Constructing Hyperbolic Polygon Tessellations & The Fundamental Group of Their Surfaces

How wallpapers and donuts are (kinda) the same thing.

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Figure 1: M. C. Escher's Circle Limit III [4]. A hyperbolic tessellation of octagons, three of which meet at every vertex, is a natural overlay to the image.

Abstract

Tessellations of the Euclidean and hyperbolic planes are a useful tool for bringing together topology, geometry, and group theory. Surfaces can be created by identifying together edges of polygons. Under specific circumstances, there exist isometry groups which act on those polygons to create tessellations of the plane that are isomorphic to the fundamental group of their corresponding surfaces. In hyperbolic space, regular polygons of any number of sides can tessellate the space. We will prove the Poincaré Polygon Theorem to show which types of regular polygons create a tessellation in hyperbolic space. Then we will determine which polygon tessellations have isometry groups that are isomorphic to their polygon's corresponding surface fundamental group. We will also investigate methods of teaching selections of these topics to a 9th grade geometry class.

Contents

1 Introduction

It is one of the great wonders of mathematics to be able to connect two seemingly unrelated items via a bridge that, when fully comprehended, causes one to wonder how such items could have ever been considered different. One such subtle and fascinating connection lies in the construction of polygon tessellations and the fundamental groups of embedded surfaces. This thesis seeks to show that, under the right circumstances, certain repeated patterns and embedded surfaces can be two different ways of looking at the same thing.

More specifically, we aim to answer a number of questions. First, how can we verify if a collection of polygons evenly covers the plane with no overlap or infinite concentrations? Second, when can we guarantee that a polygon and a group acting on that polygon will form such a tessellation? Third, what kinds of tessellations have connections to groups of loops on surfaces?

In this paper, we will address each of these questions. In particular, we will present a modified and simplified version of the Poincaré Polygon Theorem, using Alan F. Beardon's 1983 exposition of a proof for the theorem [1, Section 9.8]. The Poincaré Polygon Theorem states that, given a polygon and a group of isometries generated by a pairing of the polygon's sides, we can guarantee that the polygon under the action of that group tessellates the plane so long that its angles and side-pairing maps fulfill a short list of criteria. We will also present an original proof that connects isometries of a covering space with the fundamental group of the space being covered. At the conclusion of the paper, we will see that the group of isometries in the Poincaré Polygon Theorem is, under certain circumstances, isomorphic to that group of isometries of a covering space. As may already be apparent, it is assumed that the reader is equipped with a background knowledge of basic group theory and topology.

2 Background Information & Definitions

Before getting started, some preliminaries are in order. In this section, we will take time to familiarize ourselves with working in hyperbolic space and the subtleties that come with it. We will also discuss some algebraic topology to give a background for connections and theorems to come.

2.1 Hyperbolic Geometry

The only regular polygons that tessellate the Euclidean plane $\mathbb{R}^2$ are triangles, squares, and hexagons [3, Section 5.1]. Other, non-regular polygons will tessellate $\mathbb{R}^2$, but if we restrict our attention to tessellations by regular polygons, it is far more interesting to work within hyperbolic geometry.

We will need to understand the governing principles of hyperbolic geometry and establish a metric space to work in. The following narrative is mainly inspired by [8] and [9, Section 6.3]. The governing feature of Euclidean geometry is the Euclidean Parallel Postulate, which states that given a line l and a point P not on l, there is exactly one line through P that is parallel to l, i.e., that does not intersect l. Modifying this postulate to allow for infinitely many lines to intersect P but not l induces a different geometry, which we call hyperbolic. The two-dimensional hyperbolic plane (referred to as $\mathbb{H}^2$) can be nicely modeled by an open unit disc $D^2 \subset \mathbb{R}^2$, called the Poincaré Disc Model. In the Poincaré Disc Model, shortest curves between points are modeled as segments of circles which intersect the boundary of the disc orthogonally (they rarely resemble Euclidean lines).

We move now towards a formal definition of the terms introduced above. Over the course of the following set of definitions, the reader should become familiar with treating the unit disc as the entirety of two-dimensional hyperbolic space.

Definition 2.1 ([2] Section 2.7)**.** The **hyperbolic disc** $\mathbb{B}^2$, known as the Poincaré Disc Model of two-dimensional hyperbolic geometry, is constructed using the open unit disc in the complex plane:

$$\mathbb{B}^2 = \{(a+bi) \in \mathbb{C} : \sqrt{a^2+b^2} < 1\}.$$

In this disc, the **hyperbolic norm** of a vector $\vec{v}$ beginning at the point $(a+bi) \in \mathbb{B}^2$ is:

$$\|\vec{v}\|_{hyp} = \frac{2\|\vec{v}\|_{euc}}{1-a^2-b^2},$$

where $\|\vec{v}\|_{euc}$ is the Euclidean norm of the vector $\vec{v}$.

Note that the further $(a + bi)$ is from the origin (the center of the disc), the larger its hyperbolic norm becomes. This means that the closer the base point of a vector in the hyperbolic disc is to the boundary of the disc, the smaller its *Euclidean* norm must be in order to preserve an identical *hyperbolic* norm.

Let us now turn to constructing a hyperbolic metric in the following manner:

Definition 2.2 ([2] Section 2.7)**.** Given a piecewise differentiable curve γ from a point P to a point Q in $\mathbb{B}^2$ parameterized as $t \mapsto (x(t) + y(t)i)$ for $a \leq t \leq b$, we define **hyperbolic length** in the following way:

$$l_{hyp}(\gamma) = \int_a^b \|(x(t) + y(t)i)'\|_{hyp} dt$$

Using this definition of length, given two points $P, Q \in \mathbb{B}^2$ we can define the **hyperbolic distance** in the following way:

$$d_{hyp}(P, Q) = \inf\{l_{hyp}(\gamma) : \gamma \text{ with endpoints } P, Q\}$$

The following theorem we state without proof, since its proof is quite straightforward.

Theorem 2.3. *The distance d_{hyp} as it is defined above is a metric, and therefore $(\mathbb{B}^2, d_{hyp})$ is a metric space.*

As a brief remark, we note that the standard topology of $(\mathbb{B}^2, d_{hyp})$ is defined similarly to the standard topology of $\mathbb{R}^2$ via open balls using the hyperbolic metric.

Understanding and visualizing distance in the hyperbolic disc is not at all similar to its Euclidean counterpart. Figure 2 provides an illustration of this phenomenon. The hyperbolic distances between each of the three pairs of points in the figure are (counterintuitively) equal.

Conceptually, it is important to understand what the hyperbolic analogy to a line segment in Euclidean space is. To keep things clear, we will use a different piece of terminology. To do so, we state the following theorem without proof, from [2, Chapter 2].

Theorem 2.4. *Given two points $P, Q \in \mathbb{B}^2$, there exists a unique curve γ that intersects P and Q which has hyperbolic length between P and Q equal to $d_{hyp}(P, Q)$. This unique curve, which we call the* ***geodesic*** *between P and Q, is modeled in the Poincaré Disc Model as a segment of the circle that passes through P and Q that is orthogonal to the (possibly infinite) circle bounding $\mathbb{B}^2$.*

Geodesics, like Euclidean lines, are uniquely defined by two points. Two distinct intersecting geodesics induce angles in the same way that Euclidean lines do. All of this groundwork has paved the way to allow us to define polygons in the hyperbolic disc. By considering a collection of geodesics that intersect in a specific way, we can build a definition of hyperbolic polygons that fits with intuition.

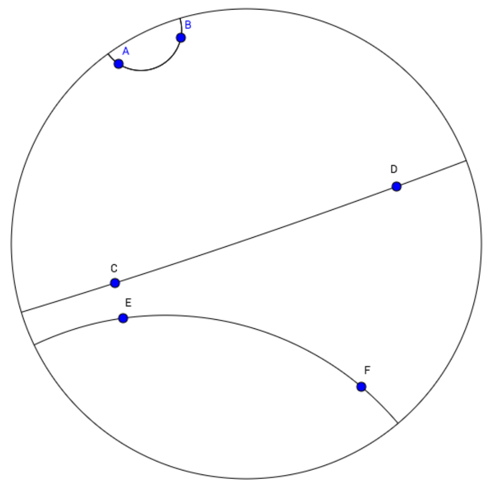


Figure 2: The Poincaré Disc Model, a unit disc in the complex plane. The hyperbolic distance between points A and B is equal to the hyperbolic distance between C and D as well as E and F.

Definition 2.5. A subset $P \subset \mathbb{B}^2$ is a **hyperbolic polygon** if P is an open region bounded by a countable set of geodesic segments, called **sides**, which meet at a countable number of endpoints called **vertices**. We say a hyperbolic polygon is **regular** if every side has equal hyperbolic length. We call the set of sides of a hyperbolic polygon ∂P, and therefore define **hyperbolic closure** of P to be the disjoint union $\overline{P} = \partial P \cup P$.

Furthermore, we say that a polygon P is **convex** if the geodesic between any two points in $\overline{P}$ is fully contained within $\overline{P}$.

For the purposes of this paper, we will only be considering bounded hyperbolic polygons with a finite number of sides and vertices. We also wish only to consider polygons with sides that do not have ideal vertices (i.e., edges which meet at infinity, the boundary of $\mathbb{B}^2$). There do exist (countably) infinite-sided polygons in $\mathbb{B}^2$, but they will not be featured in the scope of this text.

With all this talk of geodesics, it will be useful to be explicit about how a length-preserving transformation would act in this geometric setting:

Definition 2.6. In the hyperbolic disc $(\mathbb{B}^2, d_{hyp})$, an **isometry** is any bijective map $\phi : \mathbb{B}^2 \to \mathbb{B}^2$ such that $d_{hyp}(P, Q) = d_{hyp}(\phi(P), \phi(Q))$ for all points $P, Q \in \mathbb{B}^2$.

Conveniently, all isometries of the hyperbolic disc can be represented in a specific way. We will state a theorem without proof that gives the general form for hyperbolic isometries.

Theorem 2.7 ([2] Section 2.7)**.** *All hyperbolic isometries $\varphi : \mathbb{B}^2 \to \mathbb{B}^2$ can be written in one of two ways:*

$$\varphi(z) = \frac{\alpha z + \beta}{\overline{\beta} z + \overline{\alpha}}$$
$$\varphi(z) = \frac{\alpha \overline{z} + \beta}{\overline{\beta} \overline{z} + \overline{\alpha}}$$

for some $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 - |\beta|^2 = 1$.

These isometries may have some fixed points, but ultimately map most points in $\mathbb{B}^2$ to other points in $\mathbb{B}^2$. While these transformations do preserve hyperbolic length, we will see in Example 3.8 how Euclidean length is not preserved under these transformations.

2.2 Covering Spaces & The Fundamental Group

Now we will focus on establishing some properties in algebraic topology and return to considering the hyperbolic disc later. A few preliminaries are in order, namely, to define some key terms and useful theorems. Recall the definition of a covering space:

Definition 2.8 ([7] Section 11.1)**.** Let X be a topological space. The pairing of a space $\tilde{X}$ and a function $p : \tilde{X} \to X$ is a **covering space** of the space X if the following conditions are satisfied:

1. $p : \tilde{X} \to X$ is a continuous, surjective map.
2. For all $x \in X$, there exists a neighborhood U_x of x such that $p^{-1}(U_x)$ is a disjoint union of open sets $\tilde{U}_\alpha \subset \tilde{X}$, called **sheets**.
3. For each α, the sheet $\tilde{U}_\alpha$ is homeomorphic to U_x via the function p.

Recall too the definition of the fundamental group of a topological space:

Definition 2.9 ([7] Section 9.2)**.** Given a space X and a point $x \in X$, the **fundamental group** $\pi_1(X, x)$ of X is defined as the set of all homotopy classes for paths that begin and end at the point x. This is a group under path composition.

The fundamental group often provides a new perspective on a group from an algebraic topological frame. Understanding the way groups apply to classes of paths on surfaces offers us more nuance in considering groups overall.

We will now see that path-connectedness along with a trivial fundamental group combine to make an interesting type of topological space:

Definition 2.10 ([7] Section 9.2)**.** If X is a path connected space and $\pi_1(X, x)$ is trivial, then we call X a **simply connected** space.

As it turns out, simple-connectedness is a compelling feature for a covering space to possess. We will see a particularly poignant example of this later on.

Theorem 2.11 ([5] Section 1.3)**.** *If $(\tilde{X}, p)$ is a covering space for a simply connected space X, then X is homeomorphic to $(\tilde{X}, p)$. We therefore say that X is the **universal cover** for any spaces that it covers.*

When dealing with topological spaces, we talk of continuous maps and homeomorphisms and how they affect open sets within those spaces. For our current purposes, we will want to talk about maps between covering spaces that map individual points in a particular way. The next couple of definitions and propositions, which we state without proof, focus on thinking about these types of maps and how they act on covering spaces.

Definition 2.12 ([7] Section 11.2)**.** Given a space X, a covering space $(\tilde{X}, p)$, a connected space A, and a continuous function $f : A \to X$, we define a **lift** of f to be a continuous function $\tilde{f} : A \to \tilde{X}$ such that $p \circ \tilde{f} = f$.

Proposition 2.13 ([5] Section 1.3)**.** ***Unique Lifting Property.*** *Given a space X, a covering space $(\tilde{X}, p)$, a connected space A, and a continuous function $f : A \to X$, we know that two lifts $\tilde{f}_1, \tilde{f}_2 : A \to \tilde{X}$ of f are either everywhere equal ($\tilde{f}_1(a) = \tilde{f}_2(a)$ for all $a \in A$) or nowhere equal ($\tilde{f}_1(a) \neq \tilde{f}_2(a)$ for all $a \in A$).*

This is enough for now. Now that we know what we are talking about when we consider hyperbolic space and some basic principles of topology, we can move forward to Section 3: a discussion on how to work with hyperbolic polygons towards building a tessellation of the hyperbolic disc. In Section 4, we will see that coverings spaces will be useful in proving an instrumental proof related to polygon tessellations of the hyperbolic disc.

3 Tessellations of the Hyperbolic Disc

In this section, we will become familiar with what formally constitutes a tessellation of hyperbolic space, continuing to work within the Poincaré Disc Model notated as the metric space $(\mathbb{B}^2, d_{hyp})$.

3.1 Definition of a Tessellation

We will need to formally define a tessellation before we can determine what kind of polygons tessellate the hyperbolic disc. The following definition was derived from [2, Section 6.1], but has been made more specialized to hyperbolic space and defined more explicitly.

Definition 3.1. Consider the hyperbolic plane in the disc model given by $(\mathbb{B}^2, d_{hyp})$. We say that a countable collection $\{X_i\}$ of open subspaces $X_i \subset \mathbb{B}^2$ **tessellates** $\mathbb{B}^2$ if the following conditions are met:

1. **Polygons:** Each X_i is a connected hyperbolic polygon.

2. **Isometric Mapping:** Any two X_i, X_j are isometric, i.e., there exists some isometry g_{ij} of the hyperbolic disc such that $g_{ij}(X_i) = X_j$ for all i, j.

3. **Disc Cover:** $\bigcup_{i=1}^{\infty} \overline{X_i} = \mathbb{B}^2$, i.e., every point in $\mathbb{B}^2$ is contained in a set $\overline{X}_i$ for some i.

4. **No Overlap:** The intersection of any two distinct X_i, X_j is empty, and the intersection of any two $\overline{X_i}, \overline{X_j}$ can be only $\partial X_i \cap \partial X_j$.

5. **Local Finiteness:** At no point does there exist an infinite concentration of polygons X_i. In other words, given any point $P \in \mathbb{B}^2$, there exists an $\varepsilon > 0$ and epsilon-ball $B_\varepsilon(P) = \{Q \in \mathbb{B}^2 : d_{hyp}(P,Q) < \varepsilon\}$ such that:

$$\big(B_\varepsilon(P) \cap \bigcup_{i=1}^{\infty} \overline{X_i}\big) = \big(B_\varepsilon(P) \cap \bigcup_{j=1}^{n} \overline{X_j}\big)$$

for some finite sub-collection $\{X_j\}$ of $\{X_i\}$.

This is a lengthy definition but a thorough one. Before we continue, we will look at an example using a more familiar metric space than the Poincaré disc.

Example 3.2. For a moment, we depart from the usage of the hyperbolic disc $(\mathbb{B}^2, d_{hyp})$ and switch to considering the Euclidean metric space $(\mathbb{R}^2, d_{euc})$, where d_{euc} is the standard Euclidean metric. Let $R \subset \mathbb{R}^2$ be the region defined as $(0,1) \times (0,1)$. For a point $(x,y) \in \mathbb{R}^2$, let G be defined as the set of translation maps $\varphi_{n,m} : \mathbb{R}^2 \to \mathbb{R}^2$ where $\varphi_{n,m}(x,y) = (x+n, y+m)$ and $n, m \in \mathbb{Z}$. It is clear that G is a group under composition, and that $G \simeq \mathbb{Z} \times \mathbb{Z}$. Consider the orbit of R under G, which is the collection $\{g(R)\}_{g \in G}$ (the collection of all elements of G acting on R). We will show that $\{g(R)\}_{g \in G}$ fulfills Definition 3.1 for $(\mathbb{R}^2, d_{euc})$. For illustration, refer to Figure 3 to visualize this tessellation.

1. **Polygons:** It is straightforward to see that R is a connected polygon. Since every $\varphi_{n,m} \in G$ is an isometry (see Definition 2.6), it follows that for all $\varphi_{n,m} \in G$, $\varphi_{n,m}(R)$ is also a connected polygon. Therefore each element of $\{g(R)\}_{g \in G}$ is a connected polygon.

2. **Isometric Mapping:** Since G is a group of isometries, it immediately follows that any two elements in $\{g(R)\}_{g \in G}$ are isometric.

3. **Plane Cover:** First note that $\overline{\varphi_{n,m}(R)} = [n, 1+n] \times [m, 1+m]$ for some $n, m \in \mathbb{Z}$. Consider any point $(x,y) \in \mathbb{R}^2$. We know that $n_x \le x < n_x + 1$ and $m_y \le y < m_y + 1$ for some $n_x, m_y \in \mathbb{Z}$. Then it follows that $(x,y) \in \overline{\varphi_{n_x,m_y}(R)}$. We have then shown that $\bigcup_{n,m \in \mathbb{Z}} \overline{\varphi_{n,m}(R)} = \mathbb{R}^2$.

4. **No Overlap:** It is clear that for distinct $\varphi_{n_1,m_1}, \varphi_{n_2,m_2} \in G$, we have $(n_1, 1+n_1) \times (m_1, 1+m_1) \cap (n_2, 1+n_2) \times (m_2, 1+m_2) = \emptyset$. It is also easy to see that the intersection of any $[n_1, 1+n_1] \times [m_1, 1+m_1] \cap [n_2, 1+n_2] \times [m_2, 1+m_2]$ will be of a form akin to $[n_1, 1+n_1] \times \{m_2\}$

or $\{(n_2, m_2)\}$ where $n_1 + 1 = n_2$ and $m_1 + 1 = m_2$. As such, we can see that the intersection of any two distinct $\overline{\varphi_{n_1,m_1}(R)}, \overline{\varphi_{n_2,m_2}(R)}$ can only be $\partial\varphi_{n_1,m_1}(R) \cap \partial\varphi_{n_2,m_2}(R)$.

5. **Local Finiteness:** Consider any point $(x, y) \in \mathbb{R}^2$. If $x, y \notin \mathbb{Z}$, then we can let $\varepsilon = \inf\{d_{euc}((x, y), (x^*, y^*)) : x^*, y^* \in \mathbb{Z}\}$. Then:

$$\bigcup_{n,m\in\mathbb{Z}} \overline{\varphi_{n,m}(R)} \cap B_\varepsilon(x, y) = \overline{\varphi_{n^*,m^*}(R)}$$

for some n^*, m^* by the same reasoning used in proving criterion 4. Similar reasoning can be used to show that there exists an $\varepsilon > 0$ such that a finite union of elements in $\{g(R)\}_{g\in G}$ if one or both of x, y are elements of $\mathbb{Z}$.

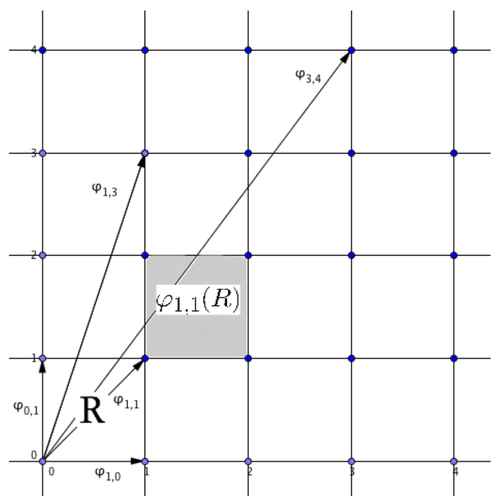


Figure 3: The tessellation of squares from Example 3.2 given by $\{g(R)\}_{g\in G}$ along with a few examples of elements $\varphi_{n,m} \in G$. The shaded box represents the transformation of the entire region R by the element $\varphi_{1,1}$.

3.2 Fuchsian Groups & Fundamental Domains

Our Euclidean example made use of our intuitive concept of Euclidean isometries, but it is not always so easy to intuit the building blocks of a hyperbolic tessellation. To understand how to construct a hyperbolic tessellation, we turn to fundamental domains and Fuchsian groups. A fundamental domain is a subset of the Poincaré disc that can be acted on by a group of hyperbolic isometries (a Fuchsian group) to form a tessellation of $\mathbb{B}^2$ via Definition 3.1. In order to understand and make use of these tools, we first need to develop some vocabulary. The following definitions were adapted from [1, Section 1.5], [1, Section 2.3], and [1, Section 8.1].

Definition 3.3. A **topological group** G is both a group and a topological space such that the following functions are continuous:

1. $f_\times : G \times G \to G$ where $f_\times(x, y) = xy$
2. $f_{\text{inv}} : G \to G$ where $f_{\text{inv}}(x) = x^{-1}$

Furthermore, two topological groups G and H are **isomorphic** if there exists a bijective map $g : G \to H$ that is both a group isomorphism and a homeomorphism between G and H.

Example 3.4. The real numbers $\mathbb{R}$ with the standard topology and the binary operation of addition forms a topological group. We state without proof that the following two functions are continuous:

1. $f_\times : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ where $f_\times(x, y) = x + y$
2. $f_{\text{inv}} : \mathbb{R} \to \mathbb{R}$ where $f_{\text{inv}}(x) = -x$

Definition 3.5. A topological group G is **discrete** if the topology on G is the discrete topology. Recall that all subsets of a set are open with respect to the discrete topology.

Example 3.6. Consider the group of integers $\mathbb{Z}$ under addition with the discrete topology. Since every subset of a space is open with respect to the discrete topology, $\mathbb{Z}$ is trivially a topological group.

Definition 3.7. We say that a topological group G is a **Fuchsian group** if it is a group of isometries of the hyperbolic disc $\mathbb{B}^2$ with the discrete topology with respect to the standard topology on $\mathbb{B}^2$.

Example 3.8. Let G be the group generated by the map $\varphi : \mathbb{B}^2 \to \mathbb{B}^2$ given by

$$\varphi(z) = \frac{3z + (2 + 2i)}{(2 - 2i)z + 3}.$$

We see that this φ fulfills Theorem 2.7 by assigning $\alpha = 3$ and $\beta = 2+2i$ so that $|\alpha|^2 - |\beta|^2 = 3^2 + 0 - 2^2 - 2^2 = 1$. Therefore, we know that G is generated by a hyperbolic isometry. We can assign the discrete topology to G and determine that it is a Fuchsian group. By solving for $\varphi(z) = z$, we find that the fixed points of this particular transformation are $\pm i$, which are not contained within $\mathbb{B}^2$. As a result, every element in G must include a translation in the hyperbolic disc since there are no points in $\mathbb{B}^2$ that are fixed by φ, the generator of G. See Figure 4 for an illustration of G acting upon the origin of $\mathbb{B}^2$.

We now turn to a discussion of important regions known as fundamental domains. Given some subset P (for our purposes that subset would be a polygon), the question to ask is whether a particular Fuchsian group G acts on that subset in such a way that every point in the hyperbolic disc can be written as $g(x)$ for some $g \in G$ and $x \in \overline{P}$. The formal definitions follow, as adapted from [1, Section 9.1].

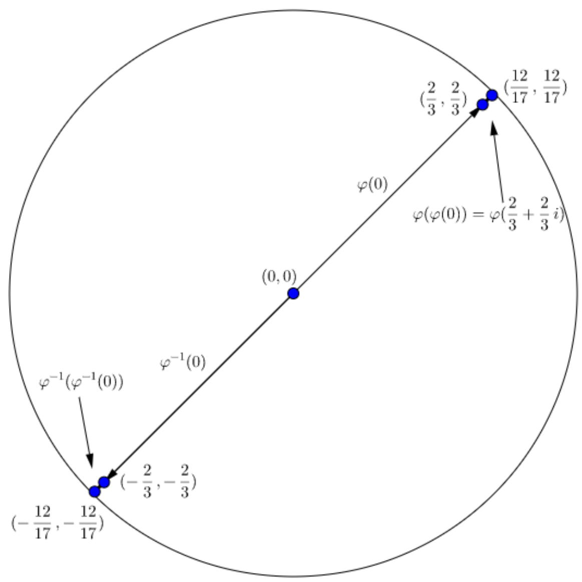


Figure 4: Diagram showing the use of the map φ and its inverse from Example 3.8 on the point $0+0i$. Note that the hyperbolic distance between 0 and $(\frac{2}{3}+\frac{2}{3}i)$ is the same as the hyperbolic distance between $(\frac{2}{3}+\frac{2}{3}i)$ and $(\frac{12}{17}+\frac{12}{17}i)$.

Definition 3.9. Let G be a Fuchsian group acting on the Poincaré disc $(\mathbb{B}^2, d_{hyp})$. A subset $F \subset \mathbb{B}^2$ is a **fundamental set** for G if F contains exactly one point from every orbit of elements in G. In more formal terms:

$$(1) \qquad \bigcup_{g \in G} g(F) = \mathbb{B}^2$$

$$(2) \qquad g(x) \neq y \text{ for all } x, y \in F, g \in G, \text{ and } x \neq y$$

To go one step further and impose a few more restrictions, we get a fundamental domain, which under the group action of a Fuchsian group G tessellates the plane.

Definition 3.10. Let G be a Fuchsian group acting on the Poincaré disc $(\mathbb{B}^2, d_{hyp})$. An open subset $E \subset \mathbb{B}^2$ is a **fundamental domain** for G if there exists a fundamental set F such that $E \subset F \subset \overline{E}$.

Furthermore, it is easy to consider a polygon that qualifies as a fundamental domain:

Definition 3.11. Given a Fuchsian group G, we say that a hyperbolic polygon $P \subset \mathbb{B}^2$ is a **fundamental polygon for** G if P is a convex fundamental domain for G.

Example 3.12. Returning to our Euclidean analog to hyperbolic geometry, the region R from Example 3.2 is a convex polygon, and as such is a fundamental polygon for the group $\mathbb{Z} \times \mathbb{Z}$.

We now have the tools to marry our concept of domains and Fuchsian groups together with Definition 3.1, showing that Fuchsian groups and fundamental domains give the necessary building blocks for a tessellation as previously defined.

Proposition 3.13. *Given a Fuchsian group G for which there exists a connected fundamental polygon $E \subset (\mathbb{B}^2, d_{hyp})$, the collection $\{g_i(E)\}$ for $g_i \in G$ tessellates $(\mathbb{B}^2, d_{hyp})$.*

Proof. There are five criteria from Definition 3.1 to fulfill in order to show that the set E as outlined above tessellates the hyperbolic disc.

1. By assumption, E is a connected polygon. Because G is a group of isometries acting on $(\mathbb{B}^2, d_{hyp})$, it follows that for all $g \in G$, $g(E)$ is also a connected polygon. Therefore each element of $\{g_i(E)\}$ is a connected polygon.

2. Any two elements (distinct or otherwise) of $\{g_i(E)\}$ are isometric since every element of $\{g_i(E)\}$ is obtained by performing some isometry upon E.

3. Because E is a fundamental domain, we know that there exists a fundamental set F such that $E \subset F \subset \overline{E}$. Because $\bigcup_{g \in G} g(F) = \mathbb{B}^2$, it follows that $\bigcup_{g \in G} g(\overline{E}) = \mathbb{B}^2$ since F is a fundamental set and $F \subset \overline{E}$.

4. We know that each $g(\overline{E})$ for all $g \in G$ is a polygon together with its sides. Because $E \subset F$ and no two distinct points $x, y \in F$ have $g(x) = y$ for any $g \in G$ (since F is a fundamental set), it follows that for all $g \in G$ not equal to the identity map, we have $E \cap g(E) = \emptyset$. Since $\overline{E} = E \cup \partial E$ and ∂E is the set of sides of E, it follows that any two distinct elements of $\{g_i(\overline{E})\}$ would share only edges and vertices and not interiors.

5. Consider a point $x \in \mathbb{B}^2$ that lies in $g_1(E)$ for some $g_1 \in G$. We want to show that there exists an $\varepsilon > 0$ such that $B_\varepsilon(x)$ intersected with $\bigcup_{g \in G} g(\overline{E})$ is equal to $B_\varepsilon(x)$ intersected with the union of a finite subcollection of $\{g_i(\overline{E})\}$. Because the set $g_1(E)$ is open, we know that there exists an $\varepsilon > 0$ such that the following holds:

$$B_\varepsilon(x) \cap \bigcup_{g \in G} g(\overline{E}) = B_\varepsilon(x) \cap g_1(E)$$

Next, consider a point $x \in \partial g(E)$ for some $g \in G$. If x in only one side of $g(E)$, then we can find a $\varepsilon > 0$ such that no other side of $g(E)$ is contained in $B_\varepsilon(x)$. We can perform a hyperbolic reflection of $g(E) \cap B_\varepsilon(x)$ over this side and see that, because of the hyperbolic distance-preservation of a hyperbolic reflection, $B_\varepsilon(x)$ is exactly the union of $g(E) \cap B_\varepsilon(x)$ and its reflection. Therefore, since every point in $\mathbb{B}^2$ can be represented as a g_i-image of some point in $\overline{E}$, we can find a $g' \in G$ such that:

$$B_\varepsilon(x) \cap \bigcup_{g \in G} g(\overline{E}) = B_\varepsilon(x) \cap (g(\overline{E}) \cup g'(\overline{E}))$$

Next, consider a point $x \in \partial g(E)$ that lies in two distinct sides of $g(E)$ (i.e. x is a vertex of $g(E)$). We can find an $\varepsilon > 0$ such that $B_\varepsilon(x)$ contains no other vertex of $g(E)$. Since we are only considering polygons with edges that do not meet at infinity, each vertex of $g(E)$ must have a nonzero angle measure. Thus, only a finite number of isometries is necessary to map $g(E) \cap B_\varepsilon(x)$ around x such that every point in $B_\varepsilon(x)$ is contained in an image of $g(E) \cap B_\varepsilon(x)$. By similar logic as before, we can see that this $B_\varepsilon(x)$ intersects a finite sub-collection of elements of $\{g_i(\overline{E})\}$. Therefore:

$$B_\varepsilon(x) \cap \bigcup_{g \in G} g(\overline{E}) = B_\varepsilon(x) \cap \bigcup_{j=1}^{n} g_j(\overline{E})$$

for some $n \in \mathbb{Z}$ and collection $\{g_j\}$.

By fulfilling the five criteria of Definition 3.1, we have shown that the collection $\{g_i(E)\}$ tessellates $\mathbb{B}^2$. □

We have seen that by starting with a Fuchsian group and an associated polygon that is a fundamental domain, we can create a tessellation. This is useful, but we want to take it a step further and begin with a fundamental polygon and see what conditions guarantee the existence of a Fuchsian group with that same polygon as a fundamental domain. This is the process that will be explored in Section 4.

4 Constructing Tessellations of the Hyperbolic Disc: The Poincaré Polygon Theorem

The central point of this section is to discuss the *Poincaré Polygon Theorem.* We have established that a Fuchsian group acting on a fundamental polygon creates a tessellation of the hyperbolic disc. In fact, given any Fuchsian group, we can guarantee the existence of a convex fundamental polygon that tessellates $\mathbb{B}^2$ under that group's action [1, Section 9.4]. We have yet to begin with a polygon and ask: when can we guarantee the existence of a Fuchsian group for which that polygon is fundamental? The Poincaré Polygon Theorem tackles this question.

Some of the logic and structure for this proof is taken from Beardon [1, Section 9.8], but I have reduced it to the hyperbolic case (Beardon proves it more generally) and reordered for clarity and relevance to my topic. A quick definition (also from Beardon) is in order first.

Definition 4.1. For a polygon P, the set of sides of P denoted by S_P, and the group of all isometries of the hyperbolic disc $G_{\mathbb{B}^2}$, a **side-pairing of P** is an injective map $\Phi : S_P \to G_{\mathbb{B}^2}$ defined as $\Phi(s) = g_s$ such that:

1. g_s is an isometry with $s \to s'$ for some pair of sides s, s'. In other words, $g_s(s) = s'$.
2. If $g_s(s) = s'$, then $g_{s'}$ is equal to g_s^{-1}, i.e., $g_{s'}(s') = s$.
3. The image $g_s(P)$ has an empty intersection with P for all sides s.

Example 4.2. Consider Figure 5, a visual of a hyperbolic square P with two side pairing maps that are hyperbolic disc translations.

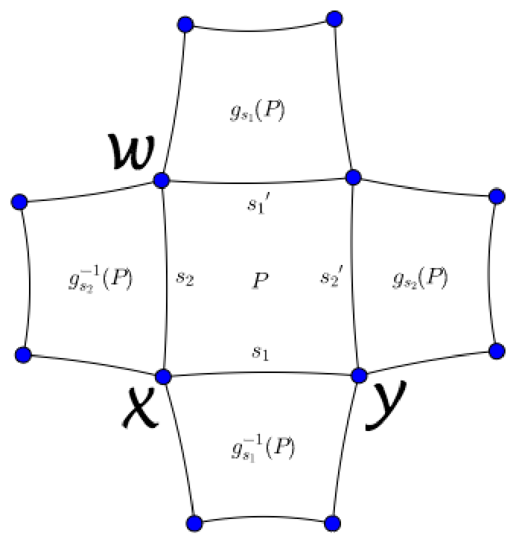


Figure 5: Diagram for Example 4.2. Specific vertices are labeled for the proof of Theorem 4.4.

Example 4.3. Consider Figure 6, a visual of a hyperbolic triangle P with three side pairing maps that are hyperbolic reflections over its three sides, i.e., the map g_i is a reflection over the geodesic segment s_i. Note that this example shows that under Definition 4.1, a side can be paired with itself.

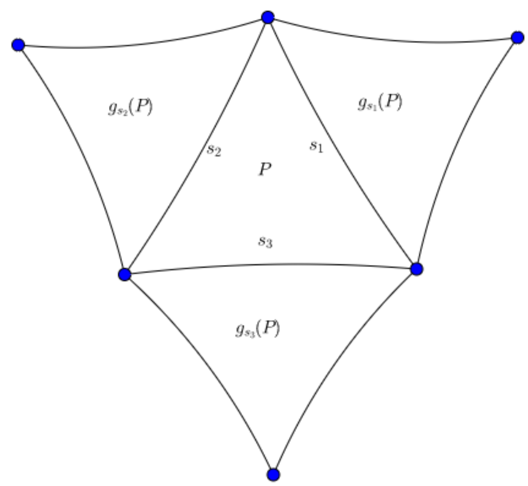


Figure 6: Diagram for Example 4.3.

Now, we state the theorem.

Theorem 4.4. ***Poincaré Polygon Theorem.*** *Let $P \subset \mathbb{B}^2$ be a compact, connected, regular hyperbolic polygon with a side pairing Φ and let G be the Fuchsian group generated by side pairing maps $g \in \Phi(S_P)$. If every angle of P is equal to $\frac{2\pi}{n}$ for some $n \in \mathbb{N}$, then P is a fundamental polygon for the group G.*

Proof. This proof will be broken down into several parts. The main idea of this proof is to build a quotient space $(G \times \overline{P})/\star$ with the following criteria:

1. $(G \times \overline{P})/\star$ has a fundamental domain for G induced by P and
2. $(G \times \overline{P})/\star$ is homeomorphic to $\mathbb{B}^2$.

This will be accomplished by showing that a $(G \times \overline{P})/\star$ with these criteria must a covering space of $\mathbb{B}^2$. Since we know that $\mathbb{B}^2$ is simply connected, any covering space of $\mathbb{B}^2$ must be homeomorphic to $\mathbb{B}^2$ by Theorem 2.11. This homeomorphism will show that $\mathbb{B}^2$ has P as a fundamental polygon for the group G, thereby satisfying a tessellation of the hyperbolic disc. The space $(G \times \overline{P})/\star$ will be built by identifying together edges of the polygon P according to the chosen side-paring maps as well as identifying all of the vertices of P together into one point.

1. **Defining the equivalence relation:**

   First, we define an equivalence relation in order to build our necessary quotient space. For $g, h \in G$ and $x, y \in \overline{P}$, we say that $(g, x) \sim (h, y)$ if and only if one of two cases hold:

(a) $x \in P$ and $x = y, g = h$

(b) x is a point contained in a side $s \in S_P$, and $y = g_s(x)$, and $g = h \circ g_s$.

Note that these conditions imply that if $(g, x) \sim (h, y)$, then $g(x) = h(y)$. As an example of this relation, we can refer back to the vertices w, x, y of the hyperbolic square P in Figure 5. We see that $(g_{s_1} \circ g_{s_2}, x) \sim (g_{s_1}, y)$ and $(g_{s_2} \circ g_{s_1}, x) \sim (g_{s_2}, w)$.

We must pause to note that $\sim$ is *not* an equivalence relation, because although it is clear that this relation is symmetric and reflexive, it is possible to have $(g_1, x_1) \sim (g_2, x_2)$ and $(g_2, x_2) \sim (g_3, x_3)$ but not $(g_1, x_1) \sim (g_3, x_3)$ (this is because the side-pairing map g_{s_3} used to map x_1 to x_3 must be defined as the composition of side-pairing maps $g_{s_2} \circ g_{s_1}$ from the first two relations, and this resulting composition may not be a generator of G). A concrete example of this breakdown in transitivity follows: consider again the hyperbolic square P in Figure 5. We can see that $(I, x) \sim (g_{s_1}^{-1}, y)$ and $(I, x) \sim (g_{s_2}^{-1}, w)$, but it is impossible for it to hold that $(g_{s_1}^{-1}, y) \sim (g_{s_2}^{-1}, w)$ because there exists no side s such that $w = g_s(y)$. In order to map y to w, we must use a composition of generators of G that is not a generator of G.

We can fix this issue by creating another relation based on $\sim$ that satisfies transitivity. Let $(g, x) \star (h, y)$ if for some finite collection (g_i, x_i) the following chain of equivalences hold:

$$(g, x) \sim (g_1, x_1) \sim \cdots \sim (g_n, x_n) \sim (h, y)$$

Here it is clear to see that transitivity holds by combining chains of equivalences. Let $\langle g, x \rangle$ be defined as the equivalence class of all elements equivalent to (g, x) via $\star$.

2. **The quotient space:**

Let $\mathbb{B}^2_\star$ be the quotient space obtained by using the equivalence relation $\star$ with the product space $(G \times \overline{P})$. In other words, let $\mathbb{B}^2_\star = (G \times \overline{P})/\star$. We can endow $(G \times \overline{P})$ with the product topology based on the discrete topology for G and the subspace topology from $\mathbb{C}$ for $\overline{P}$. This allows us to give $\mathbb{B}^2_\star$ a quotient topology based on $(G \times \overline{P})$.

For some element $f \in G$, let $f^\star$ be defined as the induced map $f^\star : \mathbb{B}^2_\star \to \mathbb{B}^2_\star$ given by $f^\star(\langle g, x \rangle) = \langle f \circ g, x \rangle$. We will see that this induced map is well-defined: Given group elements $f, g, h \in G$ and points $x, y \in \overline{P}$, suppose that $\langle g, x \rangle = \langle h, y \rangle$ in $\mathbb{B}^2_\star$. There thus exists a finite collection $\{(g_i, x_i)\}$ such that $(g, x) \sim (g_1, x_1) \sim \cdots \sim (g_n, x_n) \sim (h, y)$. Looking at the beginning of this chain of relations, we know that *either* $g = g_1$ and $x = x_1$ *or* $x_1 = g_s(x)$ and $g(x) = g_1(g_s(x))$ for some generator g_s of G. It follows that either $f \circ g = f \circ g_1$ or $f(g(x)) = f(g_1(g_s(x)))$, and therefore $(f \circ g, x) \sim (f \circ g_1, x_1)$. Continuing in this fashion and ending with showing that $(f \circ g_n, x_n) \sim (f \circ h, y)$, it holds that $(f \circ g, x) \star (f \circ h, y)$.

Therefore it follows that $\langle f \circ g, x\rangle = \langle f \circ h, y\rangle$, which proves that $f^\star$ is well-defined.

Now, let $\langle \overline{P}\rangle \subset \mathbb{B}^2_\star$ be defined in the following way:

$$\langle \overline{P}\rangle = \{\langle I, x\rangle : x \in \overline{P}\},$$

where I is the identity element of G. Notice that when we consider the action of g on the elements of $\langle \overline{P}\rangle$ for all elements $g \in G$ and $x \in \overline{P}$, we get all possible combinations of the pair g, x for $\langle g, x\rangle$. As such, the following claim holds:

$$\bigcup_{g \in G} g^\star \langle \overline{P}\rangle = \mathbb{B}^2_\star$$

Furthermore, recall that for some point $x \in P$ (the interior of the polygon) and $g, h \in G$, criterion (a) from the equivalence relation necessitates that $\langle g, x\rangle = \langle h, x\rangle$ if and only if $g = h$. Therefore, the following claim holds:

$$g^\star \langle P\rangle \cap h^\star \langle P\rangle = \emptyset \text{ for all } g \neq h \in G$$

We can now conclude that $\langle P\rangle$ satisfies a slight generalization of Definition 3.10 to qualify as a fundamental domain in $\mathbb{B}^2_\star$ for the group G. The next step of our proof will be to connect $\mathbb{B}^2_\star$ to $\mathbb{B}^2$ via a homeomorphism to show that P is a fundamental domain for G. As a brief aside, we note that we have not yet made use of the requirement that all angles of P equal $\frac{2\pi}{n}$. This assumption will be key in finding a homeomorphism between $\mathbb{B}^2_\star$ and $\mathbb{B}^2$.

3. **The induced map α:**

Now consider the maps

$$\beta : G \times \overline{P} \to \mathbb{B}^2_\star$$
$$\gamma : G \times \overline{P} \to \mathbb{B}^2$$

given by

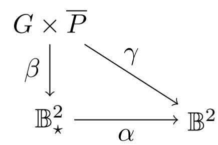
$$\beta(g, x) = \langle g, x\rangle$$
$$\gamma(g, x) = g(x)$$

We know that β is a quotient map, so β is continuous. We can also see that γ is also continuous: consider an open set $A \subset \mathbb{B}^2$. Supposing that G has the discrete topology and $(G \times \overline{P})$ has the product topology, the preimage under γ can be described as follows [1, Section 9.8]:

$$\gamma^{-1}(A) = \bigcup_{g \in G} \{g\} \times \left(g^{-1}(A) \cap \overline{P}\right)$$

which is open with respect to $(G \times \overline{P})$ since every isometry g is a continuous map and $\overline{P}$ has the subspace topology.

Notice that if we have $\beta(g_1, x_1) = \beta(g_2, x_2)$, then $\langle g_1, x_1\rangle = \langle g_2, x_2\rangle$, and thus it follows that $g_1(x_1) = g_2(x_2)$ and therefore $\gamma(g_1, x_1) = \gamma(g_2, x_2)$. Because of this particular criterion, we can employ Lemma 6.15 in [7, Section 6.2] (not stated within this text) to see that there is an induced continuous map $\alpha : \mathbb{B}^2_\star \to \mathbb{B}^2$ such that the following diagram commutes:



In other words, there exists a continuous $\alpha : \mathbb{B}^2_\star \to \mathbb{B}^2$ such that:

$$\gamma = \alpha \circ \beta.$$

This means that the map α must be defined as such:

$$\alpha(\langle g, x\rangle) = g(x).$$

Now, we connect the map α back to the wider structure of this proof. Recalling the fundamental domain $\langle P\rangle$ for G in $\mathbb{B}^2_\star$ from before, we can see that if α is a homeomorphism, then the bijective nature of α would permit that

$$\bigcup_{g \in G} g(\overline{P}) = \mathbb{B}^2$$

$$g(P) \cap h(P) = \emptyset \text{ for all } g \neq h,$$

and thus P would therefore qualify as a fundamental domain for G, thereby tessellating $\mathbb{B}^2$ along with its images under G. What remains to be shown is therefore that the map α is a homeomorphism.

4. **The map α is a covering map for $\overline{P}$:**

   We will show that α as defined above is a covering map.

   We need to show that for any point $x \in \mathbb{B}^2$, there exists a neighborhood U_x of x such that $\alpha^{-1}(U_x)$ is a disjoint union of open sets and each open set in the disjoint union $\alpha^{-1}(U_x)$ is homeomorphic to U_x via α. We will consider four different cases: (a) points in the interior of $\overline{P}$, (b) points on a side of $\overline{P}$ but not a vertex, (c) vertices of $\overline{P}$, and (d) points not contained in $\overline{P}$.

   (a) **$x \in P$, the interior of the polygon.**

   By definition of (open) interiors, there exists an $\varepsilon > 0$ such that $B(x, \varepsilon) = \{y \in \mathbb{B}^2 : d_{hyp}(x, y) < \varepsilon\}$ such that $B(x, \varepsilon) \subset P$. We also know that $B(x, \varepsilon)$ is a neighborhood of x. Notice that $\alpha^{-1}(B(x, \varepsilon))$ must consist exclusively of the set of equivalence classes $\langle I, y\rangle$, where

y is an element of $B(x, \varepsilon)$. Because the set $B(x, \varepsilon)$ does not contain any boundary points of $\overline{P}$ by construction, we see that each $\langle I, y \rangle$ contains the single point (I, y) for all $y \in B(x, \varepsilon)$. Thus, $\alpha^{-1}(B(x, \varepsilon)) = \langle I, B(x, \varepsilon) \rangle$. Furthermore, $\langle I, B(x, \varepsilon) \rangle$ is open since β is a quotient map and $\beta^{-1}(\langle I, B(x, \varepsilon) \rangle) = \{I\} \times B(x, \varepsilon)$, which is open with respect to the product topology on $G \times \overline{P}$ since G is a discrete group. Furthermore, it is clear to see that there is a one-to-one and onto correspondence via α between $\langle I, B(x, \varepsilon) \rangle$ and $B(x, \varepsilon)$. We conclude that $B(x, \varepsilon)$ is homeomorphic to $\alpha^{-1}(B(x, \varepsilon))$. We have thus shown that α acts as a quotient map for points $x \in P$.

(b) $x \in \partial P$ **but** $x \notin s \cap s'$ **for all** $s \neq s'$**. In other words,** x **is contained in exactly one side of the polygon.**

Criterion 3 of Definition 4.1 guarantees that since x lies on a side of P but not a vertex of P, we can build a neighborhood around x that intersects only $\overline{P}$ and one of its neighboring images under one of the side-pairing maps in $\Phi(S_P)$. In other words, there exists an $\varepsilon > 0$ such that we can construct an open ball $B(x, \varepsilon)$ that is the union of two partial balls laying in $\overline{P}$ centered on x and its side-paired point $g_s(x)$ (see Figure 7). Recall from Definition 4.1 that there must exist a side s' and a map g_s such that $g_s(s) = s'$ and $g_s^{-1} = g_{s'}$. Therefore, the partial ball $N_s(x, \varepsilon) = \{y \in \overline{P} : d_{hyp}(x, y) < \varepsilon\}$ has $N_s(x, \varepsilon) \cap \partial P$ equal to $g_{s'}\big(N_{s'}(g_s(x), \varepsilon) \cap \partial P\big)$ for a similarly constructed $N_{s'}(g_s(x), \varepsilon)$. Recalling that $g_{s'}(P) \cap P = \emptyset$, we can also conclude that $B(x, \varepsilon) = \{y \in \mathbb{B}^2 : d_{hyp}(x, y) < \varepsilon\}$ must be equal to $N_s(x, \varepsilon) \cup g_{s'}\big(N_{s'}(g_s(x), \varepsilon)\big)$. Refer again to Figure 7 for an illustration of this process.

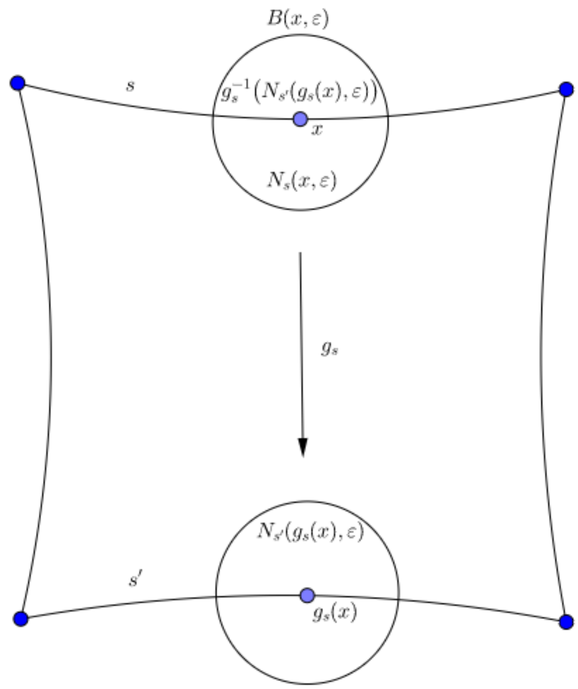


Figure 7: Illustration of the logic involved for showing that α works as a cover map for points x on a side but not a vertex of a polygon. The inner portion of the bottom ball is mapped upward to fill in the outer portion of the top ball, which is $B(x,\varepsilon)$, a neighborhood of x that is homeomorphic to $\alpha^{-1}(B(x,\varepsilon))$.

Again we are using $B(x,\varepsilon)$ as our neighborhood of x. Now we consider $\alpha^{-1}(B(x,\varepsilon))$, which must be exactly the following union:

$$\langle I, N_s(x,\varepsilon)\rangle \cup \langle g_{s'}, N_{s'}(g_s(x),\varepsilon)\rangle$$

As before, we can see that these two sets are both open by using the quotient map β. The sets constitute equivalence classes that are in one of three categories:

i. Single interior polygon points with I contained in $\overline{P} \cap N_s(x,\varepsilon)$.
ii. Single interior polygon points with $g_{s'}$ contained $\overline{P} \cap N_{s'}(x,\varepsilon)$.
iii. Pairs of points on s and s' with I and g_s^{-1} respectively. An example is the class $\langle I, x\rangle$, which contains (I,x) and $(g_{s'}, g_s(x))$.

Because there is exactly one equivalence class in $\alpha^{-1}(B(x,\varepsilon))$ for every pair of points in $s \cap N_s(x,\varepsilon)$ and $s' \cap N_{s'}(x,\varepsilon)$, we see that there is a one-to-one and onto correspondence between $\alpha^{-1}(B(x,\varepsilon))$ and $B(x,\varepsilon)$. Therefore, $\alpha^{-1}(B(x,\varepsilon))$ is homeomorphic to $B(x,\varepsilon)$ via α. We have thus shown that α acts as a quotient map for points $x \in \partial P$ that are not vertices.

(c) $x \in \partial P$ **and** $x \in s \cap s'$ **for some** $s \neq s'$**. In other words,** x **is a vertex of the polygon.**

In this part of the proof, the criterion on angle values at vertices becomes relevant. Because each vertex of $\overline{P}$ has an angle value of $\frac{2\pi}{n}$ for some integer n, we can exercise a similar process to part (b) of this proof in order to 'paste' in n portions of $\overline{P}$ that are associated with the vertices of $\overline{P}$. In other words, there is a finite number of partial balls that can be mapped via elements of G to form a union of sets with disjoint interiors that is equal to an open ball around x. So, there exists an $\varepsilon > 0$ such that we can build this $B(x, \varepsilon) = \bigcup_{j=1}^{n} g_j(N_j(x_j, \varepsilon))$, where x_j are vertices of $\overline{P}$ and $g_j \in G$ are elements such that $g_j(x_j) = x$ for all j. Note that for some $j_1 \neq j_2$ we could have $x_{j_1} = x_{j_2}$. Considering $\alpha^{-1}(B(x, \varepsilon))$, we see that it consists of an equivalence class for every interior point of each $N_j(x_j, \varepsilon)$, one class for every pair of points on shared sides (as in part (b) of this proof), and finally the class $\langle I, x \rangle$, which contains n elements (one for each time a vertex is mapped via g_j to x). It therefore follows that $\alpha^{-1}(B(x, \varepsilon))$ is again in a one-to-one and onto correspondence with the neighborhood $B(x, \varepsilon)$ of x. Therefore α is a homeomorphism between these two sets, showing that it again acts as a covering map when considering vertices of $\overline{P}$. Another conclusion we see is that it is possible to have an epsilon-ball centered at any point in $\overline{P}$ that intersects only a finite number of polygons $g_i(\overline{P})$.

(d) $x \in \mathbb{B}^2 \setminus \overline{P}$.

In order to connect a neighborhood of x to a preimage under α, we need to know whether or not it is possible to map an element of $\overline{P}$ to x via an element of G (the reason will become clear). We will borrow a line of reason used to prove a similar theorem in [2, Section 6.3]. Let $y \in P$ be a point in the interior of the polygon $\overline{P}$, and let l be the geodesic that connects y and x. We will travel along this geodesic starting at y and moving towards x. Because $x \in \mathbb{B}^2 \setminus \overline{P}$, we know that l must at some point exit $\overline{P}$. Let w_1 be the point where l leaves $\overline{P}$. Thanks to Definition 4.1 and part (c) of this proof, we know that there cannot be an infinite concentration of polygons at w_1, which means that after the point w_1 we know that l must be in either the interior of $g_1(\overline{P})$ for some $g_1 \in G$ or the intersection of a finite collection of such images. We then continue along l until it leaves the one or more images of $\overline{P}$ that it was contained in, and label that point w_2. We wish to prove that continuing this process eventually terminates after a finite number of steps, giving a point w_n where l enters an image $g_n(\overline{P})$ for some $g_n \in G$ and never leaves it (i.e., $x \in g_n(\overline{P})$). The details are lengthy but nicely laid out in the proof of Lemma 6.4 in [2, Section 6.3]. Essentially, it is a proof by contradiction: we start by assuming that the process does not terminate and that there are an infinite number of points w_j. We see that if $\mathbb{B}^2_\star$ is complete, then this sequence of points must end up converging on l eventually, contradicting the original assumption and showing that the process terminates [2, Section 6.3]. We know that $\mathbb{B}^2_\star$ must be complete since we are only dealing with bounded polygons that therefore do not have vertices on the boundary at infinity (this connection is not obvious and there exists a lengthy proof) [2, Section 6.8]. As such, we have shown that x must be contained in $g_n(\overline{P})$ for some $g_n \in G$.

Using similar methods employed in parts (a), (b), and (c) of this proof, we can now conclude that there exists an $\varepsilon > 0$ such that $B(x, \varepsilon)$ has $y \in B(x, \varepsilon)$ such that $\alpha^{-1}(B(x, \varepsilon)) = \langle g_i, y' \rangle$ where $g_i(y') = y$ for some finite collection $g_i \in G$ and $y' \in \overline{P}$. If two distinct points $y'_j, y'_k \in \overline{P}$ map to a single $y \in B(x, \varepsilon)$ via some g_j, g_k, then it must be that $\langle g_j, y'_j \rangle = \langle g_k, y'_k \rangle$. As such, we can see that there is a one-to-one and onto correspondence between $\alpha^{-1}(B(x, \varepsilon))$ and $B(x, \varepsilon)$, showing that α is a covering space for points outside of $\mathbb{B}^2$.

5. **$\mathbb{B}^2_\star$ is homeomorphic to $\mathbb{B}^2$:**

   We can now see that α acts as a covering map for any point in $\mathbb{B}^2$. As such, it follows that $(\mathbb{B}^2_\star, \alpha)$ is a covering space for all of $\mathbb{B}^2$. Since $\mathbb{B}^2$ is simply connected, however, we see by Theorem 2.11 that $\mathbb{B}^2$ must be homeomorphic to $\mathbb{B}^2$.

   This is a very useful conclusion to reach, since it allows us to make use of the fact that $\langle P \rangle$ is a fundamental domain for G that tessellates $\mathbb{B}^2_\star$. Using the homeomorphism α, we can conclude that $\alpha(\langle P \rangle) = P$ must therefore be a fundamental polygon for G that tessellates $\mathbb{B}^2$.

□

The profundity and usefulness of this proof is that it allows us to construct a tessellation out of a polygon and a collection of side-pairing isometries with only a few specific criteria. This will allow us to evaluate different polygons and compute Fuchsian groups for tessellations with ease. We will see examples of this in Section 6. First, in Section 5 we will develop another perspective on isometries of the hyperbolic disc that will be tied in with ideas from the Poincaré Polygon Theorem (also in Section 6).

5 Deck Transformations

The process of connecting the symmetries of tessellations to algebraic topology begins by thinking about the hyperbolic disc that is being tessellated as a covering space. A surface can be made by taking a quotient topology of the hyperbolic disc based on a regular hyperbolic polygon and a side-pairing with the right characteristics. Then, the tessellated hyperbolic disc is a covering space of that quotient space and the set of all **deck transformations** (ways of permuting points on the covering space) of that covering space ends up being a group that is isomorphic to the Fuchsian group that originally created the tessellation. We will see too that this same group is isomorphic to the fundamental group of the surface induced by the quotient topology.

5.1 Deck Transformations

We begin by defining a key term.

Definition 5.1 ([6] Section 5.6)**.** Given a space X and two covering spaces $(\tilde{X}_1, p_1), (\tilde{X}_2, p_2)$, we say that a **homomorphism** $\varphi : \tilde{X}_1 \to \tilde{X}_2$ is a continuous map such that for all $\tilde{x}_1 \in \tilde{X}_1$ we have $p_1(\tilde{x}_1) = p_2(\varphi(\tilde{x}_1))$. This map is an **isomorphism** if it is a homeomorphism. An isomorphism $\psi : \tilde{X}_1 \to \tilde{X}_1$ is called an **automorphism** or a **deck transformation**.

The following diagram, which commutes if φ is a homomorphism, helps in understanding the terms from Definition 5.1:

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{\varphi} & \tilde{X}_2 \\ {\scriptstyle p_1}\downarrow & \swarrow {\scriptstyle p_2} & \\ X & & \end{array}$$

The specific concept of an automorphism on a covering space is so helpful that we have given it the special name of deck transformation. It turns out that these automorphisms have several useful properties that we will soon explore.

Corollary 5.2. *Every deck transformation is uniquely determined by its action on a single point. In formal terms, given a covering space $(\tilde{X}, p)$ and any two deck transformations φ_1, φ_2, either $\varphi_1(\tilde{x}) = \varphi_2(\tilde{x})$ for all $\tilde{x} \in \tilde{X}$ or $\varphi_1(\tilde{x}) \neq \varphi_2(\tilde{x})$ for all $\tilde{x} \in \tilde{X}$.*

Proof. Deck transformations qualify as lifts under Definition 2.12, and are therefore unique. □

Proposition 5.3. *Given a space X and a cover $(\tilde{X}, p)$, the set of all deck transformations (denoted $G(\tilde{X})$) is a group under composition, and no point $\tilde{x} \in \tilde{X}$ is fixed under non-identity elements $\varphi \in G(\tilde{X})$.*

Proof. The proof that $G(\tilde{X})$ is a group under composition is plain to see, considering that every φ is a homeomorphism and we know that homeomorphisms are invertible and composition of functions is an associative operator.

Next, suppose (by contradiction) that there exists an $\tilde{x}_1 \in \tilde{X}$ and non-identity deck transformation $\varphi \in G(\tilde{X})$ such that $\varphi(\tilde{x}_1) = \tilde{x}_1$. We can rewrite $\tilde{x}_1$ as $i(\tilde{x}_1)$, where i is the identity deck transformation. Thus, because $\varphi(\tilde{x}_1) = i(\tilde{x}_1)$, by Corollary 5.2, φ must be equal to the identity map i. By contradiction, we have shown that non-trivial deck transformations have no fixed points. □

The next proposition is arrived at in a straightforward manner given what we have already covered, so we state it without proof.

Proposition 5.4. *Given a space X and a cover $(\tilde{X}, p)$, no two distinct elements in the group of deck transformations $G(\tilde{X})$ map a single point to the same image.*

This is a very useful feature of deck transformations. Proposition 5.4 gives us a tool to check quickly whether a Fuchsian group could be isomorphic to a group of deck transformations, a principle that is central to Theorem 6.1 which is exemplified in Example 6.3.

5.2 Fundamental Groups & Deck Transformations

A big connection of this thesis lies in that fact that for a universal covering space of a quotient space, the group of all deck transformations of the universal covering space is isomorphic to the fundamental group of that quotient space. This is an important connection to make: we will ultimately arrive at understanding that Fuchsian groups used to build certain tessellations of polygons are isomorphic to the fundamental group of the quotient space made with that same polygon. This essentially gives us three different perspectives on a single group, and will allow us to say a lot about a polygon and its tessellation.

First, it is necessary to state and prove an important connecting piece of this trio of isomorphic groups.

Proposition 5.5. *Given a space X and its universal cover $(\tilde{X}, p)$, the group of deck transformations $G(\tilde{X})$ is isomorphic to the fundamental group $\pi_1(X, x)$ of X.*

Proof. In order to construct this proof, we will need to find a well-defined bijective homomorphism between the two groups. We will begin by defining a map $\psi : \pi_1(X, x) \to G(\tilde{X})$ in the following way: given a homotopy class $[\gamma]$ and a point $\tilde{x}_0 \in p^{-1}(x)$, consider the unique lift $\tilde{\gamma}$ with $\tilde{\gamma}(0) = \tilde{x}_0$ (guaranteed to exist by Proposition 2.13). Since γ is a path beginning and ending at the point x in X, we know that $\tilde{\gamma}(0), \tilde{\gamma}(1)$ are both contained in $p^{-1}(x)$. Since we know that each deck transformation is uniquely determined by where it sends a single point, the deck transformation τ such that $\tau(\tilde{\gamma}(0)) = \tilde{\gamma}(1)$ must be unique. Therefore we define $\psi([\gamma]) = \tau$ in this way.

1. (**Well-defined**) Given two elements $[\gamma] = [\delta] \in \pi_1(X, x)$, can we have $\psi([\gamma]) \neq \psi([\delta])$? Consider the lifts $\tilde{\gamma}$ and $\tilde{\delta}$. Since both paths γ, δ begin and end at the point $x \in X$, we know that $\tilde{\gamma}(0), \tilde{\gamma}(1), \tilde{\delta}(0), \tilde{\delta}(1) \in p^{-1}(x)$. By construction we know that $\tilde{\gamma}(0) = \tilde{\delta}(0)$ so if we show that $\tilde{\gamma}(1) = \tilde{\delta}(1)$ then we will have shown that the unique deck transformation τ must be the same for both $\psi([\gamma]), \psi([\delta])$. Since $[\gamma] = [\delta]$, we know that a homotopy exists between γ and δ, which can be uniquely lifted to a homotopy between $\tilde{\gamma}$ and $\tilde{\delta}$ [7, Section 9.1, 11.2]. Any two paths that are homotopic have the same beginning and ending points, and thus $\tilde{\gamma}(1) = \tilde{\delta}(1)$, meaning that $\psi([\gamma]), \psi([\delta])$ are both defined by the unique deck transformation τ such that $\tau(\tilde{\gamma}(0)) = \tilde{\gamma}(1)$ and $\tau(\tilde{\delta}(0)) = \tilde{\delta}(1)$. Therefore, ψ is well-defined.

2. (**Homomorphism**) We need to know that ψ is a homomorphism, i.e., we need to show that for nontrivial $[\gamma], [\delta] \in \pi_1(X, x)$, we have $\psi([\gamma \cdot \delta]) = \psi([\gamma]) \cdot \psi([\delta])$. Let τ_γ and τ_δ be the deck transformations $\psi([\gamma]), \psi([\delta])$. Note that loops γ, δ in X lift to paths $\tilde{\gamma}, \tilde{\delta}$ in $\tilde{X}$, and that $\tilde{\gamma}(0) = \tilde{\delta}(0)$. Composing these two paths is not yet possible in $\tilde{X}$ since both start at the same point but are not loops like their corresponding parts in X. To fix this, we apply the deck transformation τ_γ to $\tilde{\delta}$ and get a path in $\tilde{X}$ that begins at the same point at which $\tilde{\gamma}$ terminates. In other words:

$$\tau_\gamma(\tilde{\delta}(0)) = \tilde{\gamma}(1)$$

This means that the lift of $\gamma \cdot \delta$ is equal to $\tilde{\gamma} \cdot \tau_\gamma(\tilde{\delta})$ since $\tilde{\delta}$ has been transformed to begin where $\tilde{\gamma}$ ends. Note that this lifted, composed path is a path beginning at $\tilde{x}_0$ and terminating at $\tau_\gamma(\tilde{\delta}(1))$. Since τ_δ is defined as the transformation that sends $\tilde{\delta}(0)$ to $\tilde{\delta}(1)$, we can rewrite $\tau_\gamma(\tilde{\delta}(1))$ as $\tau_\gamma(\tau_\delta(\tilde{x}_0))$. It follows that by definition of our mapping, we see that $\psi([\gamma \cdot \delta]) = \tau_\gamma \circ \tau_\delta$.

3. (**Surjectivity**) We need to show that any deck transformation is equal to $\psi([\gamma])$ for some $[\gamma] \in \pi_1(X, x)$. Consider some deck transformations $\tau, \tau' \in G(\tilde{X})$ and suppose that $\tilde{x}_0 \in \tilde{X}$ is a point such that $p(\tilde{x}_0) = x$. Since we know that every deck transformation is uniquely determined by where it maps a single point, we know that $\tau(\tilde{x}_0) \neq \tau'(\tilde{x}_0)$ for all $\tau \neq \tau'$. Furthermore, by definition of deck transformations, we know that $p(\tau(\tilde{x}_0)) = x$. Since $\tilde{X}$ is path connected, we can construct a path $\tilde{\gamma}$ such that $\tilde{\gamma}(0) = \tilde{x}_0$ and $\tilde{\gamma}(1) = \tau(\tilde{x}_0)$. Since we know that $\tilde{x}_0, \tau(\tilde{x}_0) \in p^{-1}(x)$, we know that $p(\tilde{\gamma}) = \gamma$ will be a loop in X beginning and ending at x. Since γ is a loop starting and ending at x, it follows that $[\gamma]$ is an element $\pi_1(X, x)$. Furthermore, $[\gamma]$ has been constructed in a way such that $\psi([\gamma])$ is equal to τ. Therefore, we have shown that the map ψ must be surjective since every element in $G(\tilde{X})$ can be written as $\psi([\gamma])$ for some $[\gamma] \in \pi_1(X, x)$.

4. (**Injectivity**) A proof that ψ is one-to-one can be accomplished by showing that the map's kernel is solely the identity homotopy class. First note that the identity of $G(\tilde{X})$ is the identity map $i : \tilde{X} \to \tilde{X}$. Therefore, the kernel of ψ is any homotopy class that lifts to a loop in $\tilde{X}$, i.e., any class $[\gamma]$ such that $\tilde{\gamma}(0) = \tilde{\gamma}(1)$ (making $[\gamma]$ an element of the fundamental group of $\tilde{X}$).

   Suppose that there exists a nontrivial homotopy class $[\delta] \in \pi_1(X, x)$ such that its lift $\tilde{\delta}$ has $\tilde{\delta}(0) = \tilde{\delta}(1)$ and therefore $[\tilde{\delta}] \in \pi_1(\tilde{X}, \tilde{x}_0)$. Since $\tilde{X}$ is simply connected, we know that $\pi_1(\tilde{X}, \tilde{x}_0)$ is trivial, and thus there exists a homotopy $\tilde{\Gamma}$ between $\tilde{\delta}$ and the trivial loop $\tilde{x}_0$. We know that $p(\tilde{x}_0) = x$ and $p(\tilde{\delta}) = \delta$, and that $p(\tilde{\Gamma})$ is an induced homotopy between x and δ. This is not possible because $[\delta]$ is not equal to the identity element of $\pi_1(X, x)$ by construction. We have shown, by contradiction, that the kernel of ψ must be exactly the identity of $\pi_1(X,x)$. Therefore ψ is an injective homomorphism.

We have thus shown that ψ is a bijective homomorphism and thus $G(\tilde{X}) \simeq \pi_1(X,x)$. □

6 Examples

Given a regular polygon in the Euclidean plane or hyperbolic disc with particular conditions, we can view a single group from three different perspectives: as an isometry group that creates a tessellation, a group of deck transformations of a universal cover, and the fundamental group of a quotient space of that universal cover. We state these conditions concretely in the form of a theorem without proof:

Theorem 6.1. *For an m-sided compact, connected, regular hyperbolic polygon $P \subset \mathbb{B}^2$ with interior angle measures $\frac{2\pi}{m}$ and a side-pairing Φ as per Definition 4.1, the following groups are isomorphic:*

1. *The Fuchsian group G generated by the maps in $\Phi(S_P)$.*
2. *The fundamental group $\pi_1(\mathbb{B}^2/G)$*
3. *The deck transformations $G(\tilde{X})$ for the covering space $(\tilde{X}, \pi)$, where π is the quotient/covering map that induces the quotient space $\mathbb{B}^2/G$.*

These conditions also hold for their analogs in Euclidean space. A core concept underlying this theorem is that each vertex of P needs to be shared among exactly m polygons in the collection $\{g(\overline{P}\}_{g\in G}$ in order for G to possess a necessary quality of deck transformations under Proposition 5.4. We will look at a series of examples in lieu of a proof of the theorem to illustrate this requirement. First, we return to Euclidean geometry briefly to look at a straightforward case.

Example 6.2. Reinstating the Euclidean plane $(\mathbb{R}^2, d_{euc})$, recall the square region R from Example 3.2. Let s_1 be the bottom of R (the region defined as $[0,1]\times\{0\}$). Let s_2 be the left-hand side of R (the region defined as $(\{0\}\times[0,1])$). We will define the side-pairing Φ as $\Phi(s_1) = \varphi_{0,1}$ and $\Phi(s_2) = \varphi_{1,0}$. Recall that $\varphi_{n,m} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined as $(x+n, y+m)$ with $n, m \in \mathbb{Z}$.

By construction of R, we conclude that the angle measures of each vertex of R is the same: $\frac{\pi}{2}$. It is clear to see that any vertex of R can be mapped to any other vertex of R using a combination of maps and their inverses in Φ. It is also clear to see that there are many points in $\overline{R}$ that get mapped by elements of G to points in $\mathbb{R}^2 \setminus R$, such as the point $(1,1) \in \overline{R}$ and the map $\varphi_{1,0}$. As such, this example does in fact fulfill the criteria of Theorem 4.4 and thus we can confirm that R is a fundamental polygon for G, and thus $\{g(R)\}_{g\in G}$ tessellates $\mathbb{R}^2$.

Notice too that the square has 4 sides and that its angle measures in $\mathbb{R}^2$ are equal to $\frac{2\pi}{4}$, which means that R fulfills the criteria of Theorem 6.1. Consider the vertex at $(0,0)$ and an $\varepsilon > 0$ such that $B_\varepsilon(0,0)$ intersects only 4 squares in the tessellation. The elements $\varphi_{-1,0}, \varphi_{0,-1}, \varphi_{-1,-1} \in G$ along with the identity element map the four vertices of R to $(0, 0)$. No single vertex is mapped to $(0, 0)$ by two different elements of G, meaning that G has the potential to be isomorphic to a group of deck transformations.

Consider the quotient space $\mathbb{R}^2/G$, which can be envisioned as the process of gluing together the side-paired edges of R. The quotient map π of this quotient space is also a covering map. This is because all integer-valued points in $\mathbb{R}^2$ map to the same point under π and every open set $U \subset \mathbb{R}^2/G$ has a $\pi^{-1}(U)$ that is a disjoint union of open sets in $\mathbb{R}^2$. As it turns out, $\mathbb{R}^2/G$ is homeomorphic to the genus-1 torus T^2, which we can see in Figure 8.

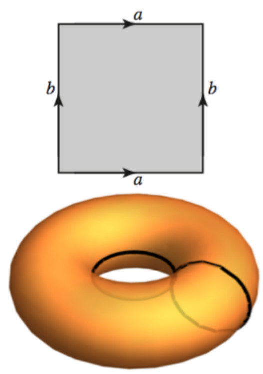


Figure 8: Illustration from [7, Section 6]. In pairing together the correspondingly labeled sides of the square so that the arrows line up, the result is the genus-1 surface T^2. This corresponds to the side-pairing maps that generate G.

Thanks to Proposition 5.5, we know that the fundamental group of T^2 is isomorphic to the group of deck transformations of its universal cover $\mathbb{R}^2$, which in this case is $\mathbb{Z} \times \mathbb{Z}$. Furthermore, this group is also isomorphic to the group of isometries used to build the tessellation $\{g(R)\}_{g \in G}$. We could have predicted this outcome by simply believing Theorem 6.1.

Example 6.3. (Non-example) Consider a regular hyperbolic square Q centered in the hyperbolic disc $\mathbb{B}^2$, such as P in Figure 5. The angle measures of Q at each vertex must each be less than $\frac{\pi}{2}$, and so the largest that the four angles of Q can be in order to fulfill Theorem 4.4 is $\frac{2\pi}{5}$. This means that to form a tessellation, at least one vertex of Q will need to be used twice in order to fill an epsilon-ball centered at a vertex of Q (recall the last part of the proof of Theorem 4.4). This implies that any Fuchsian group G which acts on Q to tessellate $\mathbb{B}^2$ would have distinct elements that send a single vertex of Q to the same image vertex. In other words, there must exist some $x \in \overline{Q}$ and distinct $g_1, g_2 \in G$ such that $g_1(x) = g_2(x)$. As such, by Proposition 5.4 it is not possible for a Fuchsian group in this example to be isomorphic to a group of deck transformations.

Example 6.4. Consider a regular hyperbolic octagon O centered in the hyperbolic disc $\mathbb{B}^2$. A regular octagon in Euclidean space has each angle equal to $\frac{6\pi}{8}$, so a hyperbolic equivalent must be strictly less than that. The biggest that these interior angles can be while fulfilling the criteria of Theorem 4.4 is $\frac{2\pi}{3}$, but we will want to impose the condition under Theorem 6.1 that the interior angles of O are each equal to $\frac{2\pi}{8}$ since O has 8 sides. This allows us to build a Fuchsian group for O that does not map distinct points to the same image under distinct maps: after fixing one vertex of O, there can be exactly 8 distinct elements of G that map each vertex of O to the fixed vertex. This means that a Fuchsian group (call it G) that has O as a fundamental polygon could be isomorphic to a group of deck transformations. Consider the quotient space obtained by identifying opposite pairs of sides of O with each other. In other words, let the side-pairing for G be a pairing of opposite sides by translations. Then we see by [2, Section 5.2] that $\mathbb{B}^2/G$ is equivalent to the genus-2 torus, pictured in Figure 9.

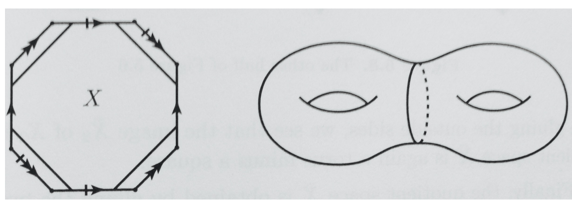


Figure 9: Illustration from [2, Section 5.2] of a process of identifying opposite edges of an octagon to get a genus-2 surface. This can be achieved with a side-pairing map that pairs sides in the indicated way.

Since we have fulfilled the criteria of Theorem 6.1, we can conclude that G must be isomorphic to the group of deck transformations for $(\mathbb{B}^2, p)$, where p is the quotient/covering map $p : \mathbb{B}^2 \to \mathbb{B}^2/G$. We can also conclude that G must be isomorphic to the fundamental group of the surface $\mathbb{B}^2/G$. To find what this group is, we can simply choose whichever of these three perspectives is easiest to consider. We select the Fuchsian group G. Since G is generated by the opposite side-pairing that we established, we can expect a generator for each pair of sides, which in this case is 4 pairs. We will use an illustration of the Fuchsian group G to show a representation of this group. Referring to Figures 10 and 11, we will use this evidence to claim that the following representation of the group at hand is valid:

$$G = \langle g_a, g_b, g_c, g_d | g_a g_b^{-1} g_c g_d^{-1} g_a^{-1} g_b g_c^{-1} g_d = 1 \rangle$$

We can write a cleaner representation by replacing the notation "g_x" with simply "x":

$$G = \langle a, b, c, d | ab^{-1}cd^{-1}a^{-1}bc^{-1}d = 1 \rangle$$

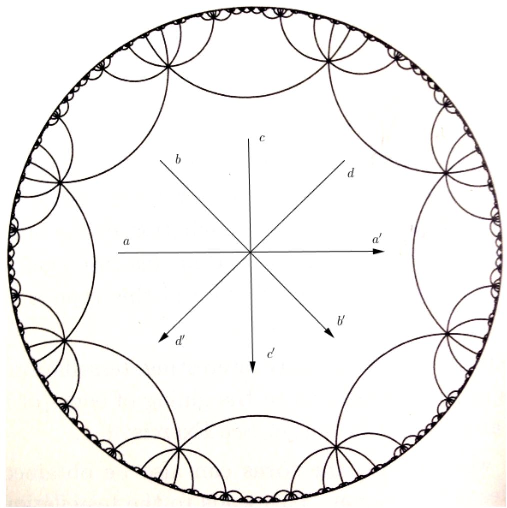


Figure 10: We let g_a be the side-pairing translation that takes a to a', and define the other three side-pairing maps similarly to create Φ. It should be clear to readers that Φ qualifies as a side-pairing under Definition 4.1, and that as such, the Fuchsian group G that is generated by elements of $\Phi(S_O)$ acts on O to tessellate $\mathbb{B}^2$ as shown. Unmodified image source: [2, Section 6.5].

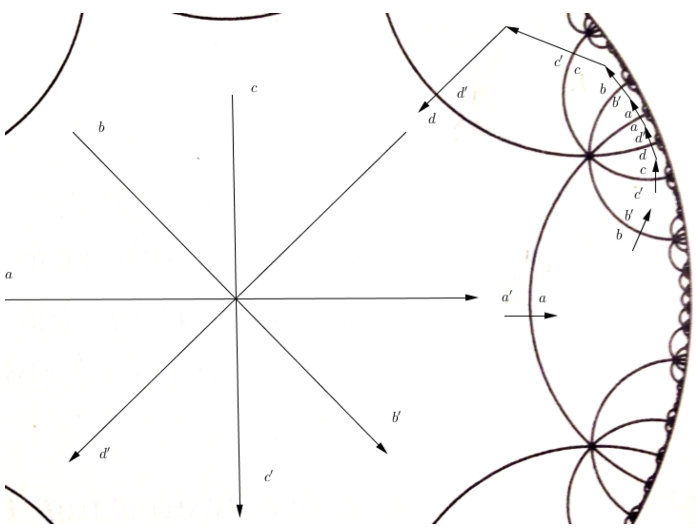


Figure 11: We use elements of G to curve around a vertex of O. This shows that the element $g_a g_b^{-1} g_c g_d^{-1} g_a^{-1} g_b g_c^{-1} g_d$ of G is equal to the identity. [2, Section 6.5]

Since the fundamental group of $\mathbb{B}^2/G$ is not easy to intuit, we can see the utility of gaining these different perspectives of different groups. Since there may exist many polygons and side-pairings that satisfy Theorem 6.1, there is great potential for further exploration of the groups that satisfy this three-pronged relationship and their associated tessellations and surfaces. We conclude with a very brief analysis of the cover image for this text.

Example 6.5. The hyperbolic octagon at the center of M.C. Escher's "Circle Limit III" (Figure 1, right-hand overlay) along with a proper side-pairing satisfies Theorem 4.4 but not Theorem 6.1 since its interior angle measures are equal to $\frac{2\pi}{3}$ instead of $\frac{2\pi}{8}$.

7 Conclusion and Acknoweledgements

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