TATA INSTITUTE OF FUNDAMENTAL RESEARCH
STUDIES IN MATHEMATICS

General Editor: M. S. NARASIMHAN

1. M. Hervé: SEVERAL COMPLEX VARIABLES
2. M. F. Atiyah and others: DIFFERENTIAL ANALYSIS
3. B. Malgrange: IDEALS OF DIFFERENTIABLE FUNCTIONS
4. S. S. Abhyankar and others: ALGEBRAIC GEOMETRY
5. D. Mumford: ABELIAN VARIETIES

ABELIAN VARIETIES

DAVID MUMFORD

Published for the
TATA INSTITUTE OF FUNDAMENTAL RESEARCH, BOMBAY
OXFORD UNIVERSITY PRESS
1970

Oxford University Press, Ely House, London W 1

GLASGOW NEW YORK TORONTO MELBOURNE WELLINGTON
CAPE TOWN SALISBURY IBADAN NAIROBI DAR ES SALAAM LUSAKA
ADDIS ABABA BOMBAY CALCUTTA MADRAS KARACHI LAHORE DACCA
KUALA LUMPUR SINGAPORE HONG KONG TOKYO

Oxford House, Apollo Bunder, Bombay 1 BR

David MUMFORD, 1937

PRINTED IN INDIA

INTRODUCTION

THIS BOOK is based on a series of lectures delivered in the winter of 1967-68 at the Tata Institute of Fundamental Research. These lectures were subsequently written up, and improved in many ways, by C. P. Ramanujam. The present text is the result of a joint effort.

To write a thorough treatise on abelian varieties would be a formidable job. This book covers roughly half of the material that I think should be in a reasonably complete treatment. We have covered:

(i) the basic material developed in the books of Weil [W1] and Lang [L],

(ii) the techniques from the theory of schemes, developed by Cartier and Grothendieck, which have given us a clear picture of the situation in characteristic p,

(iii) the basic analytic theory developed in the book of Conforto [Co].

Unfortunately, my treatment of these topics is not as elementary as it could be, and quite possibly a student will find the subject more accessible if he reads the earlier treatments of the subject instead of or as well as mine. However, I have attempted to keep the discussion as simple as was compatible with the amount of material to be covered. In particular, I recommend Chapter 2 (which is independent of Chapter 1) as the easiest. Many of the techniques which are generalized in Chapter 3 to subtler scheme situations are treated here in a more transparent classical setting. Were the book to continue, the topics which I would have liked to treat would be :

I. Jacobians,

II. Abelian schemes: deformation theory and moduli,

III. The ring of modular forms and the global structure of the moduli space,

IV. The Dieudonné theory of the "fine" characteristic p structure,

V. Arithmetic theory: abelian schemes over local, global fields.

I don't believe the word "Jacobian" is ever used in this book. Rather stubbornly I wanted to prove that the theory of abelian varieties could be developed without the crutch of "reduction to Jacobians". One of the main reasons this is possible is that I have used systematically the higher cohomology groups: I am especially fond of the proof of the main theorem of §8, which replaces Theorem 4, p. 99 of Lang [L]. But I have to admit that some people might feel Lang's argument is more geometric. For a treatment of Jacobians, the reader should look at Weil's and Lang's books: especially the very important Theorem 31, p. 117 of Weil, which Lang strangely omits. For abelian schemes, some of the basic facts can be found in my book, *Geometric Invariant Theory*, Ch. 6, [M1]. This area has been greatly clarified by recent work of Raynaud which should appear soon. The connection of modular forms with moduli spaces of abelian varieties can be found in Baily [B] and Shimura [Sh], as well as in their talks at the Boulder Summer Institute [B-M]. A purely algebro-geometric treatment of the "theta-null werte", which are special modular forms, is in my paper [M2]. It is interesting to ask whether further ties between the analytic and algebraic theories exist: e.g. an algebraic definition of the Eisenstein series as a section of a line bundle on the moduli space. For the Dieudonné theory, see Manin [Ma], Oort [O], the Séminaire Heidelberg-Strasbourg [D-G], Tate [T1], and the papers of Barsotti [Bt]. Among the vast literature on the arithmetic theory, let me only mention the Néron model [N] and the stable reduction theorem for this [G1], Kodaira [K], the Mordell-Weil theorem [L-N], the report of Cassels' [C], and Tate [T2].

Some of the material in this book is new and has not been published elsewhere. This includes the results of §16 on the index of a nondegenerate line bundle, and the results of §23 on the theta-groups

$\mathscr{G}(L)$ in the case where ϕ_L is not separable. Simplifications in §6, §13 and §16, the very elegant appendix to §4 characterizing abelian varieties as complete varieties X with arbitrary composition morphisms $X \times X \to X$ admitting a 2-sided identity, and the treatment in §21 of the local invariants of division algebras with involutions of the second kind are all due to C. P. Ramanujam. I want to thank C. P. Ramanujam for all his efforts and to thank the Tata Institute for the very pleasant and stimulating environment which encouraged these lectures. It is a pleasure to acknowledge the help of the very able staff of the Tata Institute, of the Fulbright foundation, of Mrs. Laura Schlesinger, and of the National Science Foundation.

CONTENTS

ANALYTIC THEORY

1. Complex Tori. We shall investigate in this chapter a compact connected complex Lie group X of dimension g, i.e. a compact connected complex manifold of dimension g with a group structure on the underlying set such that the maps $X \times X \to X$, $X \to X$ defined by $(x, y) \longmapsto x.y$ and $x \longmapsto x^{-1}$ are holomorphic. Let V be the tangent space to X at the identity point $e \in X$. V is a complex vector space. Recall that for *every* complex Lie group X, with tangent space V at e, for every $v \in V$ there is a unique holomorphic homomorphism

$$\phi_v : \mathbf{C} \longrightarrow X$$

such that $d\phi_v$ takes the unit tangent vector to $\mathbf{C}$ at 0 to $v \in V$. (Cf. Hochschild, *Structure of Lie Groups*, p. 79 and p. 195). Moreover the function $\phi_v(t)$ in t *and* v is a holomorphic map $\mathbf{C} \times V \to X$. The *exponential map* exp: $V \to X$ is defined by $\exp(v) = \phi_v(1)$. Because of the uniqueness property characterizing ϕ_v, $\phi_{sv}(t) = \phi_v(st)$, hence $\phi_v(t) = \exp(tv)$. Therefore, if we identify as usual the tangent space to V at 0 with V itself, the differential of exp at 0 is the identity map of V onto V. Returning to a compact connected X now, we first prove:

(1) *X is a commutative group.*

PROOF. In fact, for x in X, define C_x to be the conjugation map $X \to X, C_x(y) = xyx^{-1}$. The differential $(dC_x)_e$ is an automorphism of V and $x \longmapsto (dC_x)_e$ is a holomorphic map of X into $\mathrm{Aut}(V) \subset \mathrm{End}(V)$. Since $\mathrm{End}(V)$ is a finite-dimensional complex vector space and the only holomorphic functions on a compact connected complex manifold are constants, we deduce that $(dC_x)_e$ is independent of $x \in X$, hence $(dC_x)_e = (dC_e)_e = 1_V$. Now for any homomorphism $T: X_1 \to X_2$ of complex Lie groups,

$$T(\exp_{X_1} y) = \exp_{X_2}((dT)_e y).$$

This follows from the uniqueness property characterizing the homomorphisms $t \longmapsto \exp_{X_i}(tv)$ from $\mathbf{C}$ to X_i. It is easy to prove from this that

$$C_x(\exp y) = \exp((dC_x)_e y).$$

Since $(dC_x)_e = 1_V$, this shows that $C_x(\exp y) = \exp y$, so $\exp(V)$ is in the center of X. Since $d(\exp)$ is the identity, it follows from the implicit function theorem that exp defines a homeomorphism of a neighborhood of $0 \in V$ with a neighborhood of e in X. Since X is connected, this implies that $\exp(V)$ generates X as a group, and it follows that X is commutative.

(2) *The exponential map* $\exp\colon V \to X$ *is a surjective homomorphism of complex Lie groups with kernel a lattice*† U *in* V, *and induces an isomorphism* $V/U \xrightarrow{\sim} X$, *i.e.* X *is a complex torus.*

Let $x, y \in V$. Since X is commutative, the map $\mathbf{C} \to X$ defined by $t \longmapsto (\exp tx).(\exp ty)$ is a holomorphic homomorphism, and the image of $\left(\frac{\partial}{\partial t}\right)_0$ by the tangent map is easily seen to be $x+y$. Now for any $z \in V$, the map $t \to \exp(tz)$ is characterized as the unique holomorphic homomorphism, whose tangent map takes $\left(\frac{\partial}{\partial t}\right)_0$ to $z \in V$. Hence $(\exp tx).(\exp ty) = \exp t(x+y)$ and putting $t=1$, we find that exp is a homomorphism. It is surjective since on the one hand X is connected, while on the other hand $\exp(V)$ contains a neighborhood of e and hence an open and closed subgroup of X. The kernel U is a discrete subgroup of V, since there is a neighborhood N of 0 in V such that $\exp|_N\colon N \to X$ is injective. The induced homomorphism $V/U \to X$ is holomorphic by definition of structure of complex manifold on V/U, and is an algebraic isomorphism of groups. The tangent map at the identity of this map is an isomorphism, and hence by the inverse function theorem, the inverse is holomorphic at e and hence holomorphic everywhere on X (translations being holomorphic isomorphisms on both V/U and X). Therefore X is isomorphic to V/U. Since lattices are the only discrete subgroups of vector spaces with compact quotient, V must be a lattice.

†By definition, a *lattice* in a real vector space V is the subgroup generated by a basis of V.

From now on, we use additive notation for the group operation in X. We will fix the notation $\pi\colon V \to X$ for the exponential homomorphism for the rest of this chapter.

(3) *As an abstract group,* X *is divisible (i.e.* $nX = X$ *for* $n \in \mathbf{Z}$, $n \neq 0$*) and if for* $n \in \mathbf{Z}$, $n \neq 0$, X_n *is the subgroup of elements annihilated by* n, $X_n \simeq (\mathbf{Z}/n\mathbf{Z})^{2g}$.

Proof. By (2), we see that as a real Lie group, X is isomorphic to $(\mathbf{R}/\mathbf{Z})^{2g} = (S^1)^{2g}$, where S^1 is the circle group. Hence we have (3).

(4) *We have canonical isomorphisms*

$$H^r(X, \mathbf{Z}) \simeq \left\{ \begin{matrix} \text{group of alternating } r\text{-forms} \\ U \times \ldots \times U \longrightarrow \mathbf{Z} \end{matrix} \right\}.$$

Proof. (V, π) is clearly the universal covering space of X, hence $U = \pi^{-1}(0)$ is exactly $\pi_1(X, 0)$. Since for any good topological space X

$$H^1(X, \mathbf{Z}) \simeq \mathrm{Hom}(\pi_1(X), \mathbf{Z}),$$

the assertion is correct for $r = 1$. Then to prove it for all r, it will suffice to show that cup product induces an isomorphism

$$\Lambda^r(H^1(X, \mathbf{Z})) \xrightarrow{\sim} H^r(X, \mathbf{Z}), \text{ all } r. \qquad (*)$$

But note that if (*) is correct for spaces X_1 and X_2 with finitely generated cohomologies, then by the **Künneth** formula, (*) holds for $X_1 \times X_2$:

$$\begin{array}{ccc}
\Lambda^r(H^1(X_1 \times X_2, \mathbf{Z})) & \longrightarrow & H^r(X_1 \times X_2, \mathbf{Z}) \\
\| \wr & & \| \wr \\
\Lambda^r[H^1(X_1, \mathbf{Z}) \oplus H^1(X_2, \mathbf{Z})] & & \\
\| \wr & & \\
\sum_{p+q=r} [\Lambda^p H^1(X_1, \mathbf{Z}) \otimes \Lambda^q H^1(X_2, \mathbf{Z})] & \xrightarrow{\sim} & \sum_{p+q=r} H^p(X_1, \mathbf{Z}) \otimes H^q(X_2, \mathbf{Z}).
\end{array}$$

(Note here that (*) for X_1, X_2 implies $H^r(X_i, \mathbf{Z})$ is torsion-free, hence the tor term in Künneth disappears.) But our torus X is a product of S^1's, for which (*) is trivially valid.

(5) *Computation of the groups* $H^q(X, \Omega^p)$, *where* Ω^p = *sheaf of holomorphic p-forms on* X.

The cohomology groups $H^q(X, \Omega^p)$ are one of the most significant invariants of any compact complex manifold X, and their computation for a torus will take up the rest of this section.

Let $V = T_{0,X}$ be the tangent space to X at 0 (regarded as a complex vector space), and let $T = \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})$ be the complex cotangent space to X at 0. By translation with respect to the group law on X, every complex p-covector $\alpha \in \Lambda^p T$ extends to a translation invariant holomorphic p-form ω_α on X. In fact, let $T_x: X \to X$ be the map $T_x(y) = x+y$. Then define $(\omega_\alpha)_x = T^*_{-x}(\alpha)$. Moreover, the map $\alpha \longmapsto \omega_\alpha$ defines a homomorphism of sheaves:

$$\mathcal{O}_X \otimes_{\mathbf{C}} \Lambda^p T \longrightarrow \Omega^p \qquad (*)$$

which is easily checked to be an isomorphism. In other words, Ω^p is a *globally* free sheaf of $\mathcal{O}_X$-modules. Since the only global sections of $\mathcal{O}_X$ are constants, the global sections of Ω^p are exactly the translation-invariant p-forms ω_α. In fact, because of the isomorphism (*) we get:

$$H^q(X, \Omega^p) \simeq H^q(X, \mathcal{O}_X \otimes \Lambda^p T) \simeq H^q(X, \mathcal{O}_X) \otimes \Lambda^p T.$$

The main result that we want is

Theorem. *If* $\overline{T} = \mathrm{Hom}_{\mathbf{C}\text{-antilinear}}(V, \mathbf{C})$, *then there are natural isomorphisms*

$$H^q(X, \mathcal{O}_X) \simeq \Lambda^q \overline{T}$$

for all q, hence

$$H^q(X, \Omega^p) \simeq \Lambda^p T \otimes \Lambda^q \overline{T}.$$

Our proof of this (due to C. P. Ramanujam and related to that of Weil [W2]) depends on the well-known Dolbeault resolution:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{C}^{0,0} \xrightarrow{\overline{\partial}} \mathcal{C}^{0,1} \xrightarrow{\overline{\partial}} \mathcal{C}^{0,2} \longrightarrow \cdots$$

where $\mathcal{C}^{p,q}$ is the sheaf of C^∞ complex-valued differential forms of type-(p, q) on X, and $\overline{\partial}$ is the component of the exterior derivative d mapping $\mathcal{C}^{p,q}$ to $\mathcal{C}^{p,q+1}$. For background on this, see Gunning-Rossi, Ch. 6. The $\mathcal{C}^{p,q}$ are fine sheaves, hence the above resolution defines isomorphisms

$$H^q(X, \mathcal{O}_X) \simeq \frac{\{\overline{\partial}\text{-closed } (0, q)\text{-forms on } X\}}{\overline{\partial}\{\text{space of } (0, q-1)\text{-forms on } X\}}.$$

Moreover, if $\mathcal{C} = \mathcal{C}^{0,0}$ is the sheaf of C^∞ complex-valued functions on X, then just as with holomorphic forms, there is an isomorphism

$$\phi_{p,q} : \mathcal{C} \otimes_{\mathbf{C}} [\Lambda^p T \otimes \Lambda^q \overline{T}] \xrightarrow{\approx} \mathcal{C}^{p,q}$$

taking $\Sigma f_i \otimes \alpha_i$ to $\Sigma f_i \omega_{\alpha i}$ where ω_α is the translation invariant (p, q)-form with value $\alpha \in \Lambda^p T \otimes \Lambda^q \overline{T}$ at 0. Note that these translation-invariant forms ω_α are all closed. In fact, since $\omega_{\alpha\wedge\beta} = \omega_\alpha \wedge \omega_\beta$, it is sufficient to check this for α of degree (1, 0) or (0, 1). Since $\pi: V \to X$ is a local isomorphism, it is sufficient to check that $d(\pi^*(\omega_\alpha)) = 0$. But considering α itself (which is in $T \oplus \overline{T}$) as a function on V, $\pi^*(\omega_\alpha) = d\alpha$. Therefore $d(\pi^*\omega_\alpha) = d^2\alpha = 0$.

Now let $\Lambda^\bullet = \bigoplus_q \Lambda^q \overline{T}$ be the exterior algebra on $\overline{T}$. Let $\mathfrak{a} = \Gamma(X, \mathcal{C})$. Then via $\phi_{0,q}$, we get an isomorphism

$$\mathfrak{a} \otimes_{\mathbf{C}} \Lambda^q \xrightarrow{\approx} \Gamma(X, \mathcal{C}^{0,q}).$$

If we define a differential $\overline{\partial}$ on the set of spaces on the left by $\overline{\partial}(f \otimes \alpha) = \overline{\partial} f \wedge \alpha$, then (because the ω_α are closed), the *complexes* $\mathfrak{a} \otimes_{\mathbf{C}} \Lambda^\bullet$ and $\Gamma(X, \mathcal{C}^{0,\bullet})$ are isomorphic. Therefore

$$H^q(X, \mathcal{O}_X) \simeq H^q(\mathfrak{a} \otimes_{\mathbf{C}} \Lambda^\bullet).$$

Our aim is now to show that the inclusion $i: \Lambda^\bullet \to \mathfrak{a} \otimes_{\mathbf{C}} \Lambda^\bullet$ defines an isomorphism of cohomology, i.e. $\Lambda^q \xrightarrow{\sim} H^q(\mathfrak{a} \otimes_{\mathbf{C}} \Lambda^\bullet)$. We will do this by Fourier series. Let μ be the measure on X induced by the Euclidean measure on V, and so normalized that the volume $\mu(X)$ of X is 1. We define a $\mathbf{C}$-linear map $\mu: \mathfrak{a} \to \mathbf{C}$ by putting $\mu(f) = \int_X f\mu$. For any vector space W over $\mathbf{C}$, we denote by μ_W the map $\mu \otimes 1_W\colon \mathfrak{a} \otimes W \to W$: in particular, we get a map $\mu_\Lambda\colon \mathfrak{a} \otimes_{\mathbf{C}} \Lambda^\bullet \to \Lambda^\bullet$ which is $\Lambda^\bullet$-linear and such that $\mu_\Lambda \circ i = \mathrm{Id}_\Lambda$.

LEMMA 1. *For $\omega \in \mathfrak{a} \otimes_{\mathbf{C}} \Lambda^\bullet$, we have $\mu_\Lambda(\bar\partial\omega) = 0$.*

PROOF. Since μ_Λ is $\Lambda^\bullet$-linear, it suffices to prove that $\mu_\Lambda(\bar\partial f) = 0$ for $f \in \mathfrak{a}$. Choosing a basis $\omega_1, \ldots, \omega_n$ of $\bar T$, we can expand $\bar\partial f \in \mathfrak{a} \otimes_{\mathbf{C}} \bar T$ as $\Sigma h_i \otimes \omega_i$. The coefficients h_i are all of the form $D(f)$, where D is some invariant vector field on X. Therefore the lemma follows from the elementary fact that if f is a C^∞-function on V, periodic with respect to the lattice U, and D is a translation-invariant vector field on V, then

$$\int_{V/U} D(f)\, dx = 0$$

(dx = some Euclidean volume element).

Let $U^* = \mathrm{Hom}(U, \mathbf{Z})$. If $\lambda \in U^*$, then λ extends to an $\mathbf{R}$-linear map $\lambda\colon V \to \mathbf{R}$ and we can then form the function $x \to e^{2\pi i\lambda(x)}$ on V. This function is invariant under the action of U, hence it equals $e_\lambda \circ \pi$, where e_λ is a C^∞-function on X. Now define a $\mathbf{C}$-linear map $Q_\lambda\colon \mathfrak{a} \to \mathbf{C}$ by $Q_\lambda(f) = \mu(e_{-\lambda} f) = \int_X e_{-\lambda} . f . \mu$. More generally, for any vector space W, define $Q_\lambda\colon \mathfrak{a} \otimes_{\mathbf{C}} W \to W$ by $Q_\lambda(f \otimes w) = \mu(e_{-\lambda} f) . w$. The $Q_\lambda(f)$ are the Fourier coefficients of f: for every $f \in \mathfrak{a} \otimes_{\mathbf{C}} W$ we get the expansion

$$f = \sum_{\lambda \in U^*} e_\lambda \otimes Q_\lambda(f).$$

The Q_λ are compatible with $\mathbf{C}$-linear maps $W \to W'$ just as μ is in particular, $Q_\lambda\colon \mathfrak{a} \otimes_{\mathbf{C}} \Lambda^\bullet \to \Lambda^\bullet$ is a $\Lambda^\bullet$-linear map.

For the remainder of this proof, we choose a Hermitian norm $\|\ \|$ on the complex vector space V. As usual, this induces a norm on $\bar T$, hence on the whole exterior algebra $\Lambda^\bullet$.

Moreover, define the mapping $\bar C\colon U^* \to \bar T$ as follows:

$$U^* \longrightarrow \mathrm{Hom}_{\mathbf{R}}(V, \mathbf{R}) \subset \mathrm{Hom}_{\mathbf{R}}(V, \mathbf{C}) \simeq [T \oplus \bar T] \xrightarrow[\text{projection}]{} \bar T.$$

This makes U^* into a lattice in $\bar T$, hence by restriction we get a norm $\|\ \|$ on U^* too.

LEMMA 2. (1) *The map $f \to \{Q_\lambda(f)\}_{\lambda \in U^*}$ is an isomorphism of $\mathfrak{a}$ onto the vector space of all maps $Q\colon U^* \to \mathbf{C}$ decreasing at ∞ faster than $\|\lambda\|^{-n}$, all n, i.e.*

$$|Q(\lambda)| = O(\|\lambda\|^{-n}),\ \textit{all } n.$$

(2) *For all $\omega \in \mathfrak{a} \otimes_{\mathbf{C}} \Lambda^p$*

$$Q_\lambda(\bar\partial\omega) = (-1)^p 2\pi i [Q_\lambda(\omega) \wedge \bar C(\lambda)].$$

PROOF. (1) is standard Fourier analysis. To prove (2), note that

$$\pi^*(\bar\partial e_{-\lambda}) = \bar\partial(e^{-2\pi i\lambda}) = -2\pi i e^{-2\pi i\lambda} . \bar\partial\lambda = \pi^*[-2\pi i e_{-\lambda} \otimes \bar C(\lambda)],$$

hence $\bar\partial e_{-\lambda} = -2\pi i e_{-\lambda} \otimes \bar C(\lambda)$. Therefore, by Lemma 1, for all $\omega \in \mathfrak{a} \otimes \Lambda^p$,

$$\begin{aligned} 0 &= \mu_\Lambda(\bar\partial(\omega e_{-\lambda})) \\ &= \mu_\Lambda(e_{-\lambda} . \bar\partial\omega) + (-1)^{p-1} 2\pi i\, \mu_\Lambda(\omega e_{-\lambda} \wedge \bar C(\lambda)) \\ &= Q_\lambda(\bar\partial\omega) + (-1)^{p-1} 2\pi i\, Q_\lambda(\omega) \wedge C(\lambda). \end{aligned}$$

The following is well known.

LEMMA 3. *Let W be a complex vector space, $D \in \mathrm{Hom}_{\mathbf{C}}(W, \mathbf{C})$. Then D extends to a map $D\rfloor\colon \Lambda^p W \to \Lambda^{p-1} W$ for all p, called interior multiplication by D, such that*

(1)

$$D\rfloor(X_1 \wedge \ldots \wedge X_p) = \sum_{k=1}^{p} (-1)^{p-k} D(X_k) . X_1 \wedge \ldots \wedge \hat X_k \wedge \ldots \wedge X_p;$$

(2) *in particular, if $DX_0 = 1$, for all $\omega \in \Lambda^\bullet W$,*

$$D\rfloor(\omega \wedge X_0) + (D\rfloor\omega) \wedge X_0 = \omega.$$

We are now all set to prove that $i\colon \Lambda^\bullet \to \mathfrak{a} \otimes_{\mathbf{C}} \Lambda^\bullet$ is a homotopy equivalence for $\bar\partial$-cohomology. For every $\lambda \in U^*$, $\lambda \neq 0$ define an element $\lambda^* \in \mathrm{Hom}_{\mathbf{C}}(\bar T, \mathbf{C})$ using the Hermitian inner product $\langle , \rangle$ on $\bar T$:

$$\lambda^*(x) = \frac{\langle x, \bar C(\lambda)\rangle}{2\pi i \|\bar C(\lambda)\|^2}.$$

Then $2\pi i\, \lambda^*(\bar{C}(\lambda)) = 1$, and $\|\lambda^*\| \leq (2\pi)^{-1}.\|\lambda\|^{-1}$. For all $\omega \in \mathfrak{a} \otimes \Lambda^p$, we define $k(\omega) \in \mathfrak{a} \otimes \Lambda^{p-1}$ by means of its Fourier expansion as follows :

$$Q_\lambda(k(\omega)) = (-1)^{p-1}\lambda^* \lrcorner\, Q_\lambda(\omega), \text{ if } \lambda \neq 0$$

$$Q_0(k(\omega)) = 0.$$

It is easy to check by Lemma 2 and some easy estimates that one and only one such $k(\omega)$ exists. Then we assert:

$$\bar\partial k + k\bar\partial = 1_{\mathfrak{a}\otimes\Lambda^\cdot} - i\circ\mu_\Lambda. \qquad (*)$$

In fact, for any $\omega \in \mathfrak{a} \otimes \Lambda^p$, we can check that both sides have the same Fourier coefficients. If $\lambda \neq 0$,

$$\begin{aligned} Q_\lambda(\bar\partial k\omega + k\bar\partial\omega) &= 2\pi i.\, Q_\lambda(k\omega) \wedge \bar{C}\lambda + \lambda^* \lrcorner\, Q_\lambda(\bar\partial\omega) \\ &= 2\pi i[(\lambda^* \lrcorner\, Q_\lambda\, \omega) \wedge \bar{C}\lambda + \lambda^* \lrcorner\,(Q_\lambda(\omega) \wedge \bar{C}\lambda)] \\ &= Q_\lambda(\omega) \end{aligned}$$

and $Q_\lambda(i(\mu_\Lambda\omega)) = 0$. If $\lambda = 0$, then $Q_0(k\bar\partial\omega) = 0$, and $Q_0(\bar\partial k\omega) = 0$, while $Q_0(\omega) = Q_0(i(\mu_\Lambda\omega))$. This proves (*).

It follows immediately from (*) that μ commutes with $\bar\partial$ and that $i\circ\mu$ is homotopic to the identity. Thus $i: \Lambda^\cdot \to \mathfrak{a} \otimes_{\mathbf{C}} \Lambda^\cdot$ induces isomorphisms in $\bar\partial$-cohomology as claimed.

Remark 1. In the above isomorphism of $H^q(X, \mathcal{O}_X)$ with $\Lambda^q\bar{T}$, the cup product pairing

$$H^{q_1}(X, \mathcal{O}_X) \times H^{q_2}(X, \mathcal{O}_X) \longrightarrow H^{q_1+q_2}(X, \mathcal{O}_X)$$

corresponds to the exterior product

$$\Lambda^{q_1}\bar{T} \times \Lambda^{q_2}\bar{T} \longrightarrow \Lambda^{q_1+q_2}\bar{T}.$$

This follows from general sheaf theory, since we resolved the sheaf $\mathcal{O}_X$ of $\mathbf{C}$-algebras by a differential graded algebra $(\mathscr{C}^{0,q}, \bar\partial)$. In such a case, cup product can be computed by multiplication in the resolving sheaf (Godement, §6.6).

Corollary 1. *The natural map induced by cup product*

$$\Lambda^q(H^1(X,\mathcal{O}_X)) \longrightarrow H^q(X,\mathcal{O}_X)$$

is an isomorphism.

Remark 2. The same method used in the proof of the theorem enables one to compute the cohomology of the de Rham complex. Let $\mathscr{C}^n = \bigoplus_{p+q=n} \mathscr{C}^{p,q}$ be the sheaf of C^∞ complex-valued n-forms. Then

$$0 \longrightarrow \mathbf{C} \longrightarrow \mathscr{C}^0 \xrightarrow{d} \mathscr{C}^1 \xrightarrow{d} \cdots$$

is a fine resolution of the constant sheaf $\mathbf{C}$, hence as usual,

$$H^n(X, \mathbf{C}) \simeq \frac{\{d\text{-closed } n\text{-forms}\}}{d\{(n-1)\text{-forms}\}}.$$

Just as with $(0, q)$-forms, we obtain the result: for all d-closed n-forms ω, there is a unique translation-invariant n-form ω_α, $\alpha \in \Lambda^n\mathrm{Hom}_{\mathbf{R}}(V, \mathbf{C})$, such that

$$\omega - \omega_\alpha = d\eta, \text{ some } (n-1)\text{-form } \eta.$$

Therefore $H^n(X, \mathbf{C}) \simeq \Lambda^n[\mathrm{Hom}_{\mathbf{R}}(V, \mathbf{C})]$. Once again, these isomorphisms take cup product on the left hand side to exterior product on the right. Also, since

$$\mathrm{Hom}_{\mathbf{R}}(V, \mathbf{C}) \simeq T \oplus \bar{T},$$

this shows that

$$\begin{aligned} H^n(X, \mathbf{C}) &\simeq \Lambda^n(T \oplus \bar{T}) \\ &\simeq \bigoplus_{p+q=n} (\Lambda^p T \otimes \Lambda^q \bar{T}) \\ &\simeq \bigoplus_{p+q=n} H^q(X, \Omega^p). \end{aligned}$$

This is the famous Hodge decomposition.

Remark 3. A closer look at what we have done so far reveals that the situation is a little complicated. Consider the three sheaves on X, embedded in one another as follows :

$$\mathbf{Z} \subset \mathbf{C} \subset \mathcal{O}_X$$

($\mathbf{Z}$ and $\mathbf{C}$ being the constant sheaves). Looking at their H^1's, we have found three *independent evaluations* of these groups:

$$H^1(X,\mathbf{Z}) \simeq \mathrm{Hom}(U,\mathbf{Z}) = U^*,$$

via the isomorphism $H^1(Y,\mathbf{Z}) \simeq \mathrm{Hom}(\pi_1 Y, \mathbf{Z})$, all spaces Y.

$\Big\downarrow \alpha$ (first isom.)

$$H^1(X,\mathbf{C}) \simeq \mathrm{Hom}_{\mathbf{R}}(V,\mathbf{C}) = T \oplus \overline{T},$$

via the exact sequence $0 \longrightarrow \mathbf{C} \longrightarrow \mathscr{C}^0 \xrightarrow{d} \mathscr{C}^1_{\text{closed}} \longrightarrow 0$ and $T \oplus \overline{T} \longrightarrow H^0(\mathscr{C}^1_{\text{closed}})$.

$\Big\downarrow \beta$ (second isom.)

$$H^1(X,\mathscr{O}_X) \simeq \overline{T},$$

via the exact sequence $0 \longrightarrow \mathscr{O}_X \longrightarrow \mathscr{C}^{0,0} \xrightarrow{\bar\partial} \mathscr{C}^{0,1}_{\bar\partial\text{-closed}} \longrightarrow 0$ and $\overline{T} \longrightarrow H^0(\mathscr{C}^{0,1}_{\bar\partial\text{-closed}})$.

(third isom.)

It is only natural to assume that the vertical arrows connecting the cohomology groups correspond, under these evaluations to the canonical maps (a) $\mathrm{Hom}(U,\mathbf{Z}) \to \mathrm{Hom}(U,\mathbf{Z}) \otimes_{\mathbf{Z}} \mathbf{C} \simeq \mathrm{Hom}_{\mathbf{R}}(V,\mathbf{C})$, and (b) projection of $T \oplus \overline{T}$ onto $\overline{T}$. Let us check that this does occur.

POINT 1. The map $H^1(X,\mathbf{C}) \xrightarrow{\beta} H^1(X,\mathscr{O}_X)$. This can be computed by comparing the two resolutions. Let $C_{0,1}: \mathscr{C}^1 = \mathscr{C}^{1,0} \oplus \mathscr{C}^{0,1} \to \mathscr{C}^{0,1}$ be the projection. $C_{0,1}$ takes d-closed forms to $\bar\partial$-closed forms, hence we get a diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{C} & \longrightarrow & \mathscr{C}^0 & \xrightarrow{d} & \mathscr{C}^1_{d\text{-closed}} & \longrightarrow & 0 \\
 & & \Big\downarrow & & \Big\| & & \Big\downarrow C_{0,1} & & \\
0 & \longrightarrow & \mathscr{O}_X & \longrightarrow & \mathscr{C}^{0,0} & \longrightarrow & \mathscr{C}^{0,1}_{\bar\partial\text{-closed}} & \longrightarrow & 0.
\end{array}$$

This gives us a commutative diagram :

$$\begin{array}{ccccc}
T \oplus \overline{T} & \hookrightarrow & H^0(X,\mathscr{C}^1_{d\text{-closed}}) & \xrightarrow{\delta} & H^1(X,\mathbf{C}) \\
\text{projection}\Big\downarrow & & \Big\downarrow C_{0,1} & & \Big\downarrow \beta \\
\overline{T} & \hookrightarrow & H^0(X,\mathscr{C}^{0,1}_{\bar\partial\text{-closed}}) & \xrightarrow{\delta} & H^1(X,\mathscr{O}_X).
\end{array}$$

Thus β is the expected map.

COROLLARY 2. *β is surjective.*

POINT 2. The map $H^1(X,\mathbf{Z}) \xrightarrow{\alpha} H^1(X,\mathbf{C})$. Let $a \in H^1(X,\mathbf{Z})$. How does such an a determine a homomorphism $\tilde{a}$ from $\pi_1(X)$ to $\mathbf{Z}$? If $\phi: S^1 \to X$ is a loop in X corresponding to an element $[\phi] \in \pi_1(X)$, then $\tilde{a}([\phi])$ is found by considering $\phi^*(a) \in H^1(S^1,\mathbf{Z})$ and using the canonical isomorphism $\epsilon: H^1(S^1,\mathbf{Z}) \xrightarrow{\sim} \mathbf{Z}$:

$$\tilde{a}([\phi]) = \epsilon(\phi^*(a)).$$

In particular, for all $u \in U$, let $\phi_u: S^1 \to X$ be the loop

$$\phi_u(t) = \pi(tu) \text{ (where } S^1 \text{ is parametrized by } t \in \mathbf{R}, \text{ considered mod } \mathbf{Z}).$$

Then a determines $\tilde{a} \in U^*$ by the rule

$$\tilde{a}(u) = \epsilon(\phi_u^*(a)).$$

Now suppose we push a into the sheaf $\mathbf{C}$: we get $\alpha(a) \in H^1(X,\mathbf{C})$. According to our second evaluation, there is a unique $b \in T \oplus \overline{T}$ such that if ω_b is the invariant 1-form on X with value b at 0, then we get :

$$\begin{array}{ccc} H^0(X, \mathscr{C}^1_{\text{closed}}) & \xrightarrow{\ \delta\ } & H^1(X, \mathbf{C}) \\ \cup\!\!\!\!\cup & & \cup\!\!\!\!\cup \\ \omega_b & \longmapsto & \alpha(a). \end{array}$$

Pulling back to S^1, we find:

$$\begin{array}{ccc} H^0(S^1, \mathscr{C}^1_{\text{closed}}) & \xrightarrow{\ \delta\ } & H^1(S^1, \mathbf{C}) \\ \cup\!\!\!\!\cup & & \cup\!\!\!\!\cup \\ \phi_u^*(\omega_b) & \longrightarrow & \phi_u^*(\alpha(a)) \\ & & \| \\ & & \alpha(\phi_u^*(a)). \end{array}$$

But now it is an elementary matter to check that if η is a 1-form on S^1 (any such η is closed), if $\delta(\eta)$ is its image in $H^1(S^1, \mathbf{C})$ and if $\epsilon(\delta(\eta))$ is the image of $\delta(\eta)$ via the canonical isomorphism $\epsilon: H^1(S^1, \mathbf{C}) \xrightarrow{\sim} \mathbf{C}$, then

$$\epsilon(\delta(\eta)) = \int_{S^1} \eta.$$

Therefore

$$\begin{aligned} \tilde{a}(u) &= \epsilon(\phi_u^*(a)) \\ &= \epsilon(\delta(\phi_u^*(\omega_b))) \\ &= \int_{S^1} \phi_u^*(\omega_b) \\ &= \int_0^u \pi^*(\omega_b) \\ &= b(u). \end{aligned}$$

So $\tilde{a}$ is just the restriction of the function b on V to U.

Using compatibility of our evaluations with cup products, we have even proven now that we have the following compatibilities between the evaluations of the n^{th} cohomology groups:

$$\begin{array}{ccccl} H^n(X, \mathbf{Z}) & \underset{\text{first}}{\sim} & \Lambda^n(U^*) & & \\ \downarrow & & \downarrow & \Lambda^n \text{ of (inclusion } U^* \subset T \oplus \overline{T}) & \\ H^n(X, \mathbf{C}) & \underset{\text{second}}{\sim} & \Lambda^n(T \oplus \overline{T}) & = \bigoplus_{p+q=n} \Lambda^p T \otimes \Lambda^q \overline{T} & \\ \downarrow & & \downarrow & \text{projection onto } p=0,\ q=n \text{ factor} & \\ H^n(X, \mathcal{O}_X) & \underset{\text{third}}{\sim} & \Lambda^n(\overline{T}). & & \end{array}$$

2. Line bundles on a complex torus. We recall the well-known

THEOREM. *For all integers* $p > 0$, $H^p(\mathbf{C}^N, \mathcal{O}) = (0)$.

For a proof, cf. Gunning-Rossi, p. 28 and p. 184.

COROLLARY. $H^p(\mathbf{C}^N, \mathcal{O}^*) = (1)$, *all* $p > 0$. *In particular, all holomorphic line bundles on* $\mathbf{C}^N$ *are trivial.*

PROOF. Use the exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O} \xrightarrow{e^{2\pi i(\ )}} \mathcal{O}^* \longrightarrow 0$$

and the fact that $H^p(\mathbf{C}^N, \mathbf{Z}) = (0)$, all $p > 0$.

We wish to give a direct geometric description of every (holomorphic) line bundle L on a complex torus X. By the corollary, the line bundle $\pi^*(L)$ on V is trivial. If we choose an isomorphism

$$\chi: \pi^*(L) \xrightarrow{\sim} \mathbf{C} \times V$$

the canonical action of U on $\pi^*(L)$ (i.e. the action such that the quotient of $\pi^*(L)$ by U is just the original bundle L) carries over by means of χ into a linear action of U on the trivial bundle covering the action of U on the base V by translations. Let us denote by H^* the multiplicative group $H^0(V, \mathcal{O}_V^*)$ of nowhere vanishing holomorphic functions on V. Since the only holomorphic automorphisms of a line bundle fixing the base are given by multiplication by non-vanishing holomorphic functions, we see that the action of U on $\mathbf{C} \times V$ is given by

$$(\alpha, z) \longmapsto \phi_u(\alpha, z) = (e_u(z).\alpha, z+u), \text{ all } u \in U \qquad \text{(A)}$$

where $e_u \in H^*$. Writing down the condition that

$\phi_u(\phi_{u'}(\alpha,z)) = \phi_{u+u'}(\alpha,z)$, we see that $u \longmapsto e_u$ is a *1-cocycle* for U with coefficients in H^*:

$$e_{u+u'}(z) = e_u(z+u').e_{u'}(z).$$

Further, if the trivialization χ is altered by multiplication by a nowhere vanishing holomorphic function f on V, $\{e_u\}$ is replaced by the cohomologous cocycle :

$$e'_u(z) = e_u(z) f(z+u) f(z)^{-1}.$$

Therefore we have defined a map from $H^1(X, \mathcal{O}_X^*)$ to $H^1(U, H^*)$. But we can go in the other direction too. If we start with a 1-cocycle $\{e_u\}$ with coefficients in H^*, then define a line bundle L on X as the quotient of $\mathbf{C} \times V$ by the action of U given by $(\alpha,z) \longmapsto (e_u(z).\ \alpha, z+u)$. Therefore we have found an isomorphism

$$\phi : H^1(U, H^*) \xrightarrow{\approx} H^1(X, \mathcal{O}_X^*).$$

More generally, for any sheaf $\mathcal{F}$ on X, there is a natural map $\phi: H^1(U, \Gamma(U, \pi^*\mathcal{F})) \to H^1(X, \mathcal{F})$. The definition and properties of ϕ are recalled in an appendix to this section. Since $H^i(V, \mathcal{O}_V^*) = (1)$, all $i \geqslant 1$, the ϕ defined in the appendix is also an isomorphism. Let us check that the isomorphism just obtained and that of the appendix are the same. In fact, choose an open covering $\{V_i\}$ of X by small enough connected open sets V_i. Then

(a) $\pi^{-1}(V_i) =$ disjoint union of connected open sets $u + W_i$, all $u \in U$.

(b) If $\pi_i =$ restriction of π to W_i, $\pi_i: W_i \xrightarrow{\sim} V_i$ is a homeomorphism.

(c) If $V_i \cap V_j \neq \varnothing$, then $\exists\ u_{ij} \in V$ such that

$$\pi_j^{-1}(V_i \cap V_j) = \pi_i^{-1}(V_i \cap V_j) + u_{ij}.$$

The map ϕ of the appendix by definition takes a group 1-cocycle $\{e_u\}$ to the Čech 1-cocycle $\{f_{ij}\}, f_{ij} \in \Gamma(V_i \cap V_j, \mathcal{O}_X^*)$ defined by

$$f_{ij}(z) = e_{u_{ij}}(\pi_i^{-1}(z)).$$

But $\{f_{ij}\}$ defines the line bundle L which is the union of trivial line bundles $\mathbf{C} \times V_i$, modulo the patching

$$\begin{array}{ccc} \mathbf{C} \times V_i & & \mathbf{C} \times V_j \\ \cup & & \cup \\ \mathbf{C} \times (V_i \cap V_j) & \xrightarrow{\approx} & \mathbf{C} \times (V_i \cap V_j) \\ (\alpha, x) & \longmapsto & (\alpha f_{ij}(x), x). \end{array}$$

But π_i is an isomorphism of $\mathbf{C} \times W_i$ with $\mathbf{C} \times V_i$, so L can also be described as the union of trivial line bundles $\mathbf{C} \times W_i$, modulo the patching

$$\begin{array}{ccc} \mathbf{C} \times W_i & & \mathbf{C} \times W_j \\ \cup & & \cup \\ \mathbf{C} \times \pi_i^{-1}(V_i \cap V_j) & \xrightarrow{\approx} & \mathbf{C} \times \pi_j^{-1}(V_i \cap V_j) \\ (\alpha, x) & \longmapsto & (\alpha . f_{ij}(\pi_i(x)), x + u_{ij}) = \phi_{u_{ij}}(\alpha, x). \end{array}$$

Now the disjoint union of $\mathbf{C} \times W_i$ is just the line bundle $\mathbf{C} \times V$ pulled back to $\bigcup W_i$, and the set of above identifications is just the equivalence relation on this pull-back bundle induced by the equivalence relation on $\mathbf{C} \times V$ given by the action of the group U. Therefore, L is exactly $\mathbf{C} \times V$ modulo U.

On any complex analytic space X the exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O}_X \xrightarrow{e^{2\pi i(\ )}} \mathcal{O}_X^* \longrightarrow 0$$

† According to the conventions of the appendix, we should take the action of U on H^* as given by $(u\ h)\ (z) = h(z-u)$, $u \in U$, $z \in V$. But then, if e_u satisfies the condition above, $f_u = e_{-u}$ is a 1-cocycle for this action, and conversely. Thus, such associated 1- cocycle is given by $f_{ij} \in \Gamma(V_i \cap V_j, \mathcal{O}_X^*)$,

$$f_{ij}(z) = f_{-u_{ij}}(\pi_i^{-1}(z)) = e_{u_{ij}}(\pi_i^{-1}(z)),$$

which is the formula we have above.

defines a co-boundary $\delta : H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbf{Z})$. If a line bundle L corresponds to a cohomology class $\lambda \in H^1(X, \mathcal{O}_X^*)$, then $\delta(\lambda)$ is called the *first Chern class* of L. In our case, suppose L is defined as above by a 1-cocycle $\{e_u\}$ with values in H^*. We want to calculate the first Chern class of the corresponding line bundle. First notice that since $H^i(V, \mathbf{Z}) = (0)$ for $i > 0$, it follows from the appendix that the maps ϕ: $H^i(U, \mathbf{Z}) = H^i(U, H^0(V, \mathbf{Z})) \to H^i(X, \mathbf{Z})$ are isomorphisms. If H is the ring of holomorphic functions on V, we have an exact sequence

$$\begin{array}{ccccccccc} 0 \longrightarrow & H^0(V, \pi^*(\mathbf{Z})) & \longrightarrow & H^0(V, \pi^*(\mathcal{O}_X)) & \xrightarrow{e^{2\pi i(\ )}} & H^0(V, \pi^*(\mathcal{O}_X^*)) & \longrightarrow 0 \\ & \| & & \| & & \| & \\ & \mathbf{Z} & & H & & H^* & \end{array}$$

(since V is simply connected), so that by the compatibility of ϕ with δ (see Appendix) we get the diagram

$$\begin{array}{ccc} H^1(U, H^*) & \xrightarrow{\quad\delta\quad} & H^2(U, \mathbf{Z}) \\ \wr\!\downarrow & & \wr\!\downarrow \\ H^1(X, \mathcal{O}_X^*) & \xrightarrow{\quad\delta\quad} & H^2(X, \mathbf{Z}). \end{array}$$

Hence identifying $H^2(U, \mathbf{Z})$ and $H^2(X, \mathbf{Z})$ by the above homomorphism, the Chern class of L is simply $\delta(\mathrm{cl}\,\{e_u\})$. Write $e_u(z) = e^{2\pi i f_u(z)}$ with f_u holomorphic in V. Then by definition, $\delta(\mathrm{cl}\,\{e_u\}) \in H^2(U, \mathbf{Z})$ is given by the 2-cocycle $F(u_1, u_2)$ on U with coefficients in $\mathbf{Z}$ given by

$$F(u_1, u_2) = f_{u_2}(z + u_1) - f_{u_1+u_2}(z) + f_{u_1}(z) \in \mathbf{Z}. \qquad (*)$$

Now we use the following standard fact.

Lemma. *The map which associates to any map $F\colon U \times U \to \mathbf{Z}$ the map $AF\colon U \times U \to \mathbf{Z}$ defined by $AF(u_1, u_2) = F(u_1, u_2) - F(u_2, u_1)$ maps the group of 2-cocycles $Z^2(U, \mathbf{Z})$ into the space of alternating linear maps $U \times U \to \mathbf{Z}$, and induces an isomorphism*

$$A\colon H^2(U, \mathbf{Z}) \xrightarrow{\sim} \mathrm{Hom}\,(\Lambda^2 U, \mathbf{Z}) \simeq \Lambda^2\, \mathrm{Hom}\,(U, \mathbf{Z}).$$

Further, for $\xi, \eta \in \mathrm{Hom}(U, \mathbf{Z}) = H^1(U, \mathbf{Z})$, we have $A(\xi \cup \eta) = \xi \wedge \eta$.

Proof. First we check that if $F \in Z^2(U, \mathbf{Z})$, $E = AF$ is bilinear. We have

$$F(u_2, u_3) - F(u_1 + u_2, u_3) + F(u_1, u_2 + u_3) - F(u_1, u_2) = 0, \ u_i \in U. \qquad \text{(i)}$$

In this equation, instead of u_1, u_2 and u_3, substitute u_3, u_1 and u_2 (respectively u_1, u_3 and u_2) and call the equation so obtained (ii) (resp. (iii)). Then (i) + (ii) − (iii) gives us that

$$E(u_3, u_1 + u_2) = E(u_3, u_1) + E(u_3, u_2).$$

Since $E(u, u) = 0$ and $E(u, v) = -E(v, u)$, it follows that E is alternating bilinear. Now suppose $F = \delta G$ is a coboundary. Then

$$\begin{aligned} AF(u_1, u_2) &= (\delta G)(u_1, u_2) - (\delta G)(u_2, u_1) \\ &= [G(u_2) - G(u_1 + u_2) + G(u_1)] - \\ &\quad - [G(u_1) - G(u_1 + u_2) + G(u_2)] = 0. \end{aligned}$$

Hence A induces a homomorphism $H^2(U, \mathbf{Z}) \to \mathrm{Hom}(\Lambda^2 U, \mathbf{Z}) \simeq \Lambda^2 \mathrm{Hom}(U, \mathbf{Z})$.

Now, since we have an isomorphism ϕ of $H^*(U, \mathbf{Z})$ onto $H^*(X, \mathbf{Z})$ where X is a torus, taking cup products to cup products (see Appendix), and we know that $H^*(X, \mathbf{Z})$ is the exterior algebra on $H^1(X, \mathbf{Z})$, it follows that $H^*(U, \mathbf{Z})$ is also the exterior algebra on $H^1(U, \mathbf{Z}) = \mathrm{Hom}(U, \mathbf{Z})$. Thus, to prove that A is an isomorphism, it suffices to prove the last statement of the lemma. But now, if ξ(resp. η) is given by the homomorphism f(resp. g) of U into $\mathbf{Z}$, $\xi \cup \eta$ is given by the 2-cocycle (see Appendix) $c(s, t) = f(s).g(t)$, so that $A(\xi \cup \eta)$ is given by the map: $A(\xi \cup \eta)(s, t) = f(s)g(t) - f(t)g(s) = (f \wedge g)(s, t)$.

Remark. We have thus an isomorphism $H^2(X, \mathbf{Z}) \xleftarrow{\sim} H^2(U, \mathbf{Z}) \xrightarrow{A} \Lambda^2 \mathrm{Hom}(U, \mathbf{Z})$. This coincides with the isomorphism $H^2(X, \mathbf{Z}) \to \Lambda^2 \mathrm{Hom}(U, \mathbf{Z})$ defined in §1, using cup product in $H^*(X, \mathbf{Z})$ and the isomorphism $H^1(X, \mathbf{Z}) \xrightarrow{\approx} \mathrm{Hom}(U, \mathbf{Z})$. In fact, ϕ commutes with cup products and A has the property that it maps cup product into exterior product, by the lemma, and

$\phi: H^1(U, \mathbf{Z}) = \mathrm{Hom}(U, \mathbf{Z}) \to H^1(X, \mathbf{Z})$ is easily checked to coincide with the inverse of the isomorphism of §1 using the naturality of ϕ. Thus, in future, we can unambiguously identify $H^i(X, \mathbf{Z})$ with $\Lambda^i \mathrm{Hom}(U, \mathbf{Z})$.

Returning to the line bundle L arising from an $\{e_u\} \in Z^1(U, H^*)$ we state formally our conclusions as a

PROPOSITION. *The Chern class of the line bundle corresponding to $\{e_u\} \in Z^1(U, H^*)$ is the alternating 2-form on U with values in* $\mathbf{Z}$ *given by*

$$E(u_1, u_2) = f_{u_2}(z + u_1) + f_{u_1}(z) - f_{u_1}(z + u_2) - f_{u_2}(z), \quad (z \text{ arbitrary in } V) \qquad (**)$$

where

$$e_u(z) = e^{2\pi i f_u(z)}.$$

COROLLARY. *If we extend E $\mathbf{R}$-linearly to a map $V \times V \to \mathbf{R}$, E satisfies the identity $E(ix, iy) = E(x, y)$ for $x, y \in V$.*

PROOF. In fact, since E represents an element of $H^2(X, \mathbf{Z})$ in the image of $H^1(X, \mathcal{O}_X^*) \to H^2(X, \mathbf{Z})$, its image by $H^2(X, \mathbf{Z}) \to H^2(X, \mathcal{O}_X)$ must be zero (and conversely). Now, this last map factorises as $H^2(X, \mathbf{Z}) \xrightarrow{i} H^2(X, \mathbf{C}) \xrightarrow{j} H^2(X, \mathcal{O}_X)$. If we put $\mathrm{Hom}_{\mathbf{R}}(V, \mathbf{C}) = \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C}) \oplus \mathrm{Hom}_{\mathbf{C}\text{-anti}}(V, \mathbf{C}) = T \oplus \overline{T}$, we have established isomorphisms $H^2(X, \mathbf{C}) \simeq \Lambda^2(T \oplus \overline{T}) \simeq (\Lambda^2 T) \oplus (T \otimes \overline{T}) \oplus (\Lambda^2 \overline{T})$, and $H^2(X, \mathcal{O}_X) \simeq \Lambda^2 \overline{T}$, and j goes over into the projection $\Lambda^2(T \oplus \overline{T}) \to \Lambda^2 \overline{T}$. Further, $i(E)$ is nothing but the real linear extension of E (cf. Remark 3, §1), which again we denote by E. Write $E = E_1 + E_2 + E_3$, where $E_1 \in \Lambda^2 T$, $E_2 \in \Lambda^2 \overline{T}$, and $E_3 \in T \otimes \overline{T}$. The reality of E implies that $E_1 = \overline{E}_2$, so that $j(E) = 0$ if and only if $E = E_3$, and this holds if and only if $E(x, y) = E(ix, iy)$.

Our next aim is to give as explicitly as possible all line bundles on the complex torus X, or equivalently, to find the simplest kind of representing cocycles $\{e_u\}$ for all cohomology classes in $H^1(U, H^*)$. This is in turn equivalent to finding a system of functions $\{f_u\}_{u \in U}$ holomorphic in V and satisfying (*).

Thus we *assume* given to us an alternating form $E: U \times U \to \mathbf{Z}$, with $E(ix, iy) = E(x, y)$ and we seek to find $\{f_u\}$ satisfying (*) and (**). Let us look for solutions f_u which are linear in z (not necessarily vanishing at 0).

We use the following elementary result.

LEMMA. *Let V be a complex vector space. There is a 1-1 correspondence between the Hermitian forms H on V and the real skew-symmetric forms E on V satisfying the identity $E(ix, iy) = E(x, y)$, which is given by*

$$E(x, y) = \mathrm{Im}\, H(x, y)$$

$$H(x, y) = E(ix, y) + i\, E(x, y).$$

The proof is left to the reader.

Let H correspond to the given E; then one checks immediately that the functions

$$f_u(z) = \frac{1}{2i} H(z, u) + \beta_u$$

satisfy (**) for any constants β_u, and the reader can also check if he likes that these are the only linear solutions of (**), holomorphic in z modulo coboundaries. Substituting in (*), we get a further condition:

$$\tfrac{1}{2} H(u_1, u_2) + i\beta_{u_1} + i\beta_{u_2} - i\beta_{u_1+u_2} \in i\mathbf{Z}$$

for all $u_1, u_2 \in U$. Writing $i\beta_u = \gamma_u + \tfrac{1}{4} H(u, u)$, this reduces to

$$\gamma_{u_1} + \gamma_{u_2} - \gamma_{u_1+u_2} + \tfrac{1}{2} i E(u_1, u_2) \in i\mathbf{Z}.$$

Now it is still permissible to modify if_u by the coboundary of a $\mathbf{C}$-linear form L on V, or what is the same, we may replace γ_u by $\gamma_u - L(u)$ with $L: V \to \mathbf{C}$ being $\mathbf{C}$-linear. The above equation shows that $\mathrm{Re}\gamma_u$ is additive in U, and hence extends to an $\mathbf{R}$-linear map $\lambda: V \to \mathbf{R}$, and there is a unique $\mathbf{C}$-linear form L on V with $\mathrm{Re} L = \lambda$ (viz., the form defined by $L(v) = \lambda(v) - i\lambda(iv)$). Modifying γ by this L, we may assume that γ is pure imaginary. Writing $\alpha(u) = e^{2\pi\gamma_u}$ we see that α has to satisfy the conditions

$$|\alpha(u)| = 1$$

$$\frac{\alpha(u_1 + u_2)}{\alpha(u_1)\alpha(u_2)} = e^{i\pi E(u_1,u_2)}.$$

We can check that given E, there always exists such an α, or equivalently, that there always exists a map $\delta\colon U \to \mathbf{R}$ such that

$$\delta(u_1 + u_2) - \delta(u_1) - \delta(u_2) \equiv \tfrac{1}{2} E(u_1, u_2) \pmod 1 \text{ for all } u_1, u_2 \in U.$$

This is left as an exercise to the reader.

We have thus proved the

LEMMA. *Let H be a hermitian form on V such that if $E = \operatorname{Im} H$, $E(U \times U) \subset \mathbf{Z}$. Let $\alpha\colon U \to \mathbf{C}_1^* = \{z \in \mathbf{C}^* \mid |z| = 1\}$ be a map with*

$$\alpha(u_1 + u_2) = e^{i\pi E(u_1,u_2)}.\alpha(u_1)\alpha(u_2),\ u_i \in U.$$

Such maps α exist for any given H as above. If we put

$$e_u(z) = \alpha(u)\, e^{\pi H(z,u) + \frac{1}{2}\pi H(u,u)}$$

then $u \longmapsto e_u$ is a 1-cocycle on U with coefficients in $H^0(V, \mathcal{O}_V^*) = H^*$, the Chern class of the associated line bundle being $E \in H^2(X, \mathbf{Z})$.

DEFINITION. *$L(H,\alpha)$ is the quotient of $\mathbf{C} \times V$ for the action of U given by $\phi_u(\lambda, z) = (\alpha(u).e^{\pi H(z,u)+\frac{1}{2}\pi H(u,u)}\,.\,\lambda,\ z + u)$.*

Note that the map $(H, \alpha) \longmapsto \{e_u\}$ satisfies the condition that if $\{e_u^{(i)}\}$ corresponds to (H_i, α_i), $\{e_u^{(1)}.e_u^{(2)}\}$ corresponds to $(H_1 + H_2, \alpha_1.\alpha_2)$. Therefore we have an isomorphism of line bundles

$$L(H_1, \alpha_1) \otimes L(H_2, \alpha_2) \simeq L(H_1 + H_2, \alpha_1\alpha_2).$$

The main theorem of this section is

THEOREM OF APPELL-HUMBERT. *Any line bundle L on the complex torus X is isomorphic to an $L(H, \alpha)$ for a uniquely determined (H, α) satisfying the conditions of the above lemma. We have isomorphic exact sequences*

$$\begin{array}{ccccccccc}
0 \to & \operatorname{Hom}(U, \mathbf{C}_1^*) & \to & \{\text{Group of data } (H, \alpha)\} & \to & \left[\begin{array}{l}\text{Group of hermitian}\\ H: V \times V \to \mathbf{C} \text{ with}\\ (\operatorname{Im} H)(U \times U) \subset \mathbf{Z}\end{array}\right] & & & \to 0\\
& \wr\big\downarrow \lambda & & \wr\big\downarrow \mu & & \wr\big\downarrow \nu & & &\\
0 \to & \operatorname{Pic}^0 X & \to & \operatorname{Pic} X & \xrightarrow{C'} & \operatorname{Ker}[H^2(X, \mathbf{Z}) & \to & H^2(X, \mathcal{O}_X)] & \to 0
\end{array}$$

where $\operatorname{Pic} X$ is the group of line bundles on X, $\operatorname{Pic}^0 X$ the subgroup of those which are topologically trivial and the last vertical map is given by $H \longmapsto \operatorname{Im} H$ (with the usual identification of $H^2(X, \mathbf{Z})$ with alternating integral 2-forms on U).

PROOF. We have already shown that an alternating integral 2-form E on U, considered as an element in $H^2(X, \mathbf{Z})$, maps into 0 in $H^2(X, \mathcal{O}_X)$ if and only if $E(ix, iy) = E(x, y)$ when E is extended $\mathbf{R}$-linearly to $V \times V$; that is, if and only if it is $\operatorname{Im} H$ for H Hermitian. Thus ν is an isomorphism. By definitions and the above lemma stating existence of α for given H, the first row is exact. Since the topological triviality of a line bundle L is equivalent to the vanishing of its Chern class, and since ν is an isomorphism, the second row is also exact.

To prove the theorem, it suffices to show that λ is an isomorphism. If $\alpha \in \operatorname{Hom}(U, \mathbf{C}_1^*)$ with $\lambda(\alpha) = 1$, we can find $g \in H^* = H^0(V, \mathcal{O}_V^*)$ with

$$\frac{g(z + u)}{g(z)} = \alpha(u).$$

If K is a compact set in V with $K + U = V$, it follows that for any $z \in V$, $|g(z)| \leqslant \operatorname{Sup}_K |g(z)|$, since $|\alpha| = 1$. Hence g can only be a constant, so $\alpha = 1$, which shows that λ is injective. Consider the commutative diagram

$$\begin{array}{ccccccc}
H^1(U, \mathbf{C}) & \longrightarrow & H^1(U, H) & \xrightarrow{e^{2\pi i(\ )}} & \operatorname{Ker}[H^1(U, H^*) & \longrightarrow & H^2(U, \mathbf{Z})]\\
\wr\big\downarrow & & \wr\big\downarrow & & & \wr\big\downarrow &\\
H^1(X, \mathbf{C}) & \longrightarrow & H^1(X, \mathcal{O}_X) & \xrightarrow{e^{2\pi i(\ )}} & \operatorname{Ker}[H^1(X, \mathcal{O}_X^*) & \longrightarrow & H^2(X, \mathbf{Z})] =\\
& & & & & & = \operatorname{Pic}^0 X
\end{array}$$

where the vertical maps are isomorphisms and the maps denoted by $e^{2\pi i(\ )}$ are surjective. But we proved in §1 that $H^1(X, \mathbf{C}) \to H^1(X, \mathcal{O}_X)$ is surjective. It follows therefore that every line bundle $L \in \operatorname{Pic}^0(X)$ is presentable in the form $\mathbf{C} \times V$ modulo an action of U of form $\phi_u(\lambda, z) = (\lambda.\alpha(u),\ z + u)$, where $\alpha\colon U \to \mathbf{C}^*$ is a homomorphism.

But as we saw on p. 20, by an automorphism of $\mathbf{C} \times V$, we can always normalize such actions so that $\mathrm{Image}(\alpha) \subset \mathbf{C}_1^*$. Therefore λ is surjective.

APPENDIX TO §2

We want to study cohomology of sheaves in the situation: $Y = X/G$, where G is a discrete group, acting freely and discontinuously on a good topological space X (i.e. $\forall x \in X$, x has a neighborhood U_x such that $U_x \cap \sigma(U_x) = \emptyset$, all $\sigma \in G$, $\sigma \neq e$). Let $\pi: X \to Y$ be the projection.

First recall the definitions of the cohomology of abstract groups. Let M be a G-module and let

$C^p(G, M) = \{\text{group of functions } f: G^p \longrightarrow M\}$;

$\delta: C^p \longrightarrow C^{p+1}$ the map

$$\delta f(\sigma_0, \ldots, \sigma_p) = \sigma_0(f(\sigma_1, \ldots, \sigma_p)) + \sum_{i=0}^{p-1} (-1)^{i+1} f(\sigma_0, \ldots, \sigma_i \cdot \sigma_{i+1}, \ldots, \sigma_p) + (-1)^{p+1} f(\sigma_0, \ldots, \sigma_{p-1});$$

$Z^p(G, M) = \mathrm{Ker}(\delta)$; $B^p(G, M) = \mathrm{Im}(\delta)$;

$H^p(G, M) = Z^p(G, M)/B^p(G, M)$
$\quad$ = derived functors of $M \longmapsto H^0(G, M)$, where

$H^0(G, M) = \{m \in M \mid \sigma(m) = m, \text{ all } \sigma \in G\}$ (also written M^G).

Given a G-linear pairing $M \times N \xrightarrow{\ *\ } P$ of G-modules, get

$$\cup: H^p(G, M) \times H^q(G, N) \longrightarrow H^{p+q}(G, P) \text{ via}$$

$$f \cup g(\sigma_1, \ldots, \sigma_{p+q}) = f(\sigma_1, \ldots, \sigma_p) * (\sigma_1 \cdot \ldots \cdot \sigma_p) g(\sigma_{p+1}, \ldots, \sigma_{p+q}),$$
$$\text{all } f \in C^p(G, M),\ g \in C^q(G, N).$$

We want the result:

$\forall$ sheaves $\mathscr{F}$ on Y, there is a natural map

$$\phi: H^p(G, \Gamma(X, \pi^*\mathscr{F})) \longrightarrow H^p(Y, \mathscr{F}).$$

It has the properties:

(a) If

$$0 \longrightarrow \mathscr{F}' \longrightarrow \mathscr{F} \longrightarrow \mathscr{F}'' \longrightarrow 0$$

is an exact sequence of sheaves on Y, and

$$0 \longrightarrow \Gamma(X, \pi^*\mathscr{F}') \longrightarrow \Gamma(X, \pi^*\mathscr{F}) \longrightarrow \Gamma(X, \pi^*\mathscr{F}'') \longrightarrow 0$$

is exact, then we get a homomorphism from the cohomology sequence of $H^p(G, \cdot)$ to that of $H^p(Y, \cdot)$.

(b) The natural maps ϕ are compatible with cup product.

(c) If

$$H^i(X, \pi^*\mathscr{F}) = (0), \quad i \geqslant 1,$$

then

$$\phi: H^p(G, \Gamma(X, \pi^*\mathscr{F})) \longrightarrow H^p(Y, \mathscr{F})$$

is an isomorphism.

To define ϕ, choose a covering $\{V_i\}_{i \in I}$ of Y such that for each i,

(1) $\pi^{-1}(V_i) = \bigcup_{\sigma \in G} \sigma(U_i)$, $U_i \subset X$ open such that $\mathrm{res}\ \pi: U_i \xrightarrow{\sim} V_i$,

(2) $\forall i, j$, there exists at most one $\sigma \in G$ such that $U_i \cap \sigma U_j \neq \emptyset$; call it σ_{ij} if it exists.

Define a map from group co-chains to Čech co-chains:

$$\phi_p: C^p(G, \Gamma(\pi^*\mathscr{F})) \longrightarrow C^p(\{V_i\}, \mathscr{F})$$

by

$$(\phi_p f)_{i_0, \ldots, i_p} = \mathrm{res} \circ (\pi^*)_{i_0}^{-1}[f(\sigma_{i_0, i_1}, \ldots, \sigma_{i_{p-1}, i_p})]$$

where $(\pi^*)_i^{-1}: \Gamma(X, \pi^*\mathscr{F}) \longrightarrow \Gamma(V_i, \mathscr{F})$ is the map

$$\Gamma(X, \pi^*\mathscr{F}) \xrightarrow{\ \mathrm{res}\ } \Gamma(U_i, \pi^*\mathscr{F}) \xleftarrow[\pi^*]{\ \approx\ } \Gamma(V_i, \mathscr{F}).$$

It is easy to check that $\delta\phi_p = \phi_{p+1}\delta$, hence the ϕ_p induce a map $\phi: H^p(G, \Gamma(X, \pi^*\mathscr{F})) \to H^p(Y, \mathscr{F})$. Properties (a) and (b) follow immediately by computation. To prove (c), we use induction on p: for $p = 0$, it is obvious. In general, embed $\mathscr{F}$ in an injective $\mathscr{O}_Y$- sheaf $\mathscr{F}'$ and let $\mathscr{F}'' = \mathscr{F}'/\mathscr{F}$. Then we find

$$0 \longrightarrow \Gamma(X, \pi^*\mathscr{F}') \longrightarrow \Gamma(X, \pi^*\mathscr{F}) \longrightarrow \Gamma(X, \pi^*\mathscr{F}'') \longrightarrow H^1(X, \pi^*\mathscr{F}') = (0)$$

hence

$$\begin{array}{ccccccc} H^{p-1}(G, \Gamma(\pi^*\mathscr{F}')) & \longrightarrow & H^{p-1}(G, \Gamma(\pi^*\mathscr{F}'')) & \longrightarrow & H^p(G, \Gamma(\pi^*\mathscr{F})) & \longrightarrow & H^p(G, \Gamma(\pi^*\mathscr{F}')) \\ \downarrow \phi_1 & & \downarrow \phi_2 & & \downarrow \phi_3 & & \downarrow \\ H^{p-1}(Y, \mathscr{F}') & \longrightarrow & H^{p-1}(Y, \mathscr{F}'') & \longrightarrow & H^p(Y, \mathscr{F}) & \longrightarrow & H^p(Y, \mathscr{F}'). \end{array}$$

It suffices to prove that $\Gamma(\pi^*\mathscr{F}')$ is an injective G-module, and that $H^i(X, \pi^*\mathscr{F}') = (0)$, $i \geqslant 1$, because then it follows that $H^i(X, \pi^*\mathscr{F}'') = (0)$, $i \geqslant 1$, hence ϕ_1, ϕ_2 are isomorphisms by the induction hypothesis, hence ϕ_3 is an isomorphism. We need

Lemma. *If $\mathscr{F}$ is an injective $\mathcal{O}_Y$-sheaf, then $\pi^*\mathscr{F}$ is a flasque $\mathcal{O}_X$-sheaf and $\Gamma(\pi^*\mathscr{F})$ an injective G-module.*

Proof. For all G-modules M, let M be the constant sheaf on X with value M. There is an obvious action of G on M compatible with its action on X. Then G acts also on $\pi_*(M)$, so we can form $\pi_*(M)^G$. It is easy to check that

$$\mathrm{Hom}_G(M, \Gamma(\pi^*\mathscr{F})) \simeq \mathrm{Hom}_{\mathcal{O}_Y}(\pi_*(M)^G, \mathscr{F}).$$

So if $M_1 \subset M_2$, then $\pi_*(M_1)^G \subset \pi_*(M_2)^G$, hence

$$\mathrm{Hom}_{\mathcal{O}_Y}(\pi_*(M_2)^G, \mathscr{F}) \to \mathrm{Hom}_{\mathcal{O}_Y}(\pi_*(M_1)^G, \mathscr{F})$$

is surjective, hence $\mathrm{Hom}_G(M_2, \Gamma(\pi^*\mathscr{F})) \to \mathrm{Hom}_G(M_1, \Gamma(\pi^*\mathscr{F}))$ is surjective. This shows that $\Gamma(\pi^*\mathscr{F})$ is injective. Secondly, $\mathscr{F}$ injective implies $\mathscr{F}$ flasque, and since π is a local homeomorphism, then $\pi^*\mathscr{F}$ is flasque too. (Cf. Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (1957), esp. Ch. V, p. 195.)

3. Algebraizability of tori.

We have seen that any line bundle L on the complex torus $X = V/U$ is isomorphic to a unique line bundle of the form $L(H, \alpha)$ where $H: V \times V \to \mathbf{C}$ is hermitian with $E = \mathrm{Im}\, H$ integral on $U \times U$, and α is a map $U \to \mathbf{C}_1^*$, satisfying $\alpha(u_1 + u_2) = e^{i\pi E(u_1,u_2)}\alpha(u_1).\alpha(u_2)$. $L(H, \alpha)$ is the quotient of $\mathbf{C} \times V$ for the action of U given by

$$\phi_u(\lambda, z) = (e_u(z).\lambda, z + u)$$
$$e_u(z) = \alpha(u)\, e^{\pi H(z,u) + \frac{1}{2}\pi H(u,u)}.$$

We now investigate the sections of $L(H, \alpha)$. These sections are in a natural one-one correspondence with sections θ of the trivial bundle $\mathbf{C} \times V$ over V (i.e. holomorphic functions θ on V) which are invariant under the above action of U, that is, which satisfy the functional equation

$$\theta(z + u) = e_u(z)\,\theta(z) = \alpha(u)\,.\, e^{\pi H(z,u) + \frac{1}{2}\pi H(u,u)}\,\theta(z), \; z \in V, u \in U.$$

Such a function is called a *theta-function* for the hermitian form H and the multiplicator α.

First consider the case when H is degenerate. Since $E = \mathrm{Im}\, H$ and $H(x, y) = E(ix, y) + iE(x, y)$, we have

$$N = \{x \in V \mid H(x, y) = 0, \forall y \in V\} = \{x \in V \mid E(x, y) = 0, \forall y \in V\}.$$

It follows from the first expression for N that N is a complex subspace of V. And since E is integral on $U \times U$, it follows from the second expression for N that $N \cap U$ is a lattice in N. If θ is an associated theta-function, we must have

$$\theta(z + u) = \alpha(u)\,\theta(z), \forall u \in N \cap U.$$

Thus, if K is a compact subset of N with $N = K + (N \cap U)$, we must have

$$|\theta(z_0 + z')| \leqslant \sup_{\zeta \in K} |\theta(z_0 + \zeta)| = c(z_0),$$

for all $z' \in N$. Therefore, by the maximum principle for holomorphic functions, $\theta(z_0 + z') = \theta(z_0)$ for $z' \in N$ and θ is constant on cosets mod N. It follows from the earlier equality that if $\theta \neq 0$, then $\alpha(u) = 1$ for $u \in N \cap U$. Thus, if η: $V \to V/N$ is the natural map, we see that any theta-function for (H, α) is of the form $\bar{\theta} \circ \eta$, where $\bar{\theta}$ is a theta-function on V/N for the lattice $\eta(U)$, the hermitian form $\bar{H}$ induced by H, and the multiplicator $\bar{\alpha}$ obtained from α by passage to quotient from U to $U/N \cap U$. Now $\bar{H}$ is non-degenerate on $\bar{V} = V/N$. Thus the study of the theta-functions for (H, α) is reduced to the study of theta-functions for $(\bar{H}, \bar{\alpha})$ on the quotient $\bar{V} = V/N$, and we may restrict ourselves to the case when H is non-degenerate. In particular, we see that if H is degenerate with null space N, if θ vanishes at $z \in V$, it vanishes on the coset $z+N$, so that any section σ of $L(H, \alpha)$ which vanishes at an $x \in X = V/U$ also vanishes on the coset $x+X'$ where X' is the subtorus $N/U \cap N \subset X$. In particular, we see that if the sections of $L(H, \alpha)$ define a morphism of X into projective space at all, this morphism has to factor through the quotient torus X/X', $X' = N/U \cap N$. Thus $L(H, \alpha)$ cannot be ample if H is degenerate.

Next, suppose there is a complex subspace $W \subset V$ of positive dimension such that $H(w, w) < 0$ for $w \in W$, $w \neq 0$. Let K be a compact subset of V with $V = U + K$. Let $z_0 \in V$ and $w \in W$, and write $w = d+u$, $d \in K$, $u \in U$. We have

$$|\theta(z_0 + w)| = |\theta(z_0 + d + u)| = |\theta(z_0 + d)|\, e^{\pi \mathrm{Re} H(z_0+d,u) + \frac{1}{2}\pi H(u,u)}$$

and since

$$\begin{aligned}\mathrm{Re}H(z_0+d,u)+\tfrac{1}{2}H(u,u) &= \mathrm{Re}H(z_0+d,w) - \mathrm{Re}H(z_0+d,d) + \tfrac{1}{2}H(w,w) + \\ &\qquad \tfrac{1}{2}H(d,d) - \mathrm{Re}H(w,d) \\ &= \tfrac{1}{2}H(w,w) + \mathrm{Re}\,H(z_0,w) + c(d,z_0).\end{aligned}$$

Of the terms on the right, for fixed z_0, the first is a real negative definite quadratic form in w, the second linear in w and the third is bounded (since d stays in a compact set K), so that the expression tends to $-\infty$ as $w \to \infty$ in W, and applying the maximum principle to $\theta(z_0 + w)$ as a function of w, we conclude that $\theta(z_0 + w) = 0$, hence $\theta \equiv 0$. Thus $L(H, \alpha)$ has no non-zero sections in this case. Therefore, if H is not positive definite, $L(H, \alpha)$ cannot be ample.

From now on, we work under the assumption that H is positive definite (and $E = \mathrm{Im}\, H$ integral on $U \times U$, as always). We shall prove the following

Proposition. *When H is positive definite and $E = \mathrm{Im}\, H$ is expressed as a matrix using a basis of U over $\mathbf{Z}$, we have*

$$\dim H^0(X, L(H, \alpha)) = \dim\,[\textit{space of theta-functions with respect to } (H, \alpha)] = +\sqrt{\det E}.$$

Proof. The idea of the proof is as follows. Since in $e_u(z)$, z occurs in the exponential linearly, one might hope that by multiplying θ by $e^{Q(z)}$ where Q is a suitable quadratic function one will be able to obtain periodicity for the new function with respect to a big sublattice U' of U. We can then expand this periodic function as a Fourier series, and the behavior of θ with respect to lattice points not in U' can be expressed in terms of the Fourier coefficients. This enables one to compute the number of linearly independent solutions.

Let then $e_u(z) = \alpha(u).e^{\pi H(z,u) + \frac{1}{2}\pi H(u,u)}$ as usual, and let θ be a holomorphic function on V satisfying $\theta(z+u) = e_u(z)\,\theta(z)$. If $B\colon V \times V \to \mathbf{C}$ is any complex symmetric bilinear form, and if we put $\theta^*(z) = e^{-\frac{1}{2}\pi B(z,z)}\,\theta(z)$, $\theta^*(z)$ satisfies the modified equation

$$\theta^*(z+u) = \alpha(u)\, e^{\pi(H-B)(z,u) + \frac{1}{2}\pi(H-B)(u,u)}\, \theta^*(z)$$

for all $u \in U$. Now, we can choose a sublattice U' of U of rank g ($=\dim V$) such that (1) $E(U' \times U') = 0$, and (2) if $W = \mathbf{R}.U'$, $W \cap U = U'$. Then $W \cap iW$ is a complex subspace of V on which E and hence H is identically 0. Since H is non-degenerate, $W \cap iW = (0)$, and so $V = W \oplus iW \simeq \mathbf{C} \oplus_{\mathbf{Z}} U' \simeq \mathbf{C} \otimes_{\mathbf{R}} W$. Since $E(W \times W) = 0$, H has a real symmetric restriction to W, and by the above, there is a unique symmetric complex bilinear B on V such that $B \mid W \times W = H \mid W \times W$. By $\mathbf{C}$-linearity in the first variable, $H(z, w) = B(z, w)$ for $w \in W$, $z \in V$. Since $E \mid U' \times U' = 0$, $\alpha \mid U'\colon U' \to \mathbf{C}_1^*$ is a homomorphism, and we can find a $\mathbf{C}$-linear form λ on V with λ real on W and $\alpha(u) = e^{2\pi i \lambda(u)}$ for $u \in U'$. The functional equation for θ^* shows then that $e^{-2\pi i \lambda(z)}.\theta^*(z)$ is periodic with respect to the lattice U'.

Let us write $\widehat{U}' = \mathrm{Hom}_{\mathbf{Z}}(U', \mathbf{Z}) \subset \mathrm{Hom}_{\mathbf{C}}(V, \mathbf{C})$, and expanding $e^{-2\pi i \lambda(z)}.\theta^*(z)$ in a Fourier series, we obtain the expression

$$\theta^*(z) = \sum_{\chi \in \widehat{U}'} c_\chi .\, e^{2\pi i(\chi(z) + \lambda(z))}. \tag{1}$$

Now, for any $u \in U$ and $u' \in U'$, $(H-B)(u', u) = \overline{H(u, u')} - B(u, u') = -2i\,\mathrm{Im}\,H(u, u') = 2i\,E(u', u)$ and if $\widehat{u} \in \widehat{U}'$ is defined by $\widehat{u}(u') = E(u', u)$ and extended $\mathbf{C}$-linearly to V we deduce that $(H-B)(z, u)$

$= 2i\,\hat{u}(z)$. Substituting the Fourier series (1) in the functional equation we get for any $u \in U$,

$$\sum_{\chi\in\hat{U}'} c_\chi \cdot e^{2\pi i[\chi(u)+\lambda(u)]} \cdot e^{2\pi i[\chi(z)+\lambda(z)]} = \alpha(u) e^{i\pi\hat{u}(u)} \sum_{\chi\in\hat{U}'} c_\chi \cdot e^{2\pi i[\chi(z)+\lambda(z)+\hat{u}(z)]}$$

and comparing coefficients,

$$c_\chi = \alpha(u)\cdot e^{i\pi\hat{u}(u)-2\pi i[\chi(u)+\lambda(u)]}\cdot c_{\chi-\hat{u}}. \tag{2}$$

Thus, if M is the image of U under the homomorphism $U \to \hat{U}'$ given by $u \longmapsto \hat{u}$, we see that the c_χ are uniquely determined once they are specified for χ running through a system of representatives of $\hat{U}'/M$. (Note that if $u_1, u_2 \in U$ with $\hat{u}_1 = \hat{u}_2$, then $E(U', u_1 - u_2) = 0$ so $u_1 - u_2 \in U'$ and one checks that the relations (2) obtained with u_1 and u_2 for u are the same.) We shall check conversely that given any system $\{c_\chi\}_{\chi\in\hat{U}'}$ of constants satisfying (2), there exists a corresponding function, i.e. the series (1) is the Fourier series of a holomorphic function. It suffices to check the uniform absolute convergence of (1) on compact subsets of V. Fixing a $\chi_0 \in \hat{U}'$ it suffices to prove this for the solution c_χ such that $c_\chi = 0$ if $\chi - \chi_0 \notin M$ and $c_{\chi_0} = 1$. Writing $\chi = \chi_0 + \hat{u}$ for $\chi \in \chi_0 + M$, when z lies in a compact set $K \subset V$, the series (1) is majorized in absolute value by

$$\text{const.} \sum_{\hat{u}\in M} |\, c_{\chi_0+\hat{u}} \,| \cdot e^{2\pi|\hat{u}(z)|},$$

hence by

$$\text{const.} \sum_{\hat{u}\in M} e^{\pi\,\mathrm{Im}\,\hat{u}(u) + A\,\|\hat{u}\|}$$

where $\|u\|$ denotes a suitable norm on M, and A a positive constant determined by χ_0, K, α and H. Since the sum $V = W \oplus iW$ is direct, we can find **R**-linear maps $\phi, \psi \colon V \to W$ such that $z = \phi(z) + i\psi(z)$. Since $\hat{u}$ is real on W and $E(W \times W) = 0$, we get

$$\begin{aligned}\mathrm{Im}\,\hat{u}(u) &= \mathrm{Im}[\hat{u}(\phi(u)) + i\hat{u}(\psi(u))]\\ &= \hat{u}(\psi(u))\\ &= E(\psi(u), u)\\ &= E(\psi(u), \phi(u) + i\psi(u))\\ &= E(\psi(u), i\psi(u))\\ &= -\,H(\psi(u), \psi(u)).\end{aligned}$$

Further, $\psi(u) = 0 \iff u = \phi(u) \iff u \in W \iff \hat{u} = 0$, so that $\mathrm{Im}\,\hat{u}(u)$ is a negative definite quadratic form on M. Thus, the above series converges very rapidly.

We deduce that the dimension of the space of theta-functions for (H, α) is the index $\hat{U}'/M$.

Thus we have only to show that if U is a free abelian group of order $2g$, E a skew symmetric bilinear form on U into **Z** non-degenerate over **Q**, U' a direct summand of U of rank g on which $E \equiv 0$ and n the order of the cokernel of the map $U \to \mathrm{Hom}_{\mathbf{Z}}(\hat{U}', \mathbf{Z})$ defined by $u \longmapsto E(\cdot, u)$, then $|\det E| = n^2$. This follows from the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & U' & \longrightarrow & U & \longrightarrow & U/U' & \longrightarrow & 0\\
 & & \alpha\downarrow & & \downarrow\beta & & \downarrow\gamma & & \\
0 & \longrightarrow & (\widehat{U/U'}) & \longrightarrow & \hat{U} & \longrightarrow & \hat{U}' & \longrightarrow & 0
\end{array}$$

with exact rows and columns, the definitions of α, β, and γ being via E, using the fact that α and γ are transposes of each other up to sign, hence have cokernels of same orders, and the fact that β has a cokernel of order $|\det E|$.

We can now prove the main theorem of this section.

Theorem of Lefschetz. *Let X be a complex torus V/U, H a hermitian form on V such that $E = \mathrm{Im}\,H$ is integral on $U \times U$, α a map $U \to \mathbf{C}_1^*$, with $\alpha(u_1 + u_2) = \alpha(u_1)\,\alpha(u_2)e^{i\pi E(u_1,u_2)}$ and $L = L(H, \alpha)$*

the associated line bundle on X. Then the following statements are equivalent.

(1) *Given any complex subtorus Y of X, there is an integer $N > 0$, a section σ of $L^{\otimes N}$ and two points $x_1, x_2 \in X$, $x_1 - x_2 \in Y$ such that $\sigma(x_1) = 0$, $\sigma(x_2) \neq 0$.*

(2) *The hermitian form H is positive definite.*

(3) *The space of holomorphic sections of $L^{\otimes n}$ give an imbedding of X as a closed complex submanifold in a projective space, for each $n \geqslant 3$.*

PROOF. We have already shown earlier that (1) ⇒ (2), and (3) ⇒ (1) is clear. It remains to assume (2), and deduce (3).

We use the expressions "theta-functions for (H, α)" and "section of $L(H, \alpha)$" interchangeably, making the identifications indicated earlier. We shall prove (3) for $n = 3$ (the cases $n > 3$ being proved quite similarly).

Firstly, if θ is a section of $L(H, \alpha)$ and $a, b \in V$, then $\theta(z-a).\theta(z-b).\theta(z+a+b)$ is a section of $L(3H, \alpha^3)$. In fact, on making the substitution $z + u$ for z in this function, it acquires a factor

$$\alpha(u)^3 \exp\{\pi H(z-a, u) + \pi H(z-b, u) + \pi H(z+a+b, u) + \frac{3\pi}{2} H(u,u)\} = \alpha(u)^3 e^{\pi.3H(z,u)+\frac{1}{2}\pi 3H(u,u)}$$

which proves the assertion. Hence, if $z_0 \in V$, there is a section ϕ of L^3 not vanishing at z_0. In fact, one has only to take a non-zero section θ of $L(H, \alpha)$, which exists by the Proposition, and then to choose $a, b \in V$ such that $\theta(z_0 - a) \neq 0$, $\theta(z_0 - b) \neq 0$ and $\theta(z_0 + a + b) \neq 0$, and put ϕ to be the product $\theta(z-a).\theta(z-b).\theta(z+a+b)$. Thus, if $\theta_0, \ldots, \theta_d$ is a basis of the sections of L^3, we get a well-defined holomorphic map

$$\Theta : X \longrightarrow \mathbf{P}^d$$

given, in terms of homogeneous coordinates, by

$$\Theta(\pi(z)) = (\theta_0(z), \theta_1(z), \ldots, \theta_d(z)) \in \mathbf{P}^d, \; z \in V.$$

Next, we prove that Θ is an injective map. If not, there exist $z_1, z_2 \in V$, $z_1 - z_2 \notin U$, and a non-zero constant $\gamma \in \mathbf{C}^*$ such that for

all theta-functions ϕ for $(3H, \alpha^3)$, we have $\phi(x_2) = \gamma\phi(x_1)$. In particular, for any $a, b \in V$ and any theta-function θ for (H, α) we have

$$\theta(z_1 - a)\theta(z_1 - b)\theta(z_1 + a + b) = \gamma\theta(z_2 - a)\theta(z_2 - b)\theta(z_2 + a + b).$$

We now consider both sides as functions of a (fixing b), and take logarithmic derivatives, so as to eliminate γ. Writing ω for the (meromorphic) differential $\frac{d\theta}{\theta}$, we obtain the relation

$$-\omega(z_1 - a) + \omega(z_1 + a + b) = -\omega(z_2 - a) + \omega(z_2 + a + b), \; a\; b \in V$$

which means that the differential $\omega(z_2 + z) - \omega(z_1 + z)$ is translation invariant in z, hence of the form $dl(z)$, where l is a $\mathbf{C}$-linear form on V. But then, this is also the differential of $\log \frac{\theta(z + z_2)}{\theta(z + z_1)}$, so that we obtain an identity

$$\theta(z + z_2) = A_1 . e^{l(z)} . \theta(z + z_1)$$

for some $A_1 \in \mathbf{C}^*$. Writing $\sigma = z_2 - z_1$, this may also be written as

$$\theta(z + \sigma) = A e^{l(z)} \theta(z)$$

with a fixed $A \in \mathbf{C}^*$. Making the substitution $z \longmapsto z + u$ $(u \in U)$, using the functional equation for θ and comparing the multiplicators on both sides, we get that

$$e^{\pi H(\sigma, u)} = e^{l(u)}, \; u \in U,$$

$$\text{or } \pi H(\sigma, u) - l(u) \in 2\pi i \mathbf{Z}, \; u \in U.$$

This implies that $\pi H(\sigma, u) - l(u) = \pi H(u, \sigma) - l(u) + \pi(H(\sigma, u) - H(u, \sigma)) = \pi H(u, \sigma) - l(u) + 2\pi i E(\sigma, u)$ takes only pure imaginary values for all $u \in V$, hence the same holds for $\pi H(u, \sigma) - l(u)$, and this being complex linear in u, we must have $\pi H(u, \sigma) = l(u)$ for all $u \in V$. But then it follows that $2\pi i . E(\sigma, u) \in 2\pi i \mathbf{Z}$ for $u \in U$, hence $\sigma \in U^{\perp} = \{x \in V \mid E(x, u) \in \mathbf{Z}, \; \forall\, u \in U\}$, which is a lattice in V containing U as a sublattice of finite index. Since $\sigma \notin U$ by assumption, $U + \mathbf{Z}\sigma \supsetneq U$, and the equation

$$\theta(z + \sigma) = A . e^{l(z)} . \theta(z) = A' . e^{\pi H(z,\sigma) + \frac{1}{2}\pi H(\sigma,\sigma)} . \theta(z)$$

shows that θ is actually a θ-function for the lattice $U + \sigma\mathbf{Z}$, the hermitian form H and a suitable multiplicator α' on $U + \mathbf{Z}\sigma$ extending α. Now, this must hold for any section θ of $L(H, \alpha)$, and the dimension of the space of such θ's is $\sqrt{(\det_U E)}$, the root of the determinant of E for the lattice U. On the other hand, if $U' = U + \mathbf{Z}\sigma \supsetneq U$, the dimension of the space of theta-functions for the lattice U' and H and any multiplicator α' is $\sqrt{(\det_{U'} E)}$, the root of the determinant of E on the lattice U'. But since we have evidently

$$\det_U E > \det_{U'} E,$$

and since there are only finitely many possible α''s extending α, it follows that almost all theta-functions for H, α and U are not theta-functions for H, α', and U' for any α'. This is a contradiction. Hence $\Theta: X \to \mathbf{P}^d$ is injective.

To complete the proof of the theorem, we have only to establish that Θ induces an injective map of tangent spaces at all points of X. If not, there is a $z_0 \in V$ and a tangent vector $\sum_1^g \alpha_i \frac{\partial}{\partial z_i}$ at z_0 with not all $\alpha_i = 0$ mapped into the 0 vector at $\Theta(\pi(z_0))$ in $\mathbf{P}^d$. There is then an $\alpha_0 \in \mathbf{C}$ such that for all $\phi \in \Gamma(X, L(3H, \alpha^3))$,

$$\alpha_0 \phi(z_0) + \sum_{i=1}^{g} \alpha_i \frac{\partial \phi}{\partial z_i}(z_0) = 0,$$

i.e.

$$D(\log \phi)(z_0) = -\alpha_0$$

for all ϕ as above, where $D = \sum_1^g \alpha_i \frac{\partial}{\partial z_i}$. Take

$\phi(z) = \theta(z - a)\theta(z - b)\theta(z + a + b)$ as before, with $a, b \in V$ and $\theta \in \Gamma(L(H, \alpha))$. If we put $f(z) = D(\log \theta)(z)$, we obtain

$$f(z_0 - a) + f(z_0 - b) + f(z_0 + a + b) = -\alpha_0$$

for all a, $b \in V$. One concludes easily that f is a linear (not necessarily homogeneous) function of z. Integrating the equation $f(z) = D(\log \theta)(z)$, we obtain that there is an $\alpha \in V$, $\alpha \neq 0$ such that for all $\lambda \in \mathbf{C}$, we have

$$\theta(z + \lambda\alpha) = e^{c\lambda^2 + \lambda f(z)}.\theta(z)$$

for some constant c. One concludes as in the earlier step (by writing down the transformation formulae for both sides, for the substitution $z \longmapsto z + u, u \in U$) that for all $\lambda \in \mathbf{C}$, $\lambda\alpha$ belongs to the lattice $U^\perp = \{z \in V \mid E(u, z) \in \mathbf{Z}, \forall\, u \in U\}$. This is a contradiction.

We next recall some definitions.

Let X be an algebraic variety over $\mathbf{C}$. There is a canonically associated analytic space structure on the underlying set of X. We denote this analytic space by X_{hol}, and its structure sheaf by $\mathcal{O}_{X,\text{hol}}$. Often, we will not be so explicit, and talk of holomorphic functions on X, holomorphic maps from or into X, etc. Also, we shall say that an analytic space X is algebraic or algebraisable if there is an algebraic variety Y such that $Y_{\text{hol}} \cong X$. Note further that an algebraic variety X is complete if and only if X_{hol} is compact. (This is an easy consequence of Chow's lemma.)

We now recall the

Theorem of Chow. *Let X be a complete algebraic variety and Y a closed analytic subset of X_{hol}. Then Y is Zariski closed in X.*

Chow proved the theorem for $X = \mathbf{P}^N$, but the above version follows immediately from this and Chow's lemma.

An easy consequence is that if X and Y are complete algebraic varieties and f: $X_{\text{hol}} \to Y_{\text{hol}}$ is a holomorphic map, then f considered as a map from X to Y is an algebraic morphism. To prove this, let Γ in $X_{\text{hol}} \times Y_{\text{hol}} = (X \times Y)_{\text{hol}}$ be the graph of f; it is a closed analytic subset, hence a closed algebraic subset of $X \times Y$. For every $(x, f(x)) \in \Gamma$, the projection $\Gamma \to X$ induces a local homomorphism of the algebraic local rings $\mathcal{O}_{x,X} \to \mathcal{O}_{(x,f(x)),\Gamma}$. Let $\mathcal{O}_1 = \mathcal{O}_{x,X}$, $\mathcal{O}_2 = \mathcal{O}_{(x,f(x)),\Gamma}$. Then I claim that $\mathcal{O}_1 \to \mathcal{O}_2$ is an isomorphism. First, use the fact that the projection $\Gamma \to X$ is proper and bijective: by Zariski's Main Theorem, this means that $\mathcal{O}_2$ is a finite $\mathcal{O}_1$-module. Let $\widetilde{\mathcal{O}}_2 = \mathcal{O}_{(x,f(x)),\Gamma,\text{hol}}$ and $\widetilde{\mathcal{O}}_1 = \mathcal{O}_{x,X,\text{hol}}$. Since $\Gamma \to X$ is an analytic isomorphism, we get a diagram:

$$\begin{array}{ccc} \mathcal{O}_1 & \longrightarrow & \mathcal{O}_2 \\ \big\downarrow & & \big\downarrow \\ \widetilde{\mathcal{O}}_1 & \xrightarrow{\approx} & \widetilde{\mathcal{O}}_2. \end{array}$$

In particular, $\mathcal{O}_1 \to \mathcal{O}_2$ is injective. If m_i = maximal ideal in $\mathcal{O}_i$, then dividing by m_i^2, we get

$$\begin{array}{ccc} \mathcal{O}_1/m_1^2 & \longrightarrow & \mathcal{O}_2/m_2^2 \\ \wr\big\downarrow & & \wr\big\downarrow \\ \widetilde{\mathcal{O}}_1/m_1^2\,\widetilde{\mathcal{O}}_1 & \xrightarrow{\approx} & \widetilde{\mathcal{O}}_2/m_2^2\,\widetilde{\mathcal{O}}_2 \end{array}$$

hence $m_2 = m_1\mathcal{O}_2 + m_2^2$. Therefore the $\mathcal{O}_2$-module $m_2/m_1\mathcal{O}_2$ becomes (0) after $\otimes_{\mathcal{O}_2} \mathcal{O}_2/m_2$: hence by Nakayama's lemma, $m_2 = m_1\mathcal{O}_2$. Therefore the $\mathcal{O}_1$-module $\mathcal{O}_2/\mathcal{O}_1$ becomes (0) after $\otimes_{\mathcal{O}_1} \mathcal{O}_1/m_1$: hence by Nakayama's lemma, $\mathcal{O}_2 = \mathcal{O}_1$. This shows that $\Gamma \to X$ is an algebraic isomorphism, hence that f is an algebraic morphism.

In particular, we see that a compact, complex space has *at most one algebraic structure.*† One further fact that we will need is that if X is a complete algebraic variety, every meromorphic function f

† For non-compact complex spaces, this is quite false. For instance, Serre [S1] p. 108, has given the following example: for every 1-dimensional abelian variety X over $\mathbf{C}$, there is a unique algebraic group G which is a non-trivial extension (as *alg.* group):

$$0 \longrightarrow \mathbf{C} \longrightarrow G \longrightarrow X \longrightarrow 0.$$

It is easily checked that if $\mathcal{O}_G$ is its (algebraic) structure sheaf, then $\Gamma(\mathcal{O}_G) = \mathbf{C}$. But taking the universal covering of G, one checks that analytically,

$$G = V/U,$$

V = a 2-dimensional complex vector space, $U = \{n_1\omega_1 + n_2\omega_2 | n_i \in \mathbf{Z}\}$ where ω_1, ω_2 are a $\mathbf{C}$-basis of V. If $\mathbf{G}_m$ denotes the 1-dimensional affine algebraic group, given by $\mathbf{C} - \{0\}$ under multiplication, it follows easily that G and $\mathbf{G}_m \times \mathbf{G}_m$ are two different algebraizations of the same analytic group!

on X_{hol} is a rational function on X, i.e. in the function field $\mathbf{C}(X)$; this can be proved similarly by considering the "graph" of f, and applying Chow's theorem.

Getting back to complex tori, we have

COROLLARY. *Let $X = V/U$ be a g-dimensional complex torus. The following are equivalent.*

(1) *X is the complex space associated to a projective algebraic variety,*

(2) *X is the complex space associated to any algebraic variety,*

(3) *there exist g algebraically independent meromorphic functions on X,*

(4) *there is a positive definite hermitian form H on V such that $\operatorname{Im}(H)$ is integral on $U \times U$.*

PROOF. (1) ⇒ (2) ⇒ (3) are obvious, (4) ⇒ (1) has been proved in the theorem. It remains to prove (3) ⇒ (4). Let $f_1, \ldots, f_g$ be the independent meromorphic functions. Let D_i be the polar divisor of f_i, $D = \Sigma D_i$, and L the line bundle associated to D. Then L admits $g+1$ sections $\sigma_0, \ldots, \sigma_g$ such that whenever f_i is regular, $\sigma_i = f_i\sigma_0$. By the theorem of Appell-Humbert, $L = L(H, \alpha)$ for some hermitian form H on V with $\operatorname{Im} H(U \times U) \subset \mathbf{Z}$, some α. Since L has sections at all, by the discussion preceding the theorem, we know that H is positive semi-definite. Let V_0 be its degenerate subspace, and $X_0 = V_0/V_0 \cap U$ the corresponding subtorus of X. Then the quotient torus X/X_0 equals $(V/V_0)/\text{image}(U)$, and H is induced by a positive definite $\bar{H}$ on V/V_0, such that $\operatorname{Im}(\bar{H})$ is integral on the lattice $U/U \cap V_0$. Therefore, by the theorem, X/X_0 is a projective algebraic variety. But also, by the discussion earlier, we know that if σ is any section of L, the zeroes of σ are unions of cosets of X_0. Applying this to the sections $\sum_{i=1}^{g} \alpha_i\sigma_i$, it follows that all the analytic sets f_i = constant are unions of cosets of Y_0, hence each f_i is induced by a meromorphic function $\bar{f}_i$ on X/X_0. Let $\mathbf{C}(X/X_0)$ be the function field of X/X_0; then

$g = \text{tr. deg}_{\mathbf{C}}\,\mathbf{C}(\bar{f}_1,\ldots,\bar{f}_g) \leqslant \text{tr. deg}_{\mathbf{C}}\,\mathbf{C}(X/X_0) = \dim(X/X_0) < \dim X = g.$ Therefore $\dim X_0 = 0$, and H is non-degenerate.

As an example, notice that we get immediately the algebraizability of 1-dimensional tori: in fact, if $X = \mathbf{C}/\{n + m\omega / n, m \in \mathbf{Z}\}$, with $\operatorname{Im} \omega > 0$, then let

$$H(z, w) = \frac{1}{\operatorname{Im}(\omega)}\, z.\bar{w}.$$

One checks immediately that $\operatorname{Im} H = E$ has values $E(1,1) = E(\omega, \omega) = 0$, $E(\omega, 1) = -E(1, \omega) = 1$, hence H satisfies the hypotheses of the theorem. Moreover, several projective embeddings of X are very well-known in the classical theory (cf. Hurwitz-Courant): for example, if

$$\wp(z) = \frac{1}{z^2} + \sum_{(n,m) \neq (0,0)} \left[\frac{1}{(z - n - m\omega)^2} - \frac{1}{(n + m\omega)^2}\right]$$

is the Weierstrass $\wp$-function, then $\wp$ is a meromorphic function, periodic with respect to $1, \omega$, with double poles at the points $n + m\omega$. The map:

$$z \longmapsto (1, \wp(z), \wp'(z)) \qquad \text{(projective coordinates)}$$

$$\mathbf{C} \longrightarrow \mathbf{P}^2$$

induces an isomorphism of X with a plane cubic curve of the form $X_0 X_2^2 = 4X_1^3 + aX_0^2 X_1 + bX_0^3$ (for suitable a, b depending on ω).

On the other hand, in dimensions $\geqslant 2$, it is easy to see that almost all tori are non-algebraic. In fact, we can check that on almost all tori X, $\operatorname{Pic}(X) = \operatorname{Pic}^0(X)$ or equivalently (by the theorem of Appell-Humbert) that there is no skew-symmetric $E = V \times V \to \mathbf{R}$ which is (a) integral on $U \times U$, and (b) satisfies $E(ix, iy) = E(x, y)$. Let $X = V/U$, and put $T = \operatorname{Hom}_{\mathbf{C}}(V, \mathbf{C})$, $\bar{T} = \operatorname{Hom}_{\mathbf{C}\text{-anti}}(V, \mathbf{C})$ as before. Consider the map

$$\begin{array}{ccc} \Lambda^2(\operatorname{Hom}(U, \mathbf{Z})) & \hookrightarrow & \Lambda^2_{\mathbf{C}} \operatorname{Hom}_{\mathbf{Z}}(U, \mathbf{C}) \\ & & \| \\ & & \Lambda^2_{\mathbf{C}} \operatorname{Hom}_{\mathbf{R}}(V, \mathbf{C}) \\ & & \| \\ & & \Lambda^2_{\mathbf{C}}(T \oplus \bar{T}) \simeq (\Lambda^2_{\mathbf{C}} T) \oplus (T \otimes \bar{T}) \oplus (\Lambda^2_{\mathbf{C}} \bar{T}). \end{array}$$

We want to show that for almost all lattices $U \subset V$ no element of $\Lambda^2_{\mathbf{Z}}(\operatorname{Hom}(U, \mathbf{Z}))$ has image entirely in the middle factor $T \otimes \bar{T}$ on the right. It will suffice to show that $\Lambda^2_{\mathbf{Z}}(\operatorname{Hom}(U, \mathbf{Z})) \to \Lambda^2_{\mathbf{C}} T$ is injective. But $\operatorname{Hom}(U, \mathbf{Z})$ projects into a lattice in T, and for suitable choice of the lattice U in V, $\operatorname{Im}(\operatorname{Hom}(U, \mathbf{Z}))$ is an arbitrary lattice in T. So the conclusions follow from

Lemma. *Let V be a g-dimensional complex vector space. Then for almost all lattices $U \subset V$, the map $\Lambda^g_{\mathbf{Z}} U \to \Lambda^g_{\mathbf{C}} V \simeq \mathbf{C}$ is injective (hence all the maps $\Lambda^k_{\mathbf{Z}} U \to \Lambda^k_{\mathbf{C}} V$ are injective, $k \leqslant g$).*

If coordinates $z_1, \ldots, z_g$ are introduced in V, and U is described by giving a basis $(\omega_{i1}, \ldots, \omega_{ig})$, $1 \leqslant i \leqslant 2g$, then almost all can be interpreted to mean all $g \times 2g$-tuples (ω_{ij}) not lying on a countable union of $(g(2g) - 1)$-dimensional analytic subsets. We leave the proof of this lemma to the reader.