in the theory of supermanifolds but not always could "live" within the category of supermanifolds itself (example: orbits of smooth actions of Lie supergroups).
8.1. Geometrized Yoneda functor. In this section the natural \textbf{neglecting} functor $\text{Man} \xrightarrow{\mathfrak{V}} \text{Man}$ will be interpreted as a "geometrization" of Yoneda functor $\text{Man} \xrightarrow{\mathfrak{V}} \text{Set} \text{Man}$ composed with the functor $\text{Set} \text{Man} \xrightarrow{\text{Set}} \text{Set} \text{Point}^\circ$ of restriction to superpoints. Here $\text{Set} \text{Point}^\circ \subseteq \text{Man}$ is the full subcategory of the category $\text{Set} \text{Man}$ consisting of finite-dimensional superpoints. It is evident from Th. 3.1.2 that the category $\text{Set} \text{Point}$ is naturally equivalent to the category $\text{Gras}$ dual to the category of Grassman superalgebras.

**Proposition 8.1.1.** The functor $\text{Man} \xrightarrow{\mathfrak{V}} \text{Set} \text{Man} \xrightarrow{\text{Set}} \text{Set} \text{Point} \approx \text{Set} \text{Gras}$ is naturally isomorphic to the neglecting functor $\mathfrak{V} : \text{Man} \xrightarrow{\mathfrak{V}} \text{Man} \xrightarrow{\mathfrak{V}} \text{Set} \text{Gras}$.

This proposition gives the desired interpretation. Besides, choosing and fixing some contravariant functor

$$\mathfrak{P} : \mathfrak{P} \text{Gras} \rightarrow \text{Set} \text{Point} \tag{8.1.1}$$

establishing a natural equivalence of categories, one obtains the following important

**Corollary 8.1.2.** For every supermanifold $\mathcal{M}$ and every Grassman algebra $A$

$$\text{Gras}(A) \approx \text{Set} \text{Point}(\mathfrak{P}(A), \mathcal{M}) \tag{8.1.2}$$

natural both on $\mathcal{M}$ and $\mathcal{A}$.

8.2. Functors of supermorphisms and of supersections. Let $\mathcal{M}$ and $\mathcal{M}'$ be supermanifolds. Define the $\text{Set}$-valued functor $\mathfrak{S} \mathfrak{m}(\mathcal{M}, \mathcal{M}')$ on the category of Grassman superalgebras as follows:

$$\mathfrak{S} \mathfrak{m}(\mathcal{M}, \mathcal{M}') \approx \text{Set} \text{Point}(\mathfrak{P}(\mathcal{A}) \times \mathcal{M}, \mathcal{M}') \tag{8.2.1}$$

The functor $\mathfrak{S} \mathfrak{m}(\mathcal{M}, \mathcal{M}')$ will be called the functor of supersupermorphisms of the supermanifold $\mathcal{M}$ into the supermanifold $\mathcal{M}'$. Note that there exists the evident natural isomorphisms

$$\mathfrak{S} \mathfrak{m}(\mathcal{M}, \mathcal{M}') \approx \mathfrak{S} \mathfrak{m}(\mathcal{M}', \mathcal{M}) \approx \mathfrak{S} \mathfrak{m}(\mathcal{M}, \mathcal{M}') \tag{8.2.2}$$

Let now $\mathcal{E} \xrightarrow{\mathfrak{F}} \mathcal{M}$ be a $\mathbb{R}$-vector bundle over the base supermanifold $\mathcal{M}$.

**Proposition 8.2.1.** There exists an isomorphism of functors

$$\mathfrak{S} \mathfrak{m}(\mathcal{M}, \mathcal{V}) \approx \mathfrak{S} \mathfrak{m}(\text{Gras}(\mathcal{V}), \mathcal{M}) \tag{8.2.3}$$

natural on $\mathcal{M}$ and $\mathcal{V}$, turning $\mathfrak{S} \mathfrak{m}(\mathcal{M}, \mathcal{V})$ into a superrepresentable $\mathbb{R}$-module (in $\text{Set} \text{Gras}$).

8.3. Morphisms of composition and of evaluation. In this section it will be more convenient to work directly with the category $\text{Set} \text{Point}$ instead of equivalent to it category $\text{Gras}$. The variable $\mathfrak{P}$ will run on the set of objects of the category $\text{Set} \text{Point}$.

Let $\mathcal{M}$ and $\mathcal{M}'$ be supermanifolds. Define the evaluation morphism

$$\text{ev} : \mathfrak{S} \mathfrak{m}(\mathcal{M}, \mathcal{M}') \times \mathcal{M} \rightarrow \mathcal{M}' \tag{8.3.1}$$

as follows: for every morphism $\mathfrak{F} : \mathcal{M} \rightarrow \mathcal{M}'$ and $\mathfrak{P} : \mathcal{P} \rightarrow \mathcal{M}$ let $\text{ev}(\mathfrak{F}, \mathfrak{P})$ be the composition

$$\text{ev}(\mathfrak{F}, \mathfrak{P}) : \mathfrak{P} = (\mathfrak{P}, \mathfrak{P}) \xrightarrow{\mathfrak{P}} \mathcal{M} \xrightarrow{\mathfrak{F}} \mathcal{M}' \tag{8.3.2}$$

Let now $\mathcal{M}$, $\mathcal{M}'$ and $\mathcal{M}''$ be supermanifolds. Define the functor morphisms

$$\text{comp} : \mathfrak{S} \mathfrak{m}(\mathcal{M}, \mathcal{M}') \times \mathfrak{S} \mathfrak{m}(\mathcal{M}', \mathcal{M}'') \rightarrow \mathfrak{S} \mathfrak{m}(\mathcal{M}, \mathcal{M}'') \tag{8.3.3}$$

of composition as follows. If $\mathfrak{F} : \mathcal{M} \rightarrow \mathcal{M}'$ and $\mathfrak{F}' : \mathcal{M}' \rightarrow \mathcal{M}''$ are some morphisms, let composition $\mathfrak{F} \circ \mathfrak{F}') : \mathcal{M} \rightarrow \mathcal{M}''$.

**Proposition 8.3.1.** Morphisms $\text{comp}$ is an associative composition on the functor $\mathfrak{S} \mathfrak{m}(\mathcal{M}, \mathcal{M}')$. Besides, the point

$$\mathfrak{p} \xrightarrow{\mathfrak{P}} \mathfrak{S} \mathfrak{m}(\mathcal{M}, \mathcal{M}') \tag{8.3.4}$$
8.4. The supergroup of superdiffeomorphisms. Let \( \mathcal{M} \) be a supermanifold. Define for every \( \Lambda \) the set \( \hat{\text{Diff}}(\mathcal{M})(\Lambda) \) as the subset of all invertible elements of the semigroup \( \hat{\text{SC}}(\mathcal{M}, \mathcal{M})(\Lambda) \) (with the composition \( \circ \) defined in the preceding subsection).

Proposition 8.4.1. The family \( \{ \hat{\text{Diff}}(\mathcal{M})(\Lambda) \}_{\Lambda \in \mathbb{Z}} \) forms a subfunctor \( \hat{\text{Diff}}(\mathcal{M}) \) in \( \hat{\text{SC}}(\mathcal{M}, \mathcal{M}) \); this subfunctor coincides with the subfunctor \( \hat{\text{SC}}(\mathcal{M}, \mathcal{M}) \) of all superdiffeomorphisms (isomorphisms in \( \hat{\text{Man}}^\text{Gr} \)) of \( \mathcal{M} \). Moreover, the group structures on all \( \hat{\text{Diff}}(\mathcal{M})(\Lambda) \) produce the structure of a supergroup (a group object in \( \text{Set}^{\text{Gr}} \)) on the functor \( \hat{\text{Diff}}(\mathcal{M}) \).

The supergroup \( \hat{\text{Diff}}(\mathcal{M}) \) will be called the supergroup of superdiffeomorphisms of the supermanifold \( \mathcal{M} \). This supergroup possesses the following universal property.

Proposition 8.4.2. Let \( \mathcal{G} \times \mathcal{M} \longrightarrow \mathcal{M} \) be an action of a Lie supergroup \( \mathcal{G} \) on a supermanifold \( \mathcal{M} \). Then there exists the only morphism \( \hat{\beta} \) of supergroups (in the category \( \text{Set}^{\text{Gr}} \)) such that the diagram

\[
\begin{array}{ccc}
\mathcal{G} \times \mathcal{M} & \xrightarrow{\hat{\beta} \times \text{ev}} & \hat{\text{Diff}}(\mathcal{M}) \\
\mathcal{G} \times \mathcal{M} & \xrightarrow{\mathcal{G} \times \text{ev}} & \mathcal{M}
\end{array}
\]

is commutative. The morphism \( \hat{\beta} \) is determined as follows: for every morphism \( \rho: \mathcal{G}(\Lambda) \longrightarrow \mathcal{G} \), the morphism \( \hat{\beta}(\rho) \) is the composition

\[
\hat{\beta}(\rho): \mathcal{G}(\Lambda) \times \mathcal{M} \xrightarrow{\rho \times \text{Id}} \mathcal{G} \times \mathcal{M} \xrightarrow{\text{ev}} \mathcal{M}.
\]

8.5. Remarks on locally convex supermanifolds. One can define the category of locally convex, or Fréchet, or tame Fréchet supermanifolds, replacing simply the category \( \hat{\text{Man}}^\text{Gr} \) by the category of functors on \( \text{Gr}^\text{op} \) with values in the category of smooth locally convex, resp. Fréchet, tressed Fréchet manifolds (the corresponding theory of manifolds based on the notion of weak derivative morphisms is developed in the paper [19] of R. Hamilton).

Then one can, on the one hand, generalize the Nash-Nevièse inverse function theorem to the case of tame Fréchet supermanifolds; on the other hand, one can equip the functors \( \hat{\text{SC}}(\mathcal{M}, \mathcal{M}) \) and \( \hat{\text{F}}(\mathcal{E} \longrightarrow \mathcal{M}) \) with structures of tame Fréchet supermanifolds in the case when \( \mathcal{M} \) is compact.

The details will be considered elsewhere.

9. \( \mathbb{Z}^k \)-supermanifolds.

In this section we will construct the "iterated" category \( s^\text{Man} \) of \( \mathbb{Z}^k \)-supermanifolds such that algebras (of any polynomial type \( T \)) in this category correspond to \( \mathbb{Z}^k \)-graded Banach superalgebras of the corresponding type.

We could construct the category \( s^\text{Man} \) recursively considering \( \mathbb{Z}^k \)-supermanifolds as functors in the functor category \( s^{k-1} \text{Man}^\text{Gr} \). Instead we will do it more directly, using the functor category \( s^\text{Man}^\text{Gr} \).

9.1. \( \mathbb{Z}^k \)-Grassmann superalgebras. We will denote \( s^T_y(\mathcal{D}) \) the category of \( \mathbb{Z}^k \)-graded \( R \)-superalgebras of a polynomial type \( T \) in a category \( \mathcal{D} \) (see Sect. 2.1.10).

Let

\[
\mathbb{Z}^2 \rightarrow s^2 \mathbb{T}_R(\mathcal{D}), \quad \varepsilon \mapsto (0, \ldots, 0, \varepsilon, 0, \ldots, 0)
\]

be the canonical injection of \( \mathbb{Z}^2 \)-modules. It generates, for every commutative ring with unity in a category \( \mathcal{D} \) with finite products and for every polynomial type \( T \) of algebraic structure, some functor

\[
\mathbb{Z}^2 \longrightarrow s^T_R(\mathcal{D})
\]

from the category of \( R \)-superalgebras of the type \( T \) in \( \mathcal{D} \) to the category of \( \mathbb{Z}^k \)-graded \( R \)-superalgebras of the same type \( T \) in \( \mathcal{D} \). In particular, \( \mathbb{Z}^2 \) sends commutative superalgebras into commutative \( \mathbb{Z}^2 \)-graded superalgebras.
For every map $\mathfrak{Q} : \mathbb{Z}^k_2 \rightarrow \mathbb{N}$ denote $\Lambda_\mathfrak{Q}$ some free real $\mathbb{Z}^k_2$-graded commutative superalgebra having exactly $\mathfrak{Q}(i)$ free generators with parity $i$; the superalgebra $\Lambda_\mathfrak{Q}$ with $\mathfrak{Q}$ determined from equalities

$$\mathfrak{Q}(1_j) = n_j, \quad \text{else } \mathfrak{Q}(i) = 0 \quad (j=1, \ldots, k)$$

will be denoted as $\Lambda_{n_1, \ldots, n_k}$ and will be called $\mathbb{Z}^k_2$-Grassmann superalgebra.

Let $\Lambda_1$ and $\Lambda_2$ be $\mathbb{Z}^k_2$-graded real superalgebras. The tensor product $\Lambda_1 \otimes \Lambda_2$ of $\mathbb{Z}^k_2$-graded supermodules equipped with a multiplication defined by equalities

$$(a_1 \otimes a_2)(b_1 \otimes b_2) = \sum_{i=1}^{\infty} (-1)^{i_1 + i_2} a_1^{i_1} b_1^{i_1} a_2^{i_2} b_2^{i_2}$$

is called the tensor product of $\mathbb{Z}^k_2$-graded superalgebras $\Lambda_1$ and $\Lambda_2$.

One can easily verify that for every $\mathbb{Z}^k_2$-Grassmann superalgebra the identity

$$\Lambda_{n_1, \ldots, n_k} \cong I_{1}(A_1) \otimes \cdots \otimes I_{k}(A_k)$$

holds. It is evident, besides that $\Lambda_{n_1, \ldots, n_k}$ as an algebra (neglecting “super” structures) is the ordinary tensor product of Grassmann algebras, which coincides in this particular case with the superproduct (9.1.4).

We will denote the full subcategory of the category of $\mathbb{Z}^k_2$-graded superalgebras with unity, consisting of all $\mathbb{Z}^k_2$-Grassmann superalgebras as $\mathbb{G}_{\otimes}^k$.

One can observe that the functor

$$\mathbb{G}_{\otimes}^k \rightarrow \mathbb{G}_{\otimes}^k$$

(9.1.6)
determined by isomorphisms (9.1.5) is, in fact, an isomorphism of categories, because $\Lambda_{n_1, \ldots, n_k}$ are free. For our purposes it will be more convenient to use directly the category $\mathbb{G}_{\otimes}^k$ instead of the equivalent category $\mathbb{G}_{\otimes}^k$.

9.2. $\mathbb{Z}^k_2$-supermanifolds. Now we can literally repeat definitions and constructions of preceding sections for the functor category $\mathbb{G}_{\otimes}^k$ in place of the category $\mathbb{G}_{\otimes}^k$.

First of all if $\mathcal{V}$ is an $\mathbb{Z}^k_2$-graded $K$-module (in Top, Man or Set), one can define the $K$-module $\mathcal{V}$ in the corresponding functor category (Top, $\mathbb{G}_{\otimes}^k$; Man, $\mathbb{G}_{\otimes}^k$ or Set, $\mathbb{G}_{\otimes}^k$) just by $\mathcal{V}(i) = \mathcal{V}_i$ (where $i$ run now in $\mathbb{G}_{\otimes}^k$); similarly can be defined $\mathcal{V}$ for an even $K$-polylinear map $\mathcal{F}$ of $\mathbb{Z}^k_2$-graded algebras of some type $\mathcal{T}$ in the corresponding functor category, which are isomorphic to $\mathcal{V}$ for some $\mathbb{Z}^k_2$-graded superalgebra $\mathcal{V}$ of the type $\mathcal{T}$ (in Top, Man or Set) will be again called superrepresentable.

Define Man$_{\mathcal{V}}$ as an open subfunctor of a superrepresentable $\mathbb{G}$-module in Top, Man or Set. Every Man$_{\mathcal{V}}$-supermanifold in $\mathcal{V}$ is again of the form $\bigcup U$ for some open $U$ in $\mathcal{V}$.

The definitions of supermanifolds and of $\mathbb{Z}^k_2$-supermanifolds literally copy the corresponding definitions for the ordinary case ($k=1$). Denoting the category of Man$_{\mathcal{V}}$ as $\mathcal{S}^k_{\text{Man}}$, one can formulate the following generalization of Corollary 4.4.2:

Theorem 9.2.1. The category $\mathcal{S}^k_{\text{Top}}(\mathcal{V})$ of $\mathbb{Z}^k_2$-graded $K$-superalgebras of any type $\mathcal{T}$ in $\text{Man}$ is naturally equivalent to the category $\mathcal{T}(\mathcal{S}^k_{\text{Man}})$ of $\mathbb{Z}^k_2$-algebras of the type $\mathcal{T}$ in $\mathcal{S}^k_{\text{Man}}$.

The theory of $\mathbb{Z}^k_2$-supermanifolds could be developed further along the same lines as the theory of "ordinary" supermanifolds (with the possible exception of the theory of integration). Namely, one can define vector bundles in the category $\mathbb{S}^k_{\text{Man}}$, tangent functor $\mathcal{T}$ and Lie functor, as well as the exponential morphism following literally the corresponding definitions of the case $k=1$.

In particular, the inverse function theorem is valid in the category $\mathbb{S}^k_{\text{Man}}$ as well.

9.3. Example. We will give, in conclusion, an example showing that it is not easy (if at all possible) to reformulate the theory of finite-dimensional $\mathbb{Z}^k_2$-supermanifolds (or $n \geq 2$) in terms of spaces with sheaves of $\mathbb{Z}^k_2$-graded commutative superalgebras on them.

Define a $\mathbb{Z}^k_2$-graded commutative superalgebra $\mathcal{R}(k)$ in $\mathcal{S}^k_{\text{Man}}$ ("coordinate ring") as follows: $\mathcal{R}(k)(\mathcal{F}) = \Lambda$. The $\mathbb{Z}^k_2$-graded $\mathcal{R}$-superalgebra $\mathcal{S}^k(\mathcal{M}) := \mathcal{S}^k_{\text{Man}}(\mathcal{M}, \mathcal{R})$ will be called the superalgebra of superfields of the $\mathbb{Z}^k_2$-supermanifold $\mathcal{M}$.

Describe the structure of this superalgebra in a simple case when $k=2$ and the supermanifold $\mathcal{M}$ is a finite-dimensional $\mathbb{Z}^2_2$-superpoint, i.e., $\mathcal{M} = \mathcal{V}$, where $\dim \mathcal{V} = \{1, 2\}$, $n_1 = 0$ and $n_2$, $n_3$, $n_4 \in \mathbb{N}$.

It follows from the counterpart of Th. 3.1.2 (which generalizes to the case...
of arbitrary $k$ that in this case there exists an isomorphism

$$\mathcal{C}^\infty(\mathcal{V}) \cong \bigotimes_{n \in \mathbb{N}} \mathcal{C}^\infty(\mathcal{U}) \bigotimes \mathcal{K}[x_1, \ldots, x_n]$$

(9.3.1)

or $\mathbb{Z}^n_2$-graded algebras, where $\mathcal{K}[x_1, \ldots, x_n]$, considered as an algebra, is simply the algebra of formal power series in variables $x_1, \ldots, x_n$.

Note that if $\mathcal{U} = \mathcal{U} \times \mathcal{P}$ is a finite-dimensional $\mathbb{Z}^n_2$-superregion, such that $\mathcal{U}$ is an "ordinary" region (i.e., dimensions of $\mathcal{U}$ in "directions" $(1,0)$, $(0,1)$, and $(1,1)$ are zero) and $\mathcal{P}$ is a finite-dimensional superpoint, then, generally speaking, $\mathcal{C}^\infty(\mathcal{U}) \neq \mathcal{C}^\infty(\mathcal{U}) \bigotimes \mathcal{C}^\infty(\mathcal{P})$.

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