The Classification of Wallpaper Patterns: From Group Cohomology to Escher's Tessellations

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Preface

I first started to think seriously about symmetry groups while teaching a senior-level abstract algebra course in Fall 1999, during which we determined symmetry groups of the Platonic solids. Soon afterward I worked out for myself the classification of the symmetry groups of bounded plane figures and of strip patterns, neither of which is not very hard. From that I decided to try to understand wallpaper patterns. A big step was becoming aware of Mackiw's book [3], which contains two chapters on wallpaper patterns, I began to read through the second of these chapters, which does the classification. Fairly soon I realized that a wallpaper group is a group extension of its subgroup of translations, a group isomorphic to $\mathbb{Z}^2$, and its point group. At this point I started to think about group cohomology. This is an important tool in the theory of finite-dimensional division algebras, my primary area of research, and it is used to classify group extensions. It seemed to me likely that the classification of wallpaper groups should be the same, or nearly so, as the classification of group extensions, even though two group extensions $1 \to T \to G \to G_0 \to 1$ and $1 \to T \to G' \to G_0 \to 1$ can be inequivalent even if G and G' are isomorphic. In order to carry out the classification of group extensions, I needed to calculate several cohomology groups. After a little bit of thought, I realized that I needed to use spectral sequences to do this. At this point I decided it was time to finally buckle down and learn about these; I had some familiarity with spectral sequences from listening to various research talks, but I had never studied them in detail. After working through the spectral sequence chapter in Weibel [7], I found that calculating the cohomology groups that arise from wallpaper groups was very easy; it only took quite simple applications of the Lyndon-Hochschild-Serre spectral sequence. Doing so gives 18 inequivalent group extensions; this shows that there is very little difference between group extensions and wallpaper groups. A little more work then showed that only two inequivalent group extensions corresponded to isomorphic symmetry groups.

Armed with this cohomological classification of wallpaper groups, I proceeded to present these ideas in a series of lectures in our algebra seminar (Spring 2000). Because my enjoyment of working through this material and discussions with colleagues, I decided to write this up in a monograph. I knew that nothing I had done was original. However, I was a bit surprised that any details of the classification is not so well known, even though many people know about it. Many books on symmetry and Escher's tessellations point out the classification without proving it. Furthermore, I liked very much how tightly related the classification of the wallpaper groups is to the classification of the corresponding group extensions. I thought that doing the classification via group cohomology was a very nice application of this abstract and technical area of algebra. In fact, at the end of a graduate level course in group cohomology I taught in Fall 2000, I gave several examples of computing cohomology groups via spectral sequences by considering wallpaper groups. However, to make this monograph more accessible, I have included a section classifying the wallpaper groups without resorting to any cohomology. By ignoring the chapter on group cohomology, a well-prepared undergraduate can follow the classification given here.

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Chapter 1

Introduction

It is often said that group theory is the study of symmetry. In this book we will use group theory, along with some other fields of mathematics, to classify the symmetry of certain two-dimensional figures called wallpaper patterns. The way one classifies symmetry of geometric objects is to associate to the object a group, called its symmetry group, and then to classify the possible symmetry groups. The study of symmetry groups of wallpaper patterns began in the nineteenth century by people studying crystals, which exhibit a repeating structure in three dimensions. By the end of the nineteenth century, the classification of the so-called crystallography groups in dimensions 2 and 3 was completed by Fedorov, Schoenflies, and Barlow, building on work by several others.

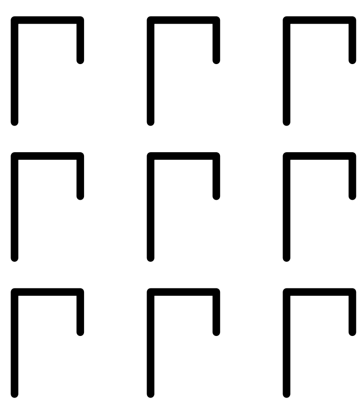
While many mathematicians know that there are exactly 17, up to isomorphism, symmetry groups of wallpaper patterns, most do not know why this is true. One of the purposes of this book is to show how to obtain the classification. While much of the classification can be understood by a good undergraduate student, our approach to the classification should be of interest to professional mathematicians, including algebraists, due to our use of group cohomology and spectral sequences. In fact, another of the book's purposes is to illustrate the use of homological techniques. Finally, a third purpose of the book is to make a connection between artistic aspects of drawing tessellations and group-theoretic concepts. By giving a brief description of group cohomology, the book is nearly self-contained, relying on other sources only for results about the cohomology of cyclic groups and spectral sequences.

The classification and organization of this book begins with the definition of wallpaper patterns, their translation lattices, and their symmetry groups, which are groups of isometries. In Chapter 2 we describe the different types of isometries of the plane and the structure of the group of isometries of the plane. Chapter 3 begins the classification in earnest; we define the point group of a wallpaper pattern and use it to describe the five lattice types of wallpaper patterns. We define group cohomology in Chapter 4, and use it to calculate the cohomology groups that classify wallpaper groups. Finally, in Chapter 5, we use our results of Chapter 4 to determine and describe the 17 wallpaper groups. We also illustrate similarities and differences between these groups, illustrating these points with examples of wallpaper patterns. In addition, we show how to obtain the classification without the use of cohomology.

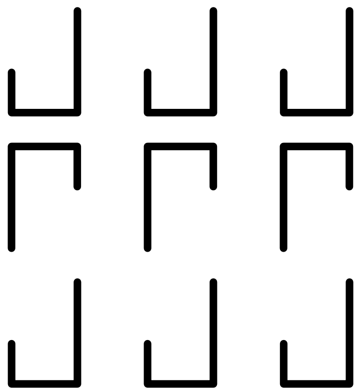
As we will see, the study of symmetry groups of wallpaper patterns involves a wonderful mix of mathematical ideas, from the very simple to the quite complex. Our basic idea is to reduce the classification of these symmetry groups, by using linear algebra, geometry, and elementary group theory, to a problem of homological algebra, notably the determination of certain cohomology groups. By making use of fundamental results from group cohomology, including the Lyndon-Hochschild-Serre spectral sequence, we are then able to calculate these cohomology groups, which then allows us to determine these symmetry groups, up to isomorphism. While the use of homological algebra is complicated for the non-specialist in algebra, much of what we do involves mathematics accessible to anyone with a good undergraduate background. If one then accepts the calculation of these cohomology groups, one can then get a good understanding of the classification of wallpaper patterns without too much difficulty. Alternatively, we show how to classify the symmetry groups of wallpaper patterns without using homological algebra.

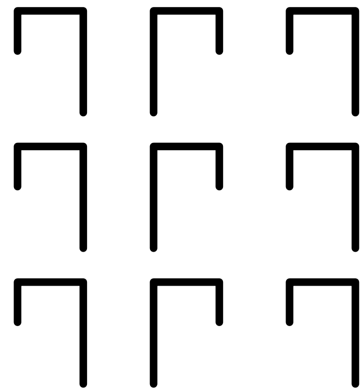
To illustrate the symmetry groups, we will look closely at many of Escher's tessellations. These beautiful works of art illustrate very nicely group-theoretic aspects of the symmetry groups. Moreover, studying the symmetry groups helps to understand the geometric restrictions Escher had to discover in order to create his tessellations. Escher gave us a large body of art with which we can use to illustrate the mathematical ideas involved in describing symmetry groups of wallpaper patterns.

We will give a formal definition shortly, but intuitively a wallpaper pattern is a design used for making wallpaper. They consist of taking a basic pattern and repeating it horizontally and vertically. The example below was made by taking a figure consisting of a single Γ, and translating it horizontally and vertically.



There are several other ways to repeat a basic figure to get different wallpaper patterns. For instance, one could repeat the figure in two other ways to obtain the following two patterns.





In the first figure, we used 180° rotations to repeat the pattern vertically. In the second, one column is a mirror image of an adjacent column.

We now become more precise. An *isometry* of $\mathbb{R}^n$ is a distance-preserving bijection. Let $\mathrm{Isom}(\mathbb{R}^n)$ be the set of isometries of $\mathbb{R}^n$. A simple argument will show that the composition of two isometries is an isometry and that the inverse of an isometry is an isometry. Therefore, $\mathrm{Isom}(\mathbb{R}^n)$ is a group under composition of functions. If W is a subset of $\mathbb{R}^n$, then the *symmetry group* of W is defined to be

$$\mathrm{Sym}(W) = \{\varphi \in \mathrm{Isom}(\mathbb{R}^n) : \varphi(W) = W\}.$$

It is clear that $\mathrm{Sym}(W)$ is a subgroup of $\mathrm{Isom}(\mathbb{R}^n)$. To work with symmetry groups of plane figures, we will need to know what are the isometries of $\mathbb{R}^2$; we will describe all isometries of $\mathbb{R}^2$ in Section 2.1. For our immediate need, we consider translations. If $v \in \mathbb{R}^n$, then we will refer to the map τ_v, given by $\tau_v(x) = x + v$ for all $x \in \mathbb{R}^n$, as translation by v. It is elementary to see that τ_v is an isometry. Furthermore, if $v, w \in \mathbb{R}^n$, then $\tau_v \circ \tau_w = \tau_{v+w}$ and $\tau_v^{-1} = \tau_{-v}$. Moreover, if $\mathbf{0}$ is the zero vector, then $\tau_{\mathbf{0}}$ is the identity map. These facts show that the set of translations $\mathbb{T}$ forms a subgroup of $\mathrm{Isom}(\mathbb{R}^n)$. Therefore,

$$\mathrm{Sym}(W) \cap \mathbb{T} = \{\tau \in \mathrm{Sym}(W) : \tau \text{ is a translation}\}.$$

is a subgroup of $\mathrm{Sym}(W)$, and we call it the *translation subgroup* of $\mathrm{Sym}(W)$.

In each of the pictures above, the symmetry group contains horizontal and vertical translations. Moreover, there is a horizontal and a vertical translation of smallest possible length. If τ_1 is the smallest horizontal translation and τ_2 the smallest vertical translation of one of the patterns, then any translation of the pattern is of the form $\tau_1^n \tau_2^m$ for some pair of integers n, m. The primary characteristic of wallpaper patterns is that there are always two translations τ_1, τ_2 of the pattern such that any other translation is of the form $\tau_1^n \tau_2^m$ for some integers n, m; we will soon formalize this in a definition.

Lemma 1.1. *The group $\mathbb{R}^n$ is isomorphic to the subgroup of $\mathrm{Isom}(\mathbb{R}^n)$ consisting of all translations of $\mathbb{R}^n$ via the map $v \mapsto \tau_v$. Therefore, the translation subgroup of the symmetry group of a figure in $\mathbb{R}^n$ is isomorphic to a subgroup of $\mathbb{R}^n$.*

Proof. We show that the map φ given by $\varphi(v) = \tau_v$ is an isomorphism. We have already pointed out that $\tau_v \circ \tau_w = \tau_{v+w}$ and $\tau_v^{-1} = \tau_{-v}$ for any $v, w \in \mathbb{R}^n$. Thus, φ is a homomorphism. It is surjective by the definition of a translation. It is injective, since if $\varphi(v) = \mathrm{id}$, then $\tau_v(x) = x$ for all $x \in \mathbb{R}^n$. However, since $\tau_v(x) = x + v$, this forces $v = \mathbf{0}$. Thus, φ is an isomorphism. The final statement is clear, since if T is the translation subgroup of some symmetry group, then $\varphi^{-1}(T)$ is a subgroup of $\mathbb{R}^n$ isomorphic to T. □

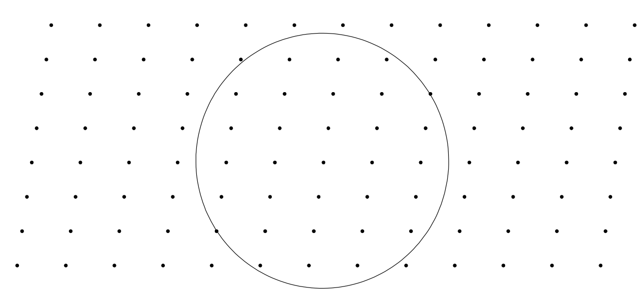
Because of this lemma, we will frequently identify a translation τ_v with the vector v; this should not cause any confusion.

Definition 1.2. *A lattice is a finitely generated subgroup of* $\mathbb{R}^n$ *for some* n.

By the fundamental theorem for finitely generated Abelian groups [1, Theorem 4.5.1], every lattice is a free Abelian group, and so is isomorphic to $\mathbb{Z}^r$ for some integer r. Therefore, a lattice has a basis as a $\mathbb{Z}$-module. The dimension of a lattice is the size of a basis. A two-dimensional lattice in $\mathbb{R}^2$ then has the form

$$T = \{nt_1 + mt_2 : n, m \in \mathbb{Z}\}$$

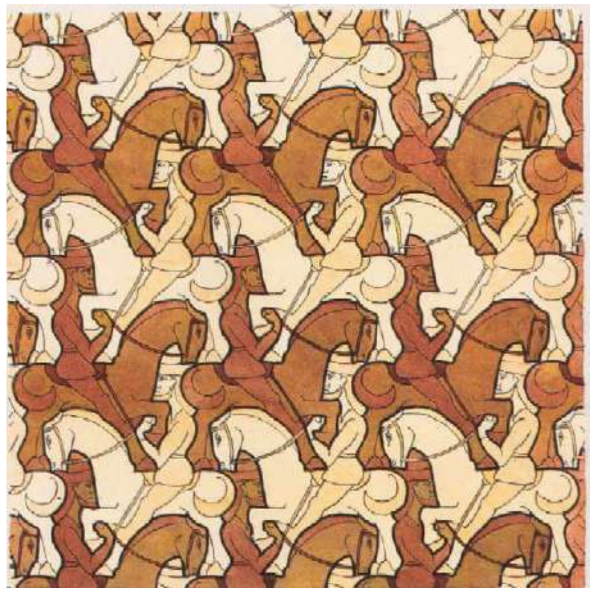
of $\mathbb{Z}$-linear combinations of t_1 and t_2 for some set $\{t_1, t_2\}$, which is then a basis of T and of $\mathbb{R}^2$. We will call a basis of T an *integral basis* to emphasize that it is a basis of T and not just a basis of $\mathbb{R}^2$. Two simple properties of lattices we will use are that (i) T contains a nonzero vector of minimal length, and (ii) T contains only finitely many vectors inside any circle. While there is always a vector v of minimal length in a lattice, it is not unique since v and $-v$ have the same length. These facts can be understood by drawing T as a subset of $\mathbb{R}^2$ as in the following picture.



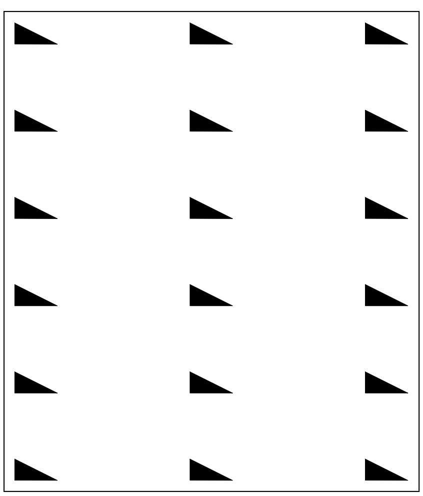
We can now give the definition of a wallpaper pattern.

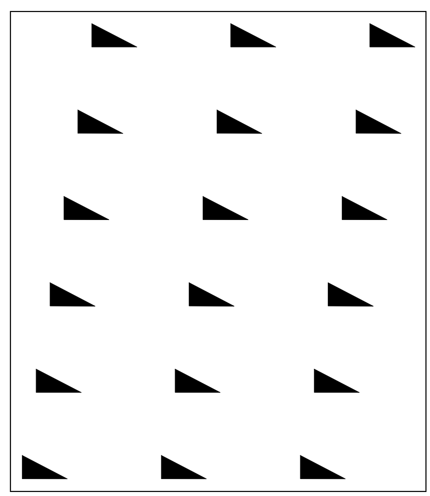
Definition 1.3. *A subset* W *of* $\mathbb{R}^2$ *is a wallpaper pattern if the translation subgroup of the symmetry group* $\mathrm{Sym}(W)$ *is a two-dimensional lattice. The symmetry group of a wallpaper pattern is said to be a wallpaper group.*

Escher's tessellations are wonderful examples of wallpaper patterns. He drew many pictures with the same symmetry as the following drawing.

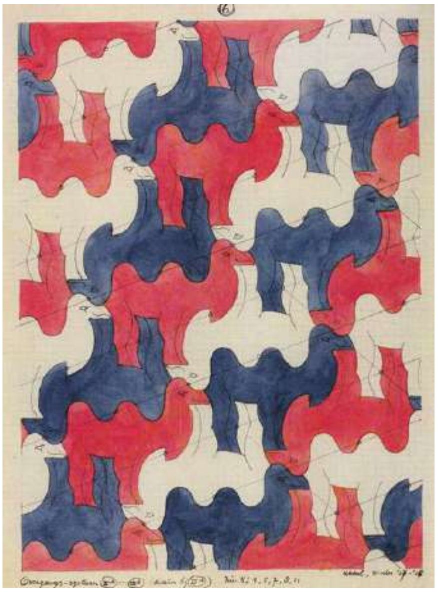


The two patterns below have isomorphic symmetry groups; both groups consist only of translations, so each is isomorphic to $\mathbb{Z}^2$.





Here are two further examples of wallpaper patterns with isomorphic wallpaper groups.

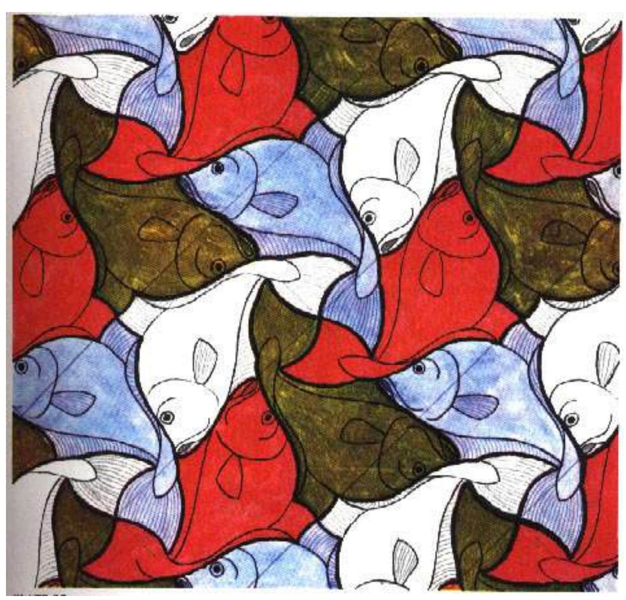


In both cases the wallpaper groups are generated by the translation lattice and a rotation of 180°; it is not hard to see that the two groups are isomorphic. In fact, each is generated by three elements t_1, t_2, r and subject to the relations

$$\begin{aligned} t_1 t_2 &= t_2 t_1, \\ r t_1 r^{-1} &= t_1^{-1}, \\ r t_2 r^{-1} &= t_2^{-1}, \\ r^2 &= 1. \end{aligned}$$

In each case $\{t_1, t_2\}$ is a basis for the translation subgroup and r is a 180° rotation. By choosing a center of a rotation to be the origin, we can then let r be that rotation.

In the following tessellation of Escher, our symmetry group is generated by two translations t_1 and t_2 and a 90° degree rotation r.



In this picture, if we keep track of color, then the rotation is not a symmetry of the pattern, and the symmetry group contains only translations. While Escher viewed color as an important part of the picture, in determining the symmetry group of a wallpaper pattern we will ignore color. One way to use our definition to find the symmetry group of a drawing is to imagine an uncolored outline of a drawing. The symmetry group is then the symmetry group of the set of points occurring in the outline.

Our classification of wallpaper groups will proceed as follows. If G is a wallpaper group, then an easy observation shows that the subgroup T of translations is a normal subgroup of G; we point this out at the end of Section 2.2. We will first determine the possible groups that arise as G/T. Second, we will determine all the possible ways that G/T can act on T; this action is described for any group G and any normal subgroup T in Section 4.1. More concretely, for a wallpaper group G, we will identify G/T with a subgroup of $\mathrm{O}_2(\mathbb{R})$ in Section 3.1, and through this identification, G/T acts on T by viewing $T \subseteq \mathbb{R}^2$, on which $\mathrm{O}_2(\mathbb{R})$ acts naturally. Finally, with the help of group cohomology, we will determine how to build G from T and G/T, along with the action of G/T on T and some cohomological information. However, we will also describe this last step without cohomology in Section 5.2. Finally, we will use this information to describe explicitly all seventeen wallpaper groups as subgroups of $\mathrm{Isom}(\mathbb{R}^2)$ and give wallpaper patterns for each group.

Chapter 2

Isometries

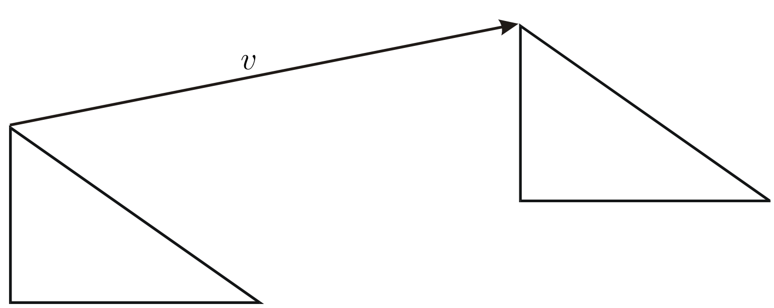
In this chapter we describe the four basic types of isometries of the plane. We then give a group-theoretic description of the group of isometries of $\mathbb{R}^n$. To describe all isometries, we will first determine the linear isometries of the plane; that is, those isometries that are linear transformations of the vector space $\mathbb{R}^2$.

2.1 Isometries of the Plane

We now describe all possible isometries of the plane. As we point out in Section 2.3, every isometry is one of the following four types.

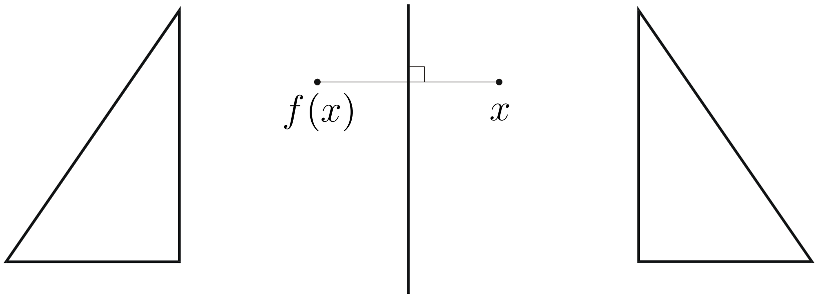
Translations.

As we saw in the previous section, translations form one type of isometry. We recall the notation τ_v for translation by v. Its inverse is τ_{-v}.



Reflections.

Let ℓ be a line in $\mathbb{R}^2$. Then the reflection across ℓ is an isometry, which one can see by a purely geometric argument, although we give a more algebraic argument below.



If ℓ is the line through the origin parallel to a vector w, then the reflection across ℓ is given by

$$f(x) = 2\left(\frac{x \cdot w}{w \cdot w}\right) w - x.$$

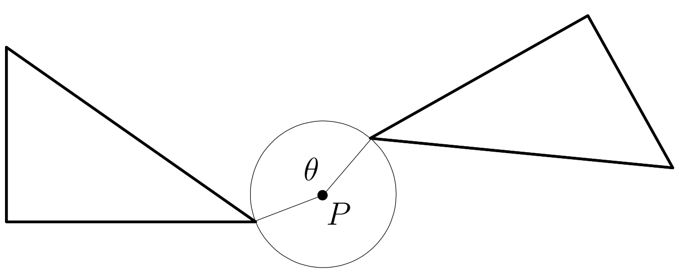
This comes from the formula for the projection of one vector onto another, which one sees in multivariable calculus. From it a straightforward calculation will show that f is an isometry. This formula also will show that f is a linear transformation. For an arbitrary reflection g, let τ be a translation that sends some fixed point on the reflection line ℓ of g to the origin. Then τ sends ℓ to a line ℓ' through the origin. If f is the reflection about ℓ', then $g = \tau^{-1} \circ f \circ \tau$, and so g is an isometry.

Rotations.

If θ is an angle, then the rotation r by an angle θ about the origin is given in coordinates by

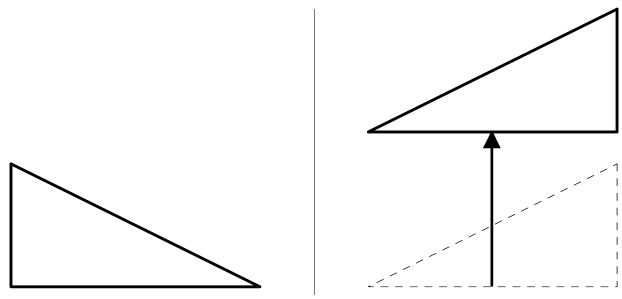
$$r\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{pmatrix}.$$

From this formula one can see that a rotation about the origin is an isometry and is a linear transformation. We can use it to describe a rotation about any point. If r' is the rotation by θ about a point $P \in \mathbb{R}^2$, and if τ is translation by P, then $r' = \tau \circ r \circ \tau^{-1}$. As a consequence, any rotation is an isometry. If a rotation r is not the identity map, then we say that r is a nontrivial rotation.



Glide Reflections.

We can produce new isometries by composition. For example, we may compose a reflection and a translation. The result may be another reflection; see Lemma 2.2 below, although it may be another type of isometry. We will call a composition of a reflection and a translation a *glide reflection*. If a glide reflection is not a reflection, then we say it is non-trivial.



We will see in Section 2.3 below that any isometry of $\mathbb{R}^2$ is a composition of a translation with either a rotation or a reflection. Therefore, we have accounted for all types of isometries of the plane.

Some Arithmetic Facts

We point out some properties of these classes of isometries. We start with some of the most simple properties. A translation has no fixed points, a rotation has a unique fixed point, and a reflection has a fixed line. A nontrivial translation has infinite order; that is, if $\tau \neq \mathrm{id}$ is a translation, then $\tau^n \neq \mathrm{id}$ for all integers $n > 0$. A reflection has order 2; thus, a reflection is its own inverse. If r is a rotation about a point P by an angle θ, then r^{-1} is rotation by $-\theta$ about the origin. Moreover, if $\theta = 2\pi/n$ for some integer n, then $r^n = \mathrm{id}$. Finally, if g is a glide reflection that is not a reflection, then we claim that g^2 is a nontrivial translation. To help us prove this, we point out a couple of facts about reflections. Let f be a reflection about a line ℓ through the origin. A vector x lies on the line through the origin that is perpendicular to ℓ if and only if $f(x) = -x$, and $x \in \ell$ if and only if $f(x) = x$. By working with an appropriate basis, every vector in $\mathbb{R}^2$ may be written in the form $u + v$ with $u \in \ell$ and v perpendicular to ℓ.

We now prove two lemmas that describe various compositions of isometries. While these results are interesting in their own right, we will use them primarily to determine when two wallpaper groups are not isomorphic.

Lemma 2.1. *If r is a nontrivial rotation about the origin by an angle θ, and if v is a vector, then $\tau_v \circ r$ is a rotation about $-(r - I)^{-1}(v)$ by θ.*

Proof. Rotations are distinguished among all isometries in that they have a unique fixed point. Suppose that $r(x) + v = x$ for some $x \in \mathbb{R}^2$. Then $v = x - r(x) = (I - r)(x)$. However, since r is a nontrivial rotation about the origin, it is a linear transformation. From this fact and the representation of a rotation by a matrix by using the standard basis for $\mathbb{R}^2$, we see that $I - r$ is invertible since $\det(I - r) \neq 0$. Thus, $x = (I - r)^{-1}(v)$ is the unique fixed point of r, which means that it is the center of the rotation. □

Lemma 2.2. *Let $f(x)$ be a reflection about a line ℓ passing through the origin, and let $v \in \mathbb{R}^2$. If g is the glide reflection $g = \tau_v \circ f$, then g is a reflection if and only if v is perpendicular to ℓ. When this occurs, the reflection line of g is $\ell + \frac{1}{2}v$. If v is not perpendicular to ℓ, then g is a non-trivial glide reflection and g^2 is translation by $v + f(v)$, a vector on the line ℓ.*

Proof. Recall that f is a linear transformation since ℓ contains the origin. The glide $g = \tau_v \circ f$ is a reflection if and only if it fixes a vector w. If $f(w)+v = w$, then $v = w - f(w)$. However, this forces

$$\begin{aligned} f(v) &= f(w - f(w)) = f(w) - f^2(w) = f(w) - w \\ &= -v. \end{aligned}$$

Therefore, if g is a reflection, then v is perpendicular to ℓ. Conversely, if v is perpendicular to ℓ, then $f(v) = -v$, so $g(\frac{1}{2}v) = \frac{1}{2}v$. This forces g to be a reflection; it fixes the line $\ell + \frac{1}{2}v$. Thus, g is a non-trivial glide reflection if and only if v is not perpendicular to ℓ. Next, we consider g^2. If $x \in \mathbb{R}^2$, then

$$g^2(x) = f(f(x) + v) + v = f^2(x) + f(v) + v = x + f(v) + v$$

since f is linear. Therefore, g^2 is translation by $f(v)+v$. Since $f(f(v)+v)) = f^2(v)+f(v) = v + f(v)$, this translation vector is fixed by f, so it is on the line ℓ. □

2.2 The Group Structure of $\mathrm{Isom}(\mathbb{R}^n)$

In this section we give a group-theoretic decomposition of the group $\mathrm{Isom}(\mathbb{R}^n)$. While we only need to consider $\mathrm{Isom}(\mathbb{R}^2)$ in our study of wallpaper groups, the analysis we give is just as simple for any n, so we consider this general situation. We will use the description of $\mathrm{Isom}(\mathbb{R}^2)$ obtained here to help us classify wallpaper groups. There are two special subgroups we will consider. One is the subgroup $\mathbb{T}$ of all translations of $\mathbb{R}^n$. The second is the *orthogonal group* $\mathrm{O}_n(\mathbb{R})$, the set of all isometries that are also linear transformations of the vector space $\mathbb{R}^n$. An alternative description, which we will prove below, is that this is the group of all isometries that fix the origin. By coordinatizing the plane, we may consider points in the plane as vectors. Following usual notation, we will write $\|u\|$ for the length of a vector u. The distance between two vectors u and v is then $\|u - v\|$. With this notation, we see that a bijection f of the plane is an isometry if $\|f(u) - f(v)\| = \|u - v\|$ for all vectors u, v.

Let g be an isometry with $g(\mathbf{0}) = \mathbf{0}$. From this condition we see for any $u \in \mathbb{R}^n$ that

$$\|g(u)\| = \|g(u) - g(\mathbf{0})\| = \|u - \mathbf{0}\| = \|u\|.$$

In other words, g preserves the length of a vector. Recall that if u and v are vectors, then there is a unique angle θ with $0 \leq \theta \leq \pi$ such that

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos\theta.$$

This fact is a restatement of the *Cauchy-Schwartz* inequality. A consequence of this inequality is that the dot product is given by the formula $u \cdot v = \|u\| \, \|v\| \cos\theta$.

Lemma 2.3. *If g is an isometry of $\mathbb{R}^n$ with $g(\mathbf{0}) = \mathbf{0}$, then g preserves angles and dot products. That is, for any $u, v \in \mathbb{R}^n$, the angle between $g(u)$ and $g(v)$ is the same as the angle between u and v, and $g(u) \cdot g(v) = u \cdot v$.*

Proof. Let g be an isometry with $g(\mathbf{0}) = \mathbf{0}$. Recall from above that this implies $\|g(u)\| = \|u\|$ for all vectors u. If θ is the angle between two vectors u and v, then

$$\|u - v\|^2 = \|u\|^2 + \|v\|^2 - 2\,\|u\|\,\|v\| \cos\theta.$$

If θ' is the angle between $g(u)$ and $g(v)$, then

$$\|g(u) - g(v)\|^2 = \|g(u)\|^2 + \|g(v)\|^2 - 2\,\|g(u)\|\,\|g(v)\| \cos\theta'.$$

However, since $\|g(u)\| = \|u\|$ and $\|g(v)\| = \|v\|$, we get

$$\|u - v\|^2 = \|g(u) - g(v)\|^2 = \|u\|^2 + \|v\|^2 - 2\,\|u\|\,\|v\| \cos\theta'.$$

This forces $\cos\theta' = \cos\theta$. Since $0 \leq \theta, \theta' \leq \pi$, we conclude that $\theta' = \theta$.

To see that g preserves dot products, we have $u \cdot v = \|u\|\,\|v\| \cos\theta$. By the previous paragraph, θ is also the angle between $g(u)$ and $g(v)$. Therefore, $g(u) \cdot g(v) = \|g(u)\|\,\|g(v)\| \cos\theta$. Since $\|g(u)\| = \|u\|$ and $\|g(v)\| = \|v\|$, this yields $g(u) \cdot g(v) = u \cdot v$. $\square$

Proposition 2.4. *If g is an isometry of $\mathbb{R}^n$ with $g(\mathbf{0}) = \mathbf{0}$, then g is a linear transformation.*

Proof. Let $\{v_1, \ldots, v_n\}$ be an orthonormal basis of $\mathbb{R}^n$, and set $w_i = g(u_i)$. First of all, $\|w_i\| = \|v_i\| = 1$ since g preserves length. Next, by Lemma 2.3, if $i \neq j$, then the angle between w_i and w_j is equal to the angle between v_i and v_j, which is $\pi/2$. Therefore, $\{w_1, \ldots, w_n\}$ is an orthonormal basis of $\mathbb{R}^n$. Recall that if $u = \sum_i \alpha_i v_i$, then the coefficients α_i are determined by the formula $\alpha_i = u \cdot v_i$. So, we have $\alpha_i = g(u) \cdot g(v_i) = g(u) \cdot w_i$ by Lemma 2.3. However, $g(u) = \sum_i \alpha_i w_i$ since $\{w_1, \ldots, w_n\}$ is an orthonormal basis, so the coefficient of w_i is $g(u) \cdot w_i$. We conclude that $g(u) = \sum_i (g(u) \cdot v_i)\, w_i$. From this formula we show that g is a linear transformation. Let $u, v \in \mathbb{R}^n$. Then

$$\begin{aligned} g(u+v) &= \sum_i ((u+v) \cdot w_i)\, w_i = \sum_i (u \cdot w_i) w_i + \sum_i (v \cdot w_i) w_i \\ &= g(u) + g(v), \end{aligned}$$

and if γ is any scalar, then

$$\begin{aligned} g(\gamma u) &= \sum_i (\gamma u \cdot w_i) w_i = \sum \gamma (u \cdot w_i) w_i = \gamma \sum_i (u \cdot w_i) w_i \\ &= \gamma g(u). \end{aligned}$$

This proves that g is a linear transformation. $\square$

Corollary 2.5. *Let f be an isometry of $\mathbb{R}^n$. Then $f(x) = g(x) + b$ for some linear isometry g and some $b \in \mathbb{R}^n$.*

Proof. Let $b = f(\mathbf{0})$ and set $g(x) = f(x) - b$. Then g is the composition of f and the translation τ_{-b}, so g is an isometry. Since $g(\mathbf{0}) = f(\mathbf{0}) - b = b - b = \mathbf{0}$, Proposition 2.4 shows that g is a linear transformation. □

If g is a linear transformation on $\mathbb{R}^n$, by viewing the elements of $\mathbb{R}^n$ as column matrices we may write $g(u) = Au$ for some $n \times n$ matrix A. The matrix an isometry g is not arbitrary. We get a restriction on A by knowing that g preserves dot products. If $g(x) = Ax$, then the (i, j)-entry of $A^T A$ is $g(e_i) \cdot g(e_j) = e_i \cdot e_j$, which is 1 for $i = j$ and 0 otherwise. This shows us that $A^T A = I_n$, the $n \times n$ identity matrix. Conversely, if A is a matrix with $A^T A = I_n$, we claim that the linear map g defined by $g(x) = Ax$ is an isometry. For,

$$\begin{aligned} g(u) \cdot g(v) &= (Au) \cdot (Av) = (Au)^T (Av) \\ &= u^T (A^T A) v = u^T v = u \cdot v. \end{aligned}$$

Therefore, by setting $v = u$, we have $\|g(u)\| = \|u\|$. Finally, since g is linear, $\|g(u) - g(w)\| = \|g(u - w)\| = \|u - w\|$.

The set of matrices that satisfy the condition $A^T A = I_n$ is called the *orthogonal group*, and is denoted $\mathrm{O}_n(\mathbb{R})$.

Corollary 2.6. *If f is an isometry of $\mathbb{R}^n$, then $f(x) = Ax + b$ for some $b \in \mathbb{R}^n$ and some $n \times n$ matrix $A \in \mathrm{O}_n(\mathbb{R})$.*

We point out that we obtained this description of isometries only from the assumption that an isometry preserves distance, not using that it is a bijection. It shows that a distance-preserving map of $\mathbb{R}^n$ is automatically a bijection.

Because of the connection between linear transformations and matrices, we get the following connection between $\mathrm{O}_n(\mathbb{R})$ and $\mathrm{Isom}(\mathbb{R}^n)$.

Proposition 2.7. *Let H be the subgroup of isometries of $\mathbb{R}^n$ that preserve the origin. Then $H \cong \mathrm{O}_n(\mathbb{R})$.*

Proof. We define a map $\sigma : \mathrm{O}_n(\mathbb{R}) \to H$ by $\sigma(A)$ is the isometry $x \mapsto Ax$. In other words, $\sigma(A)(x) = Ax$. We have

$$\begin{aligned} \sigma(AB)(x) = (AB)x = A(Bx) &= \sigma(A)(Bx) \\ &= \sigma(A)(\sigma(B)(x)) \\ &= (\sigma(A)\sigma(B))(x). \end{aligned}$$

Therefore, $\sigma(AB) = \sigma(A)\sigma(B)$. So, σ is a group homomorphism. If $\sigma(A)$ is the identity function, then $\sigma(A)(x) = x$ for all x. Then $Ax = x$ for all x. But then the matrix A defines the identity linear transformation, so $A = I_n$. Therefore, σ is injective. Finally, if $g \in H$, then $g(x) = Ax$ for some matrix A by Corollary 2.5. Corollary 2.6 shows that $A^T A = I_n$, so $A \in O_n(\mathbb{R})$. This proves that $g = \sigma(A)$, so σ is surjective. Therefore, σ is a group isomorphism. □

[1]We now show how $\mathrm{Isom}(\mathbb{R}^n)$ can be constructed from the group $\mathbb{T}$ of all translations and $O_n(\mathbb{R})$. We view $O_n(\mathbb{R})$ as both the group of linear isometries and the group of all orthogonal matrices. We write $G = \mathrm{Isom}(\mathbb{R}^n)$ and $H = O_n(\mathbb{R})$ for ease of notation. We proved above that every isometry is a composition of a linear isometry and a translation. Therefore, $G = \mathbb{T}H$. We next note that $\mathbb{T}$ is a normal subgroup of G. To see this, the equation $G = \mathbb{T}H$ shows that it is enough to prove that if f is a linear isometry and τ is a translation, then $f \circ \tau \circ f^{-1}$ is a translation. Suppose that $\tau(x) = x + v$. Then $\tau(f^{-1}(x)) = f^{-1}(x) + v$. So,

$$f\tau f^{-1}(x) = f(f^{-1}(x) + v) = x + f(v)$$

since f is linear. Therefore, $f \circ \tau \circ f^{-1}$ is translation by $f(v)$. In particular, this yields $f \circ \tau_v \circ f^{-1} = \tau_{f(v)}$. It is clear that $\mathbb{T} \cap H = \{\mathrm{id}\}$ since any nontrivial translation fixes no point. We therefore have written $G = \mathbb{T}H$ with $\mathbb{T}$ a normal subgroup, H a subgroup, and $\mathbb{T} \cap H = \{\mathrm{id}\}$. This means G is the semidirect product of $\mathbb{T}$ and H. By recalling the construction of semidirect products we can be a little more precise about the structure of G. If we denote by φ_h the restriction to $\mathbb{T}$ of the inner automorphism $\tau \mapsto h\tau h^{-1}$, then multiplication on $G = \mathbb{T}H$ is given by

$$\begin{aligned}(\tau_u h)(\tau_v h') &= (\tau_u h \tau_v h^{-1})(hh') = (\tau_u \varphi_h(\tau_v))\,(hh') \\ &= (\tau_u \tau_{h(v)})(hh') = (\tau_{u+h(v)})(hh').\end{aligned}$$

Using the isomorphism $\mathbb{T} \cong \mathbb{R}^n$ and viewing elements of H in terms of matrices, the map $\varphi : H \to \mathrm{Aut}(\mathbb{T})$ is more concretely given as

$$\varphi(A)(v) = Av,$$

where A is an orthogonal matrix and $v \in \mathbb{R}^n$. There is a surjective group homomorphism $G \to H$ that sends th to h; the kernel of this map is $\mathbb{T}$. Furthermore, G is then isomorphic to $(\mathbb{R}^n \times O_n(\mathbb{R}), \cdot)$, where the operation $\cdot$ on this Cartesian product is

$$(u, A) \cdot (v, B) = (u + Av, AB).$$

2.3 Structure of $O_2(\mathbb{R})$

As we saw in the previous section, the group $\mathrm{Isom}(\mathbb{R}^2)$ is built from the subgroup of translations and the subgroup $O_2(\mathbb{R})$ of linear isometries. We point out some facts about $O_2(\mathbb{R})$. If we use the standard basis for $\mathbb{R}^2$, then an element of $O_2(\mathbb{R})$ can be represented by a 2×2

[1]do this as a theorem?

matrix. Furthermore, any such matrix A satisfies the condition $A^T A = I_2$. Taking determinants, we obtain $\det(A) = \pm 1$. Since the determinant function is a group homomorphism from $O_2(\mathbb{R})$ to the nonzero real numbers, its kernel is a normal subgroup of $O_2(\mathbb{R})$. This subgroup is called the *special orthogonal group*, and is denoted $SO_2(\mathbb{R})$. Thus,

$$SO_2(\mathbb{R}) = \{A \in O_2(\mathbb{R}) : \det(A) = 1\}.$$

Note that $[O_2(\mathbb{R}) : SO_2(\mathbb{R})] = 2$

Let A be the matrix with respect to the standard basis for an element in $O_2(\mathbb{R})$. If

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

then the condition $A^T A = I_2$ gives

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This yields $a^2 + c^2 = 1$ and $b^2 + d^2 = 1$. Therefore, there is an angle θ with $a = \cos\theta$ and $c = \sin\theta$. Furthermore, the condition $ab + cd = 0$ says that the vector (b, d) is orthogonal to (a, c). Since $b^2 + d^2 = 1$, this forces $(b, d) = (-\sin\theta, \cos\theta)$ or $(b, d) = (\sin\theta, -\cos\theta)$. The first choice gives a matrix with determinant 1 and the second choice gives a matrix of determinant -1. From this we see that if $A \in SO_2(\mathbb{R})$, then we may write

$$A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

for some angle θ. In other words, A is a rotation about the origin by an angle θ. On the other hand, if $A \notin SO_2(\mathbb{R})$, then

$$A = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}.$$

From the formula given earlier for a reflection across a line through the origin, we can see that A is the reflection across the line $y = (\tan\theta/2)x$.

To summarize, elements of $SO_2(\mathbb{R})$ are rotations and elements of $O_2(\mathbb{R}) \setminus SO_2(\mathbb{R})$ are reflections. Since every element of $\mathrm{Isom}(\mathbb{R}^2)$ is the composition of a translation with an element of $O_2(\mathbb{R})$, this shows that every isometry of the plane is one of the four types described in Section 2.1. Furthermore, if $r \in SO_2(\mathbb{R})$ and $f \notin SO_2(\mathbb{R})$, then $rf \notin SO_2(\mathbb{R})$, so rf is a reflection. Thus, $(rf)^2 = 1$, which is equivalent to $frf = r^{-1}$. Recall that the *dihedral group* D_n is the group of symmetries of a regular n-gon. It is given by generators and relations as $D_n = \langle r, f \rangle$ with

$$r^n = f^2 = 1,$$
$$frf = r^{-1}.$$

We can identify D_n as a subgroup of $\mathrm{O}_2(\mathbb{R})$ by setting r to be the rotation by an angle of $2\pi/n$ and f any reflection. As a converse to this example, we have the following property of finite subgroups of $\mathrm{O}_2(\mathbb{R})$.

Proposition 2.8. *Let G be a finite subgroup of $\mathrm{O}_2(\mathbb{R})$. Then G is isomorphic to either a cyclic group of order n or a dihedral group of order $2n$, for some integer n.*

Proof. Let $N = G \cap \mathrm{SO}_2(\mathbb{R})$, a normal subgroup of G. Since $[\mathrm{O}_2(\mathbb{R}) : \mathrm{SO}_2(\mathbb{R})] = 2$ and G/N is isomorphic to a subgroup of $\mathrm{O}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$, we get $[G : N] \leq 2$. If $N = \{1\}$, then either $G = \{1\}$ is cyclic, or $G = \langle f \rangle$ for some reflection f, so $G \cong D_1$. Therefore, assume that $N \neq \{1\}$. The group N consists of rotations. Since it is finite, there is a nontrivial rotation $r \in N$ of minimal possible angle θ. If r' is any other nontrivial rotation in N, and if r' is a rotation by ϕ, then there is an integer m with $m\phi \leq \theta < (m+1)\phi$. The rotation $r(r')^{-m}$ is a rotation by the angle $0 \leq \theta - m\phi < \theta$. Minimality of θ then forces $m\phi = \theta$. In other words, $r' \in \langle r \rangle$. This proves that $N = \langle r \rangle$ is cyclic. If $G = N$, then G is cyclic. If $G \neq N$, then $[G : N] = 2$. If $f \in G \backslash N$, then as pointed out before, $frf = r^{-1}$. If $n = |N|$, then $|G| = 2n$, and G is generated by r and f, and satisfies the relations $r^n = f^2 = 1$ and $frf = r^{-1}$. Therefore, $G \cong D_n$. □

Chapter 3

The Point Group

Recall that a wallpaper group is a subgroup G of $\mathrm{Isom}(\mathbb{R}^2)$ that contains a two-dimensional lattice T of translations as a normal subgroup. As we indicated in the introduction, the study of the quotient group G/T will be the first step for us to determine all wallpaper groups. By choosing a basis for T we will exhibit G/T as a subgroup of $\mathrm{O}_2(\mathbb{R})$ and see how elements of G/T act on this basis. This action will be a key in describing and distinguishing wallpaper groups.

3.1 Definition and Main Properties

Let G be a wallpaper group with translation lattice T. In this section we will give an interpretation of G/T, and we will determine all possible groups, up to isomorphism, that can occur as G/T for a wallpaper group G.

We give a notation for isometries that will prove convenient. If φ is an isometry, then $\varphi(x) = A(x) + b$ for some $A \in \mathrm{O}_2(\mathbb{R})$ and $b \in \mathbb{R}^2$ by Corollary 2.5. To simplify notation, we write $\varphi = (A, b)$, and note that composition in $\mathrm{Isom}(\mathbb{R}^2)$ translates into the following formula

$$(A, b)(C, d) = (AC, A(d) + b).$$

Furthermore, inverses are given by

$$(A, b)^{-1} = (A^{-1}, -A^{-1}(b)).$$

With this notation, translation by a vector b is (I, b), and an element of $\mathrm{O}_2(\mathbb{R})$ is of the form $(A, \mathbf{0})$. We can now define the point group of G.

Definition 3.1. *Let G be a wallpaper group. The point group G_0 of G is the set*

$$\left\{A \in \mathrm{O}_2(\mathbb{R}) : (A, b) \in G \text{ for some } b \in \mathbb{R}^2\right\}.$$

From the formulas above for composition and inversion, we see that the point group is a subgroup of $\mathrm{O}_2(\mathbb{R})$. In fact, we have the following interpretation of the point group.

Proposition 3.2. *If G is a wallpaper group with translation lattice T and point group G_0, then $G_0 \cong G/T$.*

Proof. If φ is the group homomorphism $\mathrm{Isom}(\mathbb{R}^2) \to \mathrm{O}_2(\mathbb{R})$ given by $(A, b) \mapsto A$, then $\varphi(G) = G_0$. The kernel of $\varphi|_G$ is T, since T is the intersection of G and the translation subgroup of $\mathrm{Isom}(\mathbb{R}^2)$, and the translation subgroup is the kernel of φ. □

As we saw in Section 2.2, the group $\mathrm{O}_2(\mathbb{R})$ acts on the group $\mathbb{T}$ of translations by conjugation. More precisely, if $t \in \mathbb{R}^2$ and $A \in \mathrm{O}_2(\mathbb{R})$, then the equation

$$(A, \mathbf{0})(I, t)(A, \mathbf{0})^{-1} = (I, A(t))$$

shows that under the natural isomorphism $\mathbb{T} \cong \mathbb{R}^2$, this conjugation action is the natural action of $\mathrm{O}_2(\mathbb{R})$ on $\mathbb{R}^2$. This action restricts to an action of G_0 on T, for if $A \in G_0$ and $t \in T$, then there is a $b \in \mathbb{R}^2$ with $(A, b) \in G$. Then $(A, b)(I, t)(A, b)^{-1} \in G$, and is equal to (I, At); consequently, $At \in T$. The presence of this action, together with the group structure of T, will allow us to determine the groups that arise as G_0 for some wallpaper group G. To set some notation, we write C_n for the cyclic group of order n. We view C_n as a subgroup of $\mathrm{O}_2(\mathbb{R})$ by considering it to be the cyclic group generated by a rotation of $2\pi/n$. Also, let D_n be the dihedral group of order $2n$. This group is generated by two elements r, f and subject to the relations $r^n = f^2 = \mathrm{id}$ and $rfr = r^{-1}$. As we noted in Section 2.3We may view D_n as a subgroup of $\mathrm{O}_2(\mathbb{R})$ by letting r be a rotation of $2\pi/n$ and f any reflection. Note that by choosing different f we get different subgroups of $\mathrm{O}_2(\mathbb{R})$ that are isomorphic to D_n. The following lemma will be used in determining the possible point groups.

Lemma 3.3. *The point group G_0 of a wallpaper group G is finite.*

Proof. Let $\{t_1, t_2\}$ be a basis of T, and let C be a circle centered at the origin that contains t_1 and t_2 in its interior. We remarked in the introduction that there are only finitely many elements of T inside C. Since G_0 is a subgroup of $\mathrm{O}_2(\mathbb{R})$, its elements restrict to give permutations of the interior of C. There are then only finitely many pairs of elements of T that can occur as the image of $\{t_1, t_2\}$ under an element of G_0. However, since $\{t_1, t_2\}$ is a basis of $\mathbb{R}^2$, any element of G_0 is determined by its action on $\{t_1, t_2\}$. Therefore, G_0 is finite. □

We now determine the possible groups that can arise as the point group of a wallpaper group.

Theorem 3.4. *Let G_0 be the point group of a wallpaper group G. Then G_0 is isomorphic to one of the following ten groups*

$$\left\{ \begin{array}{c} C_1, C_2, C_3, C_4, C_6 \\ D_1, D_2, D_3, D_4, D_6 \end{array} \right\}.$$

Proof. By Lemma 3.3, G_0 is a finite group. From Proposition 2.8, we then know that G_0 is isomorphic to C_n or D_n for some n. It remains to determine the possible values of n. In the proof of Proposition 2.8, we saw that $N = G_0 \cap \mathrm{SO}_2(\mathbb{R})$ is a cyclic group generated by a rotation r of minimal possible angle. Moreover, $|N| = n$, so r has order n. We represent r by a matrix in two ways. First, with respect to the standard basis, if r is a rotation by an angle θ, then the matrix representing r is

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}.$$

On the other hand, let $\{t_1, t_2\}$ be an integral basis for T. Since $r(T) = T$ and $T = \mathbb{Z}t_1 \oplus \mathbb{Z}t_2$, the matrix for r with respect to this basis is of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with a, b, c, d integers. Since these matrices represent the same linear transformation but with respect to different bases, they are conjugate; therefore, they have the same trace. This yields $2\cos\theta = a + d \in \mathbb{Z}$. A simple analysis of the cosine function shows that θ or $-\theta$ is a member of $\{0, \pi/3, \pi/2, 2\pi/3, \pi\}$. Therefore, $N = \langle r \rangle$ has order $n \in \{1, 2, 3, 4, 6\}$. Since G_0 is isomorphic to either C_n or D_n for $n = |N|$, this completes the proof. □

We next verify that G_0 is uniquely determined by G. To be more precise, we show that if two wallpaper groups are isomorphic, then their point groups are isomorphic. The following lemma will do this for us by identifying the subgroup T of a wallpaper group G in a purely group-theoretic way.

Lemma 3.5. *Let G be a symmetry group with translation lattice T and set*

$$G_n = \{x \in G : xg^n = g^n x \text{ for all } g \in G\}.$$

Then $T = G_n$ whenever n is a multiple of $[G : T]$. Furthermore, if G and G' are wallpaper groups with translation lattices T and T', respectively, and if $\varphi : G \to G'$ is an isomorphism, then $\varphi(T) = T'$.

Proof. By Lemma 3.3, the group G/T is finite. Let n be a multiple of $[G : T]$. If $g \in G$, then $g^n \in T$. Since T is Abelian, we have $tg^n = g^n t$ for all g, so $t \in G_n$. Conversely, suppose that $x \in G_n$. We may write $x = (A, b)$ with $b \in \mathbb{R}^2$. Consider $g = (I, t)$ with $t \in T$. Then $g^n = (I, nt)$. Since $x \in G_n$, we have $xg^n x^{-1} = g^n$. However, by our description of the action of $\mathrm{O}_2(\mathbb{R})$ on $\mathbb{T}$ in Section 2.2, we have $xg^n x^{-1} = (I, A(nt))$. Consequently, $A(nt) = nt$ for every $t \in T$. If $\{t_1, t_2\}$ is an integral basis for T, then $\{nt_1, nt_2\}$ is a basis for $\mathbb{R}^2$. Since $A \in \mathrm{O}_2(\mathbb{R})$ is determined by its action on a basis, we must have $A = I$, so $x = (I, b) \in T$. We have thus proved that $G_n = T$.

Now, suppose G and G' are isomorphic wallpaper groups, and let $\varphi : G \to G'$ be an isomorphism. It is elementary to see that $\varphi(G_n) = G'_n$ for all n. Let $m = [G : T]$ and

$m' = [G' : T']$. If we set $n = mm'$, Lemma 3.5 shows that $T = G_n$ and $T' = G'_n$. Since $\varphi(G_n) = G'_n$, we get $\varphi(T) = T'$, as desired. □

Corollary 3.6. *If G and G' are isomorphic wallpaper groups, then their point groups G_0 and G'_0 are isomorphic.*

Proof. Let $\varphi : G \to G'$ be an isomorphism. If T (resp. T') is the translation lattice of G (resp. G'), then $\varphi(T) = T'$ by Lemma 3.5. Therefore, φ induces an isomorphism between G/T and G'/T'. Since these groups are isomorphic to G_0 and G'_0, respectively, the point groups G_0 and G'_0 are isomorphic. □

By looking more carefully at an isomorphism between wallpaper groups, we can prove a stronger statement. In the next result we give a necessary criterion on the point groups for two wallpaper groups to be isomorphic. We will use this criterion to show that certain wallpaper groups are not isomorphic. To help understand this result, if G is a wallpaper group with translation lattice T and point group G_0, the action of G_0 on T yields a group homomorphism $G_0 \to \mathrm{Aut}(T) \cong \mathrm{Aut}(\mathbb{Z}^2)$. By picking a basis $\{t_1, t_2\}$ of T, elements of $\mathrm{Aut}(\mathbb{Z}^2)$ can be represented by 2×2 matrices with integer entries. This gives an isomorphism $\mathrm{Aut}(\mathbb{Z}^2) \cong \mathrm{Gl}_2(\mathbb{Z})$, the group of units of the ring $M_2(\mathbb{Z})$ of 2×2 matrices with integer entries. For such a matrix to be invertible over $\mathbb{Z}$, its determinant must be a unit in $\mathbb{Z}$. Therefore, $\mathrm{Gl}_2(\mathbb{Z})$ consists of 2×2 integral matrices with determinant ± 1. Thus, by using this action, we can represent G_0 as a subgroup of $\mathrm{Gl}_2(\mathbb{Z})$.

Proposition 3.7. *Let $\varphi : G \to G'$ be an isomorphism of the wallpaper groups G and G'. Let T (resp. T') be the translation lattice and G_0 (resp. G'_0) the point group of G (resp. G'). By choosing integral bases for T and T', the map $\varphi|_T$ is a linear isomorphism, given by a matrix $U \in \mathrm{Gl}_2(\mathbb{Z})$, and the induced isomorphism $\overline{\varphi} : G_0 \to G'_0$ is conjugation by U.*

Proof. Suppose $\varphi : G \to G'$ is an isomorphism. By Lemma 3.5, the restriction of φ to T is an isomorphism from T to T'. Suppose that $\{t_1, t_2\}$ is a basis for T and $\{s_1, s_2\}$ is a basis for T'. We then have

$$\varphi(t_1) = \alpha s_1 + \beta s_2,$$
$$\varphi(t_2) = \gamma s_1 + \delta s_2$$

for some integers $\alpha, \beta, \gamma, \delta$. Since $\varphi|_T$ is a $\mathbb{Z}$-module isomorphism, it is determined by what it does to the basis of T and can be represented by a matrix with integer entries

$$U = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}.$$

Also, since φ^{-1} is an isomorphism that sends T' to T, we see that U^{-1} also has integer entries. Therefore, $U \in \mathrm{Gl}_2(\mathbb{Z})$. Now, take $(A, b) \in G$, and write $(C, d) = \varphi(A, b)$. For $t \in T$ we have $\varphi(I, t) = (I, Ut)$ by the definition of U. Therefore,

$$(C, d)(I, Ut)(C, d)^{-1} = \varphi\left((A, b)(I, t)(A, b)^{-1}\right),$$

or

$$(I, CUt) = \varphi\left((I, At)\right) = (I, UAt).$$

In other words, $CU = UA$, so $C = UAU^{-1}$. Thus, the induced map $G_0 \to G_0'$ is conjugation by U. $\square$

Corollary 3.8. *Let G and G' be isomorphic wallpaper groups with point groups G_0 and G_0', respectively. Identifying G_0 and G_0' as subgroups of $\mathrm{Gl}_2(\mathbb{Z})$ by choosing bases for the translation lattices of G and G', there is a matrix $U \in \mathrm{Gl}_2(\mathbb{Z})$ with $G_0' = UG_0U^{-1}$.*

The converse of this corollary is also true. If the translation lattices of two wallpaper groups are isomorphic via a map U for which conjugation by U is an isomorphism between their point groups, then we obtain an isomorphism between the groups via $(g, t) \mapsto (UgU^{-1}, U(t))$. This corollary tells us that in order for two wallpaper groups to be isomorphic, their point groups must be conjugate in $\mathrm{Gl}_2(\mathbb{Z})$, once we have represented them as subgroups of $\mathrm{Gl}_2(\mathbb{Z})$. This is a stronger condition than the point groups being isomorphic. For example, the point groups C_2 and D_1 are isomorphic. However, by the matrix representations we obtain for them in Section 3.2 below, we see that they are not conjugate in $\mathrm{Gl}_2(\mathbb{Z})$. Therefore, a wallpaper group with point group C_2 is not isomorphic to one with point group D_1. This corresponds to the geometric fact that a wallpaper pattern with a 180° rotation symmetry and no reflectional symmetry is "different" from one with a reflectional symmetry and no rotation symmetry.

3.2 The Five Lattice Types

In the previous section we proved that the point group G_0 of a wallpaper pattern is isomorphic to one of the ten groups $\{C_n, D_n : n = 1, 2, 3, 4, 6\}$. Note that this set has only nine non-isomorphic groups. However, by considering the action of G_0 on T, we will see that even though $C_2 \cong D_1$ as abstract groups, they will be distinguished by their actions on T. By fixing a basis $\{t_1, t_2\}$ of T, we have an isomorphism $T \cong \mathbb{Z}^2$, and using the basis, the action of G_0 on T induces a group homomorphism $G_0 \to \mathrm{Aut}(\mathbb{Z}^2) \cong \mathrm{Gl}_2(\mathbb{Z})$. In other words, a choice of basis together with the action of G_0 on T gives us a representation of G as a specific subgroup of $\mathrm{Gl}_2(\mathbb{Z})$.

We will see that, viewing lattices geometrically, there are five types of lattices with respect to the G_0-action; parallelogram, rectangular, square, rhombus, and hexagonal. We will be specific in what we mean as we look at the action of the ten groups above on T.

The groups C_1, C_2: parallelogram lattices.

As mentioned above, we will represent a point group as a subgroup of $\mathrm{Gl}_2(\mathbb{Z})$ by choosing a basis $\{t_1, t_2\}$ for T. The groups C_1 and C_2 are very easy to describe and their description does not depend on the basis. If $G_0 = C_1$, then

$$C_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

On the other hand, if $G_0 = C_2$, then the rotation of 180° is multiplication by -1 on T. Therefore,

$$C_2 = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle.$$

The lattice in these cases is called a *parallelogram lattice*.

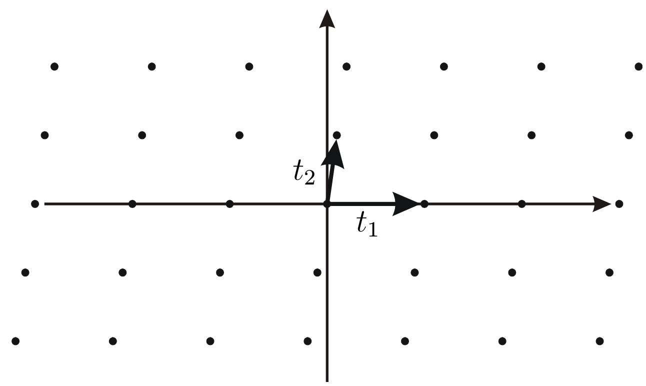


Figure 3.1: Parallelogram Lattice

We next consider the groups C_n and D_n for $n \geq 3$. The following lemma will help us find a convenient basis for T in these cases.

Lemma 3.9. *Suppose that G_0 contains a rotation r about an angle $2\pi/n$ for $n \geq 3$. If t is a nonzero element of T of minimal length, then $\{t, r(t)\}$ is a basis for T.*

Proof. Let $\{t_1, t_2\}$ be a basis for T. Then

$$\begin{aligned} t &= at_1 + bt_2 \\ r(t) &= ct_1 + dt_2 \end{aligned}$$

for some integers a, b, c, d. The set $\{t, r(t)\}$ is linearly independent because $n > 2$, so we can solve for t_1 in the two equations above; therefore, $t_1 = \alpha t + \beta rt$ for some rational numbers α, β. Write $\alpha = \alpha_0 + \varepsilon$ and $\beta = \beta_0 + \varepsilon'$ with $\alpha_0, \beta_0 \in \mathbb{Z}$ and $|\varepsilon|, |\varepsilon'| \leq 1/2$. We have $s = \alpha_0 t + \beta_0 r(t) \in T$, so $(t_1 - s) = \varepsilon t + \varepsilon' r(t) \in T$. Since t and $r(t)$ are not parallel, we see that

$$\begin{aligned} \|t_1 - s\| &= \|\varepsilon t + \varepsilon' r(t)\| < \|\varepsilon t\| + \|\varepsilon' r(t)\| \leq \frac{1}{2}(\|t\| + \|r(t)\|) \\ &= \frac{1}{2}(2\|t\|) = \|t\|, \end{aligned}$$

a contradiction to the minimality of $\|t\|$, unless $s = t_1$. Therefore, $t_1 = s$ is a $\mathbb{Z}$-linear combination of t and $r(t)$. Similarly, t_2 is a $\mathbb{Z}$-linear combination of t and $r(t)$. Since $\{t_1, t_2\}$ is a basis of T, the set $\{t, r(t)\}$ is also a basis for T. □

The groups C_4, D_4: square lattices.

Let r be a rotation by 90°. By Lemma 3.9, if $t = t_1$ is a vector in T of minimal length, then $\{t_1, r(t_1)\}$ is a basis for T. The lattice is called a *square lattice.*With respect to this basis,

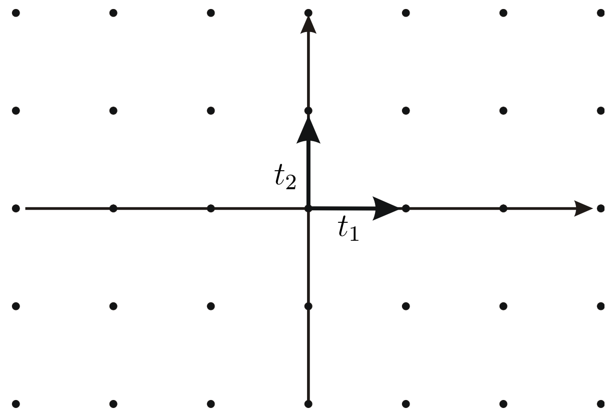


Figure 3.2: Square Lattice

we see that if $G_0 = C_4 = \langle r \rangle$, then the representation of G_0 by this basis is

$$C_4 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

On the other hand, if $G_0 = D_4$, then G_0 contains a reflection f. The four elements f, rf, r^2f, r^3f are all the reflections in G_0. These reflections must preserve the set of vectors in T of minimal length; four such vectors are $\pm t_1, \pm t_2$. However, a short argument shows that any other point on the circle of radius $\|t_1\|$ centered at the origin is a distance of less than $\|t_1\|$ from one of these four points. Figure 3.3 makes this easy to see visually. The difference of these two vectors would then be a vector in T of length less than $\|t_1\|$. Since this is impossible, we see that the four vectors above are all the vectors of minimal length in T. The four lines of reflection are then given in the following picture.Since D_4 is generated by r and any reflection, using the reflection about the line parallel to t_1, we obtain the representation

$$D_4 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle.$$

The groups C_3, D_3, C_6, D_6: hexagonal lattices.

Let r be a rotation by 120°. If t_1 is a vector in T of minimal length, then by setting $t_2 = r(t_1)$, the set $\{t_1, t_2\}$ is a basis for T, by Lemma 3.9. The lattice in this case is called a *hexagonal*

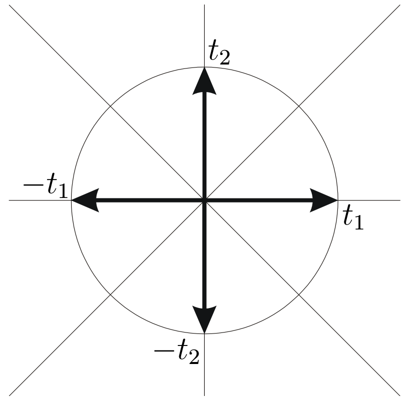


Figure 3.3: The vectors of minimal length in T when $G_0 = D_4$

lattice.

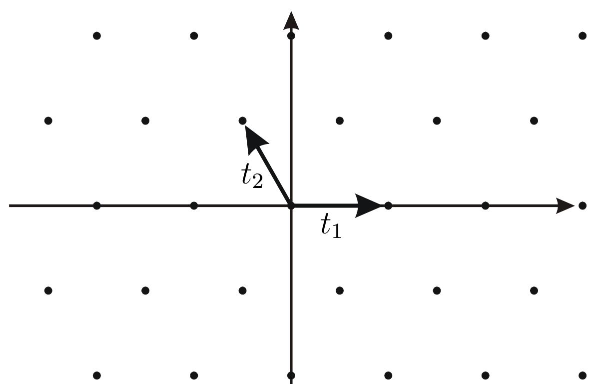


Figure 3.4: Hexagon Lattice

The group C_3 is generated by r and C_6 is generated by a 60° rotation; thus, we obtain

$$C_3 = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} \right\rangle$$

and

$$C_6 = \left\langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle .$$

Figure 3.5 indicates that we have six vectors in T of minimal length.Any point on the circle above other than the six shown is a distance less than $\|t_1\|$ from one of these six points. This shows that these six vectors are all the vectors of minimal length in T.

If $G_0 = D_3$ or D_6, then G_0 contains 6 or 12 reflections, respectively. Any reflection must permute the six vectors in Figure 3.5. For $G_0 = D_6$, we then have the following twelve lines of reflection.

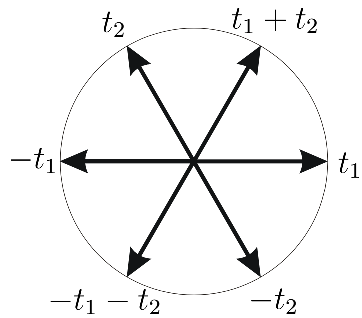
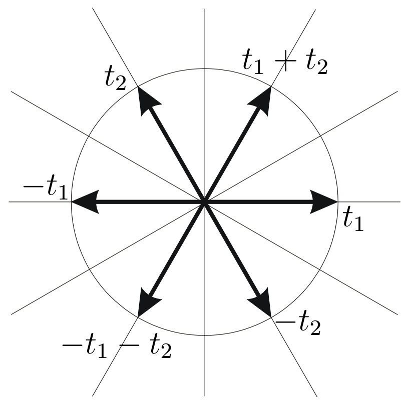


Figure 3.5: The vectors of minimal length when $G_0 = D_6$



The group D_6 is generated by C_6 and any reflection; using the reflection that fixes t_1, we have

$$D_6 = \left\langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \right\rangle.$$

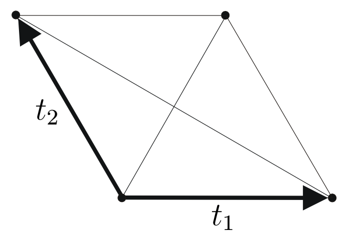
If $G_0 = D_3$, then the point group contains three reflections. The lines of reflection are separated by 60° angles; if f is a reflection in D_3, then rf is a reflection whose line of reflection makes a 60° angle with that of f. The reflection lines for D_3 must be reflection lines for D_6 since D_3 is a subgroup of D_6. We then have two possibilities: The three lines are the lines that are at angles $30^\circ, 90^\circ, 150^\circ$ with t_1 or are the lines at angles $0^\circ, 60^\circ, 120^\circ$ with t_1. This says that D_3 can act in two ways with respect to this basis. We write $D_{3,l}$ and $D_{3,s}$ to distinguish these two actions; therefore, generating $D_{3,l}$ and $D_{3,s}$ with the 120° rotation and with the reflection about the 30° and the 0° reflection lines, respectively, we have

$$D_{3,l} = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \right\rangle$$

and

$$D_{3,s} = \left\langle \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \right\rangle.$$

To give meaning to this subscript notation, we note that l and s stand for long and short, respectively. The vectors t_1 and t_2 span a parallelogram which has a long and a short diagonal. The group $D_{3,s}$ contains a reflection about the 60° line, which is the short diagonal. The group $D_{3,l}$ has a reflection across the 150° line, which is parallel to the long diagonal.



We show that the groups $D_{3,l}$ and $D_{3,s}$ are not conjugate in $\mathrm{Gl}_2(\mathbb{Z})$. This will tell us that two wallpaper groups with point groups $D_{3,l}$ and $D_{3,s}$, respectively, are not isomorphic, by Proposition 3.8. To prove this, suppose there is a matrix $U \in \mathrm{Gl}_2(\mathbb{Z})$ with $D_{3,l} = UD_{3,s}U^{-1}$. Because conjugation preserves determinants and the determinant of a reflection is -1, the three reflections of $D_{3,s}$ must be sent to the three reflections of $D_{3,l}$. We can obtain any reflection (in D_n) from any other reflection by conjugation by I, r, or r^2. Therefore, we may assume that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{Z}$ with $ad - bc = \pm 1$. Simplifying yields $d = -a$ and $c = -b$. Therefore, $ad - bc = b^2 - a^2 = (b - a)(b + a)$. Since this is ± 1, one term is 1 and the other is -1. We then have four cases, $a = \pm 1$ and $b = 0$ or $a = 0$ and $b = \pm 1$. Conjugation by $-I_2$ is the identity; therefore, we may assume that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

or

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

However, since

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$$

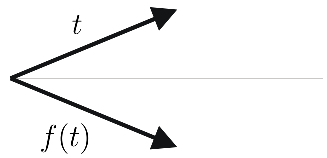
and

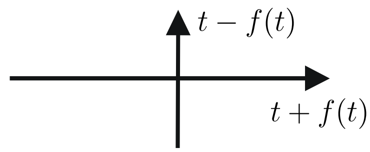
$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix},$$

neither conjugation sends $D_{3,s}$ to $D_{3,l}$ since neither of these results is an element of $D_{3,l}$. The groups $D_{3,l}$ and $D_{3,s}$ are thus not conjugate in $\mathrm{Gl}_2(\mathbb{Z})$.

The groups D_1, D_2: rectangular or rhombic lattices.

If $G_0 = D_1$ or D_2, then G_0 does not contain a rotation of order at least 3. Therefore, we cannot apply Lemma 3.9 to obtain a basis for T. We produce a basis in another way. In each of these cases we have a nontrivial reflection f in G_0. Let $t \in T$ be a nonzero vector not parallel to the line of reflection of f. Since f maps T to T, the vectors $t + f(t)$ and $t - f(t)$ are elements of T, so T contains nonzero vectors both parallel and perpendicular to the line of reflection.





Let s_1 and s_2 be nonzero vectors of minimal length parallel and perpendicular, respectively, to the reflection line. The discrete nature of T implies that such vectors exist, and that any vector parallel to (resp. perpendicular to) this line is an integer multiple of s_1 (resp. s_2). Therefore, for any $t \in T$, we have

$$\begin{aligned} t + f(t) &= m_t s_1, \\ t - f(t) &= n_t s_2 \end{aligned}$$

for some $m_t, n_t \in \mathbb{Z}$. Solving for t gives

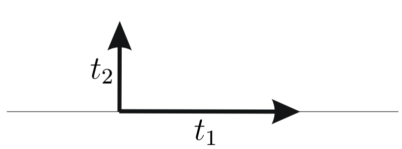
$$t = \frac{m_t}{2} s_1 + \frac{n_t}{2} s_2.$$

If, for every $t \in T$, both integers m_t, n_t are even, the set $\{s_1, s_2\}$ spans T, and so is a basis for T. On the other hand, if m_t or n_t is odd for some t, then both have to be odd, else $\frac{1}{2}s_1$ or $\frac{1}{2}s_2$ is in T, a contradiction. If we set $t_1 = \frac{1}{2}(s_1 + s_2)$ and $t_2 = \frac{1}{2}(s_1 - s_2) = f(t_1)$, then $t_1, t_2 \in T$, and

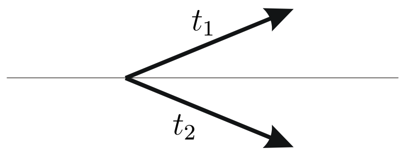
$$\begin{aligned} t &= \frac{m_t}{2} s_1 + \frac{n_t}{2} s_2 = \left(\frac{m_t + n_t}{2}\right)\left(\frac{s_1 + s_2}{2}\right) + \left(\frac{m_t - n_t}{2}\right)\left(\frac{s_1 - s_2}{2}\right) \\ &= m'_t t_1 + n'_t t_2 \end{aligned}$$

with $m'_t, n'_t \in \mathbb{Z}$. Since any t is then an integral linear combination of t_1 and t_2, the set $\{t_1, t_2\}$ is a basis for T.

To summarize these two cases, we either have a basis $\{t_1, t_2\}$ of two orthogonal vectors, one of which is fixed by a reflection in G_0,



or we have a basis of vectors of the same length with a reflection that interchanges them.



In the first case we say that T is a *rectangular lattice* and in the second case that T is a

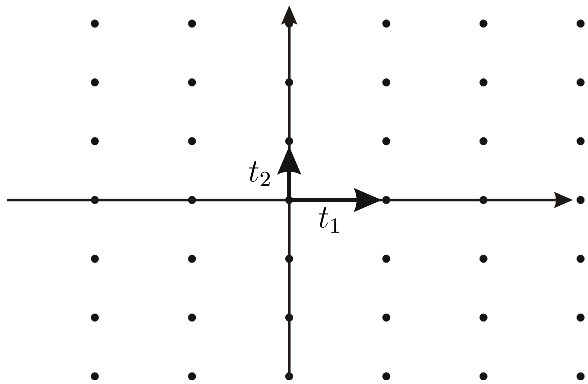


Figure 3.6: Rectangular Lattice

rhombic lattice.

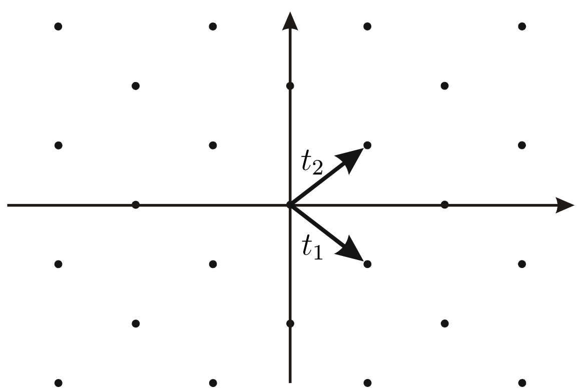


Figure 3.7: Rhombic Lattice

We can now get matrix representations for D_1 and D_2. For each group there are two possibilities, corresponding to two different actions on T. We subscript the group by p for rectangular and c for rhombic. We are using these subscripts to match common notation used for wallpaper groups that will be described in Chapter 5.1. We have

$$D_{1,p} = \left\langle \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$$

and

$$D_{1,c} = \left\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle,$$

while for D_2, which contains a rotation of 180°, we obtain

$$D_{2,p} = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\rangle$$

and

$$D_{2,c} = \left\langle \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

We prove that $D_{1,p}$ and $D_{1,c}$ are not conjugate in $\mathrm{Gl}_2(\mathbb{Z})$, nor are $D_{2,p}$ and $D_{2,c}$. This will show that no wallpaper group whose point group is one of these is isomorphic to a wallpaper group whose point group is another. For $D_{1,p}$ and $D_{1,c}$, suppose that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{Z}$ with $ad - bc = \pm 1$. Multiplying these and setting the two sides equal yields $d = -b$ and $c = -a$. Then $ad - bc = -2ab$, which is not ± 1 since a and b are integers. Therefore, $D_{1,p}$ and $D_{1,c}$ are not conjugate in $\mathrm{Gl}_2(\mathbb{Z})$. For $D_{2,p}$ and $D_{2,c}$, the previous calculation shows that we need only check if there are $a, b, c, d \in \mathbb{Z}$ with $ad - bc = \pm 1$ and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Similar calculations show that this forces $2ab = \pm 1$, again a contradiction.

Chapter 4

Group Cohomology

This chapter is the technical heart of the book. We have seen that given a wallpaper group G, we have an Abelian normal subgroup T of translations, the quotient group $G/T \cong G_0$, and an action of G_0 on T. To understand G we need to see how G is built from T and G_0. What we will see is that, given T and G_0, an Abelian group describes the possible groups that can be built from these two groups. This is the second cohomology group $H^2(G_0, T)$. In this chapter we describe building a group from a subgroup and a quotient group, we define the group $H^2(G_0, T)$ and see how it is connected with building groups, and we calculate $H^2(G_0, T)$ in all the different cases that arise for wallpaper groups. From these calculations we will be able to write down all of the isomorphism classes of wallpaper groups.

4.1 Group Extensions

To help us describe wallpaper groups, and to understand better how they are built from T and G_0, we discuss the concept of a group extension. A wallpaper group contains the Abelian normal subgroup T of translations, and G/T is a finite group isomorphic to the point group G_0. Phrasing this in another way, there is an exact sequence

$$1 \to T \to G \to G_0 \to 1$$

with T Abelian. We discuss this situation in the abstract.

Let G_0 and T be fixed groups with T Abelian. A *group extension* of T by G_0 is an exact sequence

$$1 \to T \to G \xrightarrow{\pi} G_0 \to 1.$$

For simplicity we view T as a subgroup of G. We will describe such a sequence by two additional pieces of information, an action of G_0 on T and a 2-cocycle. First, for the action, for each $g \in G_0$ choose an $x_g \in G$ with $\pi(x_g) = g$. The inner automorphism $y \mapsto x_g y x_g^{-1}$ restricts to an automorphism of T since T is normal in G. We define the action of g on T by $g(t) = x_g t x_g^{-1}$ for all $t \in T$. The first thing to note is that this is well-defined: if $\pi(y_g) = g$, then $y_g^{-1} x_g \in \ker(\pi) = T$, since T is Abelian we get $y_g^{-1} x_g t (y_g^{-1} x_g)^{-1} = t$, or $y_g t y_g^{-1} = x_g t x_g^{-1}$

for all $t \in T$. It is clear that the map $t \mapsto g(t)$ is a group automorphism of T. Furthermore, $g(ht) = (gh)t$ for all $g, h \in G_0$; in other words, the map that sends g to left multiplication by g is a group homomorphism $\varphi : G_0 \to \mathrm{Aut}(T)$. When there is such a group homomorphism, we will call T a G_0-*module*.

To give an example of a group extension, suppose that T is a G_0-module. We thus have a group homomorphism $\varphi : G_0 \to \mathrm{Aut}(T)$. We then can define the *semidirect product* $T \times_\varphi G_0$ to be, as a set, the direct product $T \times G_0$, but with the group operation defined by

$$(s, g)(t, h) = (sg(t), gh).$$

An easy calculation shows that $T \times_\varphi G_0$ is a group with this operation, that $T \cong \{(t, 1) : t \in T\}$, and that G_0 is isomorphic to the subgroup $\{(1, g) : g \in G_0\}$ of $T \times_\varphi G_0$. Furthermore, by identifying T and G_0 with these isomorphic copies, we see that $(1, g)(t, 1)(1, g)^{-1} = (g(t), 1)$, so conjugation by g induces the given action of G_0 on T. Furthermore, the homomorphisms $t \mapsto (t, 1)$ and $\pi(t, g) = g$ yield a group extension $1 \to T \to T \times_\varphi G_0 \xrightarrow{\pi} G_0 \to 1$.

Now assume that we have a group extension

$$1 \to T \to G \xrightarrow{\pi} G_0 \to 1$$

that yields a given G_0-action on T. We will view T as a subgroup of G. As before, we choose $x_g \in G$ with $\pi(x_g) = g$. While it is not necessary, we choose $x_1 = 1$; this will make the proof of Proposition 4.1 easier. The function $g \mapsto x_g$ is not necessarily a group homomorphism. We can measure the failure of this function to be a homomorphism as follows. If $c(g, h) = x_g x_h x_{gh}^{-1}$, then $g \mapsto x_g$ is a homomorphism if and only if $c(g, h) = 1$ for all $g, h \in G_0$. Note that $\pi(c(g, h)) = 1$, so $c(g, h) \in T$. This function is not arbitrary. Associativity in G gives $(x_g x_h)x_k = x_g(x_h x_k)$. Using the formula $x_g x_h = c(g, h)x_{gh}$, we obtain

$$(x_g x_h)x_k = c(g, h)x_{gh}x_k = c(g, h)c(gh, k)x_{ghk}$$

and

$$\begin{aligned} x_g(x_h x_k) &= x_g c(h, k)x_{hk} = x_g c(h, k)x_g^{-1}x_g x_{hk} \\ &= g\,(c(h, k))\,c(g, hk)x_{ghk}. \end{aligned}$$

Therefore, we have the condition

$$c(g, h)c(gh, k) = g\,(c(h, k))\,c(g, hk) \tag{4.1}$$

for all $g, h, k \in G_0$. A function $c : G_0 \times G_0 \to T$ satisfying Equation (4.1) us called a *2-cocycle*. We point out that the choice $x_1 = 1$ also implies that $c(1, g) = c(g, 1) = 1$ for all $g \in G_0$; a cocycle satisfying this condition is said to be *normalized*. By defining pointwise multiplication of 2-cocycles, we obtain an Abelian group $Z^2(G_0, T)$ of all 2-cocycles from $G_0 \times G_0$ to T. If $1 \to T \to T \times_\varphi G_0 \to G_0 \to 1$ is the group extension corresponding to the semidirect product of T by G_0, then we may choose $x_g = (1, g)$, and we find that the cocycle class representing this extension is given by the cocycle $c(g,h) = x_g x_h x_{gh}^{-1} = 1$. In other words, the trivial cocycle arises from the semidirect product of T by G_0.

Looking at the construction above of a cocycle c from a group extension $1 \to T \to G \to G_0 \to 1$, we note that c is not uniquely determined. If we make new choices y_g with $\pi(y_g) = g$, this yields a different cocycle c' given by $c'(g,h) = y_g y_h y_{gh}^{-1}$. To compare these cocycles, note that $y_g = t_g x_g$ for some $t_g \in T$ since $\pi(y_g) = \pi(x_g)$. Therefore,

$$\begin{aligned} c'(g,h) &= y_g y_h y_{gh}^{-1} = (t_g x_g)(t_h x_h)(t_{gh} x_{gh})^{-1} \\ &= t_g (x_g t x_g^{-1}) x_g x_h x_{gh}^{-1} t_{gh}^{-1} = t_g g(t_h) c(g,h) t_{gh}^{-1} \\ &= (t_g g(t_h) t_{gh}^{-1}) c(g,h). \end{aligned}$$

The function $b(g,h) = t_g g(t_h) t_{gh}^{-1}$ is then also a cocycle, being the element $c'c^{-1} \in Z^2(G_0, T)$. Cocycles of this form are called 2-*coboundaries*. The set of 2-coboundaries from $G_0 \times G_0 \to T$ is a subgroup of $Z^2(G_0, T)$, which we denote by $B^2(G_0, T)$. The quotient group

$$H^2(G_0, T) = Z^2(G_0, T)/B^2(G_0, T)$$

is called is the *second cohomology group* of G_0 with coefficients in T. By our procedure of obtaining a cocycle from a group extension $1 \to T \to G \to G_0 \to 1$, we see that while the cocycle is not uniquely determined, its coset in $H^2(G_0, T)$ is uniquely determined by the computation above.

We now describe how $H^2(G_0, T)$ determines group extensions of T by G_0. We say that two group extensions are *equivalent* if there is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T & \xrightarrow{\ i\ } & G & \xrightarrow{\ \pi\ } & G_0 & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \mathrm{id}} & & \\ 1 & \longrightarrow & T & \xrightarrow[\ i'\ ]{} & G' & \xrightarrow[\ \pi'\ ]{} & G_0 & \longrightarrow & 1 \end{array}$$

with $\varphi : G \to G'$ a group isomorphism. The connection between group extensions and $H^2(G_0, T)$ is given in the following proposition.

Proposition 4.1. *Let T be a G_0-module. Then there is a 1–1 correspondence between the elements of $H^2(G_0, T)$ and equivalence classes of group extensions of T by G_0 that induce the given G_0-action on T.*

Proof. We have shown above that, given a group extension, there is a uniquely determined cocycle class in $H^2(G_0, T)$. To go in the opposite direction, given a normalized cocycle c, we produce a group extension $1 \to T \to G \to G_0 \to 1$ whose cocycle class in $H^2(G_0, T)$ is equal to the class of c. Define G as a set by $T \times G_0$ and whose operation is given by

$$(s, g)(t, h) = (sg(t)c(g,h), gh).$$

A short calculation shows that G is a group; associativity follows exactly from the 2-cocycle condition, and inverses are given by the formula $(s,g)^{-1} = ((g^{-1}(s)c(g^{-1},g))^{-1}, g^{-1})$. Moreover, the map $T \to G$ with $t \mapsto (t,1)$ and the map $G \to G_0$ with $(t,g) \mapsto g$ are group homomorphisms, so we have a group extension $1 \to T \to G \to G_0 \to 1$. One consequence of the cocycle condition is that for the normalized cocycle c, we have $c(g,g^{-1}) = g(c(g^{-1},g))$; this follows by setting $h = g^{-1}$ and $k = g$ in Equation 4.1. If we set $x_g = (1,g)$, then

$$\begin{aligned} x_g(t,1)x_g^{-1} &= (1,g)(t,1)(c(g^{-1},g)^{-1},g^{-1}) = (g(t),g)(c(g^{-1},g)^{-1},g^{-1}) \\ &= (g(t)g(c(g^{-1},g))^{-1}c(g,g^{-1}) = (g(t),1). \end{aligned}$$

Therefore, the G_0-action on T is the same as that arising from this extension. Finally,

$$\begin{aligned} x_g x_h x_{gh}^{-1} &= (1,g)(1,h)(c((gh)^{-1},gh)^{-1},(gh)^{-1}) = (c(g,h),gh)(c((gh)^{-1},gh)^{-1},(gh)^{-1}) \\ &= (c(g,h)gh(c((gh)^{-1},gh)^{-1}c(gh,(gh)^{-1}) = (c(g,h),1). \end{aligned}$$

Therefore, the cocycle class for this extension is the same as the class of c.

Finally, we will finish the proof by showing that if two extensions have the same cocycle class, then they are equivalent. Suppose that

$$1 \to T \to G \to G_0 \to 1$$

and

$$1 \to T \to H \to G_0 \to 1$$

are two extensions giving rise to the same cocycle class. We may then assume that there are $x_g \in G$ and $y_g \in H$ with $x_g x_h x_{gh}^{-1} = y_g y_h y_{gh}^{-1}$ in T; note that we are viewing T as a subgroup of both G and H. Moreover, we may need to alter the y_g by an element of T to suppose that these group extensions give rise to the same cocycle, not just the same cocycle class. It is easy to see that $G = \{tx_g : t \in T, g \in G_0\}$ and $H = \{ty_g : t \in T, g \in G_0\}$. We then define a map $\varphi : G \to H$ by $\varphi(tx_g) = ty_g$. A short calculation shows that φ is well-defined, and that φ is a group isomorphism with $\varphi|_T = \mathrm{id}$ and the induced map $G_0 \to G_0$ is also the identity. Thus, these extensions are equivalent. □

Example 4.2. To give some examples of group extensions, consider the case $T = \mathbb{Z}$ and $G_0 = \mathbb{Z}_2$. Since $\mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$, there are two possible actions of $\mathbb{Z}_2$ on $\mathbb{Z}$. One, the *trivial action*, corresponds to the trivial homomorphism $\mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$, and so the action satisfies $gt = t$ for all $t \in \mathbb{Z}$ and $g \in \mathbb{Z}_2$. The other action arises from the nontrivial homomorphism $\mathbb{Z}_2 \to \mathrm{Aut}(\mathbb{Z})$, and so the nonidentity element of $\mathbb{Z}_2$ acts as -1 on $\mathbb{Z}$. With respect to the trivial action the following two sequences are group extensions of $\mathbb{Z}$ by $\mathbb{Z}_2$:

$$0 \to \mathbb{Z} \to \frac{1}{2}\mathbb{Z} \to \mathbb{Z}_2 \to 0$$

and

$$0 \to \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}_2 \to \mathbb{Z}_2 \to 0,$$

These extensions are not equivalent since the middle groups are not even isomorphic. In fact, one can show by the calculations of Section 4.4 below that $H^2(\mathbb{Z}_2, \mathbb{Z}) \cong \mathbb{Z}_2$ for this trivial action. The two group extensions above are exactly the two inequivalent extensions of $\mathbb{Z}_2$ by $\mathbb{Z}$ with this action.

Example 4.3. If we consider group extensions of $\mathbb{Z}$ by $\mathbb{Z}_2$ with the nontrivial action of $\mathbb{Z}_2$ on $\mathbb{Z}$, and if G is the semidirect product of $\mathbb{Z}$ and $\mathbb{Z}_2$, then

$$0 \to \mathbb{Z} \to G \to \mathbb{Z}_2 \to 0$$

is a group extension. We can see that this group extension is not equivalent to the previous two because G is not Abelian, so it is not isomorphic to either of the middle groups in those sequences. However, another way to see this is to note that if two group extensions of T by G_0 are equivalent, then the action of G_0 on T is the same for both group extensions. We do not give a proof of this fact. Instead, Corollary 3.8 will be sufficient for our needs in terms of comparing point groups acting in different ways on T.

The notion of equivalence of group extension is more subtle than that of isomorphism of the middle terms of the sequence. Consider $T = \mathbb{Z}_p$ and $G_0 = \mathbb{Z}_p$ for p an odd prime. Given a group extension

$$0 \to \mathbb{Z}_p \to G \to \mathbb{Z}_p \to 0,$$

the group G has order p^2, so is Abelian. This forces the action of $\mathbb{Z}_p$ on $\mathbb{Z}_p$ to be trivial. One can also see this from the fact that $\mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$, so there is no nontrivial group homomorphism $\mathbb{Z}_p \to \mathrm{Aut}(\mathbb{Z}_p)$. There are two isomorphism classes of groups of order p^2, the cyclic group of order p^2, and the direct product $\mathbb{Z}_p \times \mathbb{Z}_p$. However, there are p equivalence classes of extensions of $\mathbb{Z}_p$ by $\mathbb{Z}_p$. One can see this by showing that $H^2(\mathbb{Z}_p, \mathbb{Z}_p) \cong \mathbb{Z}_p$. However, to be more explicit, the direct product $\mathbb{Z}_p \times \mathbb{Z}_p$ corresponds to the trivial cocycle. On the other hand, if $G = \langle a \rangle$ is cyclic of order p^2, then for $1 \leq i < p$, we get a group extension

$$0 \to \langle a^p \rangle \to \langle a \rangle \xrightarrow{\pi_i} \mathbb{Z}_p \to 0$$

of $\mathbb{Z}_p$ by $\mathbb{Z}_p$ by defining $\pi_i(a) = i (\mathrm{mod}\, p)$. A short calculation shows that, for $i \neq j$, the corresponding group extensions are not equivalent. Nor are any of these extensions trivial since G is not isomorphic to $\mathbb{Z}_p \times \mathbb{Z}_p$. These yield $p-1$ inequivalent group extensions, all with the middle group isomorphic to $\mathbb{Z}_{p^2}$.

4.2 Group Extensions of T by G_0

Let G be a wallpaper group with translation lattice T and point group G_0. In Section 3.1, we proved that $G_0 \cong G/T$, that G_0 is isomorphic to a cyclic group C_n or a Dihedral group D_n with $n \in \{1, 2, 3, 4, 6\}$, and that, by considering the action of G_0 on T, we produced thirteen subgroups of $\mathrm{Gl}_2(\mathbb{Z})$, none of which are conjugate in $\mathrm{Gl}_2(\mathbb{Z})$, as candidates for G_0.

By Corollary 3.8, if two wallpaper groups are isomorphic, by representing their point groups as subgroups of $\mathrm{Gl}_2(\mathbb{Z})$, these two groups must be the same from this list of thirteen groups. Furthermore, from G we get a group extension $1 \to T \to G \to G_0 \to 1$, which then yields an element of the cohomology group $H^2(G_0, T)$.

We can use group extensions to limit the number of possibilities of nonisomorphic wallpaper groups. We first point out that any group extension of T by G_0 corresponds to some wallpaper group; we prove this in the following lemma. To prove the lemma we need two results from group cohomology. One is simple. If $S \subseteq T$ are G_0-modules, then there is a natural group homomorphism $H^2(G_0, S) \to H^2(G_0, T)$ induced by viewing a 2-cocycle $f \in Z^2(G_0, S)$ as a 2-cocycle $G_0 \times G_0 \to T$. The other is the Lyndon-Hochschild-Serre spectral sequence, which we describe in Section 4.4.

Lemma 4.4. *Let $1 \to T \to G \to G_0 \to 1$ be a group extension. Then, up to equivalence of group extension, G can be taken to be a subgroup of* $\mathrm{Isom}(\mathbb{R}^2)$*, and this subgroup is a wallpaper group.*

Proof. By a calculation using the Lyndon-Hochschild-Serre spectral sequence, given in Lemma 4.7 below, we have $H^2(G_0, \mathbb{R}^2) = 0$. If c is the cocycle class of the given group extension, then c goes to 0 under the natural map $H^2(G_0, T) \to H^2(G_0, \mathbb{R}^2)$. Therefore, there are $t_g \in \mathbb{R}^2$ with

$$c(g, h) = t_g + gt_h - t_{gh}$$

for all $g, h \in G_0$. We define G' by

$$G' = \{(g, t + t_g) : g \in G_0, t \in T\}.$$

An easy calculation shows that G' is a subgroup of $\mathrm{Isom}(\mathbb{R}^2)$, and that the maps $t \mapsto (\mathrm{id}, t)$ and $(g, t) \mapsto g$ from T to G and from G to G_0, respectively, yield a group extension

$$1 \to T \to G' \to G_0 \to 1.$$

Furthermore the cocycle class representing this extension is c; this latter fact can be seen by choosing $x_g = (g, t_g)$, so $x_g x_h x_{gh}^{-1} = (I, t_g + gt_h - t_{gh}) = (I, c(g, h))$. Thus, this extension is equivalent to the original extension. Finally, we note that G' is indeed a wallpaper group since G' contains the two-dimensional lattice T. □

For a given point group G_0 and an action of G_0 on T, two nonisomorphic wallpaper groups G and G', both with translation lattice T and point group G_0, correspond to inequivalent group extensions. Therefore, the number of nonisomorphic wallpaper groups is at most the number of inequivalent group extensions of T by G_0. Since $H^2(G_0, T)$ classifies the group extensions of T by G_0 for a given action of G_0 on T, the number $|H^2(G_0, T)|$ is an upper bound for the number of nonisomorphic wallpaper groups with point group G_0 with the given action. We determine these cohomology groups in all possible cases in the next section. Before we calculate them, we summarize the calculations in the following chart.

G_0	$H^2(G_0,T)$	$\lvert H^2(G_0,T)\rvert$
C_1	0	1
C_2	0	1
C_3	0	1
C_4	0	1
C_6	0	1
$D_{1,p}$	$\mathbb{Z}/2\mathbb{Z}$	2
$D_{1,c}$	0	1
$D_{2,p}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	4
$D_{2,c}$	0	1
$D_{3,l}$	0	1
$D_{3,s}$	0	1
D_4	$\mathbb{Z}/2\mathbb{Z}$	2
D_6	0	1
		18 total extensions

Table 4.1: The Cohomology Groups $H^2(G_0,T)$

We will see in Section 5.1 that determining all nonisomorphic wallpaper groups is almost the same as determining all group extensions of T by G_0 for the various G_0. In fact, in only one occasion will two inequivalent group extensions give rise to isomorphic wallpaper groups; thus, from 18 group extensions we will get 17 nonisomorphic wallpaper groups.

4.3 Higher Cohomology Groups

The second cohomology group $H^2(G_0,T)$ describes the equivalence classes of extensions of T by G_0; we need to calculate this cohomology group in order to determine the possible wallpaper groups. However, to do this we need to be able to calculate higher cohomology groups. We give an extremely brief description of what are these groups in this section. For a more complete description, see any book on homological algebra. A systematic description of cohomology groups would best involve derived functors. However, because we are being brief, we will give a more ad-hoc description. In fact, this description is essentially that given in the paper of Hochschild and Serre [2].

Let G_0 be a group and let T be a G_0-module. If n is a nonnegative integer, let $C^n(G_0,T)$ be the set of all functions from the Cartesian product $G_0^n = \prod_{i=1}^n G_0$ to T. If $n = 0$, we interpret G_0^0 as a single point and then identify $C^0(G_0,T)$ with T. Let $d^n : C^n(G_0,T) \to$

$C^{n+1}(G_0, T)$ be the map defined by

$$d^n(f)(g_1, \dots, g_{n+1}) = g_1 f(g_2, \dots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i f(g_1, \dots, g_{i-1}, g_i g_{i+1}, \dots, g_{n+1}) + (-1)^{n+1} f(g_1, \dots, g_n).$$

If $n = 0$, and if $t \in C^0(G_0, T) = T$, then we define $d^0(t)(g) = gt - t$. A tedious calculation will show that $d^{n+1} \circ d^n = 0$. That is, the collection $\{C^n(G_0, T)\}_{n \geq 0}$ together with the sequence $\{d^n\}$ of maps forms a chain complex. We obtain the cohomology groups $H^n(G_0, T)$ as the homology of this complex. In other words,

$$H^n(G_0, T) = \ker(d^n) / \operatorname{im}(d^{n-1})$$

if $n > 0$, and $H^0(G_0, T) = \ker(d^0) = T^{G_0}$. Moreover, a quick check will show that $f : G_0 \times G_0 \to T$ is a 2-cocycle if and only if $f \in \ker(d^2)$, and f is a 2-coboundary if and only if $f \in \operatorname{im}(d^1)$. This definition of $H^2(G_0, T)$ is then the same as that given in Section 4.1.

For those familiar with derived functors, we mention the connection between group cohomology and derived functors. Let $T^{G_0} = \{t \in T : gt = g \text{ for all } g \in G_0\}$. Then $H^0(G_0, T) = T^{G_0}$, and in fact the functor $H^0(G_0, -)$ is naturally equivalent to the fixed point functor $(-)^{G_0}$. Therefore, an alternative description of $H^n(G_0, T)$ is that if $F = (-)^{G_0}$, and if $R^n(F)$ is the n-th right derived functor of F, then $H^n(G_0, T) = R^n(F)(T)$. Furthermore, $H^n(G_0, T) = \operatorname{Ext}^n_{\mathbb{Z}G_0}(\mathbb{Z}, G_0)$. The complex $\{C^n(G_0, T)\}$ arises by taking the free resolution $\{P_n\}$ of $\mathbb{Z}$ as a trivial G_0-module, where P_n is the free $\mathbb{Z}G_0$-module on the set G_0^n. The differential $d : P_n \to P_{n-1}$ is given by the formula $d = \sum_{i=0}^n (-1)^i d_i$, where

$$\begin{aligned} d_0(g_1, \dots, g_n) &= g_1(g_2, \dots, g_{n-1}), \\ d_i(g_1, \dots, g_n) &= (g_1, \dots, g_{i-1}, g_i g_{i+1}, \dots, g_n), 1 < i < n \\ d_n(g_1, \dots, g_n) &= (g_1, \dots, g_{n-1}) \end{aligned}$$

Applying the contravariant functor $\hom_{\mathbb{Z}G_0}(-, T)$ to the resolution $\{P_n\}$, using the isomorphism $\hom_{\mathbb{Z}G_0}(P_n, T) \cong C^n(G_0, T)$, which holds since homomorphisms are determined by their action on a basis, we see that the chain complex $\{\hom_{\mathbb{Z}G_0}(P_n, T)\}$ is the complex $\{C^n(G_0, T)\}$.

4.4 Calculation of the Groups $H^2(G_0, T)$

In this section we calculate the cohomology groups $H^2(G_0, T)$ for the various point groups G_0. In the next chapter we will see how to determine all wallpaper groups from these calculations. The calculations of these groups for $G_0 = C_n$ are very easy. If the point group is a Dihedral group, then we have to work harder. If you do not wish to bother with spectral sequences, you should skip those calculations. However, if you are familiar with spectral sequences or if you wish to see an example of their use, these calculations are a nice and easy illustration of the power of spectral sequences.

There are various facts about cohomology we will need for our calculations. Because seven of the thirteen G_0 are cyclic groups, we start with facts about the cohomology of cyclic groups. If $C = \langle c \rangle$ is a cyclic group of order n and M is a C-module, the *norm map* N_C on M is defined as $N_C(x) = \sum_{i=0}^{n-1} c^i x$. The set $M^C = \{m \in M : cm = m\}$ is the subgroup of M fixed by all elements of C. The cohomology groups of C with coefficients in M are then

$$\begin{aligned} H^0(C, M) &\cong M^C, \\ H^{2n}(C, M) &\cong M^C / N_C(M), \\ H^{2n+1}(C, M) &\cong \ker(N_C) / \operatorname{im}(1 - c), \end{aligned}$$

for all positive integers n. This result can be found in [7, Chapter 6]. We give the description of $H^q(C, M)$ for all q because we will need to know $H^q(C_n, T)$ for all q to calculate $H^2(D_n, T)$ via the Lyndon-Hochschild-Serre spectral sequence.

$G_0 = C_n$.

Suppose that our point group G_0 is generated by a rotation. Then G_0 acts on T without any nonzero fixed points; that is, $T^{G_0} = 0$. However, since

$$H^2(G_0, T) \cong T^{G_0} / N_{G_0}(T),$$

we obtain $H^2(G_0, T) = 0$.

$G_0 = D_{1,c}$.

In this case G_0 is generated by a reflection f that interchanges the two basis vectors t_1, t_2 of T. Since G_0 is cyclic, we have

$$H^2(G_0, T) \cong T^{G_0} / \operatorname{im}(N_{G_0}).$$

Furthermore, $T^{G_0} = \mathbb{Z}(t_1 + t_2)$, and $t_1 + t_2 = N_{G_0}(t_1)$ since t_1 is sent to t_2 by f. Therefore, $T^{G_0} = \operatorname{im}(N_{G_0})$, so $H^2(G_0, T) = 0$.

$G_0 = D_{1,p}$.

Here we have $G_0 = \langle f \rangle$, where $f(t_1) = t_1$ and $f(t_2) = -t_2$. As before, $H^2(G_0, T) \cong T^{G_0} / \operatorname{im}(N_{G_0})$. In this case $T^{G_0} = \mathbb{Z}t_1$ and $\operatorname{im}(N_{G_0}) = \mathbb{Z}(2t_1)$, since

$$\begin{aligned} N_{G_0}(at_1 + bt_2) &= (at_1 + bt_2) + f(at_1 + bt_2) \\ &= 2at_1. \end{aligned}$$

Therefore, $H^2(D_{1,p}, T) \cong \mathbb{Z}/2\mathbb{Z}$. We thus have two inequivalent group extensions of T by $D_{1,p}$.

We now consider the non-cyclic point groups. For $D_{2,c}$ and D_6 we need more machinery. If G is a group with normal subgroup N, if M is a G-module, and if $H^1(N, M) = 0$, then there is a five term exact sequence [4, p. 307] of low degree terms

$$0 \to H^2(G/N, M^N) \to H^2(G, M) \to H^2(N, M)^{G/N} \to H^3(G/N, M^N) \to H^3(G, M).$$

This exact sequence arises from the Lyndon-Hochschild-Serre spectral sequence associated to the normal subgroup N of G; an introduction to spectral sequences and the description of the Lyndon-Hochschild-Serre spectral sequence is given below. We will use this exact sequence for $G = D_6$ and $D_{2,c}$.

$G_0 = D_6$.

The group D_6 contains the normal subgroup C_6. Moreover,

$$C_6 = \left\langle \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \right\rangle,$$

the matrix representing the $60°$ rotation r. From the description above of the cohomology of a cyclic group,

$$H^1(C_6, T) \cong \ker(N_{C_6}) / \operatorname{im}(1 - r),$$

where N_{C_6} is the norm map as defined above. It is easy to see that $N_{C_6} = 0$ since the image of this map is contained in T^{C_6}, which is zero. Therefore, $\ker(N_{C_6}) = T$. However, $1 - r$ is represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix},$$

which is invertible in $\mathrm{Gl}_2(\mathbb{Z})$. Therefore, $\operatorname{im}(1 - r) = T$, so $H^1(C_6, T) = 0$. The first part of the five term exact sequence mentioned above is

$$0 \to H^2(D_6/C_6, T^{C_6}) \to H^2(D_6, T) \to H^2(C_6, T)^{D_6/C_6}.$$

The first term is zero since $T^{C_6} = 0$. The third term is 0 since $H^2(C_6, T) = 0$, as we saw above. Therefore, $H^2(D_6, T) = 0$.

$G_0 = D_{2,c}$.

In this case we have $G_0 = \langle r, f \rangle$ with r the $180°$ rotation and f the reflection given by $f(t_1) = t_2$ and $f(t_2) = t_1$. Note that $r(t) = -t$ for all $t \in T$. Our argument will be slightly different from the previous case because $H^1(C_2, T) \neq 0$. Therefore, to apply the five term sequence, we need a different normal subgroup of G_0. Let $M = \langle f \rangle$. Then $H^1(M, T) = \ker(N_M)/\operatorname{im}(1 - f)$. We have $N_M(t) = t + f(t)$, so it is easy to see that $\ker(N_M) = \mathbb{Z}(t_1 - t_2)$. Also, $(1-f)(at_1 + bt_2) = a(t_1 - t_2) + b(t_2 - t_1)$, so $\operatorname{im}(1-f) = \ker(N_M)$. Therefore, $H^1(M, T) = 0$. The first part of the five term exact sequence arising from the normal subgroup N of G_0 is

$$0 \to H^2(G_0/M, T^M) \to H^2(G_0, T) \to H^2(M, T)^{G_0/M}.$$

The first term is a quotient of $(T^M)^{G_0/M} = T^{G_0} = 0$; we see that $T^{G_0} = 0$ since $r(t) = -t$ for all t. For the third term, we have $T^M = \mathbb{Z}(t_1 + t_2)$ and $\operatorname{im}(N_M) = \mathbb{Z}(t_1 + t_2)$ as $N_M(at_1 + bt_2) = (a + b)(t_1 + t_2)$. Therefore, the third term is zero, so $H^2(G_0, T) = 0$.

To finish the remaining cases we need to use the theory of spectral sequences. We give the definition of a spectral sequence and leave details to the books of Weibel [7] and Tamme [6]. A cohomological *spectral sequence* is the following collection of data: Abelian groups

$$\{E_r^{p,q} : p, q, r \in \mathbb{Z}, r \geq 2\};$$

homomorphisms

$$d_r^{p,q} : E_r^{p,q} \longrightarrow E_r^{p+r, q-r+1}$$

with $d_r^{p,q} \circ d_r^{p-r, q+r-1} = 0$ for all p, q, r; isomorphisms

$$E_{r+1}^{p,q} \cong \ker(d_r^{p,q})/\operatorname{im}(d_r^{p-r,q+r-1});$$

filtered Abelian groups

$$\{E^n : n \in \mathbb{Z}\}$$

where the filtration $\cdots F^p(E^n) \supseteq F^{p+1}(E^n) \supseteq \cdots$ satisfies $F^p(E^n) = E^n$ for $p \ll 0$ and $F^p(E^n) = 0$ for $p \gg 0$; *limit terms*

$$\{E_\infty^{p,q} : p, q \in \mathbb{Z}\};$$

and isomorphisms

$$E_\infty^{p,q} \cong F^p(E^{p+q})/F^{p+1}(E^{p+q}).$$

We will only consider the case where $E_r^{p,q} = 0$ if $p < 0$ or $q < 0$. In this case we can define $E_\infty^{p,q}$. If $r > \max\{p, q + 1\}$, consider the sequence

$$E_r^{p-r,q+r-1} \overset{d_r^{p-r,q+r-1}}{\longrightarrow} E_r^{p,q} \overset{d_r^{p,q}}{\longrightarrow} E_r^{p+r,q-r+1}.$$

The assumption on r gives $E_r^{p-r,q+r-1} = 0 = E_r^{p+r,q-r+1}$. Therefore, $\ker(d_r^{p,q}) = E_r^{p,q}$ and $\operatorname{im}(d_r^{p-r,q+r-1}) = 0$. Therefore, $E_{r+1}^{p,q} = E_r^{p,q}$. This implies that the sequence $E_r^{p,q}$ becomes constant, up to isomorphism, once r is large enough. We denote $E_\infty^{p,q}$ to be this constant value. This large amount of data is typically denoted $E_2^{p,q} \Longrightarrow E^{p+q}$ for short.

We note the following immediate consequences of the definition.

1. If $E_r^{p,q} = 0$, then $E_s^{p,q} = 0$ for all $s \geq r$, and $E_\infty^{p,q} = 0$.

2. For any $n \geq 0$, we have $F^0(E^n) = E^n$. This follows from the first fact because our assumption that $E_r^{p,q} = 0$ if $p < 0$ or $q < 0$ yields $E_\infty^{-1,n+1} = 0$.

3. For any $n \geq 0$, we have $F^{n+1}(E^n) = 0$. This follows for similar reasons as in the previous fact.

In many situations, including ours, the E^n are n-th cohomology groups. Having a spectral sequence allows one to get information about E^n since determining the limit terms $E_\infty^{p,q}$ determines the factors $F^p(E^{p+q})/F^{p+1}(E^{p+q})$ of the filtration of E^{p+q}. In several situations, including ours, the spectral sequence is simple enough to completely determine E^2 from the limit terms.

We give two general but simple examples that we will use below.

Example 4.5. Suppose that $E_r^{p,q} = 0$ for all $p, q \geq 0$. Then $E_\infty^{p,q} = 0$ for all p, q. Therefore, $F^p(E^2)/F^{p+1}(E^2) = 0$ for all $p \geq 0$. This means $F^p(E^2) = F^{p+1}(E^2)$ for all $p \geq 0$. Since $F^0(E^2) = E^2$ and $F^3(E^2) = 0$, as noted in the second and third facts above, this yields $E^2 = 0$.

Example 4.6. Suppose that $E_\infty^{0,2} = E_\infty^{2,0} = 0$ and $E_\infty^{1,1} = A$ for some group A. Then the relation to the filtration for E^2 gives

$$\begin{aligned} 0 &= F^0(E^2)/F^1(E^2) = E^2/F^1(E^2), \\ A &= F^1(E^2)/F^2(E^2), \\ 0 &= F^2(E^2)/F^3(E^3) = F^2(E^2). \end{aligned}$$

Again we are using the second and third facts above. These three equations tell us that $E^2 = F^1(E^2)$ and $F^1(E^2) = A$. Therefore, $E^2 = A$.

To determine $H^2(D_n, T)$ for $n \geq 2$, we will use the *Lyndon-Hochschild-Serre* spectral sequence. If N is a normal subgroup of a group G, and if A is a G-module, this is the spectral sequence

$$E_2^{p,q} = H^p(G/N, H^q(N, A)) \Longrightarrow H^{p+q}(G, A).$$

A proof of the existence of this sequence can be found in [7, Section 6.8]. We will apply this sequence with $G = D_n$ and $N = C_n$ for $A = T$. As a first use of this spectral sequence, we prove the result quoted above in Lemma 4.4.

Lemma 4.7. *If G_0 is a finite subgroup of $\mathrm{O}_2(\mathbb{R})$, then $H^2(G_0, \mathbb{R}^2) = 0$.*

Proof. By Proposition 2.8, G_0 is isomorphic to C_n or D_n for some integer n. If $G_0 = C_n$, then G_0 is generated by a rotation r. Since a rotation has no nonzero fixed point, we see that $(\mathbb{R}^2)^{G_0} = \{\mathbf{0}\}$. Therefore, $H^2(G_0, \mathbb{R}^2) \cong (\mathbb{R}^2)^{G_0}/\operatorname{im}(N_{G_0}) = 0$. On the other hand, if

$G_0 = D_n$, let $N = G_0 \cap \mathrm{SO}_2(\mathbb{R})$, the normal subgroup of rotations in G_0. We consider the spectral sequence

$$H^p(G_0/N, H^q(N, \mathbb{R}^2)) \Longrightarrow H^{p+q}(G_0, \mathbb{R}^2).$$

For $q = 2n$ even, we have $H^q(N, \mathbb{R}^2) = H^2(N, \mathbb{R}^2) = 0$ by the argument just given. For $q = 2n+1$ odd, $H^q(N, \mathbb{R}^2) \cong \ker(N_{G_0})/\operatorname{im}(1-r)$, where the subgroup N is generated by r. Since r has no nonzero fixed points, 1 is not an eigenvalue of r. Therefore, $1-r$ is invertible on $\mathbb{R}^2$. This forces $\operatorname{im}(1-r) = \ker(N_{G_0}) = \mathbb{R}^2$, and so $H^q(N, \mathbb{R}^2) = 0$. Therefore, the $E_2^{p,q}$ terms of the spectral sequence are $H^p(G_0/N, 0) = 0$. By Example 4.5, we get $H^n(G_0, \mathbb{R}^2) = 0$ for all n. In particular, $H^2(G_0, \mathbb{R}^2) = 0$. □

$G_0 = D_{3,l}, D_{3,s}$.

In these two cases we do not need to keep track of the action of G_0 on T except for how the 120° rotation acts, so we consider $G_0 = D_3$ without worrying about the difference between the two different actions we have. To determine $H^2(D_3, T)$, we use the Lyndon-Hochschild-Serre spectral sequence arising from the normal subgroup C_3 of D_3, which is

$$E_2^{p,q} = H^p(D_3/C_3, H^q(C_3, T)) \Longrightarrow H^{p+q}(D_3, T) = E^{p+q}.$$

If $C_3 = \langle r \rangle$, then the calculation of the cohomology of a cyclic group gives

$$H^q(C_3, T) = \begin{cases} T^{C_3}/N_C(T) = 0 & \text{if } q \equiv 0 (\operatorname{mod} 2) \\ \ker(N)/\operatorname{im}(1-r) & \text{if } q \equiv 1 (\operatorname{mod} 2) \end{cases}.$$

However, the norm map $N : T \to T$ with respect to C_3 is $N(t) = t + r(t) + r^2(t)$. Since $1 + r + r^2 = 0$ as a linear transformation, $\ker(N) = T$. The linear transformation $1 - r$ is represented by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}.$$

Therefore, $\operatorname{im}(1-r) = \{(a+b, -a+2b) : a, b \in \mathbb{Z}\}$, and it is easy to see that this is equal to $\{(x, y) : x + y \equiv 0 (\operatorname{mod} 3)\}$. Therefore, $H^q(C_3, T) \cong \mathbb{Z}_3$ if q is odd. The nontrivial element of $D_3/C_3 \cong \mathbb{Z}_2$ either acts trivially on $\mathbb{Z}_3$ or it acts as -1. In either of these cases, one of which occurs with $D_{3,l}$ and the other with $D_{3,s}$, we see that $H^n(D_3/C_3, \mathbb{Z}_3) = 0$ for all n by the calculation of the cohomology of a cyclic group. Therefore, putting all these pieces together, we see that $E_2^{p,q} = 0$ for all $p, q \geq 0$. By Example 4.5, this yields $H^2(D_3, T) = 0$.

$G_0 = D_4$.

Again, we use the Lyndon-Hochschild-Serre spectral sequence arising from the normal subgroup C_n of D_n, which in this case is

$$H^p(D_4/C_4, H^q(C_4, T)) \Longrightarrow H^{p+q}(D_4, T).$$

From the case of $C_4 = \langle r \rangle$, we saw that $H^{2n}(C_4, T) = 0$ for all n. For odd integers, if $N : T \to T$ is the norm map $N(t) = t + r(t) + r^2(t) + r^3(t)$, we have

$$H^{2n+1}(C_4, T) \cong \ker(N)/\operatorname{im}(1-r).$$

Since r^2 acts as -1 on T, we see that $N = 0$. Therefore, $\ker(N) = T$. Also,

$$(1-r)(a,b) = (a,b) - (b,-a) = (a-b, a+b).$$

Therefore, it follows that $\operatorname{im}(1-r) = \{(x,y) : x \equiv y (\operatorname{mod} 2)\}$. The two elements $(0,0)$ and $(1,0)$ then represent all cosets in $T/\operatorname{im}(N)$, so $H^{2n+1}(C_4, T) \cong \mathbb{Z}_2$. The group D_4/C_4 then acts trivially on the groups $H^q(C_4, T)$ since these groups are either trivial or have two elements. The norm map for D_4/C_4 on $H^q(C_4, T)$ is trivial in either case. Therefore, we obtain the formulas

$$E_2^{p,q} = \begin{cases} 0 & \text{if } q \equiv 0 (\operatorname{mod} 2) \\ \mathbb{Z}_2 & \text{if } q \equiv 1 (\operatorname{mod} 2) \end{cases}.$$

Since $E_2^{2,0} = 0 = E_2^{0,2}$, we have $E_\infty^{2,0} = 0 = E_\infty^{0,2}$. Also, from the relationship between $E_\infty^{p,q}$ and the $E_r^{p,q}$, we have $E_\infty^{1,1} = E_3^{1,1}$, and this is the homology of the complex $0 = E_2^{2,0} \to E_2^{1,1} \to E_2^{0,2} = 0$. Thus, $E_3^{1,1} = E_2^{1,1}$. Example 4.6 then yields

$$H^2(D_4, T) = E_\infty^{1,1} = \mathbb{Z}_2.$$

$G_0 = D_{2,p}$.

In this final case we continue to use the Lyndon-Hochschild-Serre spectral sequence, and we apply it to the normal subgroup C_2 of D_2, where we write D_2 for $D_{2,p}$. We have $D_2 = \langle r, f \rangle$, and the action on T satisfies $rt = -t$ for all t, and there is a basis $\{t_1, t_2\}$ of T with $ft_1 = t_1$ and $ft_2 = -t_2$. From our description of the cohomology of a cyclic group, it is easy to see that

$$H^q(C_2, T) = \begin{cases} 0 & \text{if } q \equiv 0 (\operatorname{mod} 2) \\ T/2T & \text{if } q \equiv 1 (\operatorname{mod} 2) \end{cases}.$$

as Abelian groups. Without giving any details, to see what is the D_2/C_2-action on $H^q(C_2, T)$, we point out that Example 6.7.10 of [7] can be modified to show that the isomorphism $H^{2n+1}(C_2, T) \cong T/2T$ sends the action of f to $(-1)^n f$. However, $\pm f$ acts as the identity on $T/2T$ by the description of f and because -1 acts as the identity on a group of exponent 2. Therefore, D_2/C_2 acts trivially on $H^{2n+1}(C_2, T)$. Because D_2/C_2 is cyclic,

$$E_2^{p,q} = H^p(D_2/C_2, H^q(C_2, T)) = \begin{cases} 0 & \text{if } q \equiv 0 (\operatorname{mod} 2) \\ T/2T & \text{if } q \equiv 1 (\operatorname{mod} 2) \end{cases}.$$

Again, by Example 4.6, we conclude that

$$H^2(D_2, T) = T/2T \cong \mathbb{Z}_2 \times \mathbb{Z}_2.$$

Chapter 5

The Wallpaper Groups

In the previous chapter we calculated the cohomology groups $H^2(G_0, T)$ for all possible point groups G_0 and all possible actions of G_0 on a two-dimensional lattice of translations T. We will now see how these calculations lead us to a determination of all wallpaper groups, up to isomorphism. We also give an explicit description of a wallpaper group in each isomorphism class as a subgroup of $\mathrm{Isom}(\mathbb{R}^2)$ given by generators along with pictures of corresponding wallpaper patterns.

5.1 Classification of Wallpaper Groups

In ten of the thirteen possibilities for the point group G_0 there is only one group extension of T by G_0, so there is only one wallpaper group for this G_0. For $D_{1,p}$ and D_4, we will see that the two inequivalent group extensions for each of these point groups give nonisomorphic groups. However, for $D_{2,p}$, while there are four inequivalent group extensions, there are only three nonisomorphic wallpaper groups. Therefore, the eighteen different group extensions noted from the chart below give us seventeen isomorphism classes of wallpaper groups. We note them now. To have a notation to refer to them, we will use the standard notation used by crystallographers.

We give a brief description of the naming scheme. First of all, we choose a basis $\{t_1, t_2\}$ for the translation lattice as in Section 3.2. We consider the direction of t_1 to be the x-axis. The full name consists of four symbols. The first symbol represents the lattice type; p for primitive and c for centered (or rhombic). The second symbol is the largest order of a rotation. The third symbol is either an m, g, or 1. An m (resp. g) means there is a reflection line (resp. glide reflection line but not a reflection line) perpendicular to the x-axis while a 1 means there is no line of either type. Finally, the fourth symbol is also either an m, a g, or a 1. In this case an m (resp. g) represents a reflection line (resp. glide reflection line) at

Wallpaper Group	**Full Name**	**Point Group**
p1	*p111*	C_1
cm	*c1m1*	$D_{1,c}$
pm	*p1m1*	$D_{1,p}$
pg	*p1g1*	
p2	*p211*	C_2
cmm	*c2mm*	$D_{2,c}$
pmm	*p2mm*	$D_{2,p}$
pmg	*p2mg*	
pgg	*p2gg*	
p3	*p311*	C_3
p3m1	*p3m1*	$D_{3,l}$
p31m	*p31m*	$D_{3,s}$
p4	*p411*	C_4
p4m	*p4m1*	D_4
p4g	*p4g1*	
p6	*p611*	C_6
p6m	*p6m1*	D_6

Table 5.1: The 17 Wallpaper Groups

an angle α with the x-axis, the angle depending on the largest order of rotation as follows: $\alpha = 180°$ for $n = 1, 2$; $\alpha = 60°$ for $n = 3, 6$; $\alpha = 45°$ for $n = 4$.

For example, the group name *p3m1* represents a group with a 120° rotation, a reflection line perpendicular to the x-axis, and no reflection or glide line at an angle of 60° with the x-axis. However, in the group *p31m*, we have the same rotation, but no reflection or glide line perpendicular to the x-axis, while there is a reflection line at an angle of 60° with the x-axis.

In Section 4.4 we showed that there are eighteen inequivalent group extensions of a point group of a symmetry group by $T \cong \mathbb{Z}^2$. As we saw from Corollary 3.8, if two groups have different point groups, when taking into account the action on T, then they are not isomorphic. Therefore, to determine whether two inequivalent group extensions represent nonisomorphic wallpaper groups, we have to consider only three point groups, $D_{1,p}$, $D_{2,p}$, and D_4, as indicated in the table at the end of Section 4.2.

In the remainder of this section, we describe explicitly the seventeen wallpaper groups. We continue to make use of some cohomological information. The next section gives this description without the use of cohomology.

To begin, we prove a simple lemma to enable us to write down elements of a wallpaper group in terms of the point group and certain vectors in $\mathbb{R}^2$. We will see that it is the determination of these vectors that will determine our wallpaper groups.

Lemma 5.1. *Let G be a wallpaper group with point group G_0. For each $g \in G_0$ there is a $t_g \in \mathbb{R}^2$ with $(g, t_g) \in G$. Furthermore, t_g is uniquely determined up to addition by an element of T. Furthermore, $G = \{(g, t_g + t) : g \in G_0, t \in T\}$.*

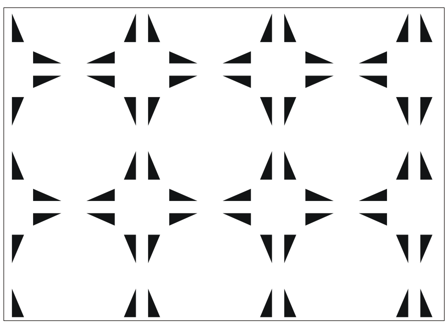
Proof. Recall from Proposition 3.2 that the map $\varphi : G \to G_0$ defined by $\varphi(g, t) = g$ is a surjective homomorphism with kernel T. Therefore, for each $g \in G_0$, there is a vector t_g with $(g, t_g) \in G$. If $(g, s_g) \in G$, then $\varphi(g, s_g) = \varphi(g, t_g)$, so $(g, s_g) \equiv (g, t_g) \bmod \ker(\varphi)$. Since $\ker(\varphi) = T$, there is a $t \in T$ with $(g, s_g) = (I, t)(g, t_g)$. Thus, $s_g = t_g + t$. In other words, t_g is uniquely determined up to addition by an element of T. Finally, since φ is a surjection onto G_0 with kernel T, and since $(g, t_g) \in \varphi^{-1}(g)$ and $G = \bigcup_{g \in G_0} \varphi^{-1}(g)$, we see that $G = \{(g, t_g + t) : g \in G_0, t \in T\}$. □

To describe explicitly a wallpaper group G, it is sufficient to determine the vectors $\{t_g\}$. To find generators of a wallpaper group, one case for each point group is easy: if G is the semidirect product of T and G_0; that is, G corresponds to the trivial group extension of T by G_0, then G is generated by t_1, t_2 and the generators of G_0. These give us thirteen nonisomorphic wallpaper groups. For the groups that represent nontrivial group extensions, we can look to Lemma 4.4 for help in determining the groups. Suppose $G_0 = D_n$ for some $n > 1$, and let r, f be generators of G_0, where r is a rotation and f a reflection. Recalling the construction of the cocycle associated to a group extension, for each $g \in G_0$ we need to find an element x_g of G projecting to g. We may choose $x_{r^i} = (r^i, \mathbf{0}) \in G$ since G contains the semidirect product of T by $\langle r \rangle$; this is a consequence of the equality $H^2(\langle r \rangle, T) = 0$ that we saw in Section 4.4. For f, let us write $x_f = (f, u)$ for some $u \in \mathbb{R}^2$ yet to be determined. As $(r^i, 0)(f, u) = (r^i f, r^i(u))$, we may then choose $x_{r^i f} = (r^i f, r^i(u))$. Therefore, we may set $t_{r^i} = \mathbf{0}$ and $t_{r^i f} = r^i(u)$ for each i. The cocycle c representing this group is given by $c(g, h) = t_g + g t_h - t_{gh}$. We must have $c(g, h) \in T$ for all $g, h \in G_0$. In particular, $c(f, f), c(rf, r) \in T$. However, $c(f, f) = u$ and $c(rf, r) = r(u) - u$ since $t_r = \mathbf{0}$ and $rfr = f$. Therefore, u is restricted by the condition $r(u) - u \in T$. Note that u is uniquely determined only up to addition by an element of T since for any $t \in T$ we have $(I, t)(f, u) = (f, u + t)$ is another choice of x_f.

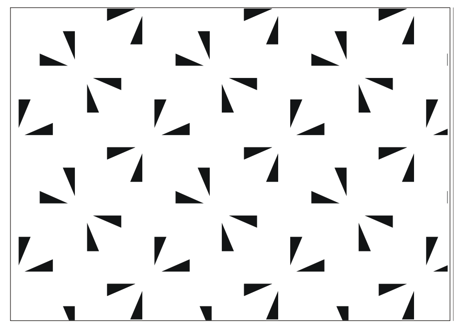
We use the restriction $r(u) - u \in T$ to determine wallpaper groups explicitly for the point groups D_4 and $D_{2,p}$. We will need a different condition to determine the wallpaper groups with point group $D_{1,p}$. In each case we will also determine when two inequivalent group extensions represent nonisomorphic wallpaper groups.

First, consider $G_0 = D_4 = \langle r, f \rangle$. From Section 3.2 we have a basis $\{t_1, t_2\}$ such that r sends t_1 and t_2 to t_2 and $-t_1$, respectively, and f sends t_1 and t_2 to t_1 and $-t_2$, respectively. If $u = \alpha t_1 + \beta t_2$ with $\alpha, \beta \in \mathbb{R}$, then $r(u) - u = (-\alpha - \beta)t_1 + (\alpha - \beta)t_2$. Therefore, $\alpha + \beta$ and $\alpha - \beta$ are integers. Since u is determined only modulo T, we may assume that $0 \le \alpha, \beta < 1$. Therefore, we have two possibilities, $u = \mathbf{0}$ if $\alpha = \beta = 0$ and $u = \frac{1}{2}(t_1 + t_2)$ if $\alpha = \beta = \frac{1}{2}$. The two inequivalent group extensions of T by D_4 correspond to these two choices of u. To see that the resulting groups are not isomorphic, the group corresponding to the trivial cocycle, which is the semidirect product of $\mathbb{Z}^2$ and D_4, contains a subgroup isomorphic to

D_4. We show that the other group does not contain such a subgroup. We can see this from the explicit description of G. The elements (r, t) and (r^3, t) for $t \in T$ are $90°$ rotations by Lemma 2.1. We recall that $u = \frac{1}{2}(t_1 + t_2)$ for this group. The elements $(fr, u + t)$ and $(fr^3, u + t)$ are reflections if t is an integral multiple of $t_1 + t_2$. If G contains a subgroup H isomorphic to D_4, then it contains a $90°$ rotation and a reflection. Moreover, the product of two such maps is again a reflection. However, the elements of the form $(f, u + t)$ and $(fr^2, u + t)$ for $t \in T$ are never reflections by Lemma 2.2. Also, the product of a rotation and a reflection in G is an element of one of these forms. This shows that G cannot contain a subgroup isomorphic to D_4. The two groups with point group D_4 are illustrated with the following wallpaper patterns.



p4m



p4g

Figure 5.1: Wallpaper patterns with groups *p4m* and *p4g*

In the case $G_0 = D_{2,p}$, the $180°$ rotation r acts on T as -1, the condition $r(u) - u \in T$ means $2b \in T$. Therefore, modulo T, we have exactly four cases: $u = \mathbf{0}, \frac{1}{2}t_1, \frac{1}{2}t_2, \frac{1}{2}(t_1 + t_2)$. This is the one case where the number of isomorphism classes of wallpaper groups is less than the number of group extensions of T by G_0. By our determination of the possible values of u, the four inequivalent group extensions correspond to groups given in terms of generators by

$$pmm = \langle t_1, t_2, r, g_1 \rangle \,, \qquad pmg = \langle t_1, t_2, r, g_2 \rangle \,,$$
$$pmg' = \langle t_1, t_2, r, g_3 \rangle \,, \qquad pgg = \langle t_1, t_2, r, g_4 \rangle \,,$$

where T is generated by t_1, t_2, and the g_i are given by

$$g_1 = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \mathbf{0} \right), \qquad g_2 = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tfrac{1}{2}t_1 \right),$$
$$g_3 = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tfrac{1}{2}t_2 \right), \qquad g_4 = \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tfrac{1}{2}(t_1 + t_2) \right).$$

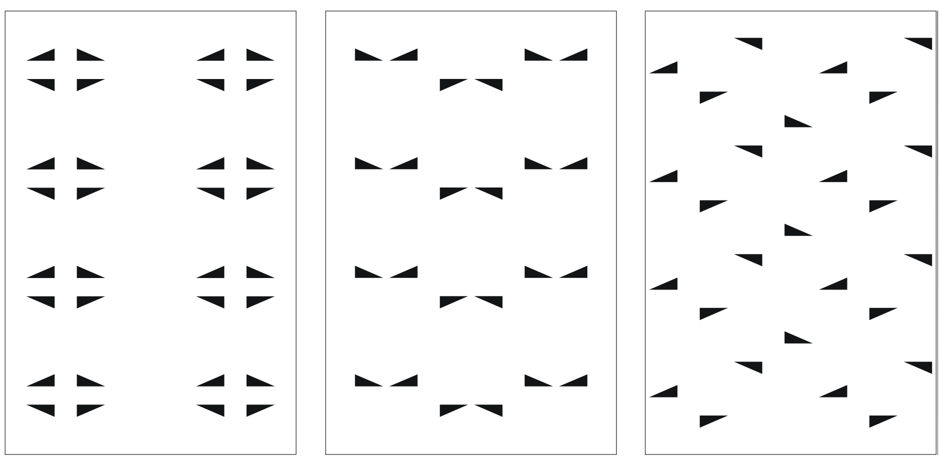
The groups *pmg* and *pmg'* are isomorphic because if

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

the matrix that represents the reflection about the line $y = x$, then conjugation by U is an isomorphism between *pmg* and *pmg*′. This map is not the identity on T, which is why this isomorphism does not give an equivalence between *pmg* and *pmg*′ as group extensions.

We thus have at most three nonisomorphic wallpaper groups in this case, *pmm*, *pmg*, and *pgg*. To see they are pairwise nonisomorphic, we argue first that *pmm* is not isomorphic to either *pmg* or *pgg*, and second that *pmg* and *pgg* are not isomorphic. First, note that *pmm* contains a subgroup of order 4, the subgroup generated by r and the horizontal reflection f. We claim that *pmg* does not contain such a subgroup. Lemmas 2.1 and 2.2 imply that the elements of *pmg* of order 2 are (r, t) for any $t \in T$ and (rf, nt_2) for any $n \in \mathbb{Z}$. However, no product of two of these elements also has order 2. This proves that *pmg* is not isomorphic to *pmm*. Similarly, the only elements of *pgg* of order 2 are of the form (r, t) and the product of any two has infinite order. So, *pmm* is not isomorphic to *pgg*. Second, if there is an isomorphism φ from *pmg* to *pgg*, Corollary 3.8 shows that there is a matrix $U \in \mathrm{Gl}_2(\mathbb{Z})$ with $\varphi(g, t) = (UgU^{-1}, Ut)$ for all $(g, t) \in pmg$. From the description above of elements of order 2 in *pmg* and *pgg*, $(rf, \mathbf{0})$ is sent to an element of the form (r, t) for some $t \in T$. This forces $UrfU^{-1} = r$, a contradiction since $\det(rf) = -1$ and $\det(r) = 1$. Therefore, *pmg* and *pgg* are not isomorphic.

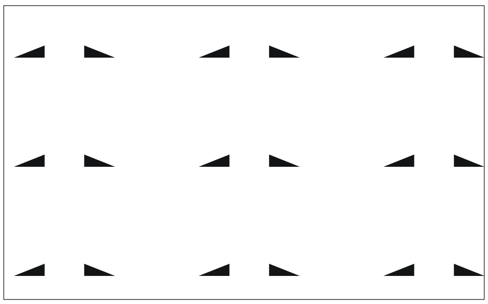
The following three pictures have groups *pmm*, *pmg*, and *pgg*, respectively.



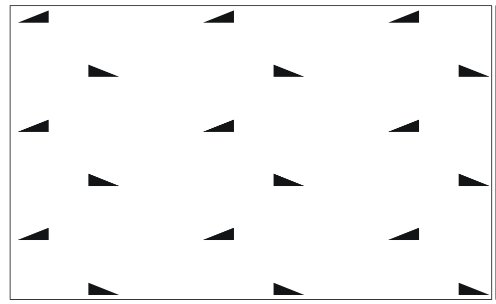
pmm *pmg* *pgg*

Figure 5.2: Wallpaper patterns with groups *pmm*, *pmg*, and *pgg*

The final case to consider is $G_0 = D_{1,p}$, which is generated by a reflection f. There is a basis $\{t_1, t_2\}$ for T with $f(t_1) = t_1$ and $f(t_2) = -t_2$.The condition $r(u) - u \in T$ does not say anything since $r = \mathrm{id}$. However, by considering the cocycle value $c(f, f) = t_f + f(t_f) - t_{\mathrm{id}}$, if we set $u = t_f$, we see that $u + f(u) \in T$. Therefore, modulo T, two possibilities are $u = \mathbf{0}$ and $u = \frac{1}{2}t_1$. Note that we obtain the same cocycle for $u = \frac{1}{2}t_1$ as with the choice $u = \frac{1}{2}t_1 + \beta t_2$ for any $\beta \in \mathbb{R}$. The independence of the group on the choice of β will be addressed in Section 5.5 below. There are two inequivalent group extensions with this point group. The two resulting groups are not isomorphic. This follows from the fact that the semidirect product of G_0 with T contains a subgroup of order 2, namely G_0. However, the other group is generated by t_1, t_2, and $g = (f, \frac{1}{2}t_1)$. In this group there are no elements of finite order, since $(f, \frac{1}{2}t_1 + t)^2 = (I, t + t_1 + f(t))$ for any $t \in T$, and $t_1 + t + f(t) \neq 0$ for any $t \in T$. The following pictures illustrate the two groups, *pm* and *pg*, respectively.



pm



pg

Figure 5.3: Wallpaper patterns with groups *pm* and *pg*

We now list explicit descriptions of the seventeen groups, given by generators, and we give two pictures of wallpaper patterns, one simple picture constructed from triangles, and one Escher tessellation, except for the case $G = pm$, since Escher did not draw a picture with symmetry group *pm*.

One thing to notice about these wallpaper groups is that those groups that contain a nontrivial glide reflection are exactly those that correspond to nontrivial group extensions. That is, the groups that contain a nontrivial glide reflection correspond to nonzero elements in a cohomology group $H^2(G_0, T)$. We remark further about this point in Section 5.5.

5.2 Classification Without Cohomology

In this section we show how to classify wallpaper groups without cohomology. We do make full use of the results from Chapter 3 but we do not need anything from Chapter 4. There is a fair amount of overlap with Section 5.1, but we repeat many ideas here to remove any reference to cohomology.

In this section, as in the previous section, we will use systematically the notation (g, t_g) to denote an element of a wallpaper group. The element t_g need not be an element of T. For example, if we view *pg* as the symmetry group of Escher's *Horseman* on Page 5, then it contains a glide reflection of the form $(f, \frac{1}{2}t_1)$ with neither $(f,0)$ nor the translation component $(I, \frac{1}{2}t_1)$ an element of *pg*.

To determine a wallpaper group G we need to know G_0 and to determine the possible choices of the elements $\{t_g\}_{g \in G}$. We will do so by considering the different possibilities of

G_0. Before we do this, we note some restrictions on the t_g. First, if $g, h \in G_0$, then

$$(g, t_g)(h, t_h) = (gh, g + g(t_h)).$$

Since (gh, t_{gh}) is another element of G mapping to gh under φ, this forces

$$g + g(t_h) - t_{gh} \in T. \tag{5.1}$$

Conversely, if we have, for each $g \in G_0$, an element $t_g \in \mathbb{R}^2$ such that Equation 5.1 is satisfied, then the set $\{(g, t + t_g) : g \in G_0, t \in T\}$ is a wallpaper group with translation lattice T and point group G_0; we leave this as a trivial exercise. For example, if $t_g = \mathbf{0}$ for all g, then Equation 5.1 is clearly satisfied, and the corresponding group is $G = \{(g, t) : g \in G_0, t \in T\}$. In this case G_0 is actually isomorphic to the subgroup $\{(g, \mathbf{0}) : g \in G_0\}$ of G. We refer to this case as the *split wallpaper group* with point group G_0 (and translation lattice T), since G is then the semidirect product of T and G_0, which corresponds to the split group extension of T by G_0, hence the terminology.

Next, we consider how the t_g may change by considering an isomorphic group. If $s \in \mathbb{R}^2$ is fixed, consider the inner automorphism $\sigma : \mathrm{Isom}(\mathbb{R}^2) \to \mathrm{Isom}(\mathbb{R}^2)$ given by $(g, t) \mapsto (I, s)(g, t)(I, s)^{-1}$. This map is given by the formula

$$\begin{aligned}(I, s)(g, t)(I, s)^{-1} &= (I, s)(g, t)(I, -s) = (g, s + t)(I, -s) \\ &= (g, s + t - g(s)).\end{aligned}$$

If we have a wallpaper group G corresponding to $\{t_g\}_{g \in G_0} \subseteq \mathbb{R}^2$, then $\sigma(G)$ is a wallpaper group isomorphic to G and corresponding to $\{(s + t_g - g(s))\}_{g \in G_0}$. Therefore, given two groups G and G' corresponding to $\{t_g\}$ and $\{t'_g\}$, respectively, then G and G' are isomorphic if there is an $v \in \mathbb{R}^2$ with

$$t'_g = t_g + v - g(v) \tag{5.2}$$

for all $g \in G_0$. While the map σ appears to be a very special type of isomorphism, it will be general enough for our purposes. For example, if $t_g = v - g(v) = (I - g)(v)$ for all $g \in G_0$, then G is isomorphic to the split wallpaper group with point group G_0. Since each t_g is only determined up to addition by an element of T, we see that G is a split wallpaper group if there is a $v \in \mathbb{R}^2$ with

$$t_g \equiv v - g(v) \pmod{T} \tag{5.3}$$

for all $g \in G_0$.

We have seen that there are thirteen possible point groups; therefore, we have thirteen split wallpaper groups. We will see that there are four more wallpaper groups once we finish the classification.

For those who wish to see how these ideas relate to the use of cohomology, the function $(g, h) \mapsto g + g(t_h) - t_{gh} \in T$ is precisely a 2-cocycle representing the group extension G, and Equation 5.2 says that two cocycles representing G differ by a 2-coboundary. Therefore, what we are doing in this section is redeveloping the connection between group extensions and cohomology for wallpaper groups.

Proposition 5.2. *Let G be a wallpaper group with point group C_n for some n. Then G is isomorphic to a split wallpaper group.*

Proof. The argument for the case $n = 1$ is different from the general argument. If $n = 1$, then $G/T = 0$, so $G = T$ is split. Thus, we only need to consider the case $G_0 = C_n$ with $n > 1$. Let r be a generator of C_n. With notation above, for each i with $1 \leq i < n$, there are $t_{r^i} \in \mathbb{R}^2$ with $(r^i, t_{r^i}) \in G$. For ease of notation, we set $u = t_r$. Note that, by induction, we have

$$(r, u)^i = (r^i, u + r(u) + \cdots + r^{i-1}(u)).$$

Since t_{r^i} is uniquely determined up to addition by an element of T, we may assume that $t_{r^i} = u + r(u) + \cdots + r^{i-1}(u)$. Now, since r is a nontrivial rotation, it fixes a unique point. As a consequence, $I - r$ is invertible, so there is a vector v with $u = v - r(v)$. Another simple induction argument yields $t_{r^i} = v - r^i(v)$. Therefore, by the argument preceding Equation 5.3, we see that G is split. □

This proposition also gives information about wallpaper groups with point group D_n.

Corollary 5.3. *Let G be a wallpaper group with point group D_n, and let $\varphi : G \to G_0 = D_n$ be the canonical homomorphism. Then $\varphi^{-1}(C_n)$ is a split wallpaper group, and if $f \in D_n - C_n$ with $(f, t_f) \in G$, then G is generated by $\varphi^{-1}(C_n)$ and (f, t_f).*

Proof. The group $\varphi^{-1}(C_n)$ contains $\ker(\varphi) = T$, so the translation subgroup of $\varphi^{-1}(C_n)$ is T, and thus it is a wallpaper group. It is split by Proposition 5.2. Finally, the group generated by $\varphi^{-1}(C_n)$ and (f, t_f) contains T and maps onto D_n; thus it must be G. □

We use this corollary to determine wallpaper groups with point group D_n. We write $D_n = \langle r, f \rangle$ with relations $r^n = e = f^2$ and $frf = r^{-1}$. Let G be a wallpaper group with point group D_n. As a consequence of the corollary, we may choose $t_{r^i} = \mathbf{0}$ for all i. Set $u = t_f$. Since $(r, \mathbf{0})(f, u) = (r^i f, r^i(u))$, we may choose $t_{r^i f} = r^i(u)$. Therefore, the choice of u completely determines G. One or two special cases of Equation 5.1 will allow us to determine u. First, if $g = h = f$, then Equation 5.1 yields

$$u + f(u) \in T \tag{5.4}$$

since $t_{f^2} = t_I = \mathbf{0}$. Next, with $g = rf$ and $h = r$, we have

$$(rf, r(u))(r, \mathbf{0}) = (rfr, r(u)) = (f, r(u)).$$

Therefore,

$$r(u) - u \in T. \tag{5.5}$$

We will use Equations 5.3, 5.4, and 5.5 to describe all wallpaper groups with a dihedral point group.

Consider $G_0 = D_{1,p}$. Then $G_0 = \langle f \rangle$, and the only condition we need to consider is $u + f(u) \in T$. We saw in Section 3.2 that T has a basis $\{t_1, t_2\}$ with $f(t_1) = t_1$ and

$f(t_2) = -t_2$. Let $u = \alpha t_1 + \beta t_2$ with $\alpha, \beta \in \mathbb{R}$. Because u is only determined up to addition by an element of T, we may modify α and β to assume that $0 \leq \alpha, \beta < 1$. We have $u + f(u) = 2\alpha t_1$. For this to be an element of T we must have $2\alpha \in \mathbb{Z}$, so $\alpha = 0$ or $\alpha = \frac{1}{2}$. Therefore, there are two wallpaper groups with point group $D_{1,r}$; one is the split group and the other corresponds to the choice of $u = \frac{1}{2}t_1 + \beta t_2$. Note that we have no restriction on β. In fact, different choices of β yield the same group, up to isomorphism. For, if G corresponds to the choice of $u = \frac{1}{2}t_1$ and G' corresponds to the choice of $u = \frac{1}{2}t_1 + \beta t_2$ for any β, then $G' \cong G$ via the isomorphism given by conjugation by $(I, \frac{1}{2}\beta t_2)$, as an exercise shows. We discuss this further in Section 5.5.

Next, we consider $G_0 = D_{1,c}$. The difference with $D_{1,p}$ is that for $D_{1,c}$, the lattice T has a basis $\{t_1, t_2\}$ with $f(t_1) = t_2$ and $f(t_2) = t_1$. As above, we write $u = t_f$. If $u = \alpha t_1 + \beta t_2$, then $u + f(u) = (\alpha + \beta)(t_1 + t_2)$. For this to be an element of T, we must have $\alpha + \beta \in \mathbb{Z}$. We may choose $\beta = -\alpha$ since u is uniquely determined modulo T. Then $u = \alpha(t_1 - t_2)$. The choice of α does not affect the group since if G corresponds to $\alpha = 0$, so $u = \mathbf{0}$, and if G' corresponds to another choice of α, then $G' \cong G$ via the isomorphism given by conjugation by $(I, \alpha t_1)$. Thus, G is split, as we see by taking $\alpha = 0$, so $u = \mathbf{0}$.

We next consider $D_{3,l}$ and $D_{3,s}$ simultaneously. For both cases T has a basis $\{t_1, t_2\}$ such that the rotation r of $120°$ satisfies $r(t_1) = t_2$ and $r(t_2) = -t_1 - t_2$. If f is any reflection, Equation 5.5 forces $r(u) - u \in T$, where $u = t_f$. If $u = \alpha t_1 + \beta t_2$, then

$$r(u) - u = \alpha t_2 + \beta(-t_1 - t_2) - (\alpha t_1 + \beta t_2) = -(\alpha + \beta)t_1 + (\alpha - 2\beta)t_2.$$

Thus, $r(u) - u \in T$ when $\alpha + \beta \in \mathbb{Z}$ and $\alpha - 2\beta \in \mathbb{Z}$. If we restrict β to the range $0 \leq \beta < 1$, then these two conditions force $\beta = 0, \frac{1}{3}, \frac{2}{3}$. Then α is determined uniquely, modulo $\mathbb{Z}$, as $-\beta$. Thus, we have three possible choices for u, either $\mathbf{0}$, $\frac{1}{3}(-t_1 + t_2)$, or $\frac{2}{3}(-t_1 + t_2)$. We wish to show that all three choices produce isomorphic wallpaper groups. To do this we use Equation 5.3. We point out that with $t_{r^i} = \mathbf{0}$ and $t_{r^i f} = r^i(u)$, Equation 5.3 is satisfied if and only if there is a $v \in \mathbb{R}^2$ with $r(v) \equiv v (\operatorname{mod} T)$ and $u \equiv v - f(v) (\operatorname{mod} T)$. We show that $v = -u$ satisfies these conditions. Since $r(u) - u \in T$, we see that $r(v) \equiv v (\operatorname{mod} T)$. Next, $v - f(v) = -u + f(u) \equiv 2u (\operatorname{mod} T)$ since Equation 5.4 yields $f(u) \equiv u (\operatorname{mod} T)$. Thus, $u - (v - f(v)) \equiv 3u (\operatorname{mod} T)$. In all three possibilities for u, we have $3u \equiv \mathbf{0} (\operatorname{mod} T)$. Thus, Equation 5.3 shows that a wallpaper group with point group $D_{3,l}$ or $D_{3,s}$ is split.

For $G_0 = D_6$ our argument is similar. Here $D_6 = \langle r, f \rangle$, there is a basis $\{t_1, t_2\}$ with $r(t_1) = t_1 + t_2$ and $r(t_2) = -t_1$. Therefore, if $t_f = u = \alpha t_1 + \beta t_2$, then

$$r(u) - u = \alpha(t_1 + t_2) - \beta t_1 - (\alpha t_1 - \beta t_2) = -\beta t_1 + (\alpha + \beta)t_2.$$

For $r(u) - u \in T$, we must have $\beta \in \mathbb{Z}$ and $\alpha + \beta \in \mathbb{Z}$. Thus, both $\alpha, \beta \in \mathbb{Z}$, so any choice of α and β yields the same group, up to isomorphism, as the choice $\alpha = \beta = 0$. Therefore, G is the split wallpaper group with point group D_6.

Let $G_0 = D_4$. There is a basis $\{t_1, t_2\}$ for T with $r(t_1) = t_2$ and $r(t_2) = -t_1$. Set $u = t_f = \alpha t_1 + \beta t_2$. The condition $r(u) - u \in T$ says $(\alpha t_2 - \beta t_1) - (\alpha t_1 + \beta t_2) \in T$, or

$(-\alpha - \beta)t_1 + (\alpha - \beta)t_2 \in T$. In other words, $\alpha + \beta \in \mathbb{Z}$ and $\alpha - \beta \in \mathbb{Z}$. Therefore, with $0 \le \alpha, \beta < 1$, we have the solutions $\alpha = \beta = 0$ and $\alpha = \beta = \frac{1}{2}$. So, either $u = \mathbf{0}$ or $u = \frac{1}{2}(t_1 + t_2)$. We then have two wallpaper groups with point group D_4.

The final point group to consider is D_2. We have two cases. First, let $G_0 = D_{2,c}$. Then T has a basis $\{t_1, t_2\}$ such that $f(t_1) = t_2$ and $f(t_2) = t_1$. The 180° rotation r satisfies $r(t) = -t$ for all $t \in T$. Let $u = t_f = \alpha t_1 + \beta t_2$ as above. The condition $r(u) - u \in T$ and $0 \le \alpha, \beta < 1$ forces $\alpha = 0, \frac{1}{2}$ and $\beta = 0, \frac{1}{2}$. Since $f(u) + u = (\alpha + \beta)(t_1 + t_2)$, this says $\alpha + \beta \in \mathbb{Z}$. Therefore, $u = \mathbf{0}$ or $u = \frac{1}{2}(t_1 + t_2)$. Remembering that we can modify u by adding an element of T, in the second case we may replace u by $u - t_2$ to assume that $u = \frac{1}{2}(t_1 - t_2)$. Then the two groups obtained by the two choices of u are isomorphic, since $\frac{1}{2}(t_1 - t_2) = (I - f)(\frac{1}{2}t_1)$.

Finally, let $G_0 = D_{2,p}$. Then T has a basis $\{t_1, t_2\}$ with $f(t_1) = t_1$ and $f(t_2) = -t_2$. As with the previous case, $r(t) = -t$ for all $t \in T$. If $u = \alpha t_1 + \beta t_2$, then $r(t) - t \in T$ says that $\alpha = 0, \frac{1}{2}$ and $\beta = 0, \frac{1}{2}$. The condition $f(u) + u \in T$ gives no further restriction. Therefore, we have four possibilities:

$$
\begin{aligned}
u &= \mathbf{0}, \\
u &= \frac{1}{2}t_1, \\
u &= \frac{1}{2}t_2, \\
u &= \frac{1}{2}(t_1 + t_2).
\end{aligned}
$$

We claim that these four cases yield three nonisomorphic groups. Instead of repeating the argument, we refer the reader to Section 5.1; this claim is verified there, and no references to cohomology are used in the argument.

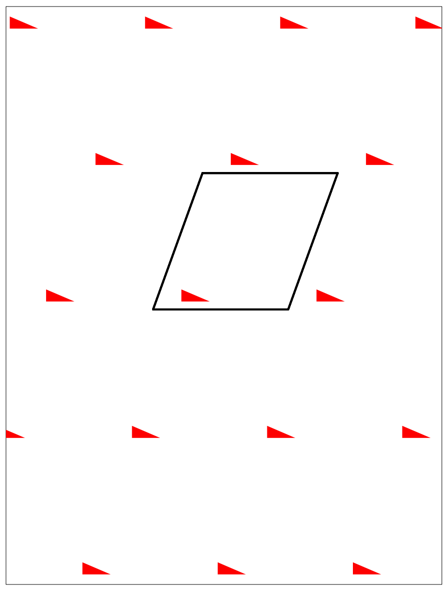
5.3 Description of the Wallpaper Groups

In this section we describe each wallpaper group in terms of generators and relations along with giving two wallpaper patterns whose symmetry group is the given group. Throughout we write $\{t_1, t_2\}$ for a basis of the translation lattice T, described for each lattice type as in Section 3.2. We recall the labelling of the groups, organized according to the lattice type.

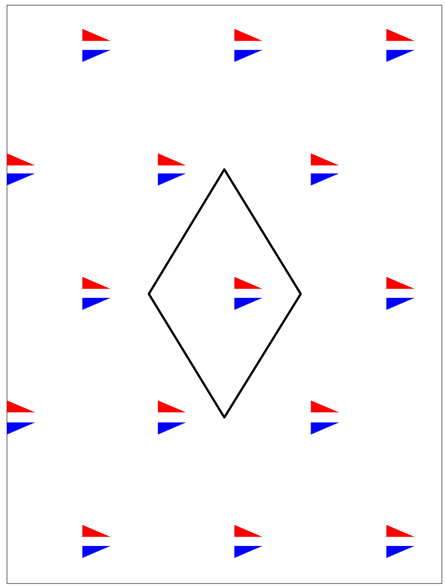
Parallelogram	**Rectangular**	**Rhombic**	**Square**	**Hexagonal**
p1 *p2*	*pm* *pg* *pmm*	*cm* *cmm*	*p4* *p4m* *p4g*	*p3* *p3m1* *p31m*
	pmg			*p6*
	pgg			*p6m*

Table 5.2: The 17 Wallpaper Groups

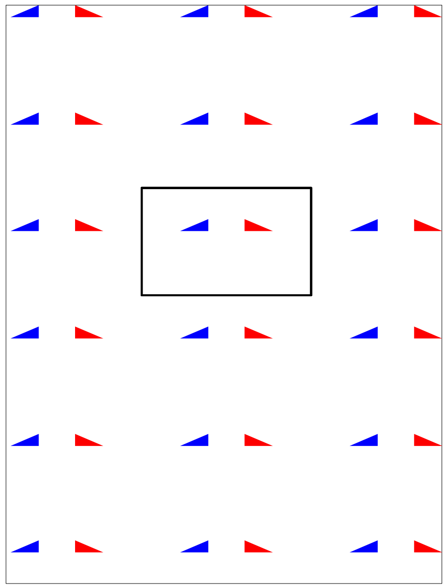
p1. Generators t_1, t_2; point group C_1.

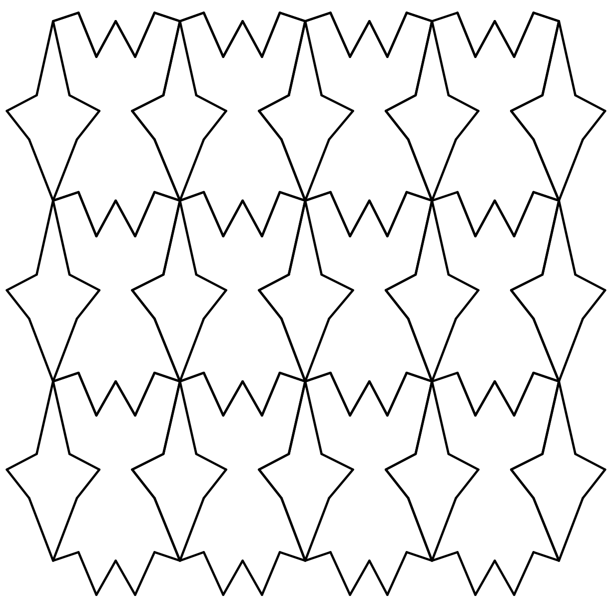


cm. Generators $t_1, t_2, \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0 \right)$; point group $D_{1,c}$.

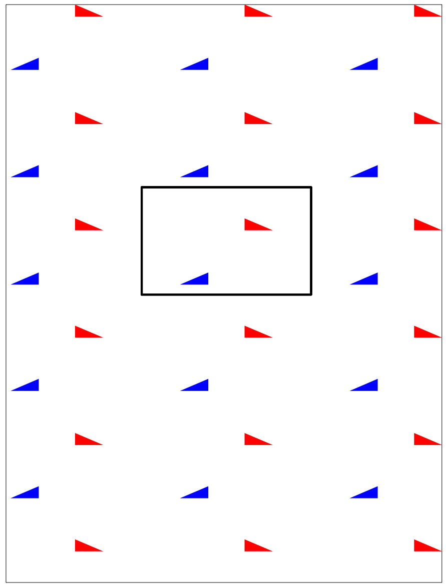


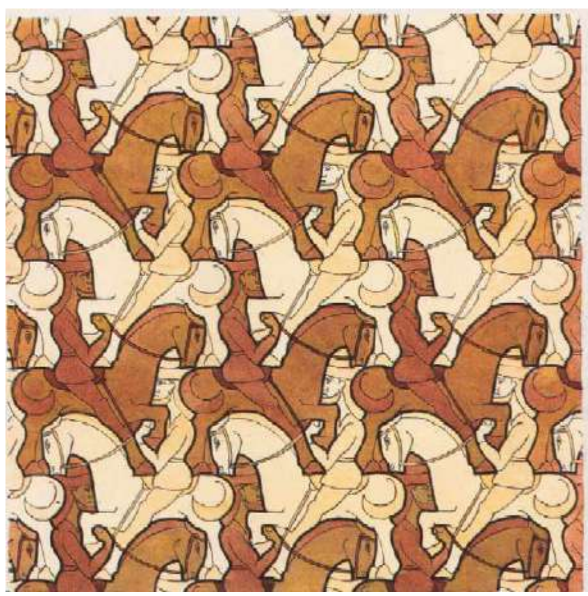
pm. Generators $t_1, t_2, \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0\right)$; point group $D_{1,p}$.



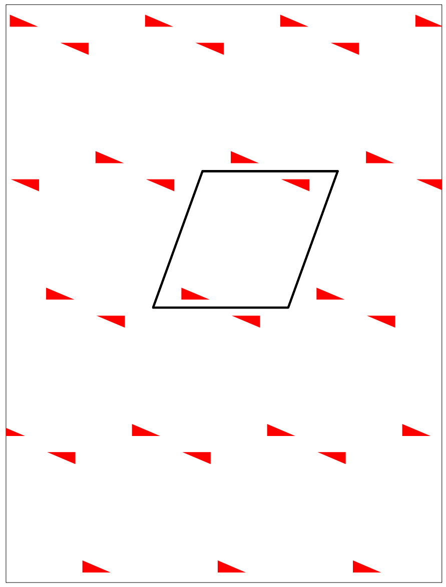


pg. Generators $t_1, t_2, \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2}t_1\right)$; point group $D_{1,p}$.

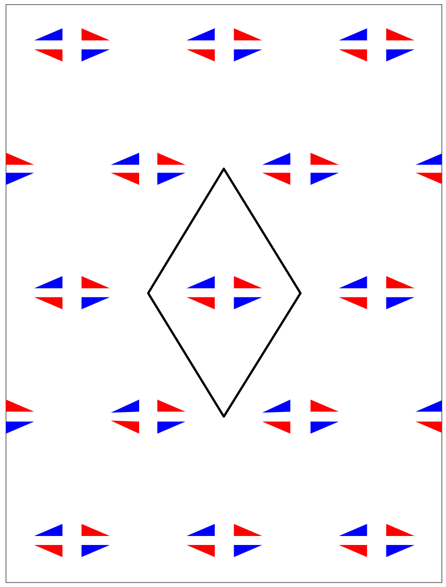


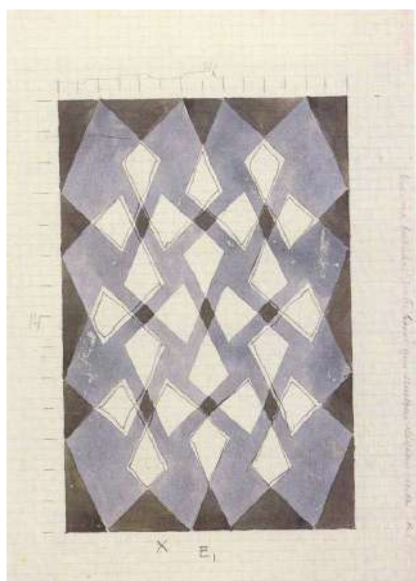


p2. Generators $t_1, t_2, \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 0\right)$; point group C_2.

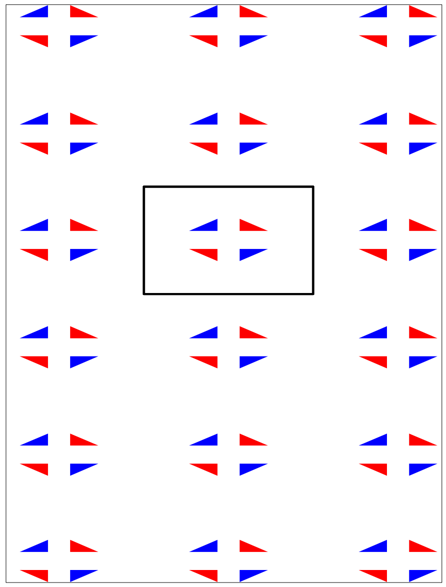


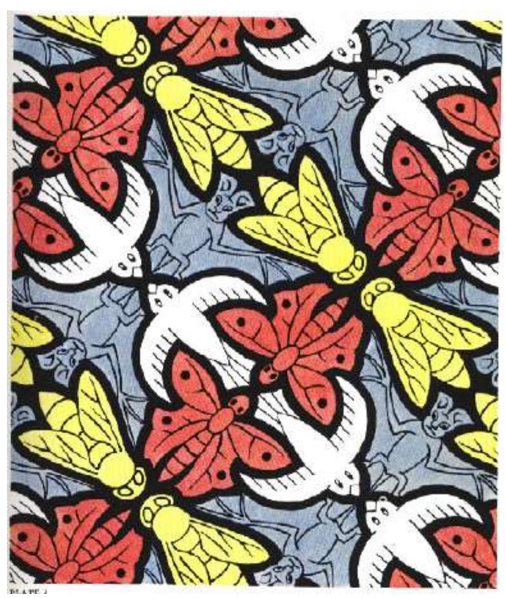
cmm. Generators $t_1, t_2, \left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 0\right), \left(\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}, 0\right)$; point group $D_{2,c}$.



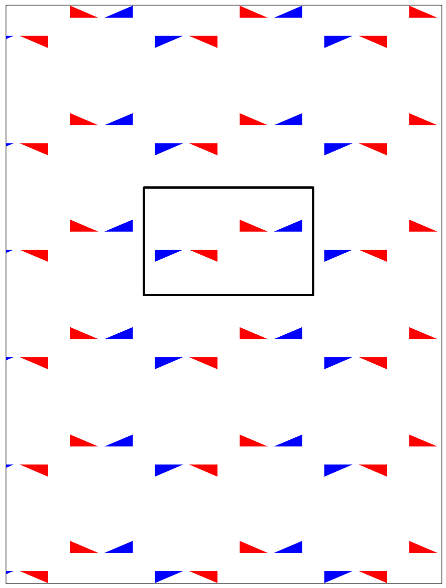


pmm. Generators $t_1, t_2, \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 0\right), \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0\right)$; point group $D_{2,p}$.

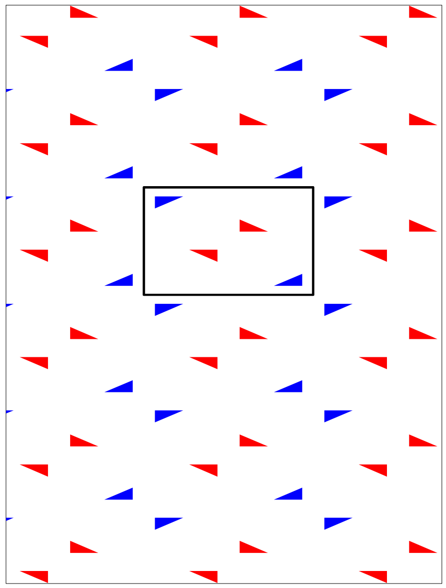


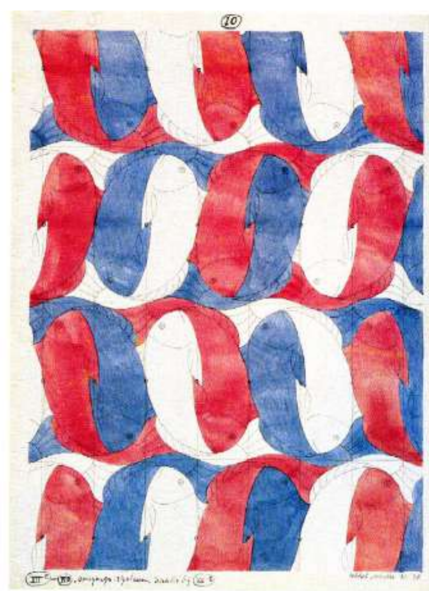


pmg. Generators $t_1, t_2, \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 0\right), \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2}t_1\right)$; point group $D_{2,p}$.

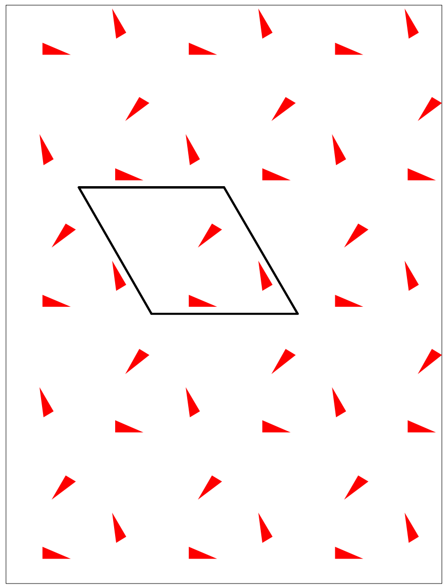


pgg. Generators $t_1, t_2, \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, 0\right), \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2}(t_1 + t_2)\right)$; point group $D_{2,p}$.

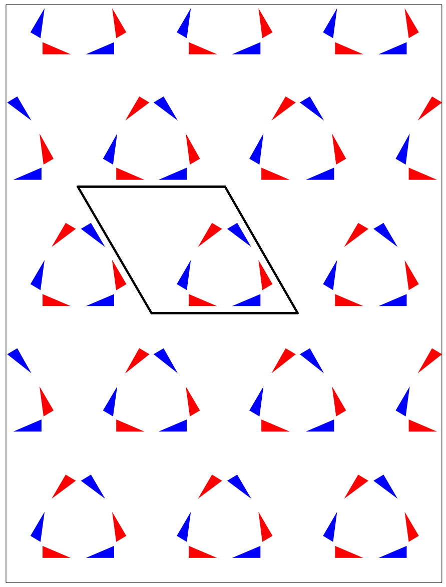




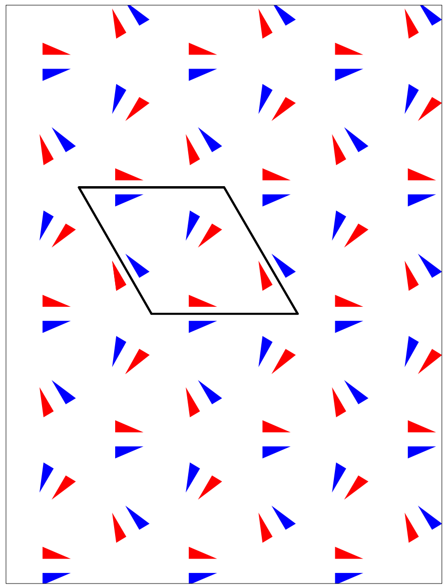
p3. Generators $t_1, t_2, \left(\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, 0 \right)$; point group C_3.



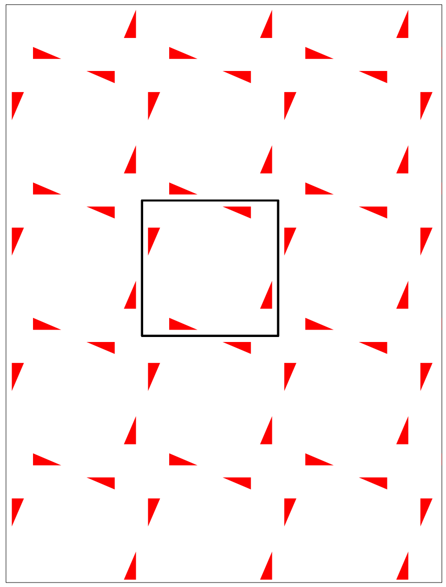
p3m1. Generators $t_1, t_2, \left(\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, 0\right), \left(\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, 0\right)$; point group $D_{3,l}$.

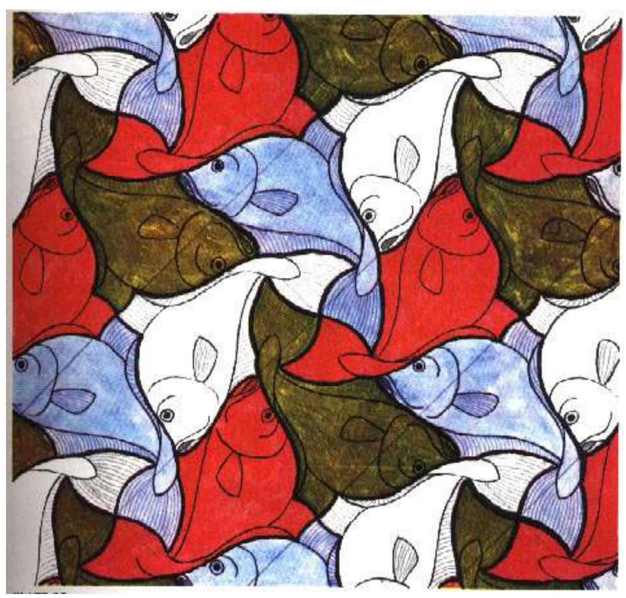


p31m. Generators $t_1, t_2, \left(\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, 0\right), \left(\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, 0\right)$; point group $D_{3,s}$.

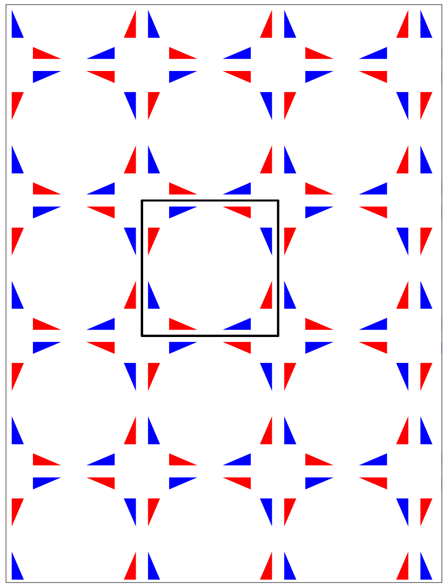


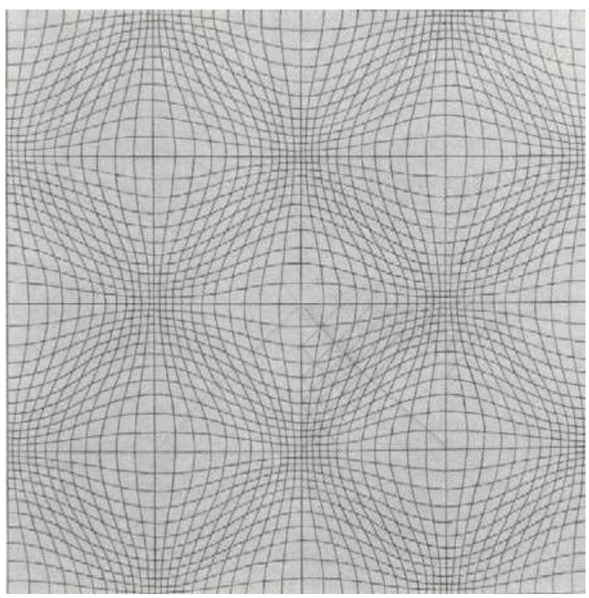
p4. Generators $t_1, t_2, \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0\right)$; point group C_4.



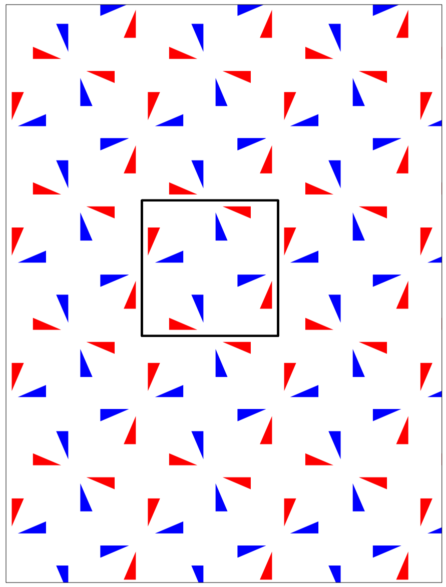


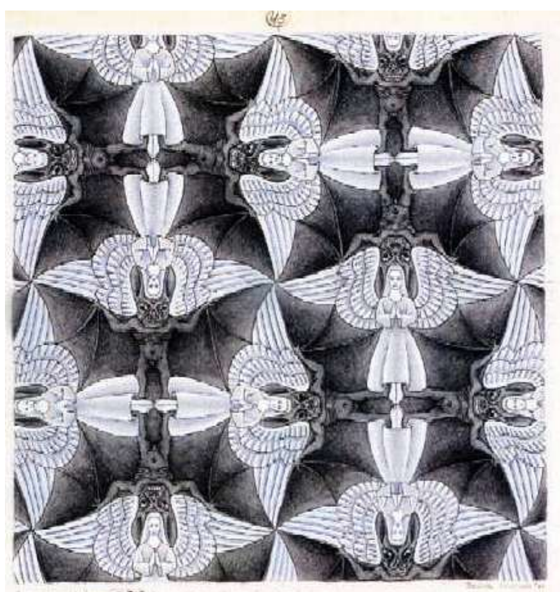
p4m. Generators $t_1, t_2, \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0\right), \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, 0\right)$; point group D_4.



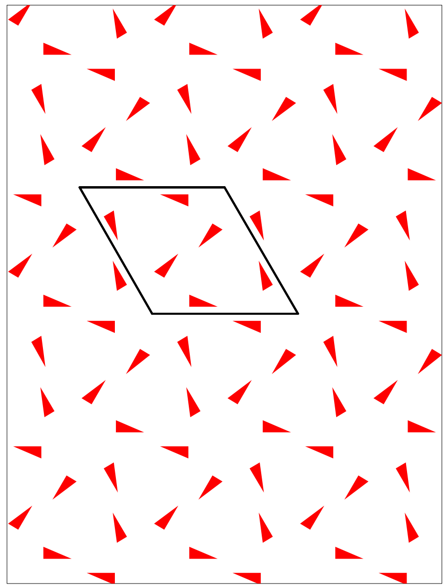


p4g. Generators $t_1, t_2, \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, 0\right), \left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \frac{1}{2}(t_1 + t_2)\right)$; point group D_4.

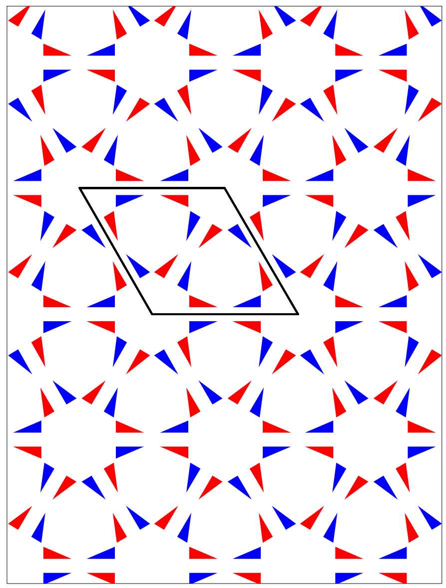


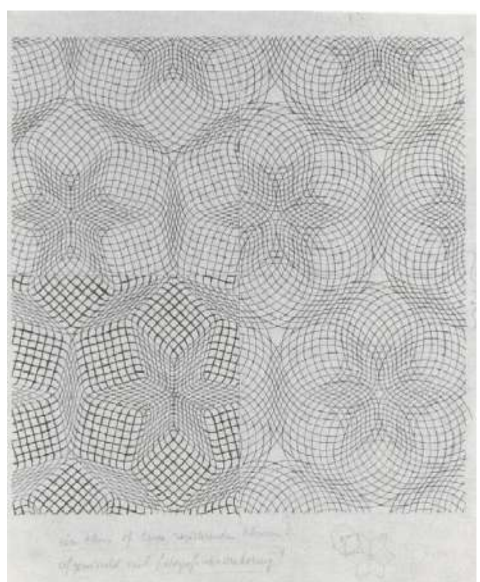


p6. Generators $t_1, t_2, \left(\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, 0 \right)$; point group C_6.



p6m. Generators $t_1, t_2, \left(\begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, 0\right), \left(\begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix}, 0\right)$; point group D_6.

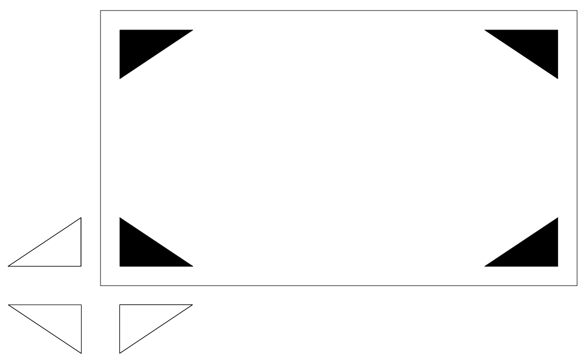




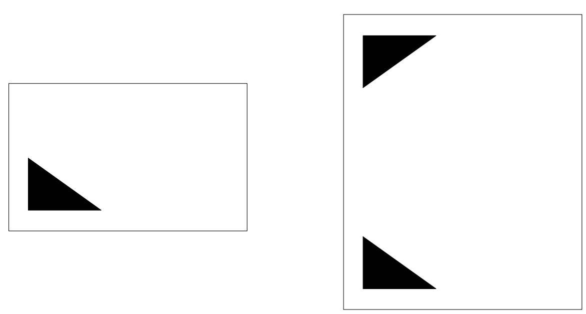
5.4 Fundamental Domains

Throughout this section we continue to write $\{t_1, t_2\}$ for a basis of the translation lattice T, described for each lattice type as in Section 3.2. In any wallpaper pattern there is a parallelogram (with half of its boundary) such that every point in the plane is uniquely obtained as a translation of one point in the parallelogram. This is a *fundamental domain* with respect to the translation subgroup. It is not uniquely determined.

For example, for the group *pmm*, if one considers the bottom left corner of the following fundamental domain to be the origin, then the point group consists of the identity, the 180° rotation about the origin, and horizontal and vertical reflections. If one starts with a triangle in the bottom left and applies the three nonidentity transformations in the point group, then one gets the three unshaded triangles of the following picture. By translating them by t_1, t_2, and $t_1 + t_2$, respectively, they are placed inside the rectangle.



For different subgroups one can get a different fundamental domain. For instance, for the entire group, one would only need one of the four triangles above; every other triangle can be obtained by some isometry in the group. On the other hand, for the subgroup generated by t_1, t_2, and the 180° rotation, one would need two triangles. The following pictures indicate fundamental domains for these two groups.



5.5 Arithmetic Aspects of the Wallpaper Groups

In this section we address some particular aspects of some of the symmetry groups, such as illustrating the differences between wallpaper groups with the same point group. The group-theoretic differences are nicely illustrated with examples of wallpaper patterns. For example, we show how, using our description of the wallpaper groups, how to determine centers of rotations and lines of reflections of various wallpaper patterns.

The first point we make is how to determine, by the wallpaper patterns, when a wallpaper group is not split. Recall that if G has translation lattice T and point group G_0, and if G is split, then $G = \{(g,t) : g \in G_0, t \in T\}$. In particular, if $(g,t) \in G$, then both $(g,\mathbf{0}) \in G$ and $t \in T$. Thus, if a wallpaper pattern with symmetry group G has a glide reflection (f,t) for which $t \notin T$, then G is not split. For example, Escher's *Horseman* on Page 5, the corresponding symmetry group has a nontrivial glide reflection, so it is not a split wallpaper group.

Before we discuss specific groups, we point out two general facts. First, if $r \in D_n$ is a rotation by an angle θ and $f \in D_n$ is a reflection about the line ℓ, then the reflection line of rf is the line ℓ' obtained by rotating ℓ by $\theta/2$. Second, by Lemma 2.2, if a wallpaper group contains a reflection across a line ℓ through the origin, and if v is a vector perpendicular to ℓ, then the group contains reflections with reflection lines $\left\{\ell + \frac{n}{2}v : n \in \mathbb{Z}\right\}$, and that these lines are precisely the reflection lines parallel to ℓ.[1]

p2.

The group *p2* consists of translations and 180° rotations. What are the centers of rotations? Let r be the 180° rotation about the origin. We see that as a consequence of Lemma 2.1 that (r,t) is a 180° rotation about the point $t/2$. Therefore, the set of centers of rotations for 180° rotations in *p2* is the lattice $\left\{\frac{1}{2}t : t \in T\right\}$. Restricting to a fundamental domain, we then have the following picture for the rotation centers.Escher's drawings of tessellations

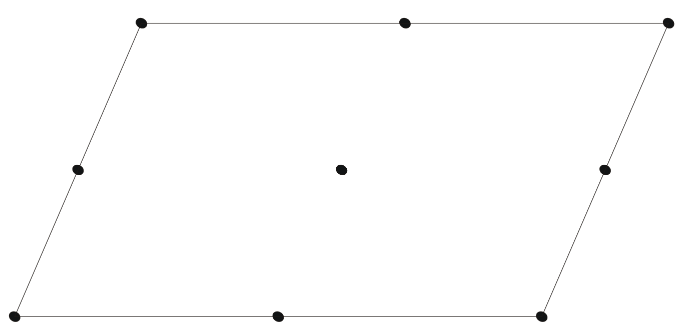


Figure 5.4: *p2* rotation centers

with symmetry group *p2* often show the rotation centers. For example both Camels and Squirrels, shown on Page 6 show all rotation centers.

The same pattern of rotation centers holds for any symmetry group for whom the rotation subgroup of the point group is of order 2, for exactly the same reason as for *p2*. These groups

[1] do we need this fact here?

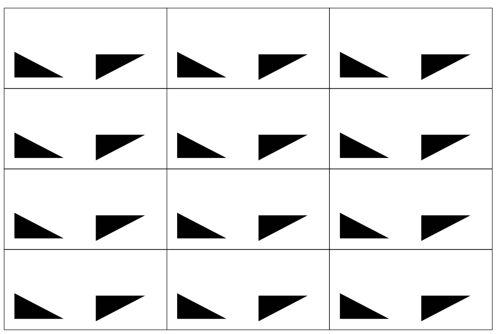
are *pmm*, *pmg*, *pgg*, and *cmm*.

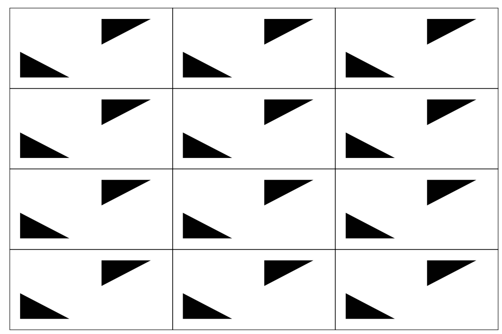
pg.

We next consider the group *pg*. This group is generated by translations and a glide reflection. We may assume that the glide is of the form $(f, \frac{1}{2}t_1 + \alpha t_2) = (f, u)$, where f is a reflection and α is an arbitrary number. We recall from Section 5.1 that the cocycle c representing the group extension for *pg* satisfies

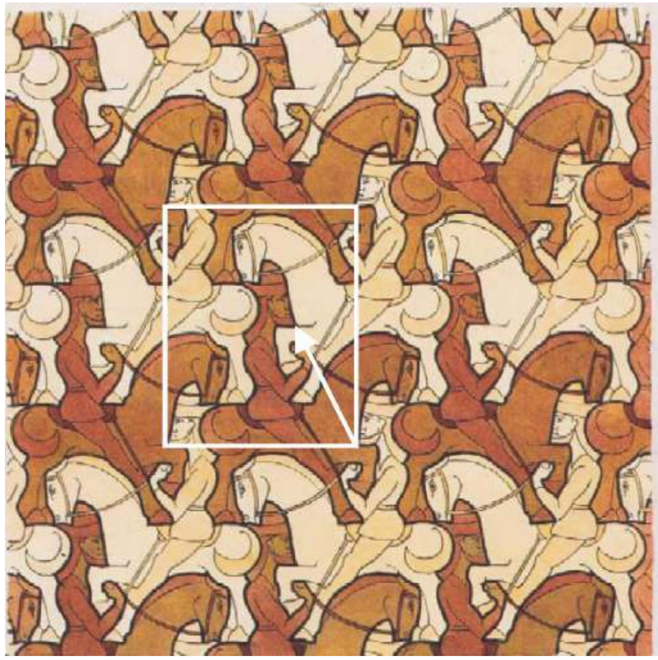
$$\begin{aligned} c(f,f) &= t_f + f(t_f) - t_{\text{id}} \\ &= u + f(u) - \mathbf{0} = t_1; \end{aligned}$$

the vector αt_2 does not show up in this value of the cocycle c. Therefore, it does not affect the equivalence class of the group extension corresponding to c. The picture below shows two wallpaper patterns of type *pg* corresponding to different choices of α.





Escher has several pictures illustrating the symmetry type *pg*. In the following two we also see how the translation component of the basic glide reflection is different in the two pictures. The translation component is marked on the pictures. In both cases the glide is a vertical reflection followed by the indicated translation.



pm, *cm*, *pmm*, and *cmm*.

Wallpaper patterns with symmetry groups *pm* and *cm* (resp. *pmm* and *cmm*) can be distinguished by considering the relationship between reflection lines and a fundamental domain. In Section 3.2, we distinguished a rectangular lattice from a rhombic lattice by finding a basis $\{t_1, t_2\}$ such that, for rectangular lattices, there is a reflection fixing t_1, and in the case of a rhombic lattice, there is a reflection interchanging t_1 and t_2. If we draw a fundamental domain appropriately, for *pm* and *pmm* there will be a reflection line parallel to a side of the domain, while for *cm* and *cmm* there will be a reflection line parallel to one of the diagonals of the domain. For example, in the left picture below, with a fundamental domain superimposed, we see that the diagonals of the domain are reflection lines; thus, the symmetry group of this wallpaper pattern is *cmm*. The picture below on the right has reflection lines parallel to the sides of the domain, so the group of this pattern is *pmm*.

p6 and *p6m*.

The groups *p6* and *p6m* contain 60° rotations, and *p6m* also has 6 basic reflections. In the picture below, we view the center of the hexagon as the origin. For either group the dark circles are rotation centers of 60° rotations, the hollow circles are centers of 120° rotations, and the remaining circles are rotation centers of 180° rotations.To see why this is so, let r

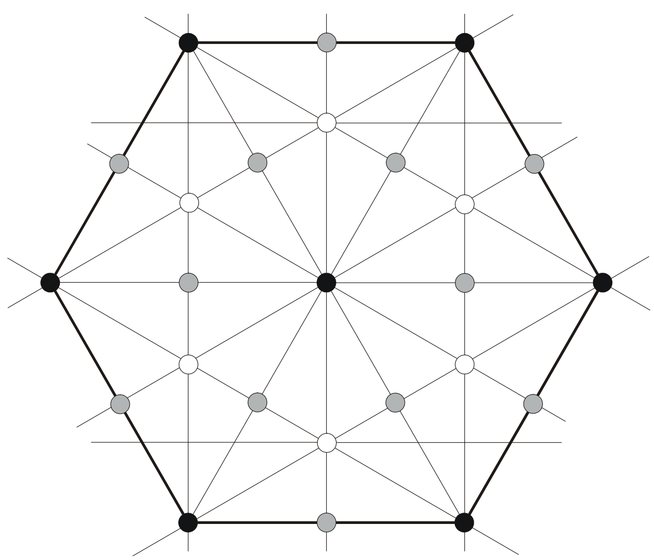


Figure 5.5: *p6/p6m* rotation centers and reflection lines

be a 60° rotation through the origin. As we noted for *p2*, since r^3 is a 180° rotation, (r^3, t) is a 180° rotation centered at $\frac{1}{2}t$. Now, $\{r^i(t_1) : 1 \leq 1 \leq 6\}$ are the six translation vectors of minimal length in T. Thus, $(r^3, r^i(t_1))$ yield the six 180° rotations centered at the six grey points inside the hexagon. For the six grey points on the hexagon's border, we point out that the vectors $r^i(t_1) + r^{i+1}(t_1)$ lie at angles 30°, 90°, 150°, 210°, 270°, and 330° to the x-axis, and each have length $\sqrt{3}\,\|t_1\|$. Thus, the six vectors $\frac{1}{2}\left(r^i(t_1) + r^{i+1}(t_1)\right)$ yield the grey points on the hexagon's border, and are rotation centers for $(r^3, r^i(t_1) + r^{i+1}(t_1))$.

To find the 60° rotation centers, we point out that P is such a center if $t = P - r(P)$ for some $t \in T$; the point P is the rotation center of (r, t). We wish to find those P with $\|P\| \leq \|t_1\|$. Now, P is determined from t by $P = -(r - I)^{-1}(t)$. However, r satisfies its characteristic equation, which seen to be $x^2 - x + 1 = 0$, and so $r^2 = r - I$. Thus, $P = -r^{-2}(t) = r(t)$ since $r^3 = -1$. If we take the six vectors $r^i(t_1)$ of minimal length in T and apply r, we obtain the six vertices of the hexagon; these then are all of the 60° rotation centers, other than the origin, inside the hexagon.

Finally, for the 120° rotations, if P is a rotation center, then $P = (I - r^2)^{-1}(t)$ for some $t \in T$. By using the matrix for r, we see that

$$(I - r^2)^{-1} = \frac{1}{3}\begin{pmatrix} 3/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 3/2 \end{pmatrix} = \frac{1}{\sqrt{3}}\begin{pmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{pmatrix}.$$

In other words, the linear transformation $(I - r^2)^{-1}$ is rotation by an angle $-60°$ followed by multiplication by $1/\sqrt{3}$. If we take the six corners of the hexagon and apply this transformation, we get the six white circles in the picture above.

The reflection lines for *p6m* consist of the six lines through the center together with the six other lines that appear inside the hexagon. To explain this, let f be the reflection that fixes t_1, the horizontal translation of minimal length. The first six are the reflection lines of $\{r^i f : 1 \leq i \leq 6\}$. To explain the other six reflection lines, we note that $\{r^i(t_1) : 1 \leq 1 \leq 6\}$ are the six translation vectors of minimal length in T. The vector $r^i(t_1)$ is perpendicular to the reflection line of exactly one of the $r^i f$. For example, t_1 is perpendicular to the reflection line of $r^3 f$. Thus, $(r^3 f, t_1)$ is a reflection whose reflection line is the translate of the reflection line of f by the vector $\frac{1}{2}t_1$. This is the rightmost vertical line in the picture above. The other five lines are translates by $\frac{1}{2}r^i(t_1)$ of the appropriate reflection line passing through the center.

p3m1 and *p31m*.

We next consider the groups whose rotation subgroup of the point group is C_3. In addition to considering rotation centers, we investigate the difference between *p3m1* and *p31m*. Perhaps the easiest way to visually distinguish wallpaper patterns with these groups is to consider the relation between lines of reflection and centers of rotation.

We first consider rotation centers. The following wallpaper pattern has symmetry type *p3*. The centers of rotations are indicated in the picture; from the analysis for *p6*, we can conclude that centers of 60° or 120° rotations in *p6* are rotation centers for *p3*. Note that in any parallelogram that represents a fundamental domain for the translation subgroup, the four corners along with the centers of the two triangular halves are centers of rotation. The parallelogram marked in the picture below is a fundamental domain for *p3*.

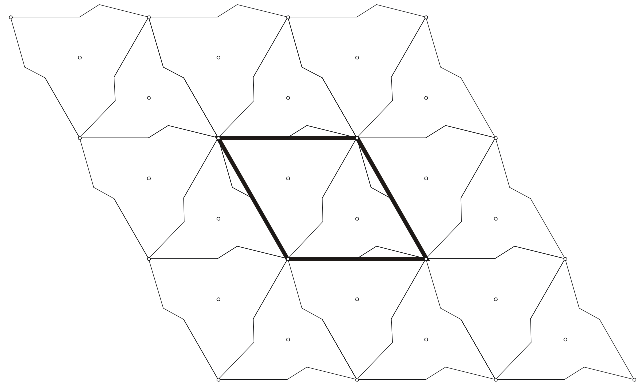


Figure 5.6: *p3* rotation centers

As we saw in Section 3.2, the difference between the symmetry groups *p3m1* and *p31m* is in the choice of reflection lines. The group *p3m1* corresponds to taking the three reflection lines that make angles of 30°, 90°, and 150° with one of the translation vectors, while *p31m* has the three reflection lines at 0°, 60°, and 120° with respect to a translation vector. By taking the picture above and reflecting the picture accordingly, we obtain the figure on the left below, whose symmetry group is *p3m1* and the figure on the right, whose symmetry group is *p31m*.

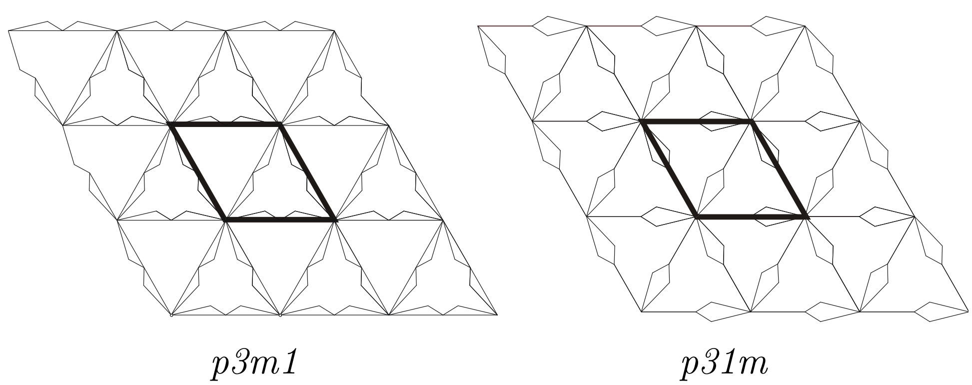


Figure 5.7: Wallpaper patterns with groups *p3m1* and *p31m*

As we can see in the pictures below, the centers of rotation are all on lines of reflection for *p3m1*, while not all centers are on lines of reflection for *p31m*. This fact yields a straightforward way to distinguish between figures whose symmetry groups are either *p3m1* or *p31m*. Another way to distinguish them is to notice that if one draws a basic hexagon, as below, then the reflection lines of *p3m1* never pass through corners of the hexagon while the reflection lines for *p31m* always pass through the corners.

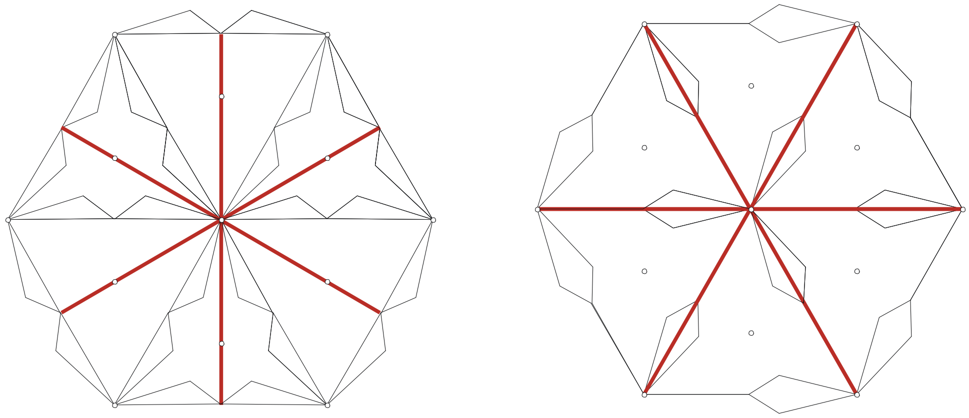


Figure 5.8: *p3m1* and *p31m* reflection lines

p4, *p4m*, and *p4g*.

The groups *p4*, *p4m* and *p4g* are those containing a 90° rotation. We first look at the rotation centers in a fundamental domain, which is a square. We place the origin at the center of one 90° rotation r. All other 90° rotations are of the form (r, t) for some $t \in T$. The center P of (r, t) satisfies $t = P - r(P) = (I - r)(P)$, so $P = (I - r)^{-1}(t)$. By using the matrix representation for r, we have

$$I - r = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix},$$

whose inverse is

$$(I - r)^{-1} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{2}/2 & -\sqrt{2}/2 \\ \sqrt{2}/2 & \sqrt{2}/2 \end{pmatrix}.$$

The point of the final form of $(I - r)^{-1}$ is to see that $(I - r)^{-1}$ is the composition of a 45° rotation followed by multiplication by $1/\sqrt{2}$. If we consider the four rotations $(r, \pm t_1)$, $(r, \pm t_2)$, then the rotation centers are the four corners of the fundamental domain, which are the images under $(I - r)^{-1}$ of $\pm t_1, t_2$. Thus, inside a fundamental domain, we have the center and the four corners as centers of 90° rotations.

To find the rotation centers of 180° rotations, we recall from the investigation of *p2* that the set of such centers is $\{t/2 : t \in T\}$. If we ignore those centers that are also centers of 90° rotations, we are left with the hollow points in the picture above.

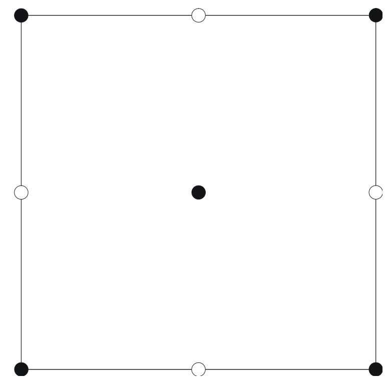


Figure 5.9: *p4* rotation centers

The groups *p4m* and *p4g*, have reflections in addition to 90° rotations. These groups can be distinguished visually by relating lines of reflection to centers of rotation. For *p4m* the group is generated by the translation subgroup and the point group. Therefore, this group contains reflections parallel to the sides of a fundamental domain, which is a square. All centers of rotations consequently lie on reflection lines. The group *p4g* does not contain the point group D_4. However, it does contain reflections in two perpendicular directions. There are centers of rotations that do not lie on reflection lines. The following pictures indicate centers of rotation and lines of reflection.

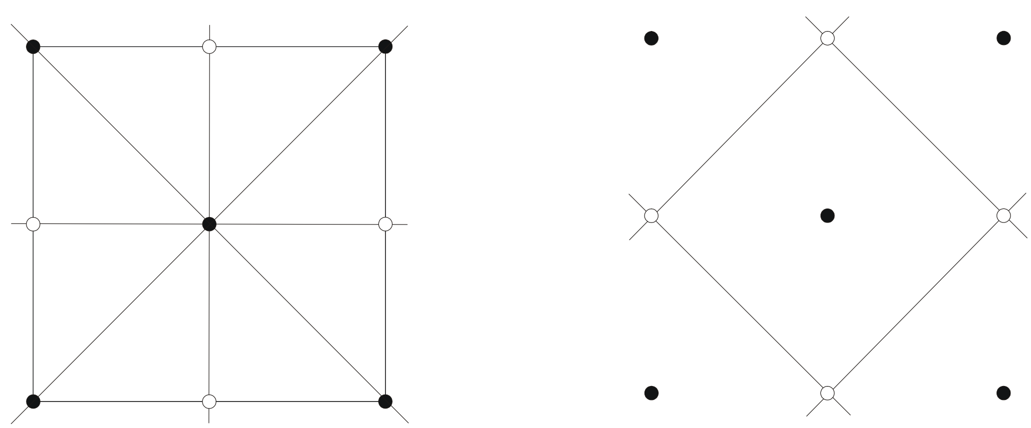


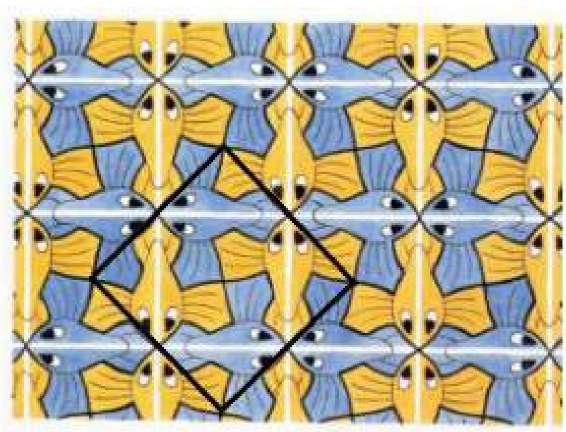
Figure 5.10: *p4m* and *p4g* reflection lines

We see that for *p4m*, all centers of rotation are on reflection lines, while the same is not true for *p4g*. To be more precise, in patterns with group *p4g*, centers of 90° rotations do not lie on rotation lines. To explain these reflection lines, recall that *p4g* is generated by the translations, r, and $g = (h, \frac{1}{2}(t_1 + t_2))$, where h is the reflection about the x-axis. The four reflections in the point group of *p4g* are h, rh, r^2h, r^3h. Therefore, any reflection in *p4g* must be of the form $(r^i h, \frac{1}{2}(t_1 + t_2) + t)$ for some $t \in T$. Recall that $(r^i h, v)$ is a reflection if and only if $r^i h(v) = -v$. To determine the reflections in *p4g*, we must find those $t \in T$ for which

$$r^i h\left(\frac{1}{2}(t_1 + t_2) + t\right) = -\left(\frac{1}{2}(t_1 + t_2) + t\right). \tag{5.6}$$

By writing $t = nt_1 + mt_2$ and recalling that $r(t_1) = t_2$ and $r(t_2) = -t_1$, a short calculation will show that Equation 5.6 holds only when $i = 1$ and $1 + n = -m$, or when $i = 3$ and $n = m$. The reflection line of rh makes a 45° with the x-axis. In the case $i = 1$, if we set $n = 1$, then we see that the reflection line of $(rh, \frac{1}{2}(t_1 + t_2) - t_2) = (rh, \frac{1}{2}t_1 - \frac{1}{2}t_2)$ is obtained by translating the line $y = x$ by the vector $\frac{1}{2}\left(\frac{1}{2}t_1 - \frac{1}{2}t_2\right)$; similarly, with $n = -1$, the reflection line of $(rh, \frac{1}{2}(t_1 + t_2) - t_1)$ is obtained by translating the line $y = x$ by $\frac{1}{2}\left(-\frac{1}{2}t_1 + \frac{1}{2}t_2\right)$. These two lines are the two 45° lines in the picture above. Similarly, the reflection lines of $(r^3 h, \frac{1}{2}(t_1 + t_2))$ and $(r^3 h, \frac{1}{2}(t_1 + t_2) - (t_1 + t_2))$ are the two 135° lines above; these reflections correspond to the choices $n = m = 0$ and $n = m = -1$, respectively.

For example, consider Escher's picture below, with a fundamental domain indicated as a diamond in the center of the picture.



The horizontal and vertical lines are reflection lines but do not represent translations. So, the lattice of lines drawn by Escher does not show the translation lattice.

pmm, *pmg*, and *pgg*.

The point group $D_{2,r}$ yields three nonisomorphic symmetry groups, *pmm*, *pmg*, and *pgg*. One way to distinguish patterns for these groups is to consider reflections. The group *pmm* contains reflections in two non-parallel directions, while *pmg* contains reflections in only one direction. The group *pgg* does not contain any reflections. For example, the leftmost picture below has symmetry group *pmm*, and there are horizontal and vertical reflections of the pattern.

In the center picture, which has symmetry group *pmg*, there are vertical reflections only, so the figure has reflections in only one direction. There is, however, a glide reflection of the

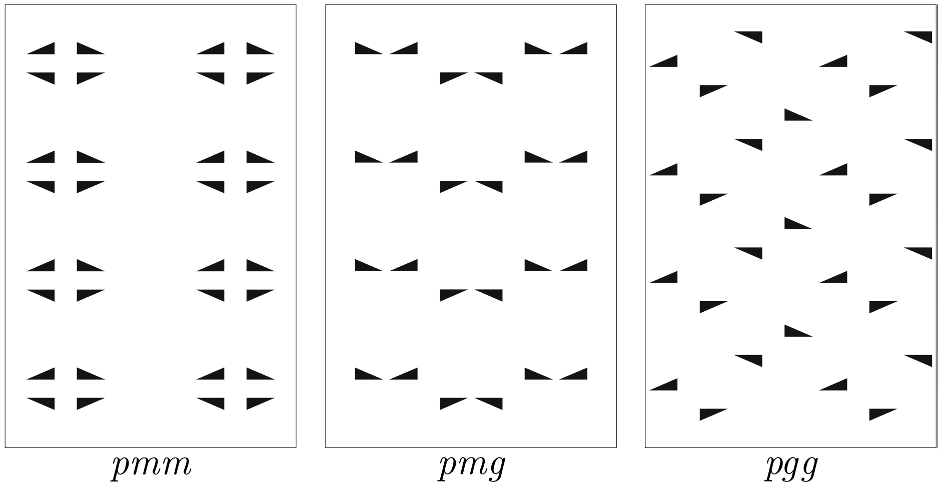


Figure 5.11: Wallpaper patterns with groups *pmm*, *pmg*, and *pgg*

form $(h, \frac{1}{2}t_1)$, where h is a horizontal reflection, and t_1 is the smallest horizontal translation of the figure. If v is a vertical reflection in $D_{2,p}$, then $v = hr$, and so $(v, \frac{1}{2}t_1) = (h, \frac{1}{2}t_1)(r, \mathbf{0}) \in G$. Now, since $\frac{1}{2}t_1$ is perpendicular to the reflection line of v, the isometry $(v, \frac{1}{2}t_1)$ is a vertical reflection.

Finally, in the rightmost figure, which has symmetry group *pgg*, there are no reflections of the pattern. There are glide reflections $(h, \frac{1}{2}(t_1 + t_2))$ and $(v, \frac{1}{2}(t_1 + t_2))$, where h and v are horizontal and vertical reflections, respectively, and $\{t_1, t_2\}$ is a basis consisting of a horizontal and a vertical translation. To see that *pgg* does not contain a reflection, we first point out that any reflection in *pgg* would have to be of the form $(h, \frac{1}{2}(t_1 + t_2) + t)$ or $(v, \frac{1}{2}(t_1 + t_2) + t)$ for some $t \in T$ since these are the only elements whose image in the point group $G_0 = D_2$ is a reflection. If $(h, \frac{1}{2}(t_1 + t_2) + t)$ is a reflection, then $h\left(\frac{1}{2}(t_1 + t_2) + t\right) = -\left(\frac{1}{2}(t_1 + t_2) + t\right)$. A short calculation shows that there is no value of t for which this is true. Similar reasoning shows that $(v, \frac{1}{2}(t_1 + t_2) + t)$ is not a reflection for any t.

Bibliography

[1] I. N. Herstein, *Topics in algebra*, Xerox College Publishing, Lexington, Mass., 1975.

[2] G. Hochschild and J.-P. Serre, *Cohomology of group extensions*, Trans. Amer. Math. Soc. **74** (1953), 110–134.

[3] G. Mackiw, *Applications of abstract algebra*, John Wiley & Sons Inc., New York, 1985.

[4] J. J. Rotman, *An introduction to homological algebra*, Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York, 1979.

[5] D. Schattschneider, *Visions of symmetry*, W. H. Freeman and Company, New York, 1990.

[6] G. Tamme, *Introduction to étale cohomology*, Springer-Verlag, Berlin, 1994.

[7] C. A. Weibel, *An introduction to homological algebra*, Cambridge University Press, Cambridge, 1994.

List of Symbols

Listed in the following table are most of the symbols used in the text, along with the meaning and a page reference for each symbol.

Symbol	**Meaning**	**Page**
$\mathbb{R}$	set of real numbers	3
$\mathbb{R}^n$	space of n-tuples of real numbers	3
$\mathrm{Isom}(\mathbb{R}^n)$	group of isometries of $\mathbb{R}^n$	3
$\mathrm{Sym}(W)$	symmetry group of W	3
$\mathbf{0}$	zero vector	3
$\mathbb{T}$	translation group	3
id	identity function	4
$\mathbb{Z}$	ring of integers	4
$\|u\|$	length of the vector u	12
$u \cdot v$	dot product of u and v	13
A^T	transpose of A	14
I_n	$n \times n$ identity matrix	14
$\mathrm{O}_n(\mathbb{R})$	orthogonal group	14
$\mathrm{Aut}(T)$	automorphism group of T	15
$\det(A)$	determinant of A	16
$\mathrm{SO}_2(\mathbb{R})$	special orthogonal group	16
D_n	dihedral group of order $2n$	16
T	translation lattice of a wallpaper group	19
(A, b)	notation for the isometry $x \mapsto Ax + b$	19
I	identity transformation	19
G_0	point group of a wallpaper group G	19
C_n	cyclic group of order n	20
$\mathrm{Gl}_2(\mathbb{Z})$	general linear group	22
$D_{3,l}$	one representation of D_3	27
$D_{3,s}$	another representation of D_3	27
$D_{1,p}$	D_1 for rectangular lattice	30
$D_{1,c}$	D_1 for rhombic lattice	30
$D_{2,p}$	D_2 for rectangular lattice	31

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