Principal Bundles over Valued Fields

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Introduction

We start with:

- a topological field $k$,
- an algebraic $k$-group $G$,
- a $k$-variety $Y$, and
- a $G$-torsor (principal $G$-bundle) $f : X \to Y$ over $Y$. 
Introduction

Taking rational points, we get

- a topological group $G(k)$,
- a continuous free action of $G(k)$ on the space $X(k)$,
- a continuous map $X(k) \to Y(k)$, invariant for this action.

This map is not surjective in general.

We will consider the following questions, in the case of a henselian valued field:

- What does the image $I$ of this map look like, as a subspace of $Y(k)$?
- Is the induced map $X(k) \to I$ a principal $G(k)$-bundle?

Remark: the answers are easy and well known in characteristic zero (and more generally if $G$ is smooth).

Principal bundles in topology

Let $G$ be a topological group. A (left) $G$-bundle consists of the following data:

- a continuous map $f : X \to Y$,
- a (left) action $G \times X \to X$ commuting with $f$ (i.e. $f(g.x) = f(x)$).

A $G$-bundle is trivial if it is isomorphic (in the obvious sense) to $G \times Y \xrightarrow{pr_2} Y$ with the action of $G$ on itself by left translation.

It is principal if it is locally trivial (on $Y$), in the obvious sense.
Principal bundles in algebraic geometry: torsors

Let $k$ be a field, $G$ an algebraic group over $k$, and $Y$ a $k$-variety.

A (left) $G$-bundle over $Y$ consists of:

- a $k$-morphism $f : X \to Y$,
- a (left) action of $G$ on $X$, compatible with $f$,

We call it a (left) $G$-torsor if it is locally trivial for the fppf (or flat) topology, i.e. there is a $k$-morphism $h : Y' \to Y$ such that:

- $h$ is flat and surjective,
- $h$ trivializes $f$, i.e. the pullback $G$-bundle $X \times_Y Y' \to Y'$ is trivial.

A simple example

Let $n$ be a positive integer. Consider the $n$-th power map

$$f : \mathbb{G}_{m,k} \to \mathbb{G}_{m,k} \quad x \mapsto x^n.$$ 

This is a $\mu_n$-torsor (with the obvious action of $\mu_n = \ker(f)$ on $\mathbb{G}_{m,k}$).

If $n$ is invertible in $k$, then $f$ is even locally trivial for the étale topology, i.e. trivialized by an étale surjective map (e.g. $f$ itself).

More generally, if $G$ is a smooth $k$-group, any $G$-torsor $f : X \to Y$ is a smooth morphism, hence locally trivial for the étale topology. This holds in particular if $\text{char}(k) = 0$.

But in our example, if $n = \text{char}(k) > 0$, then $f$ is just the Frobenius map on $\mathbb{G}_{m,k}$. 
Characterization of torsors

A $G$-bundle $f : X \to Y$ in topology (resp. in algebraic geometry) is a $G$-torsor if and only if:

- it is “formally principal” (or a “pseudo-torsor”), i.e. the natural morphism
  \[ G \times X \to X \times_Y X \]
  
  \[ (g, x) \mapsto (g \cdot x, x) \]
  
  is an isomorphism,

- $f$ has local sections on $Y$, in the obvious sense (resp. in the flat topology sense).

Characterization of torsors

The “pseudo-torsor” property

\[ G \times X \sim\to X \times_Y X \]

is completely “categorical”, and is preserved by any functor on $k$-varieties that commutes with fiber products, such as the functor of rational points $R : Z \to Z(k)$.

It follows that if $f : X \to Y$ is a $G$-torsor over $k$, then the induced map of sets (or discrete spaces)

\[ R(f) : X(k) \to Y(k) \]

(which may not be surjective) induces a principal $G(k)$-bundle over its image.
Torsors over topological fields

From now assume that $k$ is a topological field, e.g. a valued field. For every $k$-variety $Z$, the set $Z(k)$ has a natural topology. The resulting topological space will be denoted by $Z_{\text{top}}$ (or $Z(k)_{\text{top}}$).

In particular, for a $G$-torsor $f : X \rightarrow Y$:

- $G_{\text{top}}$ is a topological group, and
- $f_{\text{top}} : X_{\text{top}} \rightarrow Y_{\text{top}}$ is a $G_{\text{top}}$-bundle, in fact automatically a pseudo-torsor.

Example of the squaring map:

$$f : \mathbb{G}_m, k \rightarrow \mathbb{G}_m, k$$

$$x \mapsto x^2.$$

If $k = \mathbb{R}$, the image of $f_{\text{top}}$ is $\mathbb{R}_{>0}$ (open and closed in $\mathbb{R}^\times$), and $f_{\text{top}}$ induces a trivial $\{\pm 1\}$-bundle over this image.

If $k = \mathbb{C}$, then $f_{\text{top}}$ is surjective and induces a nontrivial principal $\{\pm 1\}$-bundle over $\mathbb{C}^\times$.

If $k = \mathbb{F}_2((t))$, then $f_{\text{top}}$ is a homeomorphism onto its image, which is closed in $k^\times$. 
Torsors over topological fields

Back to a general $G$-torsor $f : X \to Y$ over a topological field $k$:

We can factor $f_{\text{top}} : X_{\text{top}} \to Y_{\text{top}}$ as

\[
X_{\text{top}} \quad \longrightarrow \quad X_{\text{top}}/G_{\text{top}} \quad \longrightarrow \quad \text{Im}(f_{\text{top}}) \quad \longrightarrow \quad Y_{\text{top}}
\]

which gives rise to natural questions:

1. Is the image of $f_{\text{top}}$ closed (open, locally closed) in $Y_{\text{top}}$?
2. Is the middle bijection a homeomorphism? (In other words, is $f_{\text{top}}$ a strict map?)
3. Is $X_{\text{top}} \to X_{\text{top}}/G_{\text{top}}$ a principal $G_{\text{top}}$-bundle?
   Equivalently, does this map have continuous local sections everywhere?

Note that a positive answer to both Questions 2 and 3 is equivalent to a positive answer to

4. Is $X_{\text{top}} \to \text{Im}(f_{\text{top}})$ a principal $G_{\text{top}}$-bundle?
The main result

**Definition**

A valued field $(K, v)$ is **admissible** if

- $(K, v)$ is henselian;
- the completion $\hat{K}$ of $K$ is a separable extension of $K$.

**Main Theorem**

Let $(K, v)$ be an admissible valued field, $G$ an algebraic $K$-group, and $f : X \rightarrow Y$ a $G$-torsor. Then:

1. $\text{Im}(f_{\text{top}})$ is locally closed in $Y_{\text{top}}$.
2. The induced map $X_{\text{top}} \rightarrow \text{Im}(f_{\text{top}})$ is a principal $G_{\text{top}}$-bundle.

**Remark.** In some cases, we can say more about $\text{Im}(f_{\text{top}})$:

- it is open and closed in $Y_{\text{top}}$ if $G$ is smooth, or if $K$ is perfect;
- it is closed in $Y_{\text{top}}$ if $G_{\text{red}}^o$ is smooth, or if $G$ is commutative.

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**The case of homogeneous spaces**

As an example, we can take for $X$ an algebraic group and for $G$ a subgroup of $X$, and consider $f : X \rightarrow Y := X/G$.

Then the image of $f_{\text{top}}$ is the orbit $X_{\text{top}}.y$ ($y=$origin of $Y$). The theorem says that

- this orbit is locally closed in $Y_{\text{top}}$, and
- the induced map $X_{\text{top}} \rightarrow X_{\text{top}}.y$ is a principal $G_{\text{top}}$-bundle (in particular, $X_{\text{top}}/G_{\text{top}} \rightarrow X_{\text{top}}.y$ is a homeomorphism).

When $K$ is a local field, this is due to Bernstein and Zelevinsky (1976).
An example of a non-closed orbit

Assume \( \text{char} (K) = p > 0 \). Let \( S = \mathbb{G}_a \times \mathbb{G}_m \) be the affine group in dimension 1, acting on \( X = \mathbb{A}^1_K \) transitively “via Frobenius on \( S \)”,

\[
S \times \mathbb{A}^1 \longrightarrow \mathbb{A}^1 \\
((x, y), u) \mapsto (x, y).u := x^p + y^p u
\]

For \( u \in K \), consider the orbit morphism

\[
f_u : S \rightarrow \mathbb{A}^1, \quad s \mapsto s.u.
\]

This is a torsor under the stabilizer \( S_u \) of \( u \).

The image of \( f_{u,\text{top}} \) is the orbit \( S(K).u = K^p + (K^\times)^p u \subset K \). In particular:

- if \( u \in K^p \), the orbit is \( K^p \), which is closed in \( K \) if \( K \) is admissible;
- for any choice of \( u \), the orbit has 0 in its closure (consider the action of \( \mathbb{G}_m \)).

Hence, if \( u \notin K^p \), then \( \text{Im}(f_{u,\text{top}}) \) is not closed in \( K \).

Notation and conventions

- \( R \): a valuation ring,
- \( K = \text{Frac} \,(R) \),
- \( v \): the valuation,
- \( \hat{K} \): completion of \( K \),
- \( K \)-variety = \( K \)-scheme of finite type,
- algebraic \( K \)-group = \( K \)-group scheme of finite type,
- \( R \) (or \( (K, v) \)) is admissible if \( R \) is henselian and the extension \( \hat{K}/K \) is separable.
Properties of admissible valued fields

Assume \((K, v)\) is admissible. Then:

- \(K\) is algebraically closed in \(\hat{K}\).
- If \(L\) is a finite extension of \(K\), then:
  - \(L\) is admissible (for the unique extension of \(v\)),
  - as a topological \(K\)-vector space, \(L\) is free (isomorphic to \(K^{[L:K]}\)),
  - \(\hat{K} \otimes_K L \sim \hat{L}\).
- If \(\text{char}(K) > 0\), the Frobenius map \(K \rightarrow K\) is a closed topological embedding.
- \(R\) has the strong approximation property (à la Greenberg).

Admissible valuations: topological properties of morphisms

Proposition 1

Assume \((K, v)\) is admissible, and let \(f : X \rightarrow Y\) be a morphism of \(K\)-varieties. Consider the induced continuous map \(f_{\text{top}} : X_{\text{top}} \rightarrow Y_{\text{top}}\).

1. “Implicit function theorem”: If \(f\) is étale, then \(f_{\text{top}}\) is a local homeomorphism.
2. If \(f\) is smooth, then \(f_{\text{top}}\) has local sections at each point of \(X_{\text{top}}\). (In particular, it is an open map).
3. “Continuity of roots”: If \(f\) is finite, then \(f_{\text{top}}\) is a closed map (hence proper, since it has finite fibers).

Warning! If \(f\) is proper, \(f_{\text{top}}\) is not a closed map in general. But its image is closed in \(Y_{\text{top}}\).
Now let us return to the main result:

**Main Theorem**

Let \((K, v)\) be an admissible valued field, \(G\) an algebraic \(K\)-group, and \(f : X \to Y\) a \(G\)-torsor. Then:

1. \(\text{Im}(f_{\text{top}})\) is locally closed in \(Y_{\text{top}}\).

2. The induced map \(X_{\text{top}} \to \text{Im}(f_{\text{top}})\) is a principal \(G_{\text{top}}\)-bundle.

**The smooth case**

Let us explain the smooth case. If \(G\) is smooth, then:

- \(f : X \to Y\) is a smooth morphism,
- hence \(f_{\text{top}}\) has local sections at each point of \(X_{\text{top}}\).
- This proves that
  - \(\text{Im}(f_{\text{top}})\) is open, and
  - \(X_{\text{top}} \to \text{Im}(f_{\text{top}})\) is a principal \(G_{\text{top}}\)-bundle.

Next, a standard “twisting argument” shows that \(Y_{\text{top}} \setminus \text{Im}(f_{\text{top}})\) is a union of subsets similar to \(\text{Im}(f_{\text{top}})\). Hence \(\text{Im}(f_{\text{top}})\) is also closed.
Strategy for general $G$

Let $K_s$ be a separable closure of $K$. $G$ has a largest smooth subgroup $G^\dagger$, which can be defined as the Zariski closure of $G(K_s)$ in $G$.

This construction is functorial in $G$ and commutes with separable ground field extensions.

Strategy for general $G$

It is easy to check that $(G/G^\dagger)(K_s) = \{e\}$ (in particular $(G/G^\dagger)(K) = \{e\}$).

More generally, if $T$ is a $G$-torsor over $K$, then $T/G^\dagger$ has at most one rational point.

Now let $f : X \to Y$ be a $G$-torsor. We factor it as

$$X \xrightarrow{\pi} Z := X/G^\dagger \xrightarrow{h} Y.$$ 

The corresponding factorization of $f_{\text{top}}$ looks like

$$X_{\text{top}} \xrightarrow{\pi_{\text{top}}} \text{Im}(\pi_{\text{top}}) \subset Z_{\text{top}} \xrightarrow{h_{\text{top}}} Y_{\text{top}}$$

$G^\dagger_{\text{top}}$-bundle

open, closed

injective
Strategy for general $G$

$$X_{\text{top}} \xrightarrow{G^\dagger_{\text{top}}\text{-bundle}} \text{Im}(\pi_{\text{top}}) \subset Z_{\text{top}} \xrightarrow{h_{\text{top}}} Y_{\text{top}}$$

open, closed \hspace{1cm} injective

The hard part of the proof is to show that $h_{\text{top}}$ is in fact a topological embedding, with locally closed image.

This uses:

- strong approximation,
- the construction (due to Gabber) of a remarkable $G$-equivariant compactification of $G/G^\dagger$. 