Chapter 7

# Introduction to mapping class groups of surfaces and related groups

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## Contents



## 1 Introduction

Let $\Sigma_g$ denote a closed oriented surface of genus $g$. The mapping class group of $\Sigma_g$, denoted by $\mathcal{M}_g$, is the group of isotopy classes of orientation preserving diffeomorphisms of $\Sigma_g$. This group has been investigated from various points of view for many years.

First of all, this group has been one of the main objects in the combinatorial group theory, the other one being the automorphism group of a finitely generated free group. Secondly, $\mathcal{M}_g$ acts on the Teichmüller space and the quotient space is the moduli space of genus $g$ Riemann surfaces which is a very important space in both algebraic geometry and complex analysis. Thirdly, this group has been playing crucial roles in the theory of 3-manifolds in relation to Heegaard decompositions as well as the geometry of surface bundles over the circle.

Reflecting this situation, there exist already many survey papers concerning various aspects of the mapping class group. We have the famous book by Birman [7] and several survey papers such as [8], [9], [27], [10]. Ivanov's paper [34] gives a very nice introduction to the present state of the study of the mapping class group (see also [32], [33]). Also we have the survey paper [37] by Johnson on the structure of the Torelli group which is a very important subgroup of the mapping class group.

There are also survey papers on the cohomological structure of the mapping class group or the moduli space of Riemann surfaces such as Harer [26], Hain–Looijenga [24] and the author [69], [71].

The purpose of this chapter is to describe somewhat different points of view in the study of the mapping class group and to suggest possible new directions for future research. More precisely, we would like to consider this as a special case of the study of the structure of the diffeomorphism group as well as the diffeotopy group of general $C^\infty$ manifolds. We also would like to seek for similarity and difference between the structures of the mapping class group and some of its closely related groups such as the automorphism groups of free groups and the symplectomorphism groups of surfaces. We refer to Vogtmann [84] for a survey of the study of the automorphism groups of free groups.

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## 2 Diffeomorphism groups and diffeotopy groups of differentiable manifolds

Let $M$ be a closed $C^\infty$ manifold. Then it is a very important problem to determine the set of all the isomorphism classes of differentiable $M$-bundles

$$\pi : E \longrightarrow X$$

over a given $C^\infty$ manifold $X$. By a standard technique in topology, this problem can be translated into the following one in homotopy theory. We denote by Diff $M$ the diffeomorphism group of $M$ equipped with the $C^\infty$ topology and let BDiff $M$ be its classifying space. Then we have a natural identification

$$\{\text{isomorphism classes of smooth } M\text{-bundles over } X\} \cong [X, \mathrm{BDiff}\, M]$$

where the right hand side denotes the set of all the homotopy classes of continuous mappings from $X$ to BDiff $M$. Hence we meet with the problem of determining the homotopy type of Diff $M$, in particular the computation of the homotopy groups $\pi_i(\mathrm{Diff}\, M)$. However this is an extremely difficult problem for a general differentiable manifold $M$.

Let us consider the simplest case where $X = S^1$. Any smooth $M$-bundle

$$\pi : E \longrightarrow S^1$$

over $S^1$ can be described as follows. Choose a point $x \in S^1$ and cut the total space $E$ along the fiber $\pi^{-1}(x)$ over $x$. Then we obtain an $M$-bundle over the interval $I$ so that it is diffeomorphic to the product $I \times M$. Observe here that we have chosen a diffeomorphism $\pi^{-1}(x) \cong M$. Now the given bundle $E$ can be recovered as

$$E = I \times M/(1, p) \sim (0, \varphi(p)) \quad (p \in M)$$

for a certain element $\varphi \in \mathrm{Diff}\, M$. Namely $E$ is obtained from $I \times M$ by pasting $\{1\} \times M$ to $\{0\} \times M$ by the diffeomorphism $\varphi$. Note that the element $\varphi$ is well-defined up to isotopy and also note that, if we change the identification $\pi^{-1}(x) \cong M$, then the element $\varphi$ changes into its conjugate element $\psi\varphi\psi^{-1}$. Thus we obtain

$$\begin{aligned}&\{\text{isomorphism classes of smooth } M\text{-bundles over } S^1\}\\ &\quad\cong [S^1, \mathrm{BDiff}\, M]\\ &\quad\cong \{\text{isotopy classes of elements of Diff } M\}/\text{conjugacy}.\end{aligned}$$

From the above consideration, it is natural to introduce the group consisting of all the isotopy classes of elements of Diff $M$ which we denote by $\mathcal{D}(M)$ and call it the *diffeotopy group* of $M$. It can also be described as the group of path components of the topological group Diff $M$, namely

$$\mathcal{D}(M) = \pi_0(\mathrm{Diff}\, M).$$

Alternatively, we can also write

$$\mathcal{D}(M) = \mathrm{Diff}\, M/\mathrm{Diff}_0 M$$

where $\mathrm{Diff}_0 M$ denotes the identity component of Diff $M$. Thus we have an extension

$$1 \longrightarrow \mathrm{Diff}_0 M \longrightarrow \mathrm{Diff}\, M \longrightarrow \mathcal{D}(M) \longrightarrow 1.$$

This simplest case, namely the determination of $\mathcal{D}(M)$ is already a very difficult problem in general. In fact, the case where $M$ is an $n$-dimensional sphere $S^n$ was one of the most important subjects during the early years of differential topology. By virtue of the foundational work of Cerf [13], [14] as well as the solution of the generalized Poincaré conjecture due to Smale [81], there is a natural isomorphism

$$\mathcal{D}_+(S^n) \cong \theta_{n+1} \quad (n \geq 5)$$

where $\mathcal{D}_+(S^n) = \pi_0(\mathrm{Diff}_+ S^n)$ denotes the *orientation preserving* diffeotopy group of $S^n$ and $\theta_n$ denotes the group of homotopy $n$-spheres introduced and studied by Kervaire and Milnor [44].

Since $\mathcal{D}(M)$ is the quotient group of the diffeomorphism group $\mathcal{D}(M)$ divided by the equivalence relation of isotopy which is stronger than (or sometimes equal to) that of homotopy, $\mathcal{D}(M)$ acts on any homotopy invariants of $M$ such as the fundamental group $\pi_1 M$ and the homology group $H_*(M; \mathbb{Z})$. We first consider the case of $\pi_1 M$. For any abstract group $\Gamma$, let Aut $\Gamma$ denote the automorphism group of $\Gamma$. Any element $\gamma \in \Gamma$ defines that of Aut $\Gamma$ which represents the inner automorphism of $\Gamma$ by the element $\gamma$. This induces a homomorphism $\Gamma \to \mathrm{Aut}\,\Gamma$. Let Inn $\Gamma$ denote the image of this homomorphism. Clearly the group $\Gamma$ is abelian if and only if Inn $\Gamma$ is the trivial subgroup of Aut $\Gamma$. It is easy to see that Inn $\Gamma$ is a normal subgroup of Aut $\Gamma$. The quotient group Aut $\Gamma$/Inn $\Gamma$ is denoted by Out $\Gamma$ and it is called the *outer* automorphism group of $\Gamma$. In these terminologies, we can say that the action of $\mathcal{D}(M)$ on $\pi_1 M$ induces a homomorphism

$$\rho_\pi : \mathcal{D}(M) \longrightarrow \operatorname{Out} \pi_1 M.$$

Also the action of $\mathcal{D}(M)$ on the homology group $H^*(M; \mathbb{Z})$ induces another homomorphism

$$\rho_H : \mathcal{D}(M) \longrightarrow \operatorname{Aut} H_*(M; \mathbb{Z}).$$

Then there arise natural questions about these homomorphisms. For example we could ask whether the homomorphisms $\rho_\pi$, $\rho_H$ are surjective or not. We could also ask whether they are injective or not. In the cases where these questions are answered negatively, the problem of describing the images as well as the kernels of these homomorphisms arise. It turns out that these questions depend on the global topology of the manifold $M$ and the above problems are often very difficult to be settled.

If there is given a geometric structure on $M$, then the automorphism group of this structure, which is considered to be a subgroup of Diff $M$, is one of the basic objects to be studied. Here we would like to mention a few examples.

The first obvious example is the case where there is given a Riemannian metric on $M$. Then the corresponding automorphism group is nothing but the isometry group Isom $M$. If $M$ is compact, then this group is known to be a compact Lie group sitting inside Diff $M$. For example, for the $n$-sphere $S^n$ with the standard metric, we have Isom $S^n = \mathrm{O}(n+1) \subset$ Diff $S^n$. It has been one of the main problems in the theory of differentiable transformation groups to study possible subgroups of Diff $M$ which are Lie transformation groups for a given manifold $M$.

The second example is the case where there is given a volume form $\upsilon$ on $M$. Then we can consider the subgroup of Diff $M$, denoted by $\mathrm{Diff}^\upsilon M$, which consists of those diffeomorphisms which preserve the form $\upsilon$. It is usually called the *volume preserving diffeomorphism group* of $M$. Moser's theorem in [72] implies that the inclusion $\mathrm{Diff}^\upsilon M \subset \mathrm{Diff}_+ M$ is a homotopy equivalence. Hence the classifying spaces $\mathrm{BDiff}^\upsilon M$ and $\mathrm{BDiff}_+ M$ have the same homotopy type. In particular, we have a *bijection*

$$\pi_0(\mathrm{Diff}^\upsilon M) \cong \pi_0(\mathrm{Diff}_+ M) = \mathcal{D}_+ M.$$

However the two groups $\mathrm{Diff}^\upsilon M$ and $\mathrm{Diff}_+ M$ seem to have considerably different properties as abstract groups and there should be many interesting problems here.

The third example is the case where there is given a *symplectic form* $\omega$ on a $2n$ dimensional manifold $M$. A symplectic form is, by definition, a closed 2-form $\omega$ such that $\omega^n$ is a volume form on $M$. The pair $(M, \omega)$ is called a symplectic manifold and the subgroup $\mathrm{Symp}(M, \omega) \subset$ Diff $M$ consisting of those diffeomorphisms which preserve the form $\omega$ is called the *symplectomorphism group* of $(M, \omega)$. Recently there have been obtained many interesting deep results in geometry and topology of symplectic manifolds as well as those of symplectomorphism groups (see the book [61] by McDuff and Salamon which gives an excellent introduction to this field). Let $\mathrm{Symp}_0(M, \omega)$ be the identity component of $\mathrm{Symp}(M, \omega)$ which is a normal subgroup. The quotient group $\mathcal{SD}(M,\omega) = \mathrm{Symp}(M,\omega)/\mathrm{Symp}_0(M,\omega)$ is called the symplectic mapping class group of $(M,\omega)$. Thus we have the following extension.

$$1 \longrightarrow \mathrm{Symp}_0(M,\omega) \longrightarrow \mathrm{Symp}(M,\omega) \longrightarrow \mathcal{SD}(M,\omega) \longrightarrow 1.$$

We have an obvious natural homomorphism $\mathcal{SD}(M,\omega) \to \mathcal{D}_+(M)$ and there are many interesting problems concerning it. For example, we can ask about the structure of the image as well as the kernel of this homomorphism. See Banyaga [3] and McDuff [60] for basic results concerning the structures of the groups $\mathrm{Diff}^{\upsilon} M$ and $\mathrm{Symp}(M,\omega)$.

Recall that any $C^\infty$ manifold $M$ admits a real analytic structure (see Whitney [86]). Our final example here is the subgroup $\mathrm{Diff}^{\omega} M \subset \mathrm{Diff}\, M$ consisting of *real analytic* diffeomorphisms of $M$ (with respect to a fixed real analytic structure). It seems to be an interesting problem to investigate whether there exist differences in algebraic structures between the two groups $\mathrm{Diff}^{\omega} M$ and $\mathrm{Diff}\, M$.

## 3 Mapping class groups of surfaces

The *mapping class group* of a closed oriented surface $\Sigma_g$ of genus $g$, which we denote by $\mathcal{M}_g$, is by definition the *oriented* diffeotopy group of $\Sigma_g$. Namely it is the group consisting of all the isotopy classes of *orientation preserving* diffeomorphisms of $\Sigma_g$. If we denote by $\mathrm{Diff}_+\Sigma_g$ the group of orientation preserving diffeomorphisms of $\Sigma_g$ equipped with the $C^\infty$ topology, then we have

$$\mathcal{M}_g = \pi_0(\mathrm{Diff}_+\Sigma_g).$$

It is easy to see that we have an extension

$$1 \longrightarrow \mathcal{M}_g \longrightarrow \mathcal{D}(\Sigma_g) \longrightarrow \mathbb{Z}/2 \longrightarrow 1$$

where the homomorphism $\mathcal{D}(\Sigma_g) \to \mathbb{Z}/2$ is induced by the action of $\mathcal{D}(\Sigma_g)$ on the set of orientations on $\Sigma_g$ or equivalently on the group $H_2(\Sigma_g;\mathbb{Z}) \cong \mathbb{Z}$.

As is well known, the topology and also the geometry of surfaces $\Sigma_g$ can be roughly divided into three classes, namely the cases where $g = 0$, $g = 1$ and $g \geq 2$. From the topological point of view, the fundamental group $\pi_1\Sigma_g$ is trivial for $g = 0$, rank 2 abelian for $g = 1$ and non-abelian for $g \geq 2$. On the other hand, from the geometrical point of view, each surface $\Sigma_g$ admits a Riemannian metric of constant Gaussian curvature $K$ where $K \equiv 1, 0, -1$ for $g = 0$, $g = 1$, $g \geq 2$ respectively. Furthermore for the latter two cases, there exist plenty of such metrics up to isotopy and they fit together to make a nice topological space called the *Teichmüller space*. It turns out that the structure of the mapping class group $\mathcal{M}_g$ reflects this rough classification of surfaces rather closely as follows.

First of all, we consider the case where $g = 0$, namely the case of the sphere $S^2$. Then a theorem of Smale [80] implies that the inclusion

$$\mathrm{O}(3) \subset \mathrm{Diff}\, S^2$$

is a homotopy equivalence. It follows, in particular, that the subgroup $\mathrm{Diff}_+ S^2$ is connected. Therefore any orientation preserving diffeomorphism of $S^2$ is isotopic to the identity. Hence the genus 0 mapping class group $\mathcal{M}_0$ is the trivial group.

Next we assume that $g \geq 1$ and consider the action of $\mathcal{M}_g$ on $\pi_1 \Sigma_g$ as in §2 where we considered the case of a general manifold $M$. It is known that the surface $\Sigma_g$ for $g \geq 1$ is an Eilenberg–MacLane space $K(\pi_1 \Sigma_g, 1)$ meaning that the higher homotopy groups $\pi_i \Sigma_g$ vanish for all $i \geq 2$. Here for a given (abstract) group $\pi$ and a positive integer $n$, any topological space $X$ with the property that

$$\pi_i X \cong \begin{cases} \pi & (i = n) \\ 0 & (i \neq n) \end{cases}$$

is called an *Eilenberg–MacLane space* $K(\pi, n)$ (we assume that $\pi$ is an abelian group in the cases where $n \geq 2$).

In fact, in the genus 1 case, the torus $T^2$ is expressed as $\mathbb{R}^2/\mathbb{Z}^2$ so that its universal covering manifold is the plane $\mathbb{R}^2$. Hence all the higher homotopy groups of $T^2$ vanish and $T^2$ is a $K(\mathbb{Z}^2, 1)$. In the cases where $g \geq 2$, there exist Riemannian metrics on $\Sigma_g$ which have constant negative curvature $-1$. It follows that its universal covering manifold is isometric to the upper half plane $\mathbb{H} = \{(x, y) \in \mathbb{R}^2; y > 0\}$ equipped with the Poincaré metric. Hence $\Sigma_g$ is a $K(\pi_1 \Sigma_g, 1)$. Here recall the standard presentation of the fundamental group $\pi_1 \Sigma_g$ which is expressed as

$$\begin{aligned} \pi_1 \Sigma_g &= \langle \alpha_1, \ldots \alpha_g, \beta_1, \ldots, \beta_g; \zeta \rangle \quad && (2g \text{ generators}), \\ \zeta &= [\alpha_1, \beta_1] \ldots [\alpha_g, \beta_g] && (\text{defining relation}). \end{aligned}$$

Now we state a classical theorem which is usually called the *Dehn–Nielsen theorem*.

**Theorem 3.1** (Dehn–Nielsen, Baer). *The natural action of $\mathcal{M}_g$ on $\pi_1 \Sigma_g$ induces an isomorphism*

$$\mathcal{M}_g \cong \mathrm{Out}_+ \pi_1 \Sigma_g.$$

Since $\pi_1 T^2 \cong \mathbb{Z}^2$ is an abelian group, $\mathrm{Aut}\, \mathbb{Z}^2 = \mathrm{Out}\, \mathbb{Z}^2 \cong \mathrm{GL}(2, \mathbb{Z})$. The subscript $+$ in $\mathrm{Out}_+ \mathbb{Z}^2$ means, in this case, that we consider only matrices with determinant 1. Thus we can write

$$\mathcal{M}_1 \cong \mathrm{SL}(2, \mathbb{Z}).$$

In order to interpret the subscript $+$ for the general case, we briefly recall the definition of the homology group as well as the cohomology group of an abstract group $\pi$ (see Brown's book [12] for details). It is known that, there exists a $K(\pi, 1)$ which is a CW complex. Furthermore it is uniquely defined up to homotopy equivalences. Hence for any $\pi$-module $M$ (namely $M$ is a module and there is given a homomorphism $\pi \to \mathrm{Aut}\, M$), we can define the (co)homology group of $\pi$ with coefficients in $M$ by setting

$$H_*(\pi; M) = H_*(K(\pi,1); M),$$
$$H^*(\pi; M) = H^*(K(\pi,1); M)$$

where $M$ on the right hand sides denotes the local system over $K(\pi,1)$ induced by the given $\pi$ action on the module $M$. Any group homomorphism $\rho\colon \pi \to \pi'$ induces a homomorphism $\rho_*\colon H_*(\pi) \to H_*(\pi')$ and similarly for the homology with twisted coefficients as well as the cohomology group. In particular, we have a homomorphism

$$\operatorname{Aut}\pi \longrightarrow \operatorname{Aut} H_*(\pi; M).$$

It is well known (and not so difficult to see) that the inner automorphisms induce the trivial action on the homology group so that we obtain a homomorphism

$$\operatorname{Out}\pi \longrightarrow \operatorname{Aut} H_*(\pi; M).$$

Now for any $g \geq 1$, $\Sigma_g$ is a $K(\pi_1\Sigma_g, 1)$ as mentioned above. Hence we have $H_*(\pi_1\Sigma_g; \mathbb{Z}) = H_*(\Sigma_g; \mathbb{Z})$. In particular $H_2(\pi_1\Sigma_g; \mathbb{Z}) \cong \mathbb{Z}$ and we have a homomorphism

$$\operatorname{Out}\pi_1\Sigma_g \longrightarrow \operatorname{Aut} H_2(\pi_1\Sigma_g; \mathbb{Z}) \cong \operatorname{Aut}\mathbb{Z} \cong \mathbb{Z}/2.$$

The group $\operatorname{Out}_+\pi_1\Sigma_g$ in Theorem 3.1 is defined to be the kernel of the above homomorphism. Sometimes this group is called the *orientation preserving* outer automorphism group of $\pi_1\Sigma_g$ because it is the subgroup of the whole group consisting of those outer automorphisms which are induced from orientation preserving diffeomorphisms of $\Sigma_g$.

Now we would like to mention the methods of proving Theorem 3.1 somewhat historically.

First of all, for any topological space $X$, let $\mathcal{E}(X)$ denote the set of all the homotopy classes of self homotopy equivalences of $X$. The composition of mappings induces a natural group structure on $\mathcal{E}(X)$. Next for any *topological* manifold $M$, let $\operatorname{Homeo} M$ denote the group of all the homeomorphisms of $M$ equipped with the compact open topology and let $\mathcal{H}(M) = \operatorname{Homeo} M/\operatorname{Homeo}_0 M$ denote the quotient group divided by the identity component $\operatorname{Homeo}_0 M$ of $\operatorname{Homeo} M$. It is called the *homeotopy group* of $M$. Now if $M$ is a $C^\infty$ manifold, then there is a natural sequence of forgetful homomorphisms

$$\mathcal{D}(M) \longrightarrow \mathcal{H}(M) \longrightarrow \mathcal{E}(M).$$

One can also introduce the equivalence relation on $\operatorname{Diff} M$, $\operatorname{Homeo} M$ induced by the homotopy of mappings to obtain variants of diffeotopy or homeotopy groups.

In general, these groups are all different from each other and they have their own meanings and properties. However, a very important and characteristic phenomenon occurs in dimension 2 and that is the fact that they are *all* equal for surfaces. Namely we have isomorphisms

$$\mathcal{D}(\Sigma_g) \cong \mathcal{H}(\Sigma_g) \cong \mathcal{E}(\Sigma_g),$$
$$\mathcal{M}_g = \mathcal{D}_+(\Sigma_g) \cong \mathcal{H}_+(\Sigma_g) \cong \mathcal{E}_+(\Sigma_g)$$

where the subscripts $+$ means appropriate subgroups of index 2 consisting of orientation preserving elements. Now elementary homotopy theory implies that there are canonical isomorphisms

$$\mathcal{E}(\Sigma_g) \cong \operatorname{Out} \pi_1 \Sigma_g, \quad \mathcal{E}_+(\Sigma_g) \cong \operatorname{Out}_+ \pi_1 \Sigma_g.$$

In fact the isomorphism $\mathcal{E}(X) \cong \operatorname{Out} \pi_1 X$ holds for any $K(\pi, 1)$ space $X$.

Dehn and then Nielsen [77] proved that the natural map

$$\mathcal{H}_+(\Sigma_g) \longrightarrow \operatorname{Out}_+ \pi_1 \Sigma_g$$

is *surjective*. The *injectivity* of the same map was proved by Baer [2] and, much later, reproved by Epstein [19]. It may be said that Dehn and Nielsen essentially proved that the natural map

$$\mathcal{D}_+(\Sigma_g) \longrightarrow \operatorname{Out}_+ \pi_1 \Sigma_g$$

is *surjective*, although it is unclear how they recognized the concept of diffeomorphisms as well as homeomorphisms which are now strictly distinguished. The *injectivity* of the above map can be obtained by adapting the proofs of Baer and Epstein from the context of homeomorphisms to that of diffeomorphisms which are known to be possible in this low dimensional case.

There are variants of the mapping class group and analogues of Theorem 3.1 for them as follows. First, if we choose a base point $* \in \Sigma_g$, then we can consider the subgroup $\operatorname{Diff}_+(\Sigma_g, *) \subset \operatorname{Diff}_+ \Sigma_g$ consisting of all the orientation preserving diffeomorphisms of $\Sigma_g$ which fix the base point $*$. Then we set

$$\mathcal{M}_{g,*} = \pi_0(\operatorname{Diff}_+(\Sigma_g, *))$$

and call it the mapping class group of $\Sigma_g$ *relative* to the base point. There is the forgetful homomorphism $\mathcal{M}_{g,*} \to \mathcal{M}_g$ which is an isomorphism for $g = 0, 1$ and in the cases $g \geq 2$, the kernel of this homomorphism is known to be canonically isomorphic to $\pi_1 \Sigma_g$. Thus we have an extension

$$1 \longrightarrow \pi_1 \Sigma_g \longrightarrow \mathcal{M}_{g,*} \longrightarrow \mathcal{M}_g \longrightarrow 1 \quad (g \geq 2).$$

Next if we choose an embedded disk $D^2 \subset \Sigma_g$, then we can consider the subgroup $\operatorname{Diff}(\Sigma_g, D^2) \subset \operatorname{Diff}_+ \Sigma_g$ consisting of all the diffeomorphisms of $\Sigma_g$ which restrict to the identity of $D^2$. Then we set

$$\mathcal{M}_{g,1} = \pi_0(\operatorname{Diff}(\Sigma_g, D^2))$$

and call it the mapping class group of $\Sigma_g$ *relative* to $D^2$. Alternatively, we can consider the compact surface $\Sigma_g^0 = \Sigma_g \setminus \operatorname{Int} D^2$ and the diffeomorphism group $\operatorname{Diff}(\Sigma_g^0, \partial \Sigma_g^0)$ consisting of all diffeomorphisms of $\Sigma_g^0$ which restrict to the identity on the boundary.

Then we can also define

$$\mathcal{M}_{g,1} = \pi_0(\mathrm{Diff}(\Sigma_g^0, \partial\Sigma_g^0)).$$

If we choose a base point $*$ in $D^2 \subset \Sigma_g$, then we have the forgetful homomorphism $\mathcal{M}_{g,1} \to \mathcal{M}_{g,*}$, and in the case $g \geq 1$ the kernel of this homomorphism is known to be isomorphic to $\mathbb{Z}$ so that we have an extension

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{M}_{g,1} \longrightarrow \mathcal{M}_{g,*} \longrightarrow 1 \quad (g \geq 1). \tag{3.1}$$

Now we can state the analogue of Theorem 3.1 for the mapping class groups $\mathcal{M}_{g,*}$ and $\mathcal{M}_{g,1}$ as follows. This should also be considered as a classical theorem going back to Magnus [54] and Zieschang [87].

**Theorem 3.2.** *There are natural isomorphisms*

$$\mathcal{M}_{g,*} \cong \mathrm{Aut}_+\pi_1\Sigma_g,$$
$$\mathcal{M}_{g,1} \cong \{\varphi \in \mathrm{Aut}\,\pi_1\Sigma_g^0;\, \varphi(\zeta) = \zeta\}$$

*where* $\zeta = [\alpha_1, \beta_1]\dots[\alpha_g, \beta_g]$ *denotes the single defining relation of* $\pi_1\Sigma_g$ *with respect to a standard generating system* $\alpha_1, \dots, \alpha_g, \beta_1, \dots \beta_g$.

We can also consider the mapping class groups of $\Sigma_g$ *relative to* finitely many distinguished points as well as finitely many embedded disks on $\Sigma_g$. However here we omit them.

## 4 Automorphism groups of free groups and IA automorphism groups

Let $F_n$ be a free group of rank $n$. We denote by Aut $F_n$ the *automorphism group of the free group* $F_n$. If $n = 1$, then clearly $\mathrm{Aut}\,\mathbb{Z} \cong \mathbb{Z}/2$. Henceforth we assume that $n \geq 2$. Then the homomorphism $\iota\colon F_n \to \mathrm{Aut}\,F_n$ defined by

$$\iota(\alpha)\gamma = \alpha\gamma\alpha^{-1} \quad (\alpha, \gamma \in F_n)$$

is easily seen to be injective. As was already mentioned in §2 in a general setting, the image $\mathrm{Im}\,\iota$ is denoted by Inn $F_n$ and called the *inner* automorphism group of $F_n$. It can be checked that Inn $F_n$ is a normal subgroup of Aut $F_n$ so that we can consider the quotient group

$$\mathrm{Out}\,F_n = \mathrm{Aut}\,F_n/\mathrm{Inn}\,F_n$$

which is called the *outer automorphism group of the free group* $F_n$.

These groups Aut $F_n$ and Out $F_n$ have been one of the main objects of combinatorial group theory going back to the works of Nielsen and then Magnus from the late 1910s to the 1930s.

Aut $F_n$ acts naturally on the abelianization of $F_n$, which is a free abelian group of rank $n$. If we choose a system $\gamma_1, \dots, \gamma_n$ of generators for $F_n$, then we obtain a homomorphism

$$\rho_0 \colon \operatorname{Aut} F_n \longrightarrow \mathrm{GL}(n, \mathbb{Z}). \tag{4.1}$$

It is easy to see that $\rho_0$ is trivial on the subgroup Inn $F_n$. Hence we obtain a homomorphism

$$\rho_0 \colon \operatorname{Out} F_n \longrightarrow \mathrm{GL}(n, \mathbb{Z}). \tag{4.2}$$

Nielsen [74] proved that the above homomorphism is an isomorphism for $n = 2$ so that

$$\operatorname{Out} F_2 \cong \mathrm{GL}(2, \mathbb{Z}).$$

However for $n \geq 3$, Nielsen [75] also observed that $\rho_0$ is not injective and in [76] he proved that the following four elements

$$\begin{array}{lll} \text{(i)} & \gamma_1 \to \gamma_2,\ \gamma_2 \to \gamma_1,\ \gamma_i \to \gamma_i & (i = 3, \dots, n), \\ \text{(ii)} & \gamma_1 \to \gamma_1^{-1},\ \gamma_i \to \gamma_i & (i = 2, \dots, n), \\ \text{(iii)} & \gamma_1 \to \gamma_1\gamma_2,\ \gamma_i \to \gamma_i & (i = 2, \dots, n), \\ \text{(iv)} & \gamma_1 \to \gamma_2,\ \gamma_2 \to \gamma_3,\ \dots,\ \gamma_n \to \gamma_1 & \end{array}$$

generate Aut $F_n$ and hence Out $F_n$. By this he was able to prove that the homomorphims $\rho_0$ (4.1), (4.2) above are *surjective*. He also gave a finite complete set of defining relations in terms of the above generators, for both of Aut $F_n$ and Out $F_n$. Later McCool [58] gave a simpler finite presentation for Aut $F_n$. Also Gersten [21] gave a finite presentation for the subgroup $\operatorname{Aut}_+ F_n$ which is the full inverse image under $\rho_0$ of the subgroup $\mathrm{GL}^+(n, \mathbb{Z}) \subset \mathrm{GL}(n, \mathbb{Z})$ consisting of matrices with determinant 1.

The kernels of the homomorphisms $\rho_0$ are called *IA (outer) automorphism groups* of $F_n$ which we denote by $\mathrm{IAut}_n$ and $\mathrm{IOut}_n$ respectively. Thus we have group extensions

$$\begin{array}{c} 1 \longrightarrow \mathrm{IAut}_n \longrightarrow \operatorname{Aut} F_n \xrightarrow{\rho_0} \mathrm{GL}(n, \mathbb{Z}) \longrightarrow 1, \\ 1 \longrightarrow \mathrm{IOut}_n \longrightarrow \operatorname{Out} F_n \xrightarrow{\rho_0} \mathrm{GL}(n, \mathbb{Z}) \longrightarrow 1. \end{array}$$

Magnus [53] proved that the group $\mathrm{IAut}_n$ is finitely generated. On the other hand, Baumslag–Taylor [4] proved that $\mathrm{IAut}_n$ is torsion free.

There is a close connection between the mapping class group and the automorphism groups of free groups. More precisely, we have the following two explicit relations. One is the realization of the mapping class group $\mathcal{M}_{g,1}$ of $\Sigma_g$ relative to an embedded disk $D \subset \Sigma_g$ as a subgroup of Aut $F_{2g}$ described as

$$\mathcal{M}_{g,1} = \{\varphi \in \operatorname{Aut} F_{2g};\ \varphi(\zeta) = \zeta\} \subset \operatorname{Aut} F_{2g} \tag{4.3}$$

where $\zeta = [\gamma_1, \gamma_2] \dots [\gamma_{2g-1}, \gamma_{2g}] \in F_{2g}$. This follows from Theorem 3.2 because $\pi_1 \Sigma_g^0$ is isomorphic to $F_{2g}$. The other is given as follows. The subgroup $\mathbb{Z} = \operatorname{Ker}(\mathcal{M}_{g,1} \to \mathcal{M}_{g,*})$ described in (3.1) is generated by the Dehn twist (see the next section §5) along a simple closed curve parallel to the boundary of the embedded disk $D^2 \subset \Sigma_g$ and its action on $\pi_1 \Sigma_g^0$ is the conjugation by the element $\zeta$ above. Since $\mathcal{M}_{g,1}/\mathbb{Z}$ is canonically isomorphic to $\mathcal{M}_{g,*}$, we obtain a representation

$$\mathcal{M}_{g,*} \longrightarrow \text{Aut}\, F_{2g}/\text{Inn}\, F_{2g} = \text{Out}\, F_{2g}.$$

It is known that this representation is injective so that we can consider $\mathcal{M}_{g,*}$ as a subgroup of Out $F_{2g}$

$$\mathcal{M}_{g,*} \subset \text{Out}\, F_{2g}.$$

The comparison of various group theoretical properties between the mapping class groups $\mathcal{M}_{g,*}$, $\mathcal{M}_g$ and automorphism groups of free groups Aut $F_n$, Out $F_n$ have been an important subject since the very beginning of combinatorial group theory. Recently, this tendency is strengthened in a wider framework including geometric viewpoints.

Finally we mention the following result of Laudenbach [50] which shows that, up to a certain finite group, the (outer) automorphism groups of free groups are naturally isomorphic to the diffeotopy groups of certain 3-manifolds. Let $n\, S^1 \times S^2$ denote the connected sum of $n$-copies of $S^1 \times S^2$. Then there are the following exact sequences

$$1 \longrightarrow (\mathbb{Z}/2)^n \longrightarrow \text{Out}\, F_n \longrightarrow \mathcal{D}(n\, S^1 \times S^2) \longrightarrow 1,$$

$$1 \longrightarrow (\mathbb{Z}/2)^n \longrightarrow \text{Aut}\, F_n \longrightarrow \mathcal{D}(n\, S^1 \times S^2, \text{rel}\, D^3) \longrightarrow 1$$

where $\mathcal{D}(n\, S^1 \times S^2, \text{rel}\, D^3)$ denotes the group of path components of those diffeomorphisms of $n\, S^1 \times S^2$ which are the identity on an embedded disk $D^3 \subset n\, S^1 \times S^2$.

## 5 Dehn twists

So far we have not mentioned explicit examples of elements of the mapping class group $\mathcal{M}_g$. Here we describe the most important construction of such elements which is called the *Dehn twist* because it was introduced by Dehn.

Suppose that there is given a simple closed curve $C$ on $\Sigma_g$ and also recall that there is specified an orientation on $\Sigma_g$. Then we can define an element $\tau_C \in \mathcal{M}_g$, which is called the (right handed) Dehn twist along $C$, as follows. Let us choose an embedding

$$i : S^1 \times [-1, 1] \longrightarrow \Sigma_g$$

of an annulus into $\Sigma_g$ such that

(i) $i(S^1 \times \{0\}) = C$ and

(ii) $i$ preserves the orientations

where we give $S^1$ and $[-1, 1]$ the standard orientations and the annulus $S^1 \times [-1, 1]$ the product orientation of them. Then we define a diffeomorphism $\varphi_0$ of the annulus by

$$\varphi_0(\theta, t) = (\theta + f(t), t)$$

where $0 \leq \theta \leq 2\pi$ and $-1 \leq t \leq 1$ are the coordinates of $S^1$ and $[-1,1]$ respectively, and $f\colon [-1,1] \to \mathbb{R}$ is a $C^\infty$ function such that

$$f(t) = \begin{cases} 0 & (-1 \leq t \leq -\frac{1}{2}) \\ 2\pi & (\frac{1}{2} \leq t \leq 1) \end{cases}$$

and $f$ is strictly increasing on the interval $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Now recall that for any diffeomorphism $\psi$ of a $C^\infty$ manifold $M$, the closed set

$$\operatorname{supp}(\psi) = \overline{\{p \in M;\ \psi(p) \neq p\}} \subset M$$

is called the *support* of $\varphi$. By the definition of $\varphi_0$, it is clear that $\operatorname{supp}(\varphi_0)$ is contained in $\operatorname{Int}(S^1 \times [-1,1])$. Hence we can define a diffeomorphism $\varphi \in \operatorname{Diff}_+ \Sigma_g$ by setting $\varphi = \varphi_0$ on $i(S^1 \times [-1,1])$ and extend it by the identity over the whole of $\Sigma_g$. It can be checked that the isotopy class of $\varphi$ does not depend on the choice of the function $f$ nor the orientation preserving embedding $i$ (observe here that the *opposite* embedding $\bar{i}\colon S^1 \times [-1,1] \to \Sigma_g$ defined by $\bar{i}(\theta, t) = i(-\theta, -t)$ does not change the isotopy class of $\varphi$). Furthermore it depends only on the isotopy class of the simple closed curve $C$. We denote the resulting mapping class by $\tau_C \in \mathcal{M}_g$ and call it the *right handed* Dehn twist along $C$. This is because a path, which crosses the simple closed curve $C$ transversely at a point, will be transferred by $\varphi$ to a path which, after getting near to $C$, goes around $C$ once to the right direction (with respect to the orientation of $\Sigma_g$) and then goes on as before. The inverse $\tau_C^{-1}$ is called the *left handed* Dehn twist along $C$. Note here that if we reverse the orientation of $\Sigma_g$, then the right handed Dehn twist is changed into the left handed one and vice versa. Also note that the orientation of the simple closed curve $C$ itself has nothing to do with the definition of the Dehn twist.

If a simple closed curve $C$ on $\Sigma_g$ bounds a disk, then it is easy to see that the corresponding Dehn twist is the identity in $\mathcal{M}_g$. A simple closed curve on $\Sigma_g$ is called *essential* if it does not bound a disk. We define

$$\mathcal{S}(\Sigma_g) = \{\text{isotopy classes of essential simple closed curves on } \Sigma_g\}.$$

In summary, we obtain a mapping

$$\mathcal{S}(\Sigma_g) \ni [C] \longmapsto \tau_C \in \mathcal{M}_g.$$

One simple but important property of the Dehn twists is that the equality

$$\tau_{\varphi(C)} = \varphi \circ \tau_C^{\varepsilon(\varphi)} \circ \varphi^{-1} \tag{5.1}$$

holds for any simple closed curve $C$ and any element $\varphi \in \operatorname{Diff} \Sigma_g$, where $\varepsilon(\varphi) = 1$ or $-1$ if $\varphi$ preserves (or reverses) the orientation of $\Sigma_g$.

It may appear first that the definition of the Dehn twists is so simple that they will cover a relatively small part of the mapping class group. However, if one observes that there are enormously many simple closed curves on $\Sigma_g$ and two (or more) simple closed curves can meet each other in a very complicated way, one can easily understand that products of Dehn twists along various simple closed curves can express very complicated elements in $\mathcal{M}_g$. In fact, Dehn [17] proved that finitely many Dehn twists generate $\mathcal{M}_g$. Later Lickorish [51] proved that a certain system of $3g-1$ Dehn twists, which are now called the Lickorish generators, generates $\mathcal{M}_g$. Then Humphries [30] proved that $2g+1$ members among the Lickorish generators already generate $\mathcal{M}_g$. He also proved that this number $2g+1$ is the minimum of the number of Dehn twists which can generate $\mathcal{M}_g$.

As for the presentation of the mapping class group, McCool [59] proved that $\mathcal{M}_g$ is finitely presentable without giving an explicit presentation. Hatcher and Thurston [28] gave a method of obtaining a finite presentation and it was finally completed by the work of Wajnryb [85].

## 6 Mapping class groups acting on the homology of surfaces and the Torelli groups

The mapping class group $\mathcal{M}_g$ acts on the first homology group of $\Sigma_g$ naturally. Assume here that $g \geq 1$ and we denote simply by $H$ the first integral homology group $H_1(\Sigma_g;\mathbb{Z})$ of $\Sigma_g$. As an abstract group, $H$ is a free abelian group of rank $2g$. However, there exists an important additional structure on $H$ coming from the geometry of $\Sigma_g$. More precisely, the intersection numbers of elements of $H$ give rise to a bilinear mapping

$$\mu\colon H \times H \longrightarrow \mathbb{Z}.$$

We denote by $u \cdot v$ $(u, v \in H)$ the intersection number $\mu(u,v)$. Then $v \cdot u = -u \cdot v$ so that $\mu$ is *skew symmetric*. The natural action of $\mathcal{M}_g$ on $H$ comes from orientation preserving diffeomorphisms of the surface $\Sigma_g$. Hence it clearly preserves the intersection pairing $\mu$ so that we obtain a homomorphism

$$\rho_0\colon \mathcal{M}_g \longrightarrow \operatorname{Aut}(H,\mu) \tag{6.1}$$

where $\operatorname{Aut}(H,\mu)$ denotes the automorphism group of $H$ preserving $\mu$. Namely

$$\operatorname{Aut}(H,\mu) = \{f \in \operatorname{Aut} H;\ f(u)\cdot f(v) = u \cdot v \text{ for any } u, v \in H\}.$$

Let us study how this condition can be expressed in terms of that of matrices representing elements of $\operatorname{Aut}(H,\mu)$. For this, choose a basis $x_1,\dots,x_g, y_1,\dots,y_g$ of $H$ such that

$$\begin{aligned} &x_i \cdot y_j = \delta_{ij},\\ &x_i \cdot x_j = y_i \cdot y_j = 0 \quad (i,j = 1,\dots,g). \end{aligned}$$

A basis with this property is called a *symplectic basis*. It is easy to see that there exist infinitely many such bases because

$$\langle x_1, \ldots, x_g, y_1, \ldots, y_g \rangle \text{ is a symplectic basis } \Longrightarrow$$
$$\langle x_1 + n y_1, x_2, \ldots, x_g, y_1, \ldots, y_g \rangle \text{ is also a symplectic basis}$$

for any $n \in \mathbb{Z}$, for example.

Now if we fix a symplectic basis of $H$, then the automorphism group Aut $H$ can be identified as

$$\text{Aut } H = \mathrm{GL}(n, \mathbb{Z})$$

by associating to each element in Aut $H$ the corresponding matrix with respect to the given symplectic basis. More precisely, first we express any two elements $u, v \in H$ as linear combinations

$$u = u_1 x_1 + \cdots + u_g x_g + u_{g+1} y_1 + \cdots + u_{2g} y_g,$$
$$v = v_1 x_1 + \cdots + v_g x_g + v_{g+1} y_1 + \cdots + v_{2g} y_g$$

with respect to the above symplectic basis and then we identify the two elements $u, v$ with the following $2g$-dimensional column vectors

$$u = {}^t(u_1, \ldots, u_g, u_{g+1}, \ldots, u_{2g}), \quad v = {}^t(v_1, \ldots, v_g, v_{g+1}, \ldots, v_{2g})$$

in $\mathbb{R}^{2g}$. Now set

$$J = \begin{pmatrix} O & E \\ -E & O \end{pmatrix} \in \mathrm{GL}(2g, \mathbb{Z}).$$

Then we can write

$$u \cdot v = u_1 v_{g+1} + \cdots + u_g v_{2g} - u_{g+1} v_{2g} - \cdots - u_{2g} v_g = (u, Jv)$$

where $(u, Jv)$ denotes the standard Euclidean inner product of two vectors $u, Jv \in \mathbb{R}^{2g}$. Now a matrix $A \in \mathrm{GL}(2g, \mathbb{Z})$ preserves the intersection pairing $\mu$ if and only if the condition

$$Au \cdot Av = u \cdot v \quad \text{for any } u, v \in H \tag{6.2}$$

holds. On the other hand we have

$$Au \cdot Av = (Au, JAv) = (u, {}^t AJAv),$$
$$u \cdot v = (u, Jv).$$

It follows that $A$ satisfies the condition (6.2) if and only if

$${}^t AJA = J.$$

Based on the above consideration, we define a subgroup

$$\mathrm{Sp}(2g, \mathbb{Z}) = \{A \in \mathrm{GL}(2g, \mathbb{Z});\ {}^t AJA = J\}$$

of $\mathrm{GL}(2g, \mathbb{Z})$. This group is a discrete subgroup of the symplectic group $\mathrm{Sp}(2g, \mathbb{R})$ consisting of *unimodular* symplectic matrices. Sometimes the group $\mathrm{Sp}(2g, \mathbb{Z})$ is called the *Siegel modular group* because it plays a fundamental role in the theory of Siegel modular forms.

In conclusion, if we fix a symplectic basis of $H$, then we have an isomorphism

$$\mathrm{Aut}\,(H,\mu) \cong \mathrm{Sp}(2g,\mathbb{Z})$$

and (6.1) induces a homomorphism

$$\rho_0\colon \mathcal{M}_g \longrightarrow \mathrm{Sp}(2g,\mathbb{Z}).$$

It is easy to see that $\mathrm{Sp}(2,\mathbb{Z}) = \mathrm{SL}(2,\mathbb{Z})$ so that $\rho_0$ is an isomorphism for the case $g = 1$. In the cases where $g \geq 2$, it was classically known, going back to a work of Burkhardt at the end of the 19 th century and later works of Dehn and Nielsen, that $\rho_0$ is *surjective* (cf. [55]). It was recognized that $\rho_0$ has a non-trivial kernel which is a normal subgroup of $\mathcal{M}_g$. This group was named the *Torelli group* after an Italian mathematician and was known for some time among complex analysts and algebraic geometers. However it was relatively recently that the Torelli group called the attention of topologists. Probably Birman's paper [6] published in 1971 is the earliest work on this group by topologists. Then Johnson began a systematic study of this group in the late 1970s and obtained foundational results concerning the structure of the Torelli group within several years. We refer the readers to his survey paper [37] as well as [36], [38] [39], [40]. Following his notation, the Torelli group is usually denoted by $\mathcal{I}_g$. Thus we have a group extension

$$1 \longrightarrow \mathcal{I}_g \longrightarrow \mathcal{M}_g \xrightarrow{\rho_0} \mathrm{Sp}(2g,\mathbb{Z}) \longrightarrow 1.$$

It is a classical result, going back to Grothendieck and Serre, that

the Torelli group $\mathcal{I}_g$ is torsion free.

This can be shown as follows. Suppose that there exists a non-trivial element $\varphi \in \mathcal{I}_g$ which has a finite order, say $d > 0$. Then by Nielsen [78], there exists a diffeomorphism $\tilde{\varphi}\colon \Sigma_g \to \Sigma_g$ such that $\tilde{\varphi}^d = \mathrm{id}$ and the mapping class of $\tilde{\varphi}$ is the given one $\varphi$. Then consider the quotient $\Sigma_g/G_{\tilde{\varphi}}$ of $\Sigma_g$ divided by the action of the cyclic group $G_{\tilde{\varphi}} \cong \mathbb{Z}/d$ generated by the element $\tilde{\varphi}$. It is easy to see that this quotient space is homeomorphic to a closed surface of some genus $h$, because the projection $\Sigma_g \to \Sigma_g/G_{\tilde{\varphi}}$ must be a branched covering along a finite set consisting of fixed points of $\tilde{\varphi}$ on $\Sigma_g$. Now there is a general fact on the rational cohomology of the quotient space $X/G$ divided by a properly discontinuous action of a discrete group $G$ on $X$, due originally to Grothendieck, that $H^*(X/G;\mathbb{Q}) \cong H^*(X;\mathbb{Q})^G$. If we apply this to the above simplest case of a finite group action, we obtain isomorphisms

$$H^*(\Sigma_h;\mathbb{Q}) \cong H^*(\Sigma_g/G_{\tilde{\varphi}};\mathbb{Q}) \cong H^*(\Sigma_g;\mathbb{Q})^{G_{\tilde{\varphi}}} \cong H^*(\Sigma_g;\mathbb{Q})$$

where the last isomorphism comes from the assumption that $\tilde{\varphi}$ acts trivially on the homology (and hence cohomology) of $\Sigma_g$. We can now conclude that $g = h$ which is a contradiction because the condition $g \geq 2$ should imply that $h < g$. Observe here that the genus 1 surface, namely the torus $T^2$ admits a free $\mathbb{Z}/d$ action and the quotient space is again diffeomorphic to $T^2$.

Now one of the foundational results of Johnson mentioned above is that $\mathcal{I}_g$ is finitely generated for any $g \geq 3$. To prove this, he introduced the following two types of elements of $\mathcal{M}_g$. One is the *BP-map* (BP for bounding pair) defined as follows. Suppose that there are given two disjoint simple closed curves $C$ and $D$ on $\Sigma_g$ which satisfy the condition that if we cut $\Sigma_g$ along $C$ and $D$, then the resulting surface is disconnected. In other words, the disjoint union $C \cup D$ bounds a subsurface of $\Sigma_g$. We say that $C$ and $D$ are a bounding pair. In this case, we call the element

$$\tau_C \tau_D^{-1} \in \mathcal{M}_g$$

the BP-map corresponding to the above bounding pair. The other type is the *BSCC-map* (BSCC for bounding simple closed curve) defined as follows. Suppose that there is given a simple closed curve $C$ on $\Sigma_g$ such that if we cut $\Sigma_g$ along $C$, then the resulting surface is disconnected. In other words, the simple closed curve $C$ bounds a subsurface of $\Sigma_g$. We say that $C$ is a bounding simple closed curve. In this case, we call the element

$$\tau_C \in \mathcal{M}_g$$

the BSCC-map corresponding to the bounding simple closed curve $C$.

In fact, the following important fact holds:

$$\text{BP-map, BSCC-map } \in \mathcal{I}_g. \tag{6.3}$$

To see this, let us study how a Dehn twist $\tau_C$ along a simple closed curve $C$ acts on $H_1(\Sigma_g; \mathbb{Z})$. Let $u \in H_1(\Sigma_g; \mathbb{Z})$ be a homology class and choose an oriented curve $E$ on $\Sigma_g$ which represents $u$. We can assume that $E$ intersects $C$ transversely at finitely many points. Let us choose an orientation on $C$ and let $v \in H_1(\Sigma_g; \mathbb{Z})$ be the homology class represented by $C$ with this orientation. Locally $C$ divides the regular neighborhood of $C$ into two parts. If we identify a closed regular neighborhood of the oriented $C$ with $S^1 \times [-1, 1]$ according to the given orientation on the surface in such a way that the oriented $C$ is identified with the oriented $S^1 \times \{0\}$, then we can distinguish the above two pieces by calling them the negative and positive sides respectively. Now we count the number of the intersection points $C \cap E$ algebraically by giving $+1$ if the oriented curve $E$ intersects $C$ from negative to positive direction and $-1$ if $E$ intersects $C$ from positive to negative direction. Let $m$ be the totality of these $\pm 1$ numbers. Then we have

$$\tau_C(u) = u + mv.$$

Observe here that, if we reverse the orientation of $C$, then both $m$ and $v$ change signs so that the above formula remains unchanged. Now we can check the above fact (6.3) as follows. First let $C, D$ be a BP-pair. Then we can give orientations on them so that the resulting homology classes are the same. On the other hand, in the above computation, we have $\tau_C(u) = \tau_D(u)$ for any $u$ so that $\tau_C \tau_D^{-1}$ acts on the homology trivially. Next, if $C$ is a BSCC, then clearly the corresponding homology class is trivial, whence the claim follows.

The *genus* of a BP-map (or BSCC-map) is defined as follows. If we cut $\Sigma_g$ along a BP-pair $C, D$ (or a BSCC $C$), then we obtain two surfaces. The genus of the relevant map is defined to be the smaller genus of these two surfaces.

**Theorem 6.1** (Johnson [38]). *The Torelli group $\mathcal{I}_g$ is finitely generated for any $g \geq 3$.*

The method of proving this theorem was roughly as follows. Johnson constructed a certain finite set of BP-maps of all genera between 1 and $g-2$ and showed that the subgroup of $\mathcal{I}_g$ generated by them is a *normal* subgroup. Since he had already proved in [35] that $\mathcal{I}_g$ is normally generated by just one BP-map of genus 1, the proof was completed.

Johnson considered also the subgroup

$$\mathcal{K}_g = \text{the subgroup of } \mathcal{M}_g \text{ generated by all the BSCC-maps}$$

of the mapping class group. Since any BSCC-map is contained in $\mathcal{I}_g$, $\mathcal{K}_g$ is a subgroup of $\mathcal{I}_g$. Also it is easy to see that property (5.1) implies any conjugate element of a BSCC-map is again a BSCC-map. It follows that $\mathcal{K}_g$ is a normal subgroup of $\mathcal{M}_g$ (and $\mathcal{I}_g$). More strongly, it is known that $\mathcal{K}_g$ is a characteristic subgroup of $\mathcal{M}_g$ (and $\mathcal{I}_g$). This follows from a result of Ivanov [31] (see also [57]) that any automorphism of $\mathcal{M}_g$ is induced by an inner automorphism of $\mathcal{D}(\Sigma_g)$ for any $g \geq 3$ and a similar result for the case of the Torelli group $\mathcal{I}_g$ due to Farb and Ivanov [20].

It can be shown that $\mathcal{K}_g$ coincides with $\mathcal{I}_g$ for $g = 2$. However Johnson [39] proved that the quotient $\mathcal{I}_g/\mathcal{K}_g$ is an infinite group for any $g \geq 3$. More precisely, choose a symplectic basis $x_1, \dots, x_g, y_1, \dots, y_g$ of $H = H_1(\Sigma_g; \mathbb{Z})$ as before. The element

$$\omega_0 = x_1 \wedge y_1 + \cdots + x_g \wedge y_g \in \Lambda^2 H$$

is called the *symplectic class*. It is known that this element is well defined independent of the choice of symplectic bases. It is easy to see that the mapping

$$H \ni u \longmapsto u \wedge \omega_0 \in \Lambda^3 H$$

is injective. Hence $H$ can be considered as a submodule of $\Lambda^3 H$ so that we can consider the quotient module $\Lambda^3 H/H$. Now Johnson [36] constructed a homomorphism

$$\tau\colon \mathcal{I}_g \longrightarrow \Lambda^3 H/H$$

and showed that it is surjective and vanishes on the subgroup $\mathcal{K}_g$. This homomorphism is called now the Johnson homomorphism (see §7 for more details). Later he proved in [39] that $\operatorname{Ker} \tau$ is precisely the subgroup $\mathcal{K}_g$. Thus we have an extension

$$1 \longrightarrow \mathcal{K}_g \longrightarrow \mathcal{I}_g \xrightarrow{\tau} \Lambda^3 H/H \longrightarrow 1.$$

Because of these basic works, sometimes the group $\mathcal{K}_g$ is called the *Johnson subgroup* or Johnson kernel.

To conclude this section, we would like to make a list which indicates the known results concerning finite generation as well as finite presentability of groups such as the automorphism group of $F_n$, the IA automorphism group of $F_n$, the mapping class group, the Torelli group and the Johnson subgroup $\mathcal{K}_g$ (the groups Out $F_n$ and $\mathrm{IOut}_n$ are not included in the list because the results for these groups are the same as Aut $F_n$ and $\mathrm{IAut}_n$ respectively). We refer the readers to the cited original papers and also to the survey papers [84], [10], [34], [37] for details.

| **Groups** | **Generators** | **Relations** |
|---|---|---|
| Aut $F_n$ | finitely generated<br>Nielsen [76]<br>McCool [58] | finitely presented<br>Nielsen [76]<br>McCool [58] |
| $\mathrm{Aut}_+ F_n$ | Gersten [21] | Gersten [21] |
| $\mathrm{IAut}_n$ | finitely generated<br>Magnus [53] | **unknown** $(n \geq 4)$<br>not finitely presentable $(n = 3)$<br>Krstić–McCool [48] |
| $\mathcal{M}_g$ | finitely generated<br>Dehn [17] | finitely presentable<br>McCool [59]<br>Hatcher–Thurston [28] |
| | Lickorish [51]<br>Humphries [30] | finitely presented<br>Wajnryb [85]<br>Gervais [22] |
| $\mathcal{I}_g$ | finitely generated $(g \geq 3)$<br>Johnson [38] | **unknown** $(g \geq 3)$<br>infinitesimal finite presentation<br>Hain [23] |
| | infinitely generated $(g = 2)$<br>Mess [62] | free group $(g = 2)$<br>Mess [62] |
| $\mathcal{K}_g$ | infinitely generated<br>Biss–Farb [11] $(g \geq 3)$ | **unknown** $(g \geq 3)$ |

**Problem 1.** Complete the above list by filling in the "unknown" blanks.

# 7 Johnson homomorphisms

In this section, we define so called *Johnson homomorphisms* which give homomorphisms defined on a certain series of subgroups of the mapping class group into certain abelian groups.

In order to do so, we first describe a method of investigating the structure of a given abstract group $\Gamma$ by approximating it by a series of nilpotent groups. This method is due originally to Malcev [56]. The first approximation is the abelianization $\Gamma^{\mathrm{ab}}$ of $\Gamma$. This can be algebraically expressed as follows. Let $\Gamma_1$ denote the commutator subgroup of $\Gamma$. Namely, it is the subgroup of $\Gamma$ generated by the commutators $[\gamma_1, \gamma_2] = \gamma_1\gamma_2\gamma_1^{-1}\gamma_2^{-1}$ $(\gamma_1, \gamma_2 \in \Gamma)$. It is easy to see that $\Gamma_1$ is a normal subgroup of $\Gamma$ and, as is well known, we have a canonical isomorphism

$$\Gamma^{\mathrm{ab}} \cong \Gamma/\Gamma_1.$$

Next we consider the *second* commutator subgroup $\Gamma_2$ which is defined to be the subgroup of $\Gamma$ generated by the two-fold commutators

$$[\gamma_1, [\gamma_2, \gamma_3]] = \gamma_1[\gamma_2, \gamma_3]\gamma_1^{-1}[\gamma_2, \gamma_3]^{-1} \quad (\gamma_1, \gamma_2, \gamma_3 \in \Gamma).$$

Then it can be checked that $\Gamma_2$ is a normal subgroup of $\Gamma$ and $\Gamma/\Gamma_2$ is a two-step nilpotent group. More precisely, the quotient group $\Gamma_1/\Gamma_2$ is an abelian group and $\Gamma/\Gamma_2$ can be described by the following group extension

$$0 \longrightarrow \Gamma_1/\Gamma_2 \longrightarrow \Gamma/\Gamma_2 \longrightarrow \Gamma/\Gamma_1 = \Gamma^{\mathrm{ab}} \longrightarrow 1 \tag{7.1}$$

which is a *central* extension of $\Gamma/\Gamma_1 = \Gamma^{\mathrm{ab}}$ by $\Gamma_1/\Gamma_2$. We can continue this procedure to obtain a series of nilpotent groups $N_k$ $(k = 1, 2, \dots)$ which approximate $\Gamma$ as follows, where $N_1 = \Gamma^{\mathrm{ab}}$ and $N_2 = \Gamma/\Gamma_2$. We set $\Gamma_0 = \Gamma$ and for each $k = 1, 2, \dots,$ we inductively define

$$\begin{aligned}\Gamma_k &= [\Gamma, \Gamma_{k-1}]\\ &= \text{the subgroup of } \Gamma \text{ generated by } k\text{-fold commutators.}\end{aligned}$$

It can be checked that $\Gamma_k$ is a normal subgroup of $\Gamma$ and the series $\{\Gamma_k\}_k$ of normal subgroups of $\Gamma$ is called the *lower central series* of $\Gamma$. It can be easily checked that these subgroups of $\Gamma$ are all *characteristic subgroups* meaning that any automorphism $f \in \operatorname{Aut}\Gamma$ of $\Gamma$ preserves them. Now we set $N_k = \Gamma/\Gamma_k$ and call this group the $k$-th *nilpotent quotient* of $\Gamma$. In fact, the quotient $C_k = \Gamma_{k-1}/\Gamma_k$ is easily seen to be an abelian group and furthermore we have a central extension

$$0 \longrightarrow C_k \longrightarrow N_k \longrightarrow N_{k-1} \longrightarrow 1 \quad (k = 2, 3, \dots). \tag{7.2}$$

Hence $N_k$ is a $k$-step nilpotent group and we obtain an inverse system

$$\cdots \longrightarrow N_k \longrightarrow N_{k-1} \longrightarrow \cdots \longrightarrow N_3 \longrightarrow N_2 \longrightarrow N_1 = \Gamma^{\mathrm{ab}}$$

of nilpotent groups to which there is a homomorphism from the given group $\Gamma$.

Next we ignore any torsion in the above argument and consider everything over $\mathbb{Q}$. In some sense, this procedure can be understood as taking tensor products with $\mathbb{Q}$. The first step is straightforward. Namely we just take the usual tensor product of $\Gamma^{\text{ab}}$ with $\mathbb{Q}$

$$N_1 \otimes \mathbb{Q} = \Gamma^{\text{ab}} \otimes \mathbb{Q}$$

which is a vector space over $\mathbb{Q}$. For the second step, recall that any central extension of a group $\Gamma$ by an abelian group $C$ is classified by its extension class which is a second cohomology class defined in $H^2(\Gamma; C)$ (see [12]). If we apply this to the central extension (7.1), we obtain a certain element

$$\chi_2(\Gamma) \in H^2(\Gamma^{\text{ab}}; \Gamma_1/\Gamma_2).$$

Application of the natural homomorphisms $\Gamma^{\text{ab}} \to \Gamma^{\text{ab}} \otimes \mathbb{Q}$ and $\Gamma_1/\Gamma_2 \to \Gamma_1/\Gamma_2 \otimes \mathbb{Q}$ to the above element gives

$$\chi_2^{\mathbb{Q}}(\Gamma) \in H^2(\Gamma^{\text{ab}} \otimes \mathbb{Q}; \Gamma_1/\Gamma_2 \otimes \mathbb{Q}).$$

This yields a central extension

$$0 \longrightarrow \Gamma_1/\Gamma_2 \otimes \mathbb{Q} \longrightarrow N_2 \otimes \mathbb{Q} \longrightarrow \Gamma^{\text{ab}} \otimes \mathbb{Q} \longrightarrow 1 \tag{7.3}$$

by which the group $N_2 \otimes \mathbb{Q}$ is defined. We can inductively continue this procedure and we eventually obtain a series of central extensions

$$0 \longrightarrow C_k \otimes \mathbb{Q} \longrightarrow N_k \otimes \mathbb{Q} \longrightarrow N_{k-1} \otimes \mathbb{Q} \longrightarrow 1 \quad (k = 2, 3, \dots). \tag{7.4}$$

In this way, a series of nilpotent groups $\{N_k \otimes \mathbb{Q}\}_k$ is defined. The inverse system

$$\cdots \longrightarrow N_k \otimes \mathbb{Q} \longrightarrow N_{k-1} \otimes \mathbb{Q} \longrightarrow \cdots \longrightarrow N_3 \otimes \mathbb{Q} \longrightarrow N_2 \otimes \mathbb{Q} \longrightarrow N_1 \otimes \mathbb{Q}$$

of nilpotent groups is called the *Malcev completion* of the given group $\Gamma$.

**Example 7.1.** One of the most important examples of the Malcev completions which appears in the theory of free groups as well as the mapping class group is that of free groups. Here we briefly describe it. Let $F_n$ denote a free group of rank $n$ and we denote the abelianization $H_1(F_n; \mathbb{Z})$ of $F_n$ simply by $H$ which is a free abelian group of rank $n$. We consider the free graded Lie algebra generated by the elements of $H$ which we denote by

$$\mathcal{L} = \bigoplus_{k=1}^{\infty} \mathcal{L}_k$$

as follows. The degree 1 part $\mathcal{L}_1$ is defined to be $H$ itself. Then we consider the bracket $[u, v] \in \mathcal{L}_2$ of two elements $u, v \in H$. The skew commutativity $[v, u] = -[u, v]$ of the Lie algebra implies that $\mathcal{L}_2 = \Lambda^2 H$. Next we consider the bracket

$$[\ ,\ ] : \mathcal{L}_1 \otimes \mathcal{L}_2 = H \otimes \Lambda^2 H \longrightarrow \mathcal{L}_3.$$

The Jacobi identity of the Lie algebra forces that the submodule $\Lambda^3 H \subset H \otimes \Lambda^2 H$ (defined by the correspondence $u \wedge v \wedge w \to u \otimes [v, w] + v \otimes [w, u] + w \otimes [u, v]$)

must vanish under the above map. There are no other constraints so that $\mathcal{L}_3 = H \otimes \Lambda^2 H/\Lambda^3 H$. Going further, the complexity of enumerating all the relations imposed by the structure of the Lie algebra increases. However we can avoid this difficulty by embedding $\mathcal{L}$ into the tensor algebra

$$T^*(H) = \bigoplus_{k=1}^{\infty} H^{\otimes k}.$$

The degree 1 parts of $\mathcal{L}$ and $T^*(H)$ are the same, namely both are $H$. The second term can be embedded as

$$\mathcal{L}_2 = \Lambda^2 H \ni [u, v] = u \wedge v \longmapsto u \otimes v - v \otimes u \in H^{\otimes 2}.$$

As for the third term, consider the linear mapping

$$\begin{aligned} &H \otimes \Lambda^2 H \ni u \otimes [v, w] \\ &\qquad \longmapsto u \otimes v \otimes w - u \otimes w \otimes v - v \otimes w \otimes u + w \otimes v \otimes u \in H^{\otimes 3}. \end{aligned}$$

It is easy to check that the kernel of this map is precisely $\Lambda^3 H$ so that we obtain an embedding

$$\begin{aligned} &\mathcal{L}_3 = H \otimes \Lambda^2 H/\Lambda^3 H \ni u \otimes [v, w] \\ &\qquad \longmapsto u \otimes v \otimes w - u \otimes w \otimes v - v \otimes w \otimes u + w \otimes v \otimes u \in H^{\otimes 3}. \end{aligned}$$

Then we can inductively define $\mathcal{L}_k$ as the image of the linear mapping

$$H^{\otimes k} \supset H \otimes \mathcal{L}_{k-1} \ni u \otimes \xi \longmapsto u \otimes \xi - \xi \otimes u \in H^{\otimes k}.$$

Thus $\mathcal{L} = \oplus_k \mathcal{L}_k$ is realized as a submodule of $T^*(H)$. The elements in $\mathcal{L}$ are called *Lie elements* of $T^*(H)$. Now it is a classical result that $\mathcal{L}$ is isomorphic to the graded module associated to the lower central series of the free group $F_n$. Namely there exists a canonical isomorphism

$$(F_n)_{k-1}/(F_n)_k \cong \mathcal{L}_k$$

where $(F_n)_k$ denotes the $k$-th term in the lower central series of $F_n$ (the first one $F_n/(F_n)_1 \cong H$ gives the abelianization). Thus we have a series of central extensions

$$0 \longrightarrow \mathcal{L}_k \longrightarrow N_k(F_n) \longrightarrow N_{k-1}(F_n) \longrightarrow 1 \quad (k = 2, 3, \dots)$$

where $N_k(F_n)$ denotes the $k$-th nilpotent quotient of $F_n$. See [55] for details.

Now we define the Johnson homomorphisms. First we begin with the case of automorphism groups of free groups. This case was considered first by Andreadakis [1] before the works of Johnson.

We can define a series $\operatorname{Aut} F_n(k)$ $(k = 1, 2, \dots)$ of subgroups of $\operatorname{Aut} F_n$ as follows. Any member $(F_n)_k$ in the lower central series of $F_n$ is a characteristic subgroup in the sense that it is preserved by any automorphism $\varphi \in \operatorname{Aut} F_n$. Hence we obtain a series of representations

$$p_k : \operatorname{Aut} F_n \longrightarrow \operatorname{Aut} N_k(F_n) \quad (k = 1, 2, \dots).$$

The first one $p_1$ is nothing but the natural homomorphism

$$\operatorname{Aut} F_n \longrightarrow \operatorname{Aut} N_1(F_n) = \mathrm{GL}(n, \mathbb{Z}).$$

Now we set

$$\operatorname{Aut} F_n(k) = \operatorname{Ker} p_k = \{\varphi \in \operatorname{Aut} F_n; \varphi \text{ acts on } N_k(F_n) \text{ trivially}\}.$$

The first one $\operatorname{Aut} F_n(1)$ is nothing but the subgroup $\mathrm{IAut}_n$.

Now let $\varphi \in \operatorname{Aut} F_n(k)$ be any element. Then for each element $\gamma \in F_n$,

$$\varphi(\gamma)\gamma^{-1} \in (F_n)_k$$

because, by the assumption, $\varphi$ acts on $N_k(F_n) = F_n/(F_n)_k$ trivially. Consider the image of $\varphi(\gamma)\gamma^{-1}$ in $\mathcal{L}_{k+1} = (F_n)_k/(F_n)_{k+1}$ which we denote by $[\varphi(\gamma)\gamma^{-1}]$. This procedure defines a mapping

$$F_n \ni \gamma \longmapsto [\varphi(\gamma)\gamma^{-1}] \in \mathcal{L}_{k+1}.$$

It can be shown that the above mapping factors through the abelianization of $F_n$ so that we obtain a mapping

$$H = (F_n)^{\mathrm{ab}} \longrightarrow \mathcal{L}_{k+1}.$$

Hence we can define a mapping

$$\tau_k : \operatorname{Aut} F_n(k) \longrightarrow \operatorname{Hom}(H, \mathcal{L}_{k+1}) \tag{7.5}$$

by setting

$$\tau_k(\varphi)([\gamma]) = [\varphi(\gamma)\gamma^{-1}] \quad (\gamma \in F_n).$$

Finally it can be checked that the above mapping (7.5) is in fact a *homomorphism* and this is called the $k$-th Johnson homomorphism for the automorphism groups of free groups.

By the definition of the homomorphism $\tau_k$, it is easy to see that

$$\operatorname{Ker} \tau_k = \operatorname{Aut} F_n(k+1).$$

Therefore we have an injection

$$\operatorname{Aut} F_n(k)/\operatorname{Aut} F_n(k+1) \subset \operatorname{Hom}(H, \mathcal{L}_{k+1}).$$

If we make the direct sum over $k$, we obtain an injection

$$\bigoplus_{k=1}^{\infty} \operatorname{Aut} F_n(k)/\operatorname{Aut} F_n(k+1) \ \subset \ \bigoplus_{k=1}^{\infty} \operatorname{Hom}(H, \mathcal{L}_{k+1}).$$

It is known that both of the above graded modules have the structure of graded Lie algebras over $\mathbb{Z}$ and it is a very important problem to identify the left hand side as an explicit Lie subalgebra of the right hand side.

**Problem 2.** Determine the graded module

$$\bigoplus_{k=1}^{\infty} \operatorname{Aut} F_n(k)/\operatorname{Aut} F_n(k+1)$$

associated to the filtration $\{\operatorname{Aut} F_n(k)\}_k$ of the group $\operatorname{Aut} F_n$ as a Lie subalgebra of the graded Lie algebra

$$\bigoplus_{k=1}^{\infty} \operatorname{Hom}(H, \mathcal{L}_{k+1}).$$

Next we consider the case of the mapping class group. Here, for simplicity, we consider only the case of a compact surface $\Sigma_g^0 = \Sigma_g \setminus \operatorname{Int} D^2$ with one boundary component. Then $\Gamma = \pi_1 \Sigma_g^0$ is a free group of rank $2g$. As before, we denote $H_1(\Sigma_g^0; \mathbb{Z})$ simply by $H$ which is a free abelian group of rank $2g$. By Theorem (3.2), the mapping class group $\mathcal{M}_{g,1}$ of $\Sigma_g^0$ is a subgroup of $\operatorname{Aut} F_{2g}$. Hence we can define a filtration $\{\mathcal{M}_{g,1}(k)\}_k$ of $\mathcal{M}_{g,1}$ by simply restricting that of $\operatorname{Aut} F_{2g}$ to the subgroup $\mathcal{M}_{g,1}$. The first term $\mathcal{M}_{g,1}(1)$ in this filtration is nothing but the Torelli group $\mathcal{I}_{g,1}$. It turns out that, in the case of the mapping class group, there are important additional structures which do not exist in the case of $\operatorname{Aut} F_{2g}$. First notice that the Poincaré duality theorem for the (co)homology of the surface implies that there is a natural isomorphism

$$H^* = \operatorname{Hom}(H, \mathbb{Z}) = H^1(\Sigma_g^0; \mathbb{Z}) \cong H.$$

It follows that we can replace the target $\operatorname{Hom}(H, \mathcal{L}_{k+1})$ of the $k$-th Johnson homomorphism (7.5) by $H \otimes \mathcal{L}_{k+1}$. Johnson [36] proved that the image of the first Johnson homomorphism

$$\tau_1 \colon \mathcal{I}_{g,1} \longrightarrow H \otimes \Lambda^2 H$$

is precisely the submodule

$$\Lambda^3 H \subset H \otimes \Lambda^2 H.$$

Generalizing this fact, it was proved in [66] that the target of $\tau_k$ can be narrowed, for any $k$, as follows. We define a submodule $\mathcal{H}_k$ of $H \otimes \mathcal{L}_{k+1}$ by setting

$$\mathcal{H}_k = \operatorname{Ker}\big(H \otimes \mathcal{L}_{k+1} \xrightarrow{[\ ,\ ]} \mathcal{L}_{k+2}\big)$$

where $[\ ,\ ]$ denotes the bracket operation in the graded Lie algebra $\mathcal{L}$. It can be checked that the graded submodule

$$\bigoplus_{k=1}^{\infty} \mathcal{H}_k \subset \bigoplus_{k=1}^{\infty} H \otimes \mathcal{L}_{k+1}$$

is a graded Lie subalgebra.

**Problem 3.** Determine the graded module

$$\bigoplus_{k=1}^{\infty} \mathcal{M}_{g,1}(k)/\mathcal{M}_{g,1}(k+1)$$

associated to the filtration $\{\mathcal{M}_{g,1}(k)\}_k$ of the mapping class group $\mathcal{M}_{g,1}$ as a Lie subalgebra of the graded Lie algebra

$$\bigoplus_{k=1}^{\infty} \mathcal{H}_k.$$

A similar problem for the usual mapping class group $\mathcal{M}_g$ can be formulated by making use of the result of Labute [49].

## 8 Teichmüller space and Outer Space

There are two important spaces on which the mapping class group $\mathcal{M}_g$ and the outer automorphism group $\mathrm{Out}\, F_n$ of a free group act canonically. One is the classical *Teichmüller space*, introduced by Teichmüller in the 1930s, and the other is the *Outer Space* defined in the 1980s by Culler and Vogtmann [16]. Here we briefly describe the definitions of them which can be given in parallel with each other.

The Teichmüller space of $\Sigma_g$, denoted by $\mathcal{T}_g$, is defined to be the space of all the orientation preserving diffeomorphisms

$$f\colon \Sigma_g \longrightarrow M$$

from $\Sigma_g$ to compact Riemann surfaces $M$ of genus $g$ divided by a certain equivalence relation. More precisely

$$\mathcal{T}_g = \{f\colon \Sigma_g \to M;\ M \text{ is a Riemann surface of genus } g\}/\sim$$

where two orientation preserving diffeomorphisms

$$f\colon \Sigma_g \longrightarrow M, \quad f'\colon \Sigma_g \longrightarrow M'$$

are equivalent if there exists an isomorphism $h\colon M \to M'$ of Riemann surfaces (namely a biholomorphism) such that the following diagram

$$\begin{array}{ccc} \Sigma_g & \xrightarrow{\ f\ } & M \\ \| & & \downarrow h \\ \Sigma_g & \xrightarrow{\ f'\ } & M' \end{array}$$

is homotopy commutative. Let $[f\colon \Sigma_g \to M] \in \mathcal{T}_g$ denote the equivalence class represented by $f\colon \Sigma_g \to M$. If we pull back the complex structure on $M$ by the diffeomorphism $f$, we obtain a complex structure on $\Sigma_g$ whose induced orientation coincides with the given orientation on $\Sigma_g$. Since homotopic diffeomorphisms of $\Sigma_g$ are mutually isotopic, the isotopy class of the pull back complex structure on $\Sigma_g$ depends only on the element $[f\colon \Sigma_g \to M]$. Thus we can also write

$$\mathcal{T}_g = \{\text{isotopy classes of complex structure on } \Sigma_g\}.$$

In the cases where $g \geq 2$, a complex structure on $\Sigma_g$ is the same as a hyperbolic structure (that is a Riemannian structure with constant negative curvature $-1$) by the classical uniformization theorem. Therefore we have yet one more description:

$$\mathcal{T}_g = \{\text{isotopy classes of hyperbolic structure on } \Sigma_g\} \quad (g \geq 2).$$

The space of all the complex (or hyperbolic) structures on $\Sigma_g$ has a natural $C^\infty$ topology and it induces a topology on $\mathcal{T}_g$. It is a classical result of Teichmüller that $\mathcal{T}_g$ is homeomorphic to $\mathbb{R}^{6g-6}$ for any $g \geq 2$. The mapping class group $\mathcal{M}_g$ acts on $\mathcal{T}_g$ from the right by

$$\mathcal{T}_g \times \mathcal{M}_g \ni ([f\colon \Sigma_g \to M], \varphi) \longmapsto [f \circ \tilde{\varphi}\colon \Sigma_g \to M] \in \mathcal{T}_g$$

where $\tilde{\varphi} \in \mathrm{Diff}_+\Sigma_g$ is a lift of $\varphi \in \mathcal{M}_g$. This action is known to be properly discontinuous. The quotient space

$$\boldsymbol{M}_g = \mathcal{T}_g / \mathcal{M}_g$$

is called the *moduli space of Riemann surfaces* of genus $g$ which consists of all the isomorphism classes of genus $g$ Riemann surfaces.

The Outer Space $X_n$, which is an analogue of the Teichmüller space where we replace $\mathcal{M}_g$ with $\mathrm{Out}\, F_n$, was defined by Culler and Vogtmann [16] as follows. A *metric graph* $\Gamma$ is a graph (one dimensional finite complex) such that (i) the valencies at vertices are all $\geq 3$ and (ii) there is given a length on every edge such that the sum is equal to 1. Let $R_n$ denote the wedge of labeled $n$ circles $S^1_i$ $(i = 1, \ldots, n)$ so that there is given an isomorphism $\pi_1 R_n \cong F_n$. Then the Outer Space (of rank $n$) is defined as the set of homotopy equivalences $f\colon R_n \to \Gamma$ from $R_n$ to metric graphs $\Gamma$ divided by a certain equivalence relation. More precisely

$$X_n = \{f\colon R_n \to \Gamma; \Gamma \text{ is a metric graph with } \pi_1\Gamma \cong F_n\}/\sim$$

where two homotopy equivalences

$$f\colon R_n \longrightarrow \Gamma, \qquad f'\colon R_n \longrightarrow \Gamma'$$

are equivalent if there exists an isometry $h\colon \Gamma \to \Gamma'$ of metric graphs such that the following diagram

$$\begin{array}{ccc} R_n & \xrightarrow{\;f\;} & \Gamma \\ \| & & \downarrow h \\ R_n & \xrightarrow{\;f'\;} & \Gamma' \end{array}$$

is homotopy commutative. Let $[f\colon R_n \to \Gamma] \in X_n$ denote the equivalence class represented by $f\colon R_n \to \Gamma$. The group $\operatorname{Out} F_n$ acts on $X_n$ from the right by

$$X_n \times \operatorname{Out} F_n \ni ([f\colon R_n \to \Gamma], \varphi) \longmapsto [f \circ \tilde{\varphi}\colon R_n \to \Gamma] \in X_n$$

where $\tilde{\varphi}\colon R_n \to R_n$ is a homotopy equivalence which represents $\varphi \in \operatorname{Out} F_n$. There is a natural topology on $X_n$ and Culler and Vogtmann proved that $X_n$ is contractible and the above action is properly discontinuous. The quotient space

$$\boldsymbol{G}_n = X_n / \operatorname{Out} F_n$$

is called the *moduli space of metric graphs* of rank $n$ which consists of all the isometry classes of metric graphs of rank $n$. We refer to the survey paper [84] by Vogtmann and also Bestvina [5] for recent results concerning the Outer Space as well as $\operatorname{Out} F_n$.

In general, it is a very important problem to determine the (co)homology groups of the moduli spaces associated to various geometrical objects. In the above, we have the moduli spaces of Riemann surfaces and the moduli space of metric graphs. There have been obtained many results concerning the cohomology of these moduli spaces (we refer to the survey papers [26], [45], [83], [84], [71] as well as original papers [25], [73], [64], [65], [63], [52], [29], [15]). However the cohomological structures of them are far from being very well understood.

**Problem 4.** Study the (co)homology groups of the moduli space $\boldsymbol{M}_g$ of Riemann surfaces and the moduli space $\boldsymbol{G}_n$ of metric graphs.

## 9 Symplectomorphism groups of surfaces

As in §1, let $\operatorname{Diff}_+\Sigma_g$ denote the orientation preserving diffeomorphism group of $\Sigma_g$. Let us choose an area form $\omega$ on $\Sigma_g$. Then, by the dimension reason, it can be considered also as a *symplectic form* on $\Sigma_g$. We denote by

$$\operatorname{Symp}\Sigma_g = \{\varphi \in \operatorname{Diff}_+\Sigma_g;\, \varphi^*\omega = \omega\}$$

the subgroup of $\operatorname{Diff}_+\Sigma_g$ consisting of those diffeomorphisms which preserve the form $\omega$. We call it the symplectomorphism group of the symplectic manifold $(\Sigma_g, \omega)$ or the orientation and area preserving diffeomorphism group of $\Sigma_g$ with respect to the area form $\omega$.

As was already mentioned in §1, in general, the volume preserving diffeomorphism group $\operatorname{Diff}^\upsilon M$ of a $C^\infty$ manifold $M$ with respect to a given volume form $\upsilon$ and also the symplectomorphism group $\operatorname{Symp}(M, \omega)$ of a symplectic manifold $(M, \omega)$ are both very important objects of geometry and topology. Recently there has been rapid progress in a topological study of symplectic manifolds, under the name of *symplectic topology* (see [61] for foundations and generalities of this theory).

The case of surfaces is the simplest one. However it is at the same time very important because the symplectic and the volume preserving contexts are the same in this case so that we can expect very rich structures here. In the following, we would like to describe one feature of these structures, namely the one which is induced by a basic concept in symplectic topology called the flux homomorphism.

If we apply Moser's theorem in [72], mentioned in §2, to Symp $\Sigma_g$, we can conclude that the inclusion

$$\mathrm{Symp}\,\Sigma_g \subset \mathrm{Diff}_+\Sigma_g$$

is a homotopy equivalence. It follows that the symplectic mapping class group $\mathcal{SD}(\Sigma_g,\omega)$ of the symplectic manifold $(\Sigma_g,\omega)$ can be canonically identified with the usual mapping class group $\mathcal{M}_g$ and we obtain the following exact sequence

$$1 \longrightarrow \mathrm{Symp}_0\Sigma_g \longrightarrow \mathrm{Symp}\,\Sigma_g \longrightarrow \mathcal{M}_g \longrightarrow 1.$$

In particular, the natural homomorphism Symp $\Sigma_g \to \mathcal{M}_g$ is surjective. Let us see this fact somewhat more explicitly. One form of Moser's theorem cited above can be stated as follows. Let $M$ be a closed oriented $C^\infty$ manifold and let $\upsilon, \upsilon'$ be any two volume forms on $M$. Then there exists a diffeomorphism $\varphi$ of $M$, which can be chosen to be isotopic to the identity, such that $\upsilon' = c\varphi^*\upsilon$ where $c$ is a constant defined by

$$\int_M \upsilon' = c\int_M \upsilon.$$

Now let $\varphi \in \mathcal{M}_g$ be any element and let $\tilde{\varphi} \in \mathrm{Diff}_+\Sigma_g$ be its lift. Consider the form $\tilde{\varphi}^*\omega$ which is another area form on $\Sigma_g$. Hence by the above theorem of Moser, there exists an element $\psi \in \mathrm{Diff}_+\Sigma_g$, which is isotopic to the identity, such that $\tilde{\varphi}^*\omega = \psi^*\omega$. If we set $\tilde{\varphi}' = \tilde{\varphi}\psi^{-1}$, then $(\tilde{\varphi}')^*\omega = \omega$ so that $\tilde{\varphi}'$ belongs to Symp $\Sigma_g$. On the other hand, since $\psi$ is isotopic to the identity, the projection of $\tilde{\varphi}'$ to $\mathcal{M}_g$ is the same as that of $\tilde{\varphi}$ which is the given element $\varphi$. We can now conclude that the mapping Symp $\Sigma_g \to \mathcal{M}_g$ is surjective as required. It might be amusing to observe here that the Dehn twist along a simple closed curve $C$ on $\Sigma_g$, defined in §5, preserves any area form on $\Sigma_g$ whose restriction to a cylindrical neighborhood of $C$ is equal to the 2-form $d\theta \wedge dt$.

Now we describe the *flux homomorphism* briefly (see [61] for details). It is defined for a general symplectic manifold $(M,\omega)$. Let $\mathrm{Symp}_0(M,\omega)$ denote the identity component of Symp $(M,\omega)$ as before. Then the flux homomorphism is a homomorphism

$$\mathrm{Flux}\colon \widetilde{\mathrm{Symp}_0}(M,\omega) \longrightarrow H^1(M;\mathbb{R}) \tag{9.1}$$

from the universal covering group of $\mathrm{Symp}_0(M,\omega)$ to the first real cohomology group of $M$ defined as follows. For each element $\varphi \in \mathrm{Symp}_0(M,\omega)$, let $\varphi_t \in \mathrm{Symp}_0(M,\omega)$ be an isotopy such that $\varphi_0 = \mathrm{id}$ and $\varphi_1 = \varphi$. Then

$$\mathrm{Flux}(\{\varphi_t\}) = \int_0^1 i_{\dot{\varphi}_t}\omega\, dt$$

where $\dot{\varphi}_t$ denotes the vector field associated to $\varphi_t$, which is considered as a one-parameter family of transformations of $M$, and $i$ denotes the interior product. It can be checked that the above value depends only on the homotopy class of the curve $\{\varphi_t\}$

in $\mathrm{Symp}_0(M,\omega)$ with fixed endpoints. Hence we have the induced map described in (9.1). Furthermore it can be checked that Flux is a homomorphism and also that it is surjective. We have an exact sequence

$$1 \longrightarrow \pi_1 \mathrm{Symp}_0(M,\omega) \longrightarrow \widetilde{\mathrm{Symp}}_0(M,\omega) \longrightarrow \mathrm{Symp}_0(M,\omega) \longrightarrow 1$$

and the subgroup

$$\Gamma_\omega = \mathrm{Flux}(\pi_1 \mathrm{Symp}_0(M,\omega)) \subset H^1(M;\mathbb{R})$$

is called the *flux group*. Then (9.1) induces the following homomorphism which is also called the flux homomorphism

$$\mathrm{Flux}\colon \mathrm{Symp}_0(M,\omega) \longrightarrow H^1(M;\mathbb{R})/\Gamma_\omega. \tag{9.2}$$

Very recently, Ono [79] proved a long standing conjecture that the flux group is a *discrete* subgroup of $H^1(M;\mathbb{R})$ for any compact symplectic manifold $M$.

Now in our case of surfaces, by Moser's theorem $\mathrm{Symp}_0\Sigma_g$ is homotopy equivalent to $\mathrm{Diff}_+\Sigma_g$ which in turn is known by Earle and Eells [18] to be homotopy equivalent to $T^2$ for the case $g=1$ and contractible for any $g \geq 2$. Hence we obtain homomorphisms

$$\mathrm{Flux}\colon \mathrm{Symp}_0 T^2 \longrightarrow H^1(T^2;\mathbb{R})/H^1(T^2;c\,\mathbb{Z}),$$
$$\mathrm{Flux}\colon \mathrm{Symp}_0 \Sigma_g \longrightarrow H^1(\Sigma_g;\mathbb{R}) \quad (g \geq 2)$$

where $c$ denotes the total area of $T^2$ with respect to $\omega$.

## 10 Extensions of the Johnson homomorphism and the flux homomorphism

Assume that a group $G$ acts on a module $M$ by automorphisms. In other words, suppose that a homomorphism

$$G \longrightarrow \mathrm{Aut}\, M$$

is given. Then we can give the direct product $M \times G$ a natural structure of a group by setting

$$(m,g)(n,h) = (m+g(n), gh) \quad (m,n \in M,\ g,h \in G).$$

The resulting group is denoted by $M \rtimes G$ and called the *semi-direct product* of $M$ and $G$ or *split extension* of $G$ by $M$.

We mention the following three results which have a similar formal nature to each other. The first one is given in [67], where it was proved that the first Johnson homomorphism $\tau_1\colon \mathcal{I}_{g,1} \longrightarrow \Lambda^3 H$ can be extended to a homomorphism

$$\rho_1\colon \mathcal{M}_{g,1} \longrightarrow \Lambda^3 H_{\mathbb{Q}} \rtimes \mathrm{Sp}(2g,\mathbb{Z})$$

where $H_{\mathbb{Q}} = H \otimes \mathbb{Q}$. Similar results hold for the other types of the mapping class groups $\mathcal{M}_g$ and $\mathcal{M}_{g,*}$. The second one is due to Kawazumi [41] who proved, among other things, that the first Johnson homomorphism $\tau_1 \colon \mathrm{IAut}_n \longrightarrow H^* \otimes \Lambda^2 H$ can be extended to a homomorphism

$$\rho_1 \colon \mathrm{Aut}\, F_n \longrightarrow V_{\mathbb{Q}} \rtimes \mathrm{GL}(n, \mathbb{Z})$$

where $V = H^* \otimes \Lambda^2 H$ and $V_{\mathbb{Q}} = V \otimes \mathbb{Q}$. A similar result holds for $\mathrm{Out}\, F_n$. The third one is given in [46] where it was proved that the flux homomorphism $\mathrm{Flux} \colon \mathrm{Symp}_0 \Sigma_g \to H^1(\Sigma_g; \mathbb{R})$ can be extended to a homomorphism

$$\widetilde{\mathrm{Flux}} \colon \mathrm{Symp}\, \Sigma_g \longrightarrow H^1(\Sigma_g; \mathbb{R}) \rtimes \mathrm{Sp}(2g, \mathbb{Z}).$$

Thus we have the following three commutative diagrams.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{I}_{g,1} & \longrightarrow & \mathcal{M}_{g,1} & \longrightarrow & \mathrm{Sp}(2g, \mathbb{Z}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \rho_1} & & \| & & \\
1 & \longrightarrow & \Lambda^3 H & \longrightarrow & \Lambda^3 H_{\mathbb{Q}} \rtimes \mathrm{Sp}(2g, \mathbb{Z}) & \longrightarrow & \mathrm{Sp}(2g, \mathbb{Z}) & \longrightarrow & 1
\end{array}$$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{IAut}_n & \longrightarrow & \mathrm{Aut}\, F_n & \longrightarrow & \mathrm{GL}(n, \mathbb{Z}) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \tau_1} & & \downarrow{\scriptstyle \rho_1} & & \| & & \\
1 & \longrightarrow & V & \longrightarrow & V_{\mathbb{Q}} \rtimes \mathrm{GL}(n, \mathbb{Z}) & \longrightarrow & \mathrm{GL}(n, \mathbb{Z}) & \longrightarrow & 1
\end{array}$$

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathrm{Symp}_0 \Sigma_g & \longrightarrow & \mathrm{Symp}\, \Sigma_g & \longrightarrow & \mathcal{M}_g & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{Flux}} & & \downarrow{\scriptstyle \widetilde{\mathrm{Flux}}} & & \| & & \\
1 & \longrightarrow & H_{\mathbb{R}} & \longrightarrow & H_{\mathbb{R}} \rtimes \mathcal{M}_g & \longrightarrow & \mathcal{M}_g & \longrightarrow & 1
\end{array}$$

where $H_{\mathbb{R}} = H^1(\Sigma_g; \mathbb{R})$.

The first two diagrams can be extended further by considering higher nilpotent quotients of the Torelli group $\mathcal{I}_{g,1}$ and the group $\mathrm{IAut}_n$. However, as for the last one, there is no such extension because the kernel of the flux homomorphism, which is denoted by $\mathrm{Ham}\, \Sigma_g$ and called the Hamiltonian symplectomorphism group, is known to be perfect by Thurston [82] (see also Banyaga [3] for the generalization of this fact to general symplectic manifolds). There are several results which make use of the above three commutative diagrams, see [68], [42], [43], [70], [41], [46], [47] and references in them. However it seems likely that there should exist further interesting facts to be uncovered along these lines.

**Problem 5.** Give further applications as well as generalizations of the above three commutative diagrams.

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