Principal Bundles over Valued Fields

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Summary

Introduction

We start with:

- a topological field k,
- an algebraic k-group G,
- a k-variety Y, and
- a G-torsor (principal G-bundle) $f : X \to Y$ over Y.

Introduction

Taking rational points, we get

- a topological group $G(k)$,
- a continuous free action of $G(k)$ on the space $X(k)$,
- a continuous map $X(k) \to Y(k)$, invariant for this action.

This map is not surjective in general.

We will consider the following questions, in the case of a henselian valued field:

- What does the image I of this map look like, as a subspace of $Y(k)$?
- Is the induced map $X(k) \to I$ a principal $G(k)$-bundle?

Remark: the answers are easy and well known in characteristic zero (and more generally if G is smooth).

Principal bundles in topology

Let G be a topological group. A (left) G-bundle consists of the following data:

- a continuous map $f : X \to Y$,
- a (left) action $G \times X \to X$ commuting with f (i.e. $f(g.x) = f(x)$).

A G-bundle is trivial if it is isomorphic (in the obvious sense) to $G \times Y \xrightarrow{\mathrm{pr}_2} Y$ with the action of G on itself by left translation.

It is principal if it is locally trivial (on Y), in the obvious sense.

Principal bundles in algebraic geometry: torsors

Let k be a field, G an algebraic group over k, and Y a k-variety.

A (left) G-bundle over Y consists of:

- a k-morphism $f : X \to Y$,
- a (left) action of G on X, compatible with f,

We call it a (left) G-torsor if it is locally trivial for the fppf (or flat) topology, i.e. there is a k-morphism $h : Y' \to Y$ such that:

- h is flat and surjective,
- h trivializes f, i.e. the pullback G-bundle $X \times_Y Y' \to Y'$ is trivial.

A simple example

Let n be a positive integer. Consider the n-th power map

$$\begin{aligned} f : \mathbb{G}_{\mathrm{m},k} &\longrightarrow \mathbb{G}_{\mathrm{m},k} \\ x &\longmapsto x^n. \end{aligned}$$

This is a μ_n-torsor (with the obvious action of $\mu_n = \ker(f)$ on $\mathbb{G}_{\mathrm{m},k}$).

If n is invertible in k, then f is even locally trivial for the étale topology, i.e. trivialized by an étale surjective map (e.g. f itself).

More generally, if G is a smooth k-group, any G-torsor $f : X \to Y$ is a smooth morphism, hence locally trivial for the étale topology. This holds in particular if $\mathrm{char}(k) = 0$.

But in our example, if $n = \mathrm{char}(k) > 0$, then f is just the Frobenius map on $\mathbb{G}_{\mathrm{m},k}$.

Characterization of torsors

A G-bundle $f : X \to Y$ in topology (resp. in algebraic geometry) is a G-torsor if and only if:

- it is "formally principal" (or a "pseudo-torsor"), i.e. the natural morphism
$$\begin{array}{ccc} G \times X & \longrightarrow & X \times_Y X \\ (g, x) & \longmapsto & (g.x, x) \end{array}$$
is an isomorphism,
- f has local sections on Y, in the obvious sense (resp. in the flat topology sense).

Characterization of torsors

The "pseudo-torsor" property

$$G \times X \xrightarrow{\sim} X \times_Y X$$

is completely "categorical", and is preserved by any functor on k-varieties that commutes with fiber products, such as the functor of rational points $R : Z \longmapsto Z(k)$.

It follows that if $f : X \to Y$ is a G-torsor over k, then the induced map of sets (or discrete spaces)

$$R(f) : X(k) \longrightarrow Y(k)$$

(which may not be surjective) induces a principal $G(k)$-bundle over its image.

Torsors over topological fields

From now assume that k is a topological field, e.g. a valued field.

For every k-variety Z, the set $Z(k)$ has a natural topology. The resulting topological space will be denoted by Z_{top} (or $Z(k)_{\mathrm{top}}$).

In particular, for a G-torsor $f : X \to Y$:

- G_{top} is a topological group, and
- $f_{\mathrm{top}} : X_{\mathrm{top}} \to Y_{\mathrm{top}}$ is a G_{top}-bundle, in fact automatically a pseudo-torsor.

Torsors over topological fields

Example of the squaring map:

$$\begin{array}{rcl} f : \mathbb{G}_{\mathrm{m},k} & \longrightarrow & \mathbb{G}_{\mathrm{m},k} \\ x & \longmapsto & x^2. \end{array}$$

If $k = \mathbb{R}$, the image of f_{top} is $\mathbb{R}_{>0}$ (open and closed in $\mathbb{R}^\times$), and f_{top} induces a trivial $\{\pm 1\}$-bundle over this image.

If $k = \mathbb{C}$, then f_{top} is surjective and induces a nontrivial principal $\{\pm 1\}$-bundle over $\mathbb{C}^\times$.

If $k = \mathbb{F}_2((t))$, then f_{top} is a homeomorphism onto its image, which is closed in $k^\times$.

Torsors over topological fields

Back to a general G-torsor $f : X \to Y$ over a topological field k:

We can factor $f_{\mathrm{top}} : X_{\mathrm{top}} \to Y_{\mathrm{top}}$ as

$$X_{\mathrm{top}} \underset{\substack{\text{quotient map}\\ \text{(open)}}}{\longrightarrow} X_{\mathrm{top}}/G_{\mathrm{top}} \underset{\substack{\text{continuous}\\ \text{bijection}}}{\longrightarrow} \mathrm{Im}(f_{\mathrm{top}}) \underset{\substack{\text{topological}\\ \text{embedding}}}{\longrightarrow} Y_{\mathrm{top}}$$

which gives rise to natural questions:

Torsors over topological fields

$$X_{\mathrm{top}} \underset{\text{quotient}}{\longrightarrow} X_{\mathrm{top}}/G_{\mathrm{top}} \underset{\text{bijection}}{\longrightarrow} \mathrm{Im}(f_{\mathrm{top}}) \underset{\text{embedding}}{\longrightarrow} Y_{\mathrm{top}}$$

1. Is the image of f_{top} closed (open, locally closed) in Y_{top}?
2. Is the middle bijection a homeomorphism? (In other words, is f_{top} a strict map?)
3. Is $X_{\mathrm{top}} \to X_{\mathrm{top}}/G_{\mathrm{top}}$ a principal G_{top}-bundle?
   Equivalently, does this map have continuous local sections everywhere?

Note that a positive answer to both Questions 2 and 3 is equivalent to a positive answer to

4. Is $X_{\mathrm{top}} \to \mathrm{Im}(f_{\mathrm{top}})$ a principal G_{top}-bundle?

The main result

Definition

A valued field (K, v) is admissible if

- *(K, v) is henselian;*
- *the completion $\widehat{K}$ of K is a separable extension of K.*

Main Theorem

Let (K, v) be an admissible valued field, G an algebraic K-group, and $f : X \to Y$ a G-torsor. Then:

1. *$\mathrm{Im}(f_{\mathrm{top}})$ is locally closed in Y_{top}.*
2. *The induced map $X_{\mathrm{top}} \to \mathrm{Im}(f_{\mathrm{top}})$ is a principal G_{top}-bundle.*

Remark. In some cases, we can say more about $\mathrm{Im}(f_{\mathrm{top}})$:

- it is open and closed in Y_{top} if G is smooth, or if K is perfect;
- it is closed in Y_{top} if G°_{red} is smooth, or if G is commutative.

The case of homogeneous spaces

As an example, we can take for X an algebraic group and for G a subgroup of X, and consider $f : X \to Y := X/G$.

Then the image of f_{top} is the orbit $X_{\mathrm{top}}.y$ (y=origin of Y). The theorem says that

- this orbit is locally closed in Y_{top}, and
- the induced map $X_{\mathrm{top}} \to X_{\mathrm{top}}.y$ is a principal G_{top}-bundle (in particular, $X_{\mathrm{top}}/G_{\mathrm{top}} \to X_{\mathrm{top}}.y$ is a homeomorphism).

When K is a local field, this is due to Bernstein and Zelevinsky (1976).

An example of a non-closed orbit

Assume $\operatorname{char}(K) = p > 0$. Let $S = \mathbb{G}_a \rtimes \mathbb{G}_m$ be the affine group in dimension 1, acting on $X = \mathbb{A}^1_K$ transitively "via Frobenius on S":

$$\begin{array}{ccc} S \times \mathbb{A}^1 & \longrightarrow & \mathbb{A}^1 \\ ((x,y),u) & \longmapsto & (x,y).u := x^p + y^p\, u \end{array}$$

For $u \in K$, consider the orbit morphism

$$f_u : S \to \mathbb{A}^1, \quad s \mapsto s.u\,.$$

This is a torsor under the stabilizer S_u of u.

The image of $f_{u,\mathrm{top}}$ is the orbit $S(K).u = K^p + (K^\times)^p\, u \subset K$. In particular:

- if $u \in K^p$, the orbit is K^p, which is closed in K if K is admissible;
- for any choice of u, the orbit has 0 in its closure (consider the action of $\mathbb{G}_m$).

Hence, if $u \notin K^p$, then $\mathrm{Im}(f_{u,\mathrm{top}})$ is not closed in K.

Notation and conventions

- R: a valuation ring,
- $K = \mathrm{Frac}\,(R)$,
- v: the valuation,
- $\widehat{K}$: completion of K,
- K-variety = K-scheme of finite type,
- algebraic K-group = K-group scheme of finite type,
- R (or (K, v)) is admissible if R is henselian and the extension $\widehat{K}/K$ is separable.

Properties of admissible valued fields

Assume (K, v) is admissible. Then:

- K is algebraically closed in $\widehat{K}$.
- If L is a finite extension of K, then:
  - L is admissible (for the unique extension of v),
  - as a topological K-vector space, L is free (isomorphic to $K^{[L:K]}$),
  - $\widehat{K} \otimes_K L \xrightarrow{\sim} \widehat{L}$.
- If $\mathrm{char}\,(K) > 0$, the Frobenius map $K \to K$ is a closed topological embedding.
- R has the strong approximation property (à la Greenberg).

Admissible valuations: topological properties of morphisms

Proposition 1

Assume (K, v) is admissible, and let $f : X \to Y$ be a morphism of K-varieties. Consider the induced continuous map $f_{\mathrm{top}} : X_{\mathrm{top}} \longrightarrow Y_{\mathrm{top}}$.

1. "Implicit function theorem": *If f is étale, then f_{top} is a local homeomorphism.*
2. *If f is smooth, then f_{top} has local sections at each point of X_{top}. (In particular, it is an open map).*
3. "Continuity of roots": *If f is finite, then f_{top} is a closed map (hence proper, since it has finite fibers).*

Warning! If f is proper, f_{top} is not a closed map in general. But its image is closed in Y_{top}.

Now let us return to the main result:

Main Theorem

Let (K, v) be an admissible valued field, G an algebraic K-group, and $f : X \to Y$ a G-torsor. Then:

1. $\mathrm{Im}(f_{\mathrm{top}})$ *is locally closed in* Y_{top}.
2. *The induced map* $X_{\mathrm{top}} \to \mathrm{Im}(f_{\mathrm{top}})$ *is a principal* G_{top}*-bundle.*

The smooth case

Let us explain the smooth case. If G is smooth, then:

- $f : X \to Y$ is a smooth morphism,
- hence f_{top} has local sections at each point of X_{top}.
- This proves that
  - $\mathrm{Im}(f_{\mathrm{top}})$ is open, and
  - $X_{\mathrm{top}} \to \mathrm{Im}(f_{\mathrm{top}})$ is a principal G_{top}-bundle.

Next, a standard "twisting argument" shows that $Y_{\mathrm{top}} \smallsetminus \mathrm{Im}(f_{\mathrm{top}})$ is a union of subsets similar to $\mathrm{Im}(f_{\mathrm{top}})$. Hence $\mathrm{Im}(f_{\mathrm{top}})$ is also closed.

Strategy for general G

Let K_s be a separable closure of K. G has a largest smooth subgroup $G^\dagger$, which can be defined as the Zariski closure of $G(K_s)$ in G.

This construction is functorial in G and commutes with separable ground field extensions.

Strategy for general G

It is easy to check that $(G/G^\dagger)(K_s) = \{e\}$ (in particular $(G/G^\dagger)(K) = \{e\}$).

More generally, if T is a G-torsor over K, then $T/G^\dagger$ has at most one rational point.

Now let $f : X \to Y$ be a G-torsor. We factor it as

$$X \xrightarrow{\pi} Z := X/G^\dagger \xrightarrow{h} Y.$$

The corresponding factorization of f_{top} looks like

$$X_{\mathrm{top}} \underset{G^\dagger_{\mathrm{top}}\text{-bundle}}{\longrightarrow} \mathrm{Im}(\pi_{\mathrm{top}}) \underset{\text{open, closed}}{\subset} Z_{\mathrm{top}} \underset{\text{injective}}{\xrightarrow{h_{\mathrm{top}}}} Y_{\mathrm{top}}$$

Strategy for general G

$$X_{\rm top} \underset{G^{\dagger}_{\rm top}\text{-bundle}}{\longrightarrow} \mathrm{Im}(\pi_{\rm top}) \underset{\text{open, closed}}{\subset} Z_{\rm top} \underset{\text{injective}}{\xrightarrow{h_{\rm top}}} Y_{\rm top}$$

The hard part of the proof is to show that $h_{\rm top}$ is in fact a topological embedding, with locally closed image.

This uses:

- strong approximation,
- the construction (due to Gabber) of a remarkable G-equivariant compactification of $G/G^{\dagger}$.