II

ALGEBRAIC THEORY VIA VARIETIES

4. Definition of abelian varieties.

We now turn to the study of abelian varieties over an arbitrary algebraically closed field k.

DEFINITION. *An abelian variety X is a complete algebraic variety*[†] *over k with a group law $m\colon X \times X \to X$ such that m and the inverse map are both morphisms of varieties.*

Note that if $k = \mathbf{C}$, then the underlying complex analytic space of an abelian variety is a compact complex analytic group, hence by the results of §1, it is a complex torus. When $k \neq \mathbf{C}$, the first aim of the theory of abelian varieties is to show that an abelian variety has properties analogous to those enjoyed by a complex torus. Of course, when char $k = 0$, many of these results can be proven by reduction to the case $k = \mathbf{C}$ (Lefschetz's principle), but when char $k \neq 0$, this is by no means possible. We shall want to answer the following basic questions.

QUESTION 1. Structure of X as an abstract group.

We will show that X is a commutative and divisible group. We will also show that if n_X denotes multiplication by n (n an integer > 0) on X, the kernel X_n of n_X, or what is the same, the group of elements $x \in X$ such that $nx = 0$, has the following structure :

$$X_n \simeq (\mathbf{Z}/n\mathbf{Z})^{2g} \text{ if char } k \nmid n,$$
$$X_{p^m} \simeq (\mathbf{Z}/p^m\mathbf{Z})^{i} \text{ if } p = \text{char } k,\ m > 0,$$

where i can take every value in the range $0 \leqslant i \leqslant g = \dim X$. This integer i will be called the *p-rank* of X.

QUESTION 2. Calculate the cohomology group $H^q(X, \Omega^p)$ (Ω^p being the sheaf of p-forms on X).

As in the classical cases we have canonical isomorphisms

$$H^q(X, \Omega^p) \simeq \overset{p}{\Lambda}\,[H^0(X, \Omega^1)] \otimes_k \overset{q}{\Lambda}[H^1(X, \mathcal{O}_X)],$$

[†] This means, in particular, that it is irreducible.

and

$$\dim H^1(X, \mathcal{O}_X) = \dim H^0(X, \Omega^1) = g.$$

We also show that $\pi_1(X)$ (in the algebraic sense, i.e. the projective limit of finite groups of unramified Galois coverings) is isomorphic to $\prod_l (\mathbf{Z}_l)^{2g}$ in char 0 and to $\prod_{l \neq p} (\mathbf{Z}_l)^{2g} \times \mathbf{Z}_p^i$ in char p. More precisely, we shall show that if $Y \xrightarrow{f} X$ is any morphism such that a finite group G acts freely on Y in such a way that X becomes the quotient of Y for this action, there is an integer $n > 0$ and a commutative diagram

$$\begin{tikzcd} & Y \arrow[dr, "f"] & \\ X \arrow[ur, "g"] \arrow[rr, "n_X"] & & X. \end{tikzcd}$$

Further, Y carries a structure of abelian variety such that f and g are homomorphisms.

Question 3. The structure of Pic X.

We will show that there is an exact sequence

$$0 \longrightarrow \mathrm{Pic}^0 X \longrightarrow \mathrm{Pic}\, X \longrightarrow NS(X) \longrightarrow 0$$

where $\mathrm{Pic}^0 X$ has a natural structure of an abelian variety, and $NS(X)$ is a finitely generated free abelian group, whose rank ρ is called the *base number of* X.

We will also try to find analogues of the classical description of $NS(X)$ by Riemann forms E.

Related questions are: (a) for a pair of abelian varieties X, Y, show that Hom(X, Y) is a free abelian group on a finite number of generators, (here Hom means the set of maps which are both morphisms of varieties and homomorphisms of groups); and (b) give a matricial representation of this group of homomorphisms (in the classical case, we have the representation induced in Hom($H_1(X)$, $H_1(Y)$)).

Question 4. Characterize ample line bundles. More generally, compute the cohomology groups of arbitrary line bundles, and in particular the dimension of the space of sections–this is the Riemann-Roch problem.

(i) We start with the observation that *an abelian variety is everywhere non-singular.* In fact, there has to exist a non-singular point $x_0 \in X$, and for $x \in X$, the translation morphism $T_{(x.x_0^{-1})}$: $X \to X$, given by $T_{(x.x_0^{-1})}(y) = x.x_0^{-1}.y$, is an automorphism of X carrying x_0 to x, so that x is again a non-singular point of X.

(ii) *As a group, X is commutative.* We give two proofs, one here and the second a little later. The first proof generalizes the proof we gave in the classical case. We consider not only the adjoint representation of X in the tangent space at the identity e, or in the space of differentials at e, but in each of the spaces $(\mathcal{O}_{X,e}/\mathfrak{M}^n_{X,e})$ where $\mathcal{O}_{X,e}$ is the local ring of X at e and $\mathfrak{M}_{X,e}$ its maximal ideal. For $x \in X$, let $C_x: X \to X$ be defined by $C_x(y) = x\,y\,x^{-1}$, so that $C_x(e) = e$. Then C_x induces an automorphism $C^*_{x,n}$ of the vector space $\mathcal{O}_{X,e}/\mathfrak{M}^n_{X,e}$, deduced by passage to quotient from the automorphism $C^*_x: \mathcal{O}_{X,e} \to \mathcal{O}_{X,e}$ of the local ring. This induces a set theoretic map $\gamma: X \to \mathrm{Aut}(\mathcal{O}_{X,e}/\mathfrak{M}^n_{X,e})$, $x \longmapsto C^*_{x,n}$ and if we put on the latter group the natural structure of an algebraic variety (viz. that induced from the inclusion $\mathrm{Aut}(\mathcal{O}_{X,e}/\mathfrak{M}^n_{X,e}) \subset \mathrm{End}(\mathcal{O}_{X,e}/\mathfrak{M}^n_{X,e})$, the last being a finite-dimensional vector space over k), one checks easily that this is a morphism of varieties. Since the latter is an affine variety and X is complete and connected, γ must be a constant map! Since $\gamma(e)$ is the identity, we see that $C^*_{x,n}$ is the identity for all $x \in X$ and $n > 0$. But since $\bigcap_n \mathfrak{M}^n_{X,e} = (0)$ this means that $C^*_x: \mathcal{O}_{X,e} \to \mathcal{O}_{X,e}$ is the identity, so that C_x reduces to the identity in a neighbourhood of e in X. Since X is irreducible, C_x is the identity on X for every x, that is, X is commutative.

From now on, we write the group law in X additively. Moreover, we will use the following notations: for $x \in X$, we denote by

$T_x\colon X \to X$ the translation morphism $T_x(y) = x + y$; and the map $x \longmapsto n.x$ will be denoted by n_X.

(iii) *If $T = T_{X,0}$ is the tangent space at 0 to X, Ω_0 is the dual space $T^*_{X,0}$ of differentials, then there is a natural isomorphism*

$$\Omega_0 \otimes_k \mathcal{O}_X \xrightarrow{\sim} \Omega^1_X,$$

where Ω^1_X is the sheaf of regular 1-forms on X. One defines this mapping as follows. To each $\theta \in \Omega_0$, consider the 1-form ω_θ on X defined by $(\omega_\theta)_x = T^*_{-x}(\theta)$, that is, the unique translation invariant 1-form on X whose value at 0 is θ. It is checked easily that ω_θ is a regular 1-form on X. Thus, we have a natural homomorphism as above. Since at any point x, T^*_{-x} induces an isomorphism of the space of differentials at x onto the space Ω_0, it follows that the above homomorphisms of sheaves induces an isomorphism of fibers at x reduced modulo the maximal ideal $\mathfrak{M}_{X,x}$ at x. It follows by Nakayama's lemma that it is an isomorphism of sheaves.

Since X is complete and connected and $H^0(X, \mathcal{O}_X) = k$, it follows from the above isomorphism that the everywhere regular forms on X are precisely the invariant forms.

(iv) *For every n not divisible by the characteristic p of k, the endomorphism n_X is surjective.*

For the proof, we make the following observation. The tangent space $T_{X\times X,\,(0,\,0)}$ to $X \times X$ at $(0, 0)$ splits canonically into a direct sum $T_1 \oplus T_2$, where T_1 (resp. T_2) is the isomorphic image of $T_{X,0}$ under the immersion $X \to X \times X$ given by $x \overset{i_0}{\longmapsto} (x, 0)$ (resp. $x \overset{i_1}{\longmapsto} (0, x)$). Identifying T_i with $T_{X,0} = T$ by these isomorphisms, we note that the differential $d(m)\colon T \oplus T \to T$ of the addition map $m\colon X \times X \to X$, $m(x, y) = x + y$, is nothing but addition of components: $d(m)\,(t_1, t_2) = t_1 + t_2$. In fact, it is sufficient to check this (by linearity) on the two summands T of $T \oplus T$, and for these, it follows from the fact that the composites $X \xrightarrow{i_0} X\times X \xrightarrow{m} X$ and $X \xrightarrow{i_1} X \times X \longrightarrow X$ are the identity.

It follows by induction on n that for any $n > 0$ (and hence also for $n < 0$), $(dn_X)_0$ is multiplication by n. Thus, if $p \nmid n$, dn_X is an isomorphism. If n_X were not surjective, by the dimension theorem, $\dim_0 n_X^{-1}(0) > 0$, and we can therefore find a non-zero $t \in T$ tangent to $n_X^{-1}(0)$ at 0. But then, we would have $dn_X(t) = 0$ (since $n_X^{-1}(0)$ is mapped into the single point 0 by n_X), which is a contradiction.

The following lemma, besides having other important applications, gives a second proof of the commutativity of X.

RIGIDITY LEMMA. (Form I.) *Let X be a complete variety, Y and Z any varieties, and $f\colon X \times Y \to Z$ a morphism such that for some $y_0 \in Y$, $f(X \times \{y_0\})$ is a single point z_0 of Z. Then there is a morphism $g\colon Y \to Z$ such that if $p_2\colon X \times Y \to Y$ is the projection, $f = g \circ p_2$.*

PROOF. Choose any point $x_0 \in X$, and define $g : Y \to Z$ by $g(y) = f(x_0, y)$. Since $X \times Y$ is a variety, to show that $f = g \circ p_2$, it is sufficient to show that these morphisms coincide on some open subset of $X \times Y$. Let U be an affine open neighbourhood of z_0 in Z, $F = Z - U$, and $G = p_2\,(f^{-1}(F))$; then G is closed in Y since X is complete and hence p_2 is a closed map. Further $y_0 \notin G$ since $f(X \times \{y_0\}) = \{z_0\}$. Therefore $Y - G = V$ is a non-empty open subset of Y. For each $y \in V$, the complete variety $X \times \{y\}$ gets mapped by f into the affine variety U, and hence to a single point of U. But this means that for any $x \in X, y \in V, f(x, y) = f(x_0, y) = g \circ p_2(x, y)$, and this proves our assertion.

COROLLARY 1. *If X and Y are abelian varieties and $f\colon X \to Y$ is any morphism, $f(x) = h(x) + a$ where h is a homomorphism of X into Y and $a \in Y$.*

PROOF. Replacing f by $f - f(0)$, we may assume $f(0) = 0$ and we have to show under this assumption that f is a homomorphism.

Consider the morphism $X \times X \xrightarrow{\phi} Y$ defined by $\phi(x, y) = f(x + y) - f(y) - f(x)$. Then $\phi(X \times \{0\}) = \phi(\{0\} \times X) = 0$, so that it follows by the above lemma that $\phi \equiv 0$ on $X \times X$, or what is the same, that f is a homomorphism.

Note that in the proof of the above corollary, despite the additive notation used, no use of the commutativity of X was made. Thus, we may use it to give a second proof of the commutativity of X.

Corollary 2. *X is a commutative group.*

In fact, the morphism of X into itself mapping each element onto its inverse is a homomorphism by Corollary 1. Hence, for $x, y \in X$, $(xy)^{-1} = x^{-1}y^{-1} = y^{-1}x^{-1}$, and X is commutative.

Corollary 3. *Let X be an abelian variety (with base point 0). Then on the category of complete varieties with base point, the functor $S \longmapsto \mathrm{Hom}(S, X)$ (where* Hom *denotes morphisms preserving base point) is linear; that is, for S, T in this category, the natural map*

$$\mathrm{Hom}\,(S, X) \times \mathrm{Hom}\,(T, X) \longrightarrow \mathrm{Hom}\,(S \times T, X)$$

given by $(f, g) \longmapsto h$, $h(s,t) = f(s) + g(t)$ is a bijection.

Proof. That this map is injective follows by fixing s and t in turn to be the base points s_0 and t_0 of S and T, respectively, in the equation $h(s, t) = f(s) + g(t)$. Next, take any $h \in \mathrm{Hom}\,(S \times T, X)$, and put $f(s) = h(s, t_0)$, $g(t) = h(s_0, t)$, $k(s, t) = h(s, t) - f(s) - g(t)$. Then $k(S \times \{t_0\}) = k(\{s_0\} \times T) = 0$, and it follows from the rigidity lemma that $k \equiv 0$.

APPENDIX TO §4

We want to prove the following result.

Theorem. *Let X be a complete variety, $e \in X$ a point, and*

$$m: X \times X \longrightarrow X$$

a morphism such that $m(x, e) = m(e, x) = x$ for all $x \in X$. Then X is an abelian variety with group law m and identity e.

Proof. We shall denote $m(x, y)$ simply by xy. Introduce the morphism

$$\psi: X \times X \longrightarrow X \times X$$
$$\psi: (x, y) = (xy, y).$$

Then $\psi^{-1}(e, e) = \{(e, e)\}$, so that by the dimension theorem, $\dim\,(\mathrm{Image}\,\psi) = \dim\,(X \times X)$. Since $X \times X$ is complete, this implies that ψ is surjective. In particular, given $x \in X$, there is an $x' \in X$ with $x'x = e$. Thus, if $\Gamma' = \{(x, y) \in X \times X \mid xy = e\}$, and $p_i (i = 1, 2)$ is the i^{th} projection of $X \times X$, $p_2(\Gamma') = X$. Choose an irreducible component Γ of Γ' with $p_2(\Gamma) = X$. Note that $\dim \Gamma \geqslant \dim X$. If $p'_i = p_i \mid \Gamma$, $p_1'^{-1}(e) = \{(e, e)\}$, so that again by the dimension theorem, $\dim\,(\mathrm{Image}\ p'_1) = \dim X$. Since Γ is complete, this implies that p'_1 is surjective.

Let $\phi: \Gamma \times X \to X$ be defined by $\phi((x', x), y) = x'(xy)$. Then $\phi(\Gamma \times \{e\}) = \{e\}$, so by the rigidity lemma, $\phi((x', x), y) = \phi((e, e), y)) = y$, that is,

$$x'(xy) = y, \quad \forall\ (x', x) \in \Gamma, \quad y \in X.$$

In particular, if $(x', x) \in \Gamma$, then $x'(x.x') = x'$. Choose an $(x'', x') \in \Gamma$, and multiply the last equation on the left by x'', to obtain

$$x''(x'(xx')) = x''x'.$$

But $x''x' = e$, and by (1), $x''(x'(xx')) = xx'$. Therefore if $(x', x) \in \Gamma$, then not only is $x'x = e$, but also $xx' = e$.

Let $\chi: \Gamma \times X \times X \to X$ be the map $\chi((x', x), y, z) = x((x'.y)z)$. Since $\chi(\Gamma \times \{e\} \times \{e\}) = e$, by the rigidity lemma,

$$x((x'y)z) = e((ey)z) = yz.$$

Multiplying on the left by x' and using (1), we get

$$(x'y)\,z = x'(x((x'y)z)) = x'(yz).$$

Since x' is arbitrary in X (p'_1 being surjective), this shows that multiplication is associative. Thus X is a group with group law m. In particular, for any $x_0 \in X$, the translation $x \longmapsto x_0 x$ is an automorphism of X as a variety, and we deduce that X is non-singular. This also shows that ψ is bijective. But also the tangent map of ψ at (e, e) is an isomorphism since $\psi(x, e) = (x, e)$ and $\psi(e, x) = (x, x)$. Thus ψ cannot be inseparable, so that by Zariski's Main Theorem, ψ is an isomorphism. The inverse of ψ is given by $(x, y) \longmapsto (xy^{-1}, y)$, so this shows that $y \longmapsto y^{-1}$ is also a morphism, hence X is an abelian variety.

5. Cohomology and base change. At this point, we have to digress to prove a theorem of Grothendieck on the behavior of cohomology groups of a family of vector bundles E_y on a family of varieties X_y, parametrized by points $y \in Y$ where X_y is assumed to be a flat family of varieties. An important consequence of this result is the semicontinuity of the dimensions of the cohomology groups of the E_y.

We assume the following basic result.

Theorem. *If $f: X \to Y$ is a proper morphism of locally noetherian preschemes and $\mathscr{F}$ a coherent sheaf of $\mathcal{O}_X$-modules on X, for all $p \geq 0$ the direct image sheaves $R^p f_*(\mathscr{F})$ are coherent sheaves of $\mathcal{O}_Y$-modules.*

We recall the following definition. If $f: X \to Y$ is a morphism of preschemes and $\mathscr{F}$ a quasicoherent sheaf on X, $\mathscr{F}$ is said to be *flat* over Y or *f-flat* if for each $x \in X$, $\mathscr{F}_x$(for its natural structure of $\mathcal{O}_{Y,f(x)}$-module) is $\mathcal{O}_{Y,f(x)}$-flat. It is easily shown that this condition is equivalent to requiring that for $U \subset X$, $V \subset Y$ with U and V affine open, and $f(U) \subset V$, $\mathscr{F}(U)$ is a flat $\mathcal{O}_Y(V)$-module.

The main result of this section is the following

Theorem. *Let $f: X \to Y$ be a proper morphism of noetherian schemes with $Y = \operatorname{Spec} A$ affine, and $\mathscr{F}$ a coherent sheaf on X, flat over Y. There is a finite complex $K^\cdot: 0 \to K^0 \to K^1 \to \ldots \to K^n \to 0$ of finitely generated projective A-modules and an isomorphism of functors*

$$H^p(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B) \simeq H^p(K^\cdot \otimes_A B), \quad (p \geq 0)$$

on the category of A-algebras B.

Proof. Choose a finite affine covering $\mathfrak{A} = \{U_i\}_{i \in I}$ of X by affine open subsets. Then the Čech complex $C^\cdot = C^\cdot(\mathfrak{A}, \mathscr{F}) = \oplus\, C^p(\mathfrak{A}, \mathscr{F})$ of alternating Čech cochains on $\mathfrak{A}$ with coefficients in $\mathscr{F}$ is a finite complex of A-flat modules, whose cohomologies are isomorphic to the cohomology groups $H^p(X, \mathscr{F})$.

Moreover, for all A-algebras B, $\{U_i \times_Y \operatorname{Spec} B\}$ is an affine covering of $X \times_Y \operatorname{Spec}(B)$, and $C^p(\mathfrak{A}, \mathscr{F}) \otimes_A B$ is the module of Čech p-cochains of $\mathscr{F} \otimes_A B$ for this covering. Therefore

$$H^p(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B) \simeq H^p(C^\cdot \otimes_A B)$$

for all B, and, in fact, functorially in B.

We need the following basic lemma

Lemma 1. *Let $C^\cdot$ be a complex of A-modules (A any noetherian ring) such that the $H^i(C^\cdot)$ are finitely generated A-modules and such that $C^p \neq (0)$ only if $0 \leq p \leq n$. Then there exists a complex $K^\cdot$ of finitely generated A-modules such that $K^p \neq (0)$ only if $0 \leq p \leq n$ and K^p is free if $1 \leq p \leq n$ and a homomorphism of complexes $\phi: K^\cdot \to C^\cdot$ such that ϕ induces isomorphisms $H^i(K^\cdot) \xrightarrow{\sim} H^i(C^\cdot)$, all i. Moreover if the C^p are A-flat, then K^0 will be A-flat too.*

Proof. We define, by descending induction on m, diagrams:

$$\begin{tikzcd}
 & & K^m \arrow[r, "\partial^m"] \arrow[d, "\phi_m"] & K^{m+1} \arrow[r, "\partial^{m+1}"] \arrow[d, "\phi_{m+1}"] & K^{m+2} \arrow[r] \arrow[d, "\phi_{m+2}"] & \cdots \\
\cdots \arrow[r] & C^{m-1} \arrow[r] & C^m \arrow[r, "\partial^m"'] & C^{m+1} \arrow[r, "\partial^{m+1}"'] & C^{m+2} \arrow[r] & \cdots
\end{tikzcd}$$

Put $K^p = 0$ for $p > n$. Suppose we have defined $(K^p, \phi_p, \partial^p)$ for $p \geq m+1$ such that the following conditions hold:

- (i) $\partial^p \phi_p = \phi_{p+1} \partial^p$, $(p \geq m+1)$.
- (ii) $\partial^{p+1} \circ \partial^p = 0$, $(p \geq m+1)$.
- (iii) The ϕ^p induces isomorphisms in cohomology $H^q(K^\cdot) \xrightarrow{\sim} H^q(C^\cdot)$ for $q \geq m+2$, and a surjection $\ker \partial^{m+1} \to H^{m+1}(C^\cdot)$.
- (iv) The K^p are A-free and finitely generated, $(p \geq m+1)$.

We then construct K^m, ∂^m and ϕ_m so as to satisfy (i)-(iii) with $m+1$ replaced by m.

Suppose first that $m \geq 0$. Let B^{m+1} be the kernel of the homomorphism $\ker \partial^{m+1} \to H^{m+1}(C^\cdot)$. Since B^{m+1} is finitely generated over A (A being noetherian), we can find a finitely generated free module K'^m and a surjection $\partial' : K'^m \to B^{m+1}$. Further, since $H^m(C^\bullet)$ is a finitely generated A-module, we can find a surjection $K''^m \xrightarrow{\lambda} H^m(C^\bullet)$ with K''^m finitely generated and free. Let $\mu: K''^m \to Z^m(C^\bullet)$ be any lift of λ, and $\phi''_m\ K''^m \to C^m$ the composite of μ with the inclusion $Z^m(C^\bullet) \to C^m$. We then put $K^m = K'^m \oplus K''^m$, and define $\partial^m: K^m \to K^{m+1}$ by putting it equal to zero on K''^m and equal to ∂' on K'^m. Since $\phi_{m+1} \circ \partial'(K'^m) \subset \partial C^m$, we can find $\phi'_m: K'^m \to C^m$ such that $\partial \circ \phi'_m = \phi_{m+1} \circ \partial'$. We then define $\phi_m: K^m \to C^m$ as being equal to ϕ'_m on K'^m and ϕ''_m on K''^m. The conditions (i)-(iii) are evidently fulfilled with m instead of $m + 1$.

Suppose then that $m = -1$, that is, that $\{K^p, \phi_p, \partial^p\}$ have been defined for $p \geqslant 0$ satisfying (i)-(iii). We then replace K^0 by $K^0/\ker \partial^0 \cap \operatorname{Ker} \phi_0$, and we take $\phi_0: K^0 \to C^0$ and $\partial^0: K^0 \to K^1$ to be the induced mappings. Putting $K^p = 0$ for $p < 0$, we get a complex

$$0 \to K^0 \to K^1 \to K^2 \to K^3 \to \ldots \to K^n \to 0$$

and a homomorphism $\phi: K^\bullet \to C^\bullet$ which by construction induces isomorphisms in cohomology. We have only to check that K^0 is A-flat when all the C^p are A-flat. Consider the 'mapping cylinder' complex L defined by $L^p = K^p \oplus C^{p-1}$ for $p \in \mathbf{Z}$, and $\partial: L^p \to L^{p+1}$ defined by $\partial(x, 0) = (\partial x, \phi(x))$, $\partial(0, y) = (0, -\partial y)$. If $C'^\bullet$ is the complex obtained from $C^\bullet$ by shifting degrees by one (and making a sign change in ∂), $C'^p = C^{p-1}$, we have an exact sequence of complexes $0 \to C'^\bullet \to L^\bullet \to K^\bullet \to 0$, and hence an exact cohomology sequence

$$\begin{tikzcd}
 & H^p(C^\bullet) \arrow[d, equal] & & & H^{p+1}(C^\bullet) \arrow[d, equal] \\
H^p(K^\bullet) \arrow[r] & H^{p+1}(C'^\bullet) \arrow[r] & H^{p+1}(L^\bullet) \arrow[r] & H^{p+1}(K^\bullet) \arrow[r] & H^{p+2}(C'^\bullet)
\end{tikzcd}$$

and one sees from the definition that the cohomology maps $H^p(K^\bullet) \to H^{p+1}(C'^\bullet) \simeq H^p(C^\bullet)$ are the ones induced by $\phi: K^\bullet \to C^\bullet$. Since these are all isomorphisms, $H^p(L^\bullet) = (0)$ for all $p \in \mathbf{Z}$. But then $0 \to K^0 = L^0 \to L^1 \to L^2 \to \ldots \to L^{n+1} \to 0$ is exact and the modules L^i are flat for $i \geqslant 1$, hence K^0 is A-flat.

Applying the lemma to our case, we have a complex $K^\bullet$, and a homomorphism $K^\bullet \to C^\bullet$ such that

$$H^p(K^\bullet) \xrightarrow{\sim} H^p(C^\bullet) \simeq H^p(X, \mathscr{F}), \text{ all } p.$$

Note that K^0 is A-projective, since it is A-flat and finitely generated over a noetherian A. It remains to check that for all A-algebras B, $H^p(K^\bullet \otimes_A B) \to H^p(C^\bullet \otimes_A B)$ is an isomorphism too. This is a consequence of

Lemma 2. *Let $C^\bullet$, $K^\bullet$ be any finite complexes of flat A-modules, and let $C^\bullet \to K^\bullet$ be a homomorphism of complexes inducing isomorphisms $H^p(C^\bullet) \xrightarrow{\sim} H^p(K^\bullet)$ for all p. Then for every A-algebra B, the maps $H^p(C^\bullet \otimes_A B) \xrightarrow{\sim} H^p(K^\bullet \otimes_A B)$ are isomorphisms.*

Proof. Construct the 'mapping cylinder' $L^\bullet$ exactly as in the proof of Lemma 1. As before, we see that $L^\bullet$ is an exact finite complex of flat A-modules. Then it is easy to see that all the modules $Z^p = \operatorname{Ker}(L^p \xrightarrow{\partial^p} L^{p+1})$ are flat too, hence

$$0 \longrightarrow Z^p \longrightarrow L^p \longrightarrow Z^{p+1} \longrightarrow 0$$

is a short exact sequence of flat A-modules. Therefore

$$0 \longrightarrow Z^p \otimes_A B \longrightarrow L^p \otimes_A B \longrightarrow Z^{p+1} \otimes_A B \longrightarrow 0$$

is exact, from which it follows that $L^\bullet \otimes_A B$ is exact. But now $L^\bullet \otimes_A B$ is the mapping cylinder of the map $K^\bullet \otimes_A B \to C^\bullet \otimes_A B$. So using the cohomology sequence in reverse, it follows that $H^p(K^\bullet \otimes_A B) \to H^p(C^\bullet \otimes_A B)$ are isomorphisms.

For any morphism $f: X \to Y$ and $y \in Y$, we denote by X_y the fiber over y of f (i. e., the fiber product $X \times_Y \operatorname{Spec} k(y)$, considered as a scheme over $k(y)$), and for $\mathscr{F}$ quasi-coherent on X, we denote by $\mathscr{F}_y$ the sheaf $\mathscr{F} \otimes_{\mathcal{O}_Y} k(y)$ on X_y.

We have then the following important corollary.

COROLLARY. *Let X, Y, f and $\mathcal{F}$ be as in the theorem (except that Y need not be affine). Then we have:*

(a) *For each $p \geqslant 0$, the function $Y \to \mathbf{Z}$ defined by*

$$y \longmapsto \dim_{k(y)} H^p(X_y, \mathcal{F}_y) \text{ is upper semicontinuous on } Y.$$

(b) *The function $Y \to \mathbf{Z}$ defined by*

$$y \to \chi(\mathcal{F}_y) = \sum_{p=0}^{\infty} (-1)^p \dim_{k(y)} H^p(X_y, \mathcal{F}_y)$$

is locally constant on Y.

PROOF. The problem being local on Y, we may assume Y affine. Let $K^\bullet$ be a complex as in the proposition; by further localization, we may assume $K^\bullet$ to be a free complex. Denote by $d^p\colon K^p \to K^{p+1}$ the coboundary of K. We then have

$$\begin{aligned}\dim_{k(y)} H^p(X_y, \mathcal{F}_y) &= \dim_{k(y)}[\ker(d^p \otimes_A k(y))] - \\ &\qquad - \dim_{k(y)}[\operatorname{Im}(d^{p-1} \otimes_A k(y))] \\ &= \dim_{k(y)}[K^p \otimes k(y)] - \dim_{k(y)}[\operatorname{Im}(d^p \otimes k(y))] - \\ &\qquad - \dim_{k(y)}[\operatorname{Im}(d^{p-1} \otimes k(y))]. \quad (*)\end{aligned}$$

The first term being constant on Y, (b) follows on taking alternating sum of (*) over all p. We assert that for any $p \geqslant 0$, the function $\rho_p(y) = \dim_{k(y)}[\operatorname{Im}(d^p \otimes k(y))]$ is lower semi-continuous on Y. In fact, if r is any integer $\geqslant 0$, and $d^p_r\colon \Lambda^r K^p \to \Lambda^r K^{p+1}$ is the map induced by d^p,

$$\{y \in Y \mid \rho_p(y) < r\} = \{y \in Y \mid d^p_r \otimes k(y) = 0\},$$

and this set is closed since d^p_r is a homomorphism of free finitely generated modules, and hence is described by a matrix in A, and the above set is the set of common zeros of all entries of the matrix. This proves (a).

Moreover, the theorem gives the following criterion for putting together the cohomology groups of $\mathcal{F}$ along the fibres of f into a vector bundle on Y.

COROLLARY 2. *Let X, Y, f and $\mathcal{F}$ be as above. Assume Y is reduced and connected. Then for all p the following are equivalent:*

(i) *$y \longmapsto \dim_{k(y)} H^p(X_y, \mathcal{F}_y)$ is a constant function,*

(ii) *$R^p f_*(\mathcal{F})$ is a locally free sheaf $\mathcal{E}$ on Y, and for all $y \in Y$, the natural map*

$$\mathcal{E} \otimes_{\mathcal{O}_Y} k(y) \longrightarrow H^p(X_y, \mathcal{F}_y)$$

is an isomorphism.

If these conditions are fulfilled, we have further that

$$R^{p-1} f_*(\mathcal{F}) \otimes_{\mathcal{O}_Y} k(y) \longrightarrow H^{p-1}(X_y, \mathcal{F}_y)$$

is an isomorphism for all $y \in Y$.

PROOF. Again assume Y affine, $K^\bullet$ as in the proposition. (ii) $\Rightarrow$ (i) is obvious. To prove (i) $\Rightarrow$ (ii), we need two lemmas.

LEMMA 1. *If Y is reduced and $\mathcal{F}$ a coherent sheaf on Y such that $\dim_{k(y)}[\mathcal{F} \otimes_{\mathcal{O}_Y} k(y)] = r$, all $y \in Y$, then $\mathcal{F}$ is locally free of rank r on Y.*

PROOF. For any $y \in Y$, let $\sigma_1, \ldots, \sigma_r \in \mathcal{F}_y$ lift generators of $\mathcal{F}_y \otimes k(y)$. Since $\sigma_1, \ldots, \sigma_r$ are extendable to sections in a neighborhood of y, we have a homomorphism $\sigma\colon \mathcal{O}_Y{}^r|_V \to \mathcal{F}|_V$ defined in a neighborhood V of y. Then σ is surjective on the stalks at y, by Nakayama's lemma, so coker(σ) is zero at y and hence in a neighborhood of y. Thus, we may assume σ to be surjective. Then by assumption, for every $y' \in V$, the map

$$\sigma \otimes k(y')\colon k(y')^r \to \mathcal{F}_{y'} \otimes_{\mathcal{O}_{y'}} k(y')$$

is an isomorphism. Thus, if $\mathfrak{D}$ is the kernel of σ, we have $\mathfrak{D}_{y'} \subset \mathfrak{M}_{y'}\, \mathcal{O}^r_{y'}$ for each $y' \in V$. Since Y is reduced, this means that $\mathfrak{D} = (0)$. Thus σ is an isomorphism.

We apply this in the following

LEMMA 2. *Let Y be a reduced, noetherian affine scheme, and let*

$$\mathcal{F} \xrightarrow{\phi} \mathfrak{D}$$

be a homomorphism of coherent locally free $\mathcal{O}_Y$-sheaves. If $\dim_{k(y)}[\mathrm{Im}(\phi\otimes k(y))]$ is locally constant, then there are splittings:

$$\mathscr{F}\simeq\mathscr{F}_1\oplus\mathscr{F}_2$$

$$\mathfrak{D}\simeq\mathfrak{D}_1\oplus\mathfrak{D}_2$$

such that $\phi|_{\mathscr{F}_1}=(0)$, $\mathrm{Im}(\phi)\subset\mathfrak{D}_1$, and $\phi\colon\mathscr{F}_2\to\mathfrak{D}_1$ is an isomorphism, i.e.

$$\phi=\begin{bmatrix}0 & \text{isom.}\\ 0 & 0\end{bmatrix}.$$

PROOF. By Lemma 1, $\mathfrak{D}/\phi(\mathscr{F})$ is locally free. If $Y=\mathrm{Spec}\,(A)$, $M=\Gamma(Y,\mathscr{F})$, $N=\Gamma(Y,\mathfrak{D})$, then this means that $N/\phi(M)$ is A-projective. Therefore N splits into the direct sum of $\phi(M)$ and a second submodule isomorphic to $N/\phi(M)$. Or, in sheaves, $\mathfrak{D}\simeq\mathfrak{D}_1\oplus\mathfrak{D}_2$, where $\mathfrak{D}_1=\mathrm{Im}(\phi)$. Moreover, this shows that $\phi(M)$ is A-projective, too, so M splits into the direct sum of $\mathrm{Ker}(\phi)$ and a second submodule isomorphic to $\phi(M)$. Or, in sheaves, $\mathscr{F}\simeq\mathscr{F}_1\oplus\mathscr{F}_2$, where $\phi(\mathscr{F}_1)=(0)$, $\phi\colon\mathscr{F}_2\xrightarrow{\sim}\mathfrak{D}_1$.

Now assume (i) holds. Let $K^\bullet$ be the complex given by the theorem. As in the proof of Corollary 1, $\dim[\mathrm{Im}(d^{p-1}\otimes k(y))]$ and $\dim[\mathrm{Im}(d^p\otimes k(y))]$ are locally constant. By Lemma 2, applied first to $d_p\colon K^p\to K^{p+1}$, and second to $d_{p-1}\colon K^{p-1}\to\mathrm{Ker}(d_p)$, we get splittings into projective modules:

$$\begin{tikzcd}
Z_{p-1}\oplus K'_{p-1} \arrow[d, equal] & B_p\oplus H_p\oplus K'_p \arrow[d, equal] & B_{p+1}\oplus K'_{p+1} \arrow[d, equal] \\
K_{p-1} \arrow[r] & K_p \arrow[r] & K_{p+1}
\end{tikzcd}$$

where $Z_{p-1}=\mathrm{Ker}(d_{p-1})$, $d_{p-1}\colon K'_{p-1}\to B_p$ is an isomorphism, $B_p\oplus H_p=\mathrm{Ker}(d_p)$, and $d_p\colon K'_p\to B_{p+1}$ is an isomorphism. It follows immediately that

$$H^p(K^\bullet\otimes_A B)\simeq H_p\otimes_A B\simeq H^p(K^\bullet)\otimes_A B,\text{ all }B$$

and $H^{p-1}(K^\bullet\otimes_A B)\simeq Z_{p-1}\otimes_A B/\mathrm{Im}(d_{p-2}\otimes B)\simeq H^{p-1}(K^\bullet)\otimes_A B$, all B. This proves (ii).

COROLLARY 3. *Let X, Y, f and $\mathscr{F}$ be as above (unlike Corollary 2, Y need not be reduced). Assume for some p that $H^p(X_y,\mathscr{F}_y)=(0)$, all $y\in Y$. Then the natural map*

$$R^{p-1}f_*(\mathscr{F})\otimes_{\mathcal{O}_Y}k(y)\to H^{p-1}(X_y,\mathscr{F}_y)$$

is an isomorphism for all $y\in Y$.

PROOF. Again assume $Y=\mathrm{Spec}(A)$, $K^\bullet$ as in the theorem. For all $y\in Y$, we know that

$$K^{p-1}\otimes k(y)\xrightarrow{d^{p-1}}K^p\otimes k(y)\xrightarrow{d^p}K^{p+1}\otimes k(y)$$

is exact. Split the vector space $K^p\otimes k(y)$ into $\overline{W}_1\oplus\overline{W}_2$, where $\overline{W}_1=\text{Image of }K^{p-1}\otimes k(y)$, and $\overline{W}_2$ is mapped injectively to $K^{p+1}\otimes k(y)$. To prove the corollary at y, we can replace A by any localization A_f, ($f\in A$, $f(y)\neq 0$). If we do this for a suitable f, we may assume that K^p itself splits into a direct sum of free modules $W_1\oplus W_2$ such that (a) $\overline{W}_i=W_i\otimes k(y)$, and (b) $W_1\subset\mathrm{Im}(d^{p-1})$. To do this, just lift a basis of $\overline{W}_1$ to any elements in the image of d^{p-1}, and lift a basis of $\overline{W}_2$ arbitrarily. But then since $W_2\otimes k(y)\to K^{p+1}\otimes k(y)$ is injective, it follows that $W_2\to K^{p+1}$ is also injective if A is replaced again by a suitable localization A_f. But then $\mathrm{Im}(d^{p-1})\cap W_2=(0)$, hence $W_1=\mathrm{Im}(d^{p-1})$. Since W_1 is a projective module the surjection $K^{p-1}\to W_1\to 0$ splits, and $K^{p-1}\simeq\mathrm{Ker}(d^{p-1})\oplus W_1$. It follows that we have exact sequences

$$\begin{tikzcd}
K^{p-2} \arrow[r] & \mathrm{Ker}(d^{p-1}) \arrow[r] & H^{p-1}(X,\mathscr{F}) \arrow[r] & 0 \\
K^{p-2}\otimes k(y) \arrow[r] & \mathrm{Ker}(d^{p-1})\otimes k(y) \arrow[r] & H^{p-1}(X_y,\mathscr{F}_y) \arrow[r] & 0.
\end{tikzcd}$$

Therefore $H^{p-1}(X_y,\mathscr{F}_y)\simeq H^{p-1}(X,\mathscr{F})\otimes k(y)$ as required.

COROLLARY 4. *Let X, Y, and $\mathscr{F}$ be as above. If $R^kf_*(\mathscr{F})=(0)$ for $k>k_0$, then $H^k(X_y,\mathscr{F}_y)=(0)$ for all $y\in Y$, and for $k>k_0$.*

PROOF. Use Corollary 3 and decreasing induction on k_0.

COROLLARY 5. *Let X, Y, f and $\mathscr{F}$ be as above. Then if B is a flat A-algebra,*

$$H^p(X \times_Y \operatorname{Spec} B, \mathscr{F} \otimes_A B) \simeq H^p(X, \mathscr{F}) \otimes_A B.$$

PROOF. This follows immediately, from the fact that for B flat over A, and *any* complex $K^\bullet$,

$$H^p(K^\bullet \otimes_A B) \simeq H^p(K^\bullet) \otimes_A B.$$

COROLLARY 6. (Seesaw Theorem—provisional form). *Let X be a complete variety, T any variety and L a line bundle on $X \times T$. Then the set*

$$T_1 = \{t \in T \mid L|_{X \times \{t\}} \text{ is trivial on } X \times \{t\}\}$$

is closed in T, and if on $X \times T_1$, $p_2 \colon X \times T_1 \to T_1$ is the projection, then $L|_{X \times T_1} \simeq p_2^* M$ for some line bundle M on T_1.

PROOF. We first make the remark that a line bundle M on a complete variety X is trivial if and only if $\dim H^0(X, \underline{M}) > 0$ and $\dim H^0(X, \underline{M}^{-1}) > 0$ where $\underline{M}$ denotes the sheaf of sections of M. In fact, the necessity of these conditions is clear. Suppose conversely that they hold. The first implies the existence of a non-zero homomorphism $\mathcal{O}_X \xrightarrow{\sigma} \underline{M}$, and the second implies a non-zero homomorphism $\mathcal{O}_X \to \underline{M}^{-1}$, hence on dualizing, a non-zero homomorphism $\underline{M} \xrightarrow{\tau} \mathcal{O}_X$. Hence $\tau(\sigma(1))$ is a non-zero section of $\mathcal{O}_X$, and since X is complete and connected, $\tau(\sigma(1))$ is a non-zero scalar. This implies that $\tau \circ \sigma$ is an isomorphism, hence σ and τ are isomorphisms.

It follows that T_1 is the set of points t of T such that $\dim H^0(X \times \{t\}, \underline{L} | X \times \{t\}) > 0$ *and* $\dim H^0(X \times \{t\}, \underline{L}^{-1} | X \times \{t\}) > 0$, and it follows from Corollary 1 that T_1 is closed. Replacing T by T_1 (so T is now merely a reduced scheme of finite type over k) and L by its restriction to $X \times T_1$, we may assume that $L \mid X \times \{t\}$ is trivial for each $t \in T$. Hence $\dim H^0(X \times \{t\}, L \mid X \times \{t\}) = 1$ for all $t \in T$, so that by Corollary 2, $p_{2*}(\underline{L}) = \underline{M}$ is an invertible sheaf on T and

$$\underline{M} \otimes_{\mathcal{O}_T} (kt) \leftarrow H^0(X \times \{t\}, \underline{L} | X \times \{t\})$$

is an isomorphism. It clearly follows from the triviality of $L \mid X \times \{t\}$ that the natural map $p_2^*(\underline{M}) \to \underline{L}$ is an isomorphism. Since $\underline{M}$ is the sheaf of sections of M, then $p_2^* M \simeq L$.

6. The theorem of the cube: I

THEOREM. *Let X, Y be complete varieties, Z any variety and x_0, y_0 and z_0 base points on X, Y, and Z, respectively. If L is any line bundle on $X \times Y \times Z$ whose restrictions to each of $\{x_0\} \times Y \times Z$, $X \times \{y_0\} \times Z$ and $X \times Y \times \{z_0\}$ are trivial, L is trivial.*

REMARK. Let T be a contravariant functor on the category of complete varieties into the category $\underline{\mathrm{Ab}}$ of abelian groups. Let $X_0, \ldots, X_n$ be any system of complete varieties, x_i^0 a base point of X_i, and let $\pi_i \colon X_0 \times \ldots \times X_n \to X_0 \times \ldots \times \hat{X}_i \times \ldots \times X_n$ ($\hat{X}_i$ indicating the omission of the i-th factor X_i) be the projection map, and $\sigma_i \colon X_0 \times \ldots \times \hat{X}_i \times \ldots \times X_n \to X_0 \times \ldots \times X_n$ the 'inclusion' defined by

$$\sigma_i(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = (x_0, \ldots, x_{i-1}, x_i^0, x_{i+1}, \ldots, x_n).$$

Consider the homomorphisms

$$\alpha_T^n \colon \prod_{i=0}^{n} T(X_0 \times \ldots \times \hat{X}_i \times \ldots \times X_n) \to T(X_0 \times \ldots \times X_n),$$

$$\beta_T^n \colon T(X_0 \times \ldots \times X_n) \to \prod_{i=1}^{n} T(X_0 \times \ldots \times \hat{X}_i \times \ldots \times X_n)$$

defined by

$$\alpha_T^n(\xi_0, \ldots, \xi_n) = \sum_0^n \pi_i^*(\xi_i), \quad \beta_T^n(\eta) = (\sigma_0^*(\eta), \sigma_2^*(\eta), \ldots, \sigma_n^*(\eta)).$$

One then proves by an easy induction on n that we have a natural splitting $T(X_0 \times \ldots \times X_n) = \operatorname{Im} \alpha \oplus \operatorname{Ker} \beta$. The functor T is said to be of *order n* (*linear* if $n=1$, *quadratic* if $n=2$, etc.) if α is surjective, or equivalently β is injective. (Note that the definition of α is independent of base points.)

Thus, the above theorem (when Z is also assumed complete) may be paraphrased as saying that the functor $\operatorname{Pic} X$ is a quadratic functor on the category of complete varieties.

Now, if T_i $(1 \leqslant i \leqslant 3)$ are contravariant functors on complete varieties into $\underline{\text{Ab}}$ and $T_1 \xrightarrow{f} T_2$ and $T_2 \xrightarrow{g} T_3$ are natural transformations such that $T_1 \xrightarrow{f} T_2 \xrightarrow{g} T_3$ is an exact sequence, and if T_1 and T_3 are of order n, so is T_2, as follows from the exactness of

$$0 = \operatorname{Ker} \beta^n_{T_1}(X_0, \ldots, X_n) \to \operatorname{Ker} \beta^n_{T_2}(X_0, \ldots, X_n) \to \operatorname{Ker} \beta^n_{T_3}(X_0, \ldots, X_n) = 0.$$

Thus we get a proof of the theorem of the cube when the base field is **C** by observing that we have an exact sequence

$$H^1(X, \mathcal{O}) \to H^1(X, \mathcal{O}^*) \to H^2(X, \mathbf{Z}),$$

functorial in X, and $H^1(X, \mathcal{O})$ is linear (hence quadratic) and $H^2(X, \mathbf{Z})$ is quadratic in X, by Künneth formulas.

PROOF OF THE THEOREM (following Weil and Murre). By the 'Seesaw theorem', it is sufficient to prove that for every $(x, z) \in X \times Z$, the restriction of L to $\{x\} \times Y \times \{z\}$ is trivial, since it is already given that L restricted to $X \times \{y_0\} \times Z$ is trivial. The following enables us to reduce the proof of the theorem to the case when X is a complete non-singular curve.

LEMMA. *Let X be any variety and $x_0, x_1 \in X$. Then there is an irreducible curve C on X containing x_0 and x_1.*

PROOF. We assume that $\dim X > 1$. By the lemma of Chow, we may assume X projective. Moreover, by induction on $\dim X$, it is sufficient to find a subvariety Y of codimension one in X containing x_0 and x_1. We can find an $X' \xrightarrow{f} X$ birational, with X' projective and $\dim f^{-1}(x_i) \geqslant 1$. (In fact, if h is any meromorphic function on X with indeterminacies at x_0 and x_1, X' can be taken to be the closure of the graph of h in $X \times \mathbf{P}^1$). If $X' \subset \mathbf{P}^N$ is an imbedding, there is a hyperplane H of $\mathbf{P}^N$ such that $H \cap X' = Y'$ is irreducible, by a theorem of Bertini, and $H \cap f^{-1}(x_i) \neq \varnothing$ since $\dim f^{-1}(x_i) \geqslant 1$. Then $Y = f(Y')$ is irreducible in X and contains x_0 and x_1, and the lemma is proved.

Resuming the proof of the theorem, we can find for any $x \in X$ an irreducible complete curve C_1 in X joining x_0 to x. Let $\pi\colon C \to C_1$ be the normalization of C_1 and $\pi'\colon C \times Y \times Z \to X \times Y \times Z$ the induced map. The hypotheses of the theorem are clearly fulfilled for the bundle $\pi'^*(L)$ on $C \times Y \times Z$ (with X replaced by C and x_0 by any point of C lying over x_0), and it is sufficient to prove the triviality of this bundle, since it would then follow that L restricted to $\{x\} \times Y \times \{z\}$ is trivial for any $x \in X$ and $z \in Z$.

Thus, we assume X to be a complete non-singular curve, and it is even sufficient to show the existence of a non-void open subset Z' of Z such that L restricted to $X \times Y \times Z'$ is trivial, since we would then have proved the triviality of $L \mid X \times Y \times \{z\}$ for $z \in Z'$, and it would follow by continuity that this holds for all $z \in Z$.

Let Ω^1 be the sheaf of regular 1-forms on X and let $g = \dim H^0(X, \Omega^1)$ be the genus of X. We can clearly find g points $P_1, \ldots, P_g$ on X such that if $D = \sum\limits_1^g P_i$, $\dim H^0(X, \Omega^1 \otimes \mathcal{O}_X(-D)) = 0$. Denoting by p_1 the first projection $X \times Y \times Z \to X$, let L' be the line bundle $L' = L \otimes p_1^*(L_X(D))$ (where $L_X(D)$ is the line bundle associated to $\mathcal{O}_X(D)$) on $X \times Y \times Z$, and for any $(y, z) \in Y \times Z$, let $L'_{(y,z)}$ be the restriction of L' to $X \times \{y\} \times \{z\}$. Since $L'_{(y,z_0)} = L_X(D)$, we have $\dim H^1(X, \underline{L}'_{(y,z_0)}) = \dim H^0(X, \Omega^1 \otimes \mathcal{O}_X(-D)) = 0$ by Riemann-Roch so that the closed set $F = \{(y, z) \in Y \times Z \mid \dim H^1(X, \underline{L}'_{(y,z)}) \geqslant 1\}$ of $Y \times Z$ does not encounter $Y \times \{z_0\}$. But Y being complete, we can find Z' open in Z and containing z_0 such that $Y \times Z' \cap F = \varnothing$, so that by restricting ourselves to Z', we may assume $H^1(X, \underline{L}'_{(y,z)}) = 0$ for all $(y,z) \in Y \times Z$. But this means that for all $(y,z) \in Y \times Z$,

$$\dim H^0(X, \underline{L}'_{(y,z)}) = \chi(\underline{L}'_{(y,z)}) = \chi(\underline{L}'_{(y_0,z_0)}) = \chi(\mathcal{O}_X(D)) = 1 - g + \deg D = 1.$$

In view of Corollary 2 to the proposition if $p_{23}\colon X \times Y \times Z \to Y \times Z$ is the projection, $p_{23*}(\underline{L}')$ is an invertible sheaf on $Y \times Z$ of rank one and for any (y, z), the natural map $p_{23*}(\underline{L}') \otimes k(y,z) \to H^0(X, \underline{L}'_{(y,z)})$ is an isomorphism. Let U be any open subset of $Y \times Z$ on which $p_{23*}(\underline{L}')$ is trivial, and $\sigma_U \in \Gamma(U, p_{23*}(\underline{L}')) = \Gamma(p_{23}^{-1}(U), \underline{L}')$ a generating section. Let $\widetilde{D}_U$ be the divisor of zeros of σ_U in $p_{23}^{-1}(U)$. Since for U,U' open in $Y \times Z$, σ_U and $\sigma_{U'}$ differ on $U \cap U'$ by a nowhere vanishing function, we have $\widetilde{D}_U \cap p_{23}^{-1}(U \cap U') = \widetilde{D}_{U'} \cap p_{23}^{-1}(U \cap U')$, so that we have an effective divisor $\widetilde{D}$ on $X \times Y \times Z$ such that $\widetilde{D} \mid p_{23}^{-1}(U) = \widetilde{D}_U$. For each $(y,z) \in Y \times Z$, the restriction of $\widetilde{D}$ to $X \times \{y\} \times \{z\}$ is the divisor of zeros of a non-zero section of $H^0(\underline{L}'_{(y,z)})$. In particular, $\widetilde{D}$ restricted to $X \times \{y\} \times \{z_0\}$ and $X \times \{y_0\} \times \{z\}$ for any $y \in Y$, $z \in Z$ must coincide with $D = \Sigma P_i$. Hence, if $P \in X$, $P \neq P_i\,(i = 1, \ldots, g)$, the restriction of $\widetilde{D}$ to $\{P\} \times Y \times Z$ has a support S not meeting $\{P\} \times Y \times \{z_0\}$ or $\{P\} \times \{y_0\} \times Z$. The projection T of S on Z is therefore a proper closed subset of Z, and since S is pure of codimension one in $\{P\} \times Y \times Z$, S must be of the form $\bigcup_{i=1}^{m} \{P\} \times Y \times T_i$, T_i closed and of codimension 1 in Z. But since $S \cap \{P\} \times \{y_0\} \times Z = \emptyset$, it follows that $S = \emptyset$, that is, the support of $\widetilde{D}$ does not meet $\{P\} \times Y \times Z$ for $P \in X$, $P \neq P_i$. Hence $\widetilde{D}$ must be of the form $\sum_1^g n_i(\{P_i\} \times Y \times Z)$, and restricting to $X \times \{y_0\} \times \{z_0\}$, we see that $\widetilde{D} = \sum_1^g (\{P_i\} \times Y \times Z)$. Hence for any $(y,z) \in Y \times Z$, $L'_{(y,z)}$ is the line bundle $L_X(D)$, and therefore L restricted to $X \times \{y\} \times \{z\}$ is trivial.

Corollary 1. *With X, Y, and Z as in the proposition, any line bundle on $X \times Y \times Z$ is isomorphic to $p_{12}^*(L) \otimes p_{13}^*(M) \otimes p_{23}^*(P)$ where p_{ij} is the projection of $X \times Y \times Z$ onto the product of the i^{th} and j^{th} factors, and L,M,P are line bundles on $X \times Y$, $X \times Z$, and $Y \times Z$, respectively.*

Proof. This is a consequence of the remark preceding the proof of the theorem.

Corollary 2. *Let X be any variety, Y an abelian variety, and f,g,h: $X \to Y$ morphisms. Then for all $L \in \mathrm{Pic}(Y)$, we have*

$$(f+g+h)^*L \simeq (f+g)^*L \otimes (f+h)^*L \otimes (g+h)^*L \otimes f^*L^{-1} \otimes g^*L^{-1} \otimes h^*L^{-1}.$$

Proof. Let $p_i: Y \times Y \times Y \to Y$ be the projection onto the i^{th} factor, put $m_{ij} = p_i + p_j: Y \times Y \times Y \to Y$ and $m = p_1 + p_2 + p_3$: $Y \times Y \times Y \to Y$.

Consider the line bundle

$$M = m^*L \otimes m_{12}^*L^{-1} \otimes m_{13}^*L^{-1} \otimes m_{23}^*L^{-1} \otimes p_1^*L \otimes p_2^*L \otimes p_3^*L$$

on $Y \times Y \times Y$. If q: $Y \times Y \to Y \times Y \times Y$ is the map $q(y,y') = (0,y,y')$, we have

$$q^*M = n^*L \otimes q_1^*L^{-1} \otimes q_2^*L^{-1} \otimes n^*L^{-1} \otimes 0^*L \otimes q_1^*L \otimes q_2^*L$$

where 0, q_1, q_2, n: $Y \times Y \to Y$ are the 0 map, the projections, and addition. Therefore q^*M is trivial. By symmetry, M is trivial on $Y \times (0) \times Y$ and $Y \times Y \times (0)$ too. By the theorem, M must be trivial on $Y \times Y \times Y$. Pulling back M by the map (f,g,h): $X \to Y \times Y \times Y$, the result follows.

Corollary 3. *If X is an abelian variety, and $n \in \mathbf{Z}$, then for all line bundles L,*

$$n_X^*L \simeq L^{\left(\frac{n^2+n}{2}\right)} \otimes (-1_X)^*L^{\left(\frac{n^2-n}{2}\right)}.$$

Proof. By Corollary 2 with $f = (n+1)_X$, $g = 1_X$ and $h = (-1)_X$, it follows that the "second difference",

$$(n+2)_X^*L \otimes (n+1)_X^*L^{-2} \otimes n_X^*L \simeq 1_X^*(L) \otimes (-1_X)^*L,$$

hence for some line bundles M_1, M_2, we must have

$$n_X^*L \simeq [L \otimes (-1_X)^*L]^{\frac{n(n-1)}{2}} \otimes M_1^n \otimes M_2.$$

Putting $n = 0$ shows that M_2 is trivial, and putting $n = 1$ shows $M_1 \simeq L$.

Corollary 4. (Theorem of the square.) *For all line bundles L $x,y \in X$,*

$$T_{x+y}^*L \otimes L \simeq T_x^*L \otimes T_y^*L.$$

Therefore if $\phi_L(x) =$ *isom. class of* $T^*_x L \otimes L^{-1}$ *in* $\mathrm{Pic}(X)$, ϕ_L *is a homomorphism from* X *to* $\mathrm{Pic}(X)$.

Proof. Apply Corollary 2 with $X = Y$, f and g constant maps with images x, y respectively, and $h =$ identity.

In terms of divisors, Corollary 4 asserts that for any divisor D on X, and $x, y \in X$,

$$T^*_{x+y}D + D \equiv T^*_x D + T^*_y D$$

(where $\equiv$ means linear equivalence).

In the rest of this book, we will always keep the notation ϕ_L for this very important map. Note that

(a) $\phi_{L_1 \otimes L_2} = \phi_{L_1} + \phi_{L_2}$ (+ standing for the group law induced by $\otimes$ in $\mathrm{Pic}(X)$),

(b) $\phi_{T^*_x L} = \phi_L$.

Definition. $K(L) = \mathrm{Ker}(\phi_L) = \{x \in X \mid T^*_x L \simeq L\}$.

Proposition. *$K(L)$ is a Zariski-closed subgroup of X.*

Proof. Apply the Seesaw Theorem to the line bundle $m^*L \otimes p_2^* L^{-1}$ on $X \times X$ ($m: X \times X \to X$ being addition). It follows that the set of $x \in X$ such that $m^*L \otimes p_2^*L^{-1}$ is trivial on $\{x\} \times X$ is Zariski closed. But $m^*L \otimes p_2^* L^{-1} |_{\{x\}\times X} \simeq T^*_x L \otimes L^{-1}$, so this set is $K(L)$.

Application 1. Let D be an effective divisor on an abelian variety X and $L = L(D)$ the associated line bundle. The following conditions are equivalent.

(i) The subgroup $H = \{x \in X \mid T^*_x(D) = D\}$ of X is finite (equality of divisors, not divisor classes).

(ii) $K(L)$ is finite.

(iii) The linear system $|2D|$ has no base points, and defines a finite morphism $X \to \mathbf{P}^N$.

(iv) L is ample on X.

Proof. The implication (iii) ⇒ (iv) is a general fact (*EGA* Ch. III, (2.6.1) or (4.4.2)). We show next that (iv) ⇒ (ii). If $K(L)$ is not finite, let Y be the connected component of 0 of $K(L)$, so that Y is an abelian variety of positive dimension, and the restriction L_Y of L to Y is ample on Y. Further, $T^*_y(L_Y) \simeq L_Y$ for all $y \in Y$. Hence, by the Seesaw theorem if $m: Y \times Y \to Y$ is the addition and $p_i: Y \times Y \to Y$ the projections, the line bundle $m^*(L_Y) \otimes p_1^*(L_Y^{-1}) \otimes p_2^*(L_Y^{-1})$ is trivial on $Y \times Y$. Pulling back by the morphism $Y \to Y \times Y$, $y \longmapsto (y, -y)$ gives us that $L_Y \otimes (-1_Y)^*(L_Y)$ is trivial on Y. But L_Y is ample, and so is $(-1_Y)^*(L_Y)$ since -1_Y is an automorphism of Y, so that $L_Y \otimes (-1_Y)^*(L_Y)$ is again ample. This is a contradiction since $\dim Y > 0$, which proves that (iv) ⇒ (ii). The implication (ii) ⇒ (i) is trivial, since $K(L) \supset H$.

We now show that (i) ⇒ (iii). The linear system $|2D|$ contains the divisors $T^*_x(D) + T^*_{-x}(D)$, by Corollary 4. For any $u \in X$, we can find an $x \in X$ such that $u \pm x \notin \mathrm{Supp}\, D$, and this means that $u \notin \mathrm{Supp}\,(T^*_x(D) + T^*_{-x}(D))$. Thus, the linear system $|2D|$ has no base points, and defines a morphism $\phi: X \to \mathbf{P}^N$. If ϕ is not a finite morphism, we can find an irreducible curve C such that $\phi(C) =$ one point. It follows that for all $E \in |2D|$, either E contains C or is disjoint from C. In particular, for almost all $x \in X$, C and $T^*_x(D) + T^*_{-x}(D)$ are disjoint. Now note the general fact.

Lemma. *If C is a curve on X and E is an irreducible divisor on X such that $C \cap E = \emptyset$, then E is invariant under translation by $x_1 - x_2$, all $x_i \in C$.*

Proof. If $L = L(E)$, then L is trivial on C since C and E are disjoint. Therefore, $T^*_x L$, restricted to C, has degree 0 for all $x \in X$. But then $T_x(C)$ and E can never intersect in a non-empty finite set of points, since this would imply that $T^*_x(L)|_C$ had positive degree; i.e. for all x, either $T_x(C)$ and E are disjoint, or $T_x(C) \subset E$. Let $x_1, x_2 \in C$, $y \in E$. Then $T_{y-x_2}(C)$ and E meet at y. Therefore $T_{y-x_2}(C) \subset E$, hence $y - x_2 + x_1 \in E$. This proves the lemma.

If $D = \Sigma n_i D_i$, D_i irreducible, then by the lemma, each D_i is invariant under translation by all points $x_1 - x_2$, $x_i \in C$. This contradicts (i), hence we have proved that (i) $\Rightarrow$ (iii).

This enables us to show trivially that an abelian variety X is projective. In fact, let U be any affine open subset of X, $D_1, \dots, D_t$ the components of $X - U$ and D the divisor $D = \sum_1^t D_i$. We will show that D verifies (i) above. We may assume after a translation that $0 \in U$. Then $H = \{x \in X \mid T_x^*(D) = D\}$ is a closed subgroup, and for $x \in H$, U is stable for T_x. Since $0 \in U$, it follows that $H \subset U$, and H being complete and U affine, H is finite.

Application 2. An abelian variety X is a divisible group, and for all $n \geqslant 1$, X_n is finite.

Considering the homomorphism $n_X\colon X \to X$, it is clear that $\dim(\ker n_X) > 0$ if and only if $\dim(\operatorname{Im}(n_X)) < \dim X$. Hence to prove n_X surjective, it suffices to check that $X_n = \ker(n_X)$ is finite. But let L be an ample line bundle on X. Then

$$n_X^* L \simeq L^{\frac{n(n+1)}{2}} \otimes (-1_X)^* L^{\frac{n(n-1)}{2}}.$$

Since (-1_X) is an automorphism of X, $(-1_X)^*L$ is also ample, and since $\frac{1}{2}n(n+1) > 0$, $\frac{1}{2}n(n-1) \geqslant 0$, we see that n_X^*L is also ample. But then n_X^*L cannot be trivial on any positive dimensional subvariety. Since $n_X^*L|_{\operatorname{Ker}(n_X)}$ is trivial, $\ker(n_X)$ must be finite.

Application 3. We can go even further and compute the order of X_n, when the characteristic p of k does not divide n. We first recall some general facts. Let X and Y be complete varieties both of dimension n, and let $f\colon X \to Y$ be a surjective morphism. Then via f^*, $k(X)$ is a finite algebraic extension of $k(Y)$, and we define the *degree* d (resp. *separable degree*, *inseparable degree*) of f to be the degree $[k(X)\colon k(Y)]$ of this extension (resp. $[k(X):k(Y)]_s$, $[k(X)\colon k(Y)]_i$). If f is separable, i.e. $k(X)$ is separable over $k(Y)$, then d is the cardinality of $f^{-1}(y)$ for almost all $y \in Y$. If f is inseparable, the separable degree of $k(X)$ over $k(Y)$ instead is the cardinality of $f^{-1}(y)$ for almost all y. Moreover, a basic fact is that if $D_1, \dots, D_n$ are Cartier divisors on Y, then we get the relation between intersection numbers:

$$(f^*D_1 \cdot \ldots \cdot f^*D_n)_X = d(D_1 \cdot \ldots \cdot D_n)_Y.$$

Now suppose X and Y are abelian varieties. A homomorphism $f\colon X \to Y$ is called an *isogeny* if it is surjective, with finite kernel. We have just seen that $n_X\colon X \to X$ is an isogeny. Then every isogeny f has a degree d, and since the cardinality of the kernel of f, $\#[\ker f]$, is the cardinality of $f^{-1}(y)$ for all $y \in Y$, we see that

$$\#[\ker f] = \text{separable degree } (f). \tag{*}$$

Now take the case $f = n_X$. Let D be an ample, *symmetric* divisor (i.e. $(-1_X)^*D = D$) on X: we have seen that ample D's exist, and then $D + (-1_X)^*D$ is both ample and symmetric. Then by Corollary 3, n_X^*D is linearly equivalent to n^2D. Therefore, if $g = \dim X$,

$$\begin{aligned} \text{degree } (n_X) \cdot \overbrace{(D \cdot \ldots \cdot D)_X}^{g\times} &= \overbrace{(n_X^*D \cdot \ldots \cdot n_X^*D)_X}^{g\times} \\ &= n^{2g}\overbrace{(D \cdot \ldots \cdot D)_X}^{g\times}, \end{aligned}$$

hence degree $(n_X) = n^{2g}$.

When is n_X separable? If $p \nmid n$, then by the result above, $p \nmid$ degree (n_X) so n_X must be separable. On the other hand, if $p \mid n$, then we saw in § 4 that the differential $d(n_X)$ mapping $T_{X,0}$ to $T_{X,0}$ is 0. Therefore, if ω is any invariant differential form on X, $n_X^*(\omega)$ is (a) still translation invariant, and (b) has value 0 in the cotangent space to X at 0, hence it is zero. Since the invariant differentials on X generate the sheaf Ω_X^1 over $\mathcal{O}_X$, they generate the $k(X)$-module of $k(X)/k$-differentials. Therefore we find that the induced map n_X^* on rational differentials $\Omega^1_{k(X)/k}$ is 0 if $p \mid n$. This implies that the induced map on $k(X)$ maps $k(X)$ into $k(X)^p$, and hence the inseparable degree of p_X is at least p^g.

It follows that if $p \nmid n$, $\#(X_n) = n^{2g}$. Thus X_n is a finite abelian group killed by n such that for all $m \mid n$, X_n contains exactly m^{2g}

elements of order dividing m. It is elementary group theory that the only such group is $(\mathbf{Z}/n\mathbf{Z})^{2g}$. On the other hand, X_p is annihilated by p and is of order equal to the separable degree of p_X, which is p^i for some i with $0 \leqslant i \leqslant g$, so $X_p \simeq (\mathbf{Z}/p\,\mathbf{Z})^i$. Since X is divisible, it follows by induction on m that for any $m \geqslant 1$, $X_{p^m} \simeq (\mathbf{Z}/p^m\mathbf{Z})^i$. Summarizing, we have proved

PROPOSITION. (1) *degree* $(n_X) = n^{2g}$.

(2) n_X *separable* $\iff p \nmid n$.

(3) *If* $p \nmid n$, $X_n \simeq (\mathbf{Z}/n\mathbf{Z})^{2g}$.

(4) *There is an integer* i *with* $0 \leqslant i \leqslant g$ *such that for all* $m \geqslant 1$, $X_{p^m} \simeq (\mathbf{Z}/p^m\,\mathbf{Z})^i$.

APPENDIX TO §6

We give an alternative proof of the fact that $\deg n_X = n^{2g}$ for $n \geqslant 1$, avoiding intersection theory.

For an invertible sheaf $\underline{L}$ on a complete variety X of dimension g, and any coherent sheaf $\mathscr{F}$ on X, $P_{\mathscr{F}}(n) = \chi(\mathscr{F} \otimes \underline{L}^n)$ is a polynomial in n of degree $\leqslant g$. We shall denote the coefficient of n^g in this polynomial by $d_{\underline{L}}(\mathscr{F})/g!$, so that $d_{\underline{L}}(\mathscr{F})$ is an integer $\geqslant 0$. We call $d_{\underline{L}}(\mathscr{O}_X)$ the degree of $\underline{L}$, and denote it by $\deg \underline{L}$. The basic result is the following

PROPOSITION. (1) *Let X be a complete variety with an invertible sheaf $\underline{L}$. For any coherent sheaf $\mathscr{F}$ on X, let* rank $(\mathscr{F})$ *be the dimension over the function field of X of the generic stalk of $\mathscr{F}$, or equivalently, of the space of rational sections of $\mathscr{F}$. Then we have*

$$d_{\underline{L}}(\mathscr{F}) = \text{rank}\,(\mathscr{F}).\ \deg \underline{L}.$$

(2) *Let $f: Y \to X$ be a surjective morphism of complete varieties of the same dimension g, and $\underline{L}$ an invertible sheaf on X. Then*

$$\deg f^*(L) = (\deg f).(\deg L).$$

PROOF. (1) It is a standard fact that we can find a non-zero coherent sheaf of ideals $\mathscr{I}$ and an exact sequence

$$0 \longrightarrow \mathscr{I}^{\text{rank}\,\mathscr{F}} \longrightarrow \mathscr{F} \longrightarrow \mathscr{T} \longrightarrow 0$$

with $\mathscr{T}$ a torsion sheaf. Since $\mathscr{T}$ has support of dimension $<\dim X$, $\chi(\mathscr{T} \otimes \underline{L}^n)$ is a polynomial of degree $< \dim X$, and the additivity of gives us that

$$d_{\underline{L}}(\mathscr{F}) = d_{\underline{L}}(\mathscr{I}^{\text{rank}\,\mathscr{F}}) = (\text{rank}\,\mathscr{F}).\ d_{\underline{L}}(\mathscr{I}),$$

and using the exact sequence $0 \to \mathscr{I} \to \mathscr{O}_X \to \mathscr{O}_X/\mathscr{I} \to 0$, we see that $d_{\underline{L}}(\mathscr{I}) = d_{\underline{L}}(\mathscr{O}) = \deg \underline{L}$.

(2) For each $p \geqslant 0$, we have a canonical isomorphism $R^pf_*(f^*(\underline{L}^n)) = R^pf_*(\mathscr{O}_Y) \otimes \underline{L}^n$. Taking alternating sums in the Leray spectral sequence $H^p(X, R^qf_*(f^*(\underline{L}^n))) \Rightarrow H^n(Y, f^*(\underline{L}^n))$, we obtain that

$$\chi(f^*(\underline{L}^n)) = \sum_{p=0}^{\infty} (-1)^p\, \chi(R^pf_*(\mathscr{O}_Y) \otimes \underline{L}^n).$$

Since there is a non-void open subset U of X such that $f|f^{-1}(U): f^{-1}(U) \to U$ is a finite morphism, $R^pf_*(\mathscr{O}_Y)$ have supports in a proper closed subset of X for $p > 0$. Further, $R^0f_*(\mathscr{O}_Y)$ is clearly a coherent sheaf of rank $= \deg f$. The assertion now follows on comparing coefficients of n^g on both sides.

Now let X be an abelian variety, n a positive integer. Let L be an ample symmetric line bundle on X; for any ample L, $L \otimes (-1_X)^*L$ is both ample and symmetric. Then by Corollary 3, n_X^*L is isomorphic to L^{n^2}. Therefore, if $g = \dim X$, we get by the above proposition that

$$\deg n_X.\ \deg \underline{L} = \deg n_X^*(\underline{L}) = \deg \underline{L}^{n^2} = n^{2g}.\ \deg \underline{L},$$

and since $\deg \underline{L} > 0$, we get that $\deg n_X = n^{2g}$.

7. **Dividing varieties by finite groups.** Let $f: X \to Y$ be a morphism of algebraic varieties (over an algebraically closed field k). f is said to be *étale* if

(i) f is flat,

(ii) for all $x \in X$, $y = f(x) \in Y$, if m_x and m_y are the maximal ideals in $\mathcal{O}_x$ and $\mathcal{O}_y$, then $f^*(m_y)\mathcal{O}_x = m_x$.

This is equivalent to assuming that f is a "formal isomorphism" in the sense:

(i)′ for all $x \in X$, $y = f(x) \in Y$, if $\hat{\mathcal{O}}_x$, $\hat{\mathcal{O}}_y$ are the completions of $\mathcal{O}_x$, $\mathcal{O}_y$, then the natural map

$$\hat{f}^*\colon \hat{\mathcal{O}}_y \to \hat{\mathcal{O}}_x$$

is an isomorphism.

(Cf. Mumford, *Intro. Alg. Geom.*, p.353). When $k = \mathbf{C}$, it is also equivalent to assuming that f is a local isomorphism of analytic spaces. Our main result is

THEOREM. *Let X be an algebraic variety, and G a finite group of automorphisms of X. Suppose that for any $x \in X$, the orbit G_x of x is contained in an affine open subset of X. Then there is a pair (Y, π), where Y is a variety and $\pi\colon X \to Y$ a morphism, satisfying the following conditions*:

(i) *as a topological space, (Y, π) is the quotient of X for the G-action,*

(ii) *if $\pi_*(\mathcal{O}_X)^G$ denotes the subsheaf of G-invariants of $\pi_*(\mathcal{O}_X)$ for the action of G on $\pi_*(\mathcal{O}_X)$ deduced from* (i), *the natural homomorphism $\mathcal{O}_Y \to \pi_*(\mathcal{O}_X)^G$ is an isomorphism.*

The pair (Y, π) is determined up to an isomorphism by these conditions. The morphism π is finite, surjective and separable. Y is affine if X is affine.

If further G acts freely on X (that is, if $gx \neq x$ for any $x \in X$ and any $g \in G$ with $g \neq e$), π is an étale morphism.

PROOF. Since the conditions (i) and (ii) determine the topology and structure sheaf of Y, the uniqueness assertion is trivial. Also, the problem of existence reduces to proving that if Y is the quotient X/G as a topological space, and is given the structure sheaf $\mathcal{O}_Y = \pi_*(\mathcal{O}_X)^G$, it is an algebraic variety. Suppose we knew the theorem (in its entirety) to be valid when X is affine. For any $x \in X$, let U' be an affine open subset of X containing Gx. Then $U = \bigcap_{g \in G} gU'$ is an affine open subset of X containing x and stable for G. Thus X is covered by G-stable affine open sets U. Then each $\pi(U)$ is open in Y, and $\pi^{-1}(\pi(U)) = U$, so by the affine case of the theorem $\pi(U)$, with the restriction of $\mathcal{O}_Y$, is an affine variety. But the open subsets $\pi(U)$ cover Y, so the theorem would follow for X.

We may therefore assume $X = \operatorname{Spec}(A)$. Let $A = k[x_1, \ldots, x_n]$. Then G acts on A by the law $g(f)(x) = f(g^{-1}x)$, $g \in G$, $f \in A$, $x \in X$. Let $\nu =$ order of G. For $f \in A$ and $1 \leqslant k \leqslant \nu$, denote by $\sigma_k(f)$ the elementary symmetric function of degree k in $\{g(f)\}_{g \in G}$, and put $B' = k[\sigma_i(x_j)]_{\substack{1 \leqslant i \leqslant \nu \\ 1 \leqslant j \leqslant n}}$, so that B' is a finitely generated k-algebra, contained in the algebra $B = A^G$ of G invariants of A. But the x_j are integral over B since they satisfy the equation

$$X^\nu - \sigma_1(x_j)X^{\nu-1} + \ldots + (-1)^\nu\sigma_\nu(x_j) = 0,$$

so A is a finite B'-module. Since $B' \subset B \subset A$ and B' is noetherian, B is a finite B'-module too and hence a finitely generated k-algebra, and A is a finite B-module. If $Y = \operatorname{Spec} B$, then Y is a variety and we get a morphism $\pi\colon X \to Y$ corresponding to the inclusion $B \subset A$, and this morphism is finite and surjective. Next if R is the quotient field of A, the action of G on A extends uniquely to an action on R. If $a/b \in R^G$, $a, b \in A$, $b \neq 0$, then

$$\frac{a}{b} = a \prod_{g \neq e} g(b) \Big/ \prod_{g \in G} g(b)$$

hence $a. \prod_{g \neq e} g(b) \in A^G$, so that R^G is the quotient field of $A^G = B$. This proves that R is a Galois extension of the quotient field of B. In particular, π is separable. Next, note that $\pi_*(\mathcal{O}_X)^G$ is a coherent sheaf on Y since it is the intersection of kernels of

$$\phi_g\colon \pi_*(\mathcal{O}_X) \to \pi_*(\mathcal{O}_X),\ \phi_g(f) = gf,\ \ f, g \in G.$$

The natural homomorphism $\mathcal{O}_Y \to \pi_*(\mathcal{O}_X)^G$ induces an isomorphism of global sections, so it is an isomorphism. Next if $x, x' \in X$ have distinct orbits $Gx \neq Gx'$ under G, we can find an $f \in A$ with

$f(gx) = 1$ for all $g \in G$, $f(gx') = 0$ for all $g \in G$, and if $\phi = \prod_{g \in G} g(f)$, $\phi \in B$ and $\phi(\pi(x)) = 1$, $\phi(\pi(x')) = 0$. This shows that $\pi(x) \neq \pi(x')$. Thus as a set, Y is the quotient of X by G. But $\pi \colon X \to Y$ is a finite and hence a closed and continuous map, so Y has the quotient topology too.

Only the last assertion remains to be checked. Let $x \in X$, $y = f(x)$, and $\mathfrak{M}$ the maximal ideal of y in B. Considering A as a B-module of finite type, let $\hat{B}$ and $\hat{A}$ be the completions of B and A respectively for the $\mathfrak{M}$-adic topology. Then, the natural homomorphism $\hat{B} \otimes_B A \to \hat{A}$ is an isomorphism. Moreover, $\hat{B}$ is also the completion $\hat{\mathcal{O}}_{Y,y}$ of $\mathcal{O}_{Y,y}$ with respect to its maximal ideal. On the other hand, since the only prime ideals of A containing $\mathfrak{M}.A$ are the maximal ideals of the points $g(x)$, $g \in G$, we have by the Chinese remainder theorem a natural isomorphism

$$\hat{A} \xrightarrow{\sim} \prod_{g \in G} \hat{\mathcal{O}}_{X,gx},$$

where $\hat{\mathcal{O}}_{X,gx}$ is the completion of $\mathcal{O}_{X,gx}$ with respect to its maximal ideal. The group G acts on A, and hence on the two other rings occurring above. On $\hat{B} \otimes_B A$, the action is given by $h(b \otimes a) = b \otimes h(a)$ for $h \in G$, $b \in B$, $a \in A$. The fact that B is the ring of G-invariants in A can be expressed by the exactness of the sequence of B-modules

$$0 \longrightarrow B \longrightarrow A \longrightarrow \prod_{h \in G} A$$

$$a \longmapsto (\ldots, h(a) - a, \ldots).$$

Since $\hat{B}$ is a flat B-module, it follows that

$$0 \longrightarrow \hat{B} \longrightarrow \hat{B} \otimes_B A \longrightarrow \prod_{h \in G} (\hat{B} \otimes_B A)$$

$$b \otimes a \longrightarrow (\ldots, b \otimes (h(a) - a), \ldots)$$

is exact, hence $\hat{B}$ is the subring of G-invariants in $\hat{B} \otimes_B A \simeq \hat{A}$. On the other hand, the action of $h \in G$ induces an isomorphism $\hat{\mathcal{O}}_{X,x} \xrightarrow{\sim} \hat{\mathcal{O}}_{X,hx}$, and if we identify $\prod_{g \in G} \hat{\mathcal{O}}_{X,gx}$ with $\prod_{g \in G} \hat{\mathcal{O}}_{X,x}$ by means of these isomorphisms, the action of G on this ring may be described by simply permuting the factors, i.e. $h(\{\alpha_g\}_{g \in G}) = \{\alpha_{(h^{-1}g)}\}_{g \in G}$, for any $h \in G$ and $\{\alpha_g\}_{g \in G} \in \prod_{g \in G} \hat{\mathcal{O}}_{X,x}$. Thus the invariants for this action can be identified with $\hat{\mathcal{O}}_{X,x}$, for its diagonal immersion in $\prod_{g \in G} \hat{\mathcal{O}}_{X,x}$. We thus deduce that the natural homomorphism $\hat{\mathcal{O}}_{Y,y} \to \hat{\mathcal{O}}_{X,x}$ is an isomorphism, i.e. f is étale at x.

REMARK. The condition that any G-orbit in X be contained in an affine open subset is always verified when X is quasi-projective. In fact, if X is a locally closed subset of $\mathbf{P}^N$ and if $\bar{X}$ is its closure in $\mathbf{P}^N$ and $x_i (1 \leqslant i \leqslant n)$ is any finite set of points of X, we can always find a hypersurface S in $\mathbf{P}^N$ containing $\bar{X} - X$ but not any of the x_i. Then $\bar{X} - (\bar{X} \cap S) = X - (X \cap S)$ is affine and open in X and contains all the x_i.

When X and G are as above and (Y, π) is the pair given by the theorem, Y is called the *quotient* of X by G, and is denoted by X/G.

Now let G act on X and let (Y, π) be the quotient, and let $\mathcal{F}$ be a coherent sheaf on Y. Since for any $g \in G$, we have the commutative triangle

$$\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr, "\pi"'] & & X \arrow[dl, "\pi"] \\
& Y &
\end{tikzcd}$$

we deduce that there is a natural automorphism $g^* \colon \pi^*(\mathcal{F}) \to \pi^*(\mathcal{F})$ over the action of g on X. Thus, G acts on $\pi^*(\mathcal{F})$ in a manner compatible with its action on X. By a coherent *G-sheaf* on X, we shall mean a coherent $\mathcal{O}_X$-module on which G acts in a way compatible with its action on X.

PROPOSITION 2. *Let G act freely on X, and $Y = X/G$. Then the functor $\mathscr{F} \longmapsto \pi^*(\mathscr{F})$ is an equivalence between the category of coherent $\mathcal{O}_Y$-modules and that of coherent G-sheaves on X, whose inverse is given by $\mathfrak{G} \to \pi_*(\mathfrak{G})^G$. Locally free sheaves correspond to locally free sheaves of the same rank.*

PROOF. There are natural homomorphisms

$$S(\mathscr{F})\colon \mathscr{F} \to \pi_*(\pi^*\mathscr{F})^G, \ \mathscr{F} \text{ a sheaf on } Y;$$

$$T(\mathfrak{G})\colon \pi^*(\pi_*(\mathfrak{G})^G) \to \mathfrak{G}, \ \mathfrak{G} \text{ a } G\text{-sheaf on } X.$$

We will show that S and T are isomorphisms. We can again assume X and Y are affine. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, where $B = A^G$. We must show that the natural maps

$$S(M)\colon M \to (M \otimes_B A)^G, \qquad M \text{ a } B\text{-module},$$

$$T(N)\colon (N^G) \otimes_B A \to N, \qquad N \text{ a } G\text{-}A\text{-module}$$

are isomorphisms. But for all B-modules M the composition

$$M \otimes_B A \xrightarrow{S(M) \otimes 1_A} (M \otimes_B A)^G \otimes A \xrightarrow{T(M \otimes_B A)} M \otimes_B A$$

is the identity. Since A is faithfully flat over B, $S(M) \otimes 1_A$ is an isomorphism if and only if $S(M)$ is an isomorphism; therefore it will suffice to prove that all the T's are isomorphisms.

Now in the case in which A is isomorphic as a ring to $B \times \ldots \times B$ and in which G acts on $B \times \ldots \times B$ by a simply transitive group of permutations, it is quite obvious that $T(N)$ is an isomorphism for every G-A-module N. On the other hand, we can reduce the proposition to this case by taking completions. Let $x \in X$, $y = f(x)$, and $\hat{B} = \hat{\mathcal{O}}_{y,Y}$. To show that $T(N)$ is an isomorphism, it will suffice to show that

$$[(N^G) \otimes_B A] \otimes_B \hat{B} \xrightarrow{T(N) \otimes 1_{\hat{B}}} N \otimes_B \hat{B}$$

is an isomorphism for every $y \in Y$. But since $\hat{B}$ is flat over B, the module of G-invariants in $N \otimes_B \hat{B}$ equals $(N^G) \otimes_B \hat{B}$ (cf. proof of theorem), hence we get a diagram

$$\begin{tikzcd}
{[N^G \otimes_B A] \otimes_B \hat{B}} \arrow[r, "T(N) \otimes 1_{\hat{B}}"] \arrow[d, "\wr", "\|"'] & N \otimes_B \hat{B} \arrow[dd, equal] \\
(N^G \otimes_B \hat{B}) \otimes_{\hat{B}} (A \otimes_B \hat{B}) \arrow[d, "\wr", "\|"'] & \\
(N \otimes_B \hat{B})^G \otimes_{\hat{B}} (A \otimes_B \hat{B}) \arrow[r, "T(N \otimes_B \hat{B})"] & N \otimes_B B.
\end{tikzcd}$$

Since $A \otimes_B \hat{B}$ is isomorphic to $\prod_{g \in G} \hat{\mathcal{O}}_{g(x),X}$, hence to $\hat{B} \times \ldots \times B$, $T(N \otimes_B \hat{B})$ is an isomorphism.

We want to study in more detail the case when X is a complete variety and G acts freely on X. We shall denote by $\hat{G}$ the group $\operatorname{Hom}(G, k^*)$ of k^*-valued characters of G.

PROPOSITION 3. *In the above case, for all characters $\alpha\colon G \to k^*$, let*

$$\underline{L}_\alpha = \{a \in \pi_*(\mathcal{O}_X) \mid g(a) = \alpha(g).a, \text{ all } g \in G\}.$$

Then $\underline{L}_\alpha$ is an invertible sheaf on Y, and the multiplication in $\pi_*(\mathcal{O}_X)$ induces an isomorphism $\underline{L}_\alpha \otimes \underline{L}_\beta \to \underline{L}_{\alpha+\beta}$. The assignment $\alpha \longmapsto \underline{L}_\alpha$ defines an isomorphism

$$\hat{G} \xrightarrow{\sim} \operatorname{Ker}[\operatorname{Pic} Y \to \operatorname{Pic} X].$$

PROOF. By Proposition 2, $\operatorname{Ker}[\operatorname{Pic} Y \to \operatorname{Pic} X]$ can be identified as a set with actions of G on the trivial sheaf $\mathcal{O}_X$ covering the action of G on X. Given any such action, the image of the unit section by $g \in G$ is a nowhere vanishing section of $\mathcal{O}_X$, and since X is complete, a non-zero scalar $\alpha^{-1}(g) \in k^*$; clearly $\alpha\colon G \to k^*$ is a homomorphism. Conversely, given any such homomorphism, we can define an action of G on $\mathcal{O}_X$ covering the action on the base by $g(f) = \alpha^{-1}(g).(f \circ g^{-1})$. Thus, as a set, we have a bijection $\hat{G} \xrightarrow{\sim} \operatorname{Ker}[\operatorname{Pic} Y \to \operatorname{Pic} X]$. It is easy to see that this is a group homomorphism.

Given an action σ of G on $\mathcal{O}_X$ corresponding to a character α, if we denote the natural action of G on $\pi_*(\mathcal{O}_X)$ by $(g, f) \longmapsto g(f) = f \circ g^{-1}$, the action of G on $\pi_*(\mathcal{O}_X)$ induced by the action σ is described by

$$\sigma(g)(f) = \alpha^{-1}(g) \cdot g(f).$$

Since the corresponding invertible sheaf $\underline{L}_\alpha$ is the set of invariants of $\pi_*(\mathcal{O}_X)$ for this action, we get that

$$\underline{L}_\alpha \simeq \{a \in \pi_*(\mathcal{O}_X) \mid g(a) = \alpha(g).a\}.$$

Considered as subsheaves of $\pi_*(\mathcal{O}_X)$, we have evidently $\underline{L}_\alpha \cdot \underline{L}_\beta \subset \underline{L}_{\alpha+\beta}$. On the other hand, since a nowhere zero section on an open set of a line bundle on Y induces a nowhere zero section of the induced bundle on the inverse image of this open set, we see that any generating section $f \in \underline{L}_\alpha(U) \subset \Gamma(\pi^{-1}(U), \mathcal{O}_X)$ admits an inverse f^{-1} in $\pi_*(\mathcal{O}_X)(U)$, which clearly proves that $\underline{L}_\alpha \otimes \underline{L}_\beta \to \underline{L}_{\alpha+\beta}$ is surjective. Since both sides are invertible sheaves, this is an isomorphism.

Remarks. (1) Suppose G is of order prime to the characteristic. Since the representations of G in all the k-vector spaces $\pi_*(\mathcal{O}_X)(V)$ are completely reducible for every open V in Y, it is easy to check that

$$\pi_*(\mathcal{O}_X) \simeq \bigoplus_{\alpha \in \hat{G}} \underline{L}_\alpha \oplus \mathcal{E}$$

where the representation of G in all the vector spaces $\mathcal{E}(V)$ contains no 1-dimensional subrepresentation. If G is also commutative, then we have simply

$$\pi_*(\mathcal{O}_X) \simeq \bigoplus_{\alpha \in \hat{G}} \underline{L}_\alpha.$$

Since $\pi_*\pi^*\mathcal{F} \simeq \mathcal{F} \otimes_{\mathcal{O}_Y} \pi_*\mathcal{O}_X$ for all $\mathcal{O}_Y$-modules $\mathcal{F}$, this proves also

Corollary. *If G has order prime to the characteristic, then for all coherent $\mathcal{O}_Y$-modules $\mathcal{F}$, $\mathcal{F}$ is a direct summand of $\pi_*(\pi^*\mathcal{F})$.*

Let us apply our results to abelian varieties. The main consequence is something like the fundamental theorem of Galois theory.

Theorem 4. *Let X be an abelian variety. Then there is a 1-1 correspondence between the two sets of objects:*

(a) *finite subgroups $K \subset X$,*

(b) *separable isogenies $f: X \to Y$, where two isogenies $f_1: X \to Y_1$, $f_2: X \to Y_2$ are considered equal if there is an isomorphism $h: Y_1 \to Y_2$ such that $f_2 = h \circ f_1$, which is set up by $K = \ker(f)$, and $Y = X/K$.*

Proof. First start with a finite subgroup $K \subset X$. Then K acts freely on X by translations, so we can form the quotient $(X/K, f)$, and f is an étale surjective finite morphism $X \to X/K$. On the other hand, X/K as a set is the quotient of the abstract group X by the subgroup K, hence it has a group structure. The group law is in fact a morphism. This follows by considering the diagram:

$$\begin{tikzcd}
X \times X \arrow[r, "m"] \arrow[d, "f \times f"'] & X \arrow[d, "f"] \\
X/K \times X/K \arrow[r, "n"'] & X/K
\end{tikzcd}$$

where m is the group law of X (a morphism) and n is the group law of X/K (so far, just a map). But it is easy to check that $X/K \times X/K = X \times X/K \times K$, and since the morphism $f \circ m$: $X \times X \to X/K$ collapses the action of $K \times K$, it factors through $X \times X/K \times K$ i.e. n is also a morphism. Similarly, it can be checked that the inverse map on X/K is a morphism. Therefore X/K is an algebraic group. Finally, X/K is the image of a complete variety and therefore is complete. Thus X/K is an abelian variety, and $f: X \to X/K$ is a separable isogeny. Clearly the kernel of f is K.

Second, start with a separable isogeny $f: X \to Y$. Let K be its kernel, and as above form a new separable isogeny $g: X \to X/K$. A morphism h in the diagram:

$$\begin{tikzcd} & X \arrow[dl, "g"'] \arrow[dr, "f"] & \\ X/K \arrow[rr, "h"'] & & Y \end{tikzcd}$$

exists, since f collapses the action of K, hence it factors through the quotient X/K. But h is obviously bijective, and the separability of f implies the separability of h. Therefore h is birational too. Therefore by Zariski's Main Theorem, h is an isomorphism.

Corollary 1. *A separable isogeny $f: X \to Y$ is an étale morphism.*

Corollary 2. *Let $f: X \to Y$ be an isogeny of order prime to p. Then the kernel of f and the kernel of f^*: $\mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ are dual finite abelian groups.*

Proof. Apply Proposition 3 and Theorem 4.

8. The dual abelian variety: char 0. We will use the hypothesis of char 0 only towards the end of this section.

Definition. *$\mathrm{Pic}^0(X)$ is the subgroup of $\mathrm{Pic}(X)$ consisting of line bundles L such that the homomorphism ϕ_L is identically zero.*

By the theorem of the square, the *image* of each ϕ_L is contained in $\mathrm{Pic}^0(X)$, so we get an exact sequence:

$$0 \longrightarrow \mathrm{Pic}^0(X) \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Hom}(X, \mathrm{Pic}^0(X)).$$
$$L \longmapsto \phi_L.$$

The main purpose of this section is to show (in char 0) that $\mathrm{Pic}^0(X)$ is naturally isomorphic to another abelian variety $\widehat{X}$, called the dual of X. We make some general observations about $\mathrm{Pic}^0(X)$.

(i) $L \in \mathrm{Pic}^0(X) \iff T_x^* L \simeq L$, all $x \in X$

$$\iff m^*L \simeq p_1^* L \otimes p_2^* L \text{ on } X \times X.$$

Proof. By the See-saw theorem, $m^*L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$ is trivial if and only if it is trivial on $X \times \{a\}$ and on $\{0\} \times X$. But it always is trivial on $\{0\} \times X$ and its restriction to $X \times \{a\}$ is isomorphic to $T_a^* L \otimes L^{-1}$.

(ii) If $L \in \mathrm{Pic}^0(X)$, then for all schemes S and all morphisms $f, g: S \to X$, $(f+g)^*L \simeq f^*L \otimes g^*L$.

Proof. Consider the last isomorphism in (i) and pull it back to S by $(f, g): S \to X \times X$.

(iii) If $L \in \mathrm{Pic}^0(X)$, $n_X^* L \simeq L^n$.

Proof. Apply induction to (ii).

(iv) For all $L \in \mathrm{Pic}(X)$, $n_X^* L \simeq L^{n^2} \otimes$ (something in $\mathrm{Pic}^0(X)$).

Proof. In fact, by §6, $n_X^* L \simeq L^{n^2} \otimes [L \otimes (-1_X)^* L^{-1}]^{(n-n^2)/2}$ so it suffices to prove that $L \otimes (-1_X)^* L^{-1} \in \mathrm{Pic}^0(X)$. By translating by x, we get

$$\begin{aligned} T_x^*(L \otimes (-1_X)^* L^{-1}) &\simeq T_x^* L \otimes (-1_X)^* T_{-x}^* L^{-1} \\ &\simeq T_x^* L \otimes (-1_X)^* \underbrace{[L \otimes T_{-x}^* L^{-1}]}_{\text{in } \mathrm{Pic}^0(X)} \otimes (-1_X)^* L^{-1} \\ &\simeq T_x^* L \otimes L^{-1} \otimes T_{-x}^* L \otimes (-1_X)^* L^{-1}, \text{ by (iii)} \\ &\simeq L \otimes (-1_X)^* L^{-1} \end{aligned}$$

by theorem of the square.

(v) If $L \in \mathrm{Pic}(X)$ has finite order, then $L \in \mathrm{Pic}^0(X)$.

Proof. If L^n is trivial, $0 = \phi_{L^n}(x) = n\phi_L(x) = \phi_L(nx)$, all $x \in X$. Since X is divisible, this shows that $\phi_L \equiv 0$.

(vi) For all varieties S, and all line bundles L on $X \times S$, if

$$L_s = L|_{X \times \{s\}}, \text{ then } L_{s_1} \otimes L_{s_0}^{-1} \in \mathrm{Pic}^0(X), \ (s_0, s_1 \in S).$$

Proof. Replacing S by open sets belonging to a covering of S, we can assume that $L|_{\{0\} \times S}$ is trivial. Further, replacing L by $L \otimes p_1^*(L_{s_0}^{-1})$, we can assume that L_{s_0} is trivial, and we must then prove that $L_s \in \mathrm{Pic}^0(X)$, all $s \in S$. We shall show that

$m^*(L_s)\otimes p_1^*(L_s^{-1})\otimes p_2^*(L_s^{-1})$ is trivial for all s. In fact, construct a line bundle M on $X\times X\times S$

$$M=\mu^*L\otimes p_{13}^*L^{-1}\otimes p_{23}^*L^{-1},$$

$$\mu(x,y,s)=(x+y,s)$$

$$p_{13}(x,y,s)=(x,s)$$

$$p_{23}(x,y,s)=(y,s).$$

Then M is trivial on $X\times\{0\}\times S$, $\{0\}\times X\times S$ and $X\times X\times\{s_0\}$. Therefore by the theorem of the cube, M is trivial. But M, restricted to $X\times X\times\{s\}$ is $m^*(L_s)\otimes p_1^*(L_s^{-1})\otimes p_2^*(L_s^{-1})$.

(vii) If $L\in\mathrm{Pic}^0(X)$ and L is not trivial, then $H^i(X,\underline{L})=(0)$, all i.

PROOF. If $H^0(\underline{L})\neq 0$, then $\underline{L}\simeq\mathcal{O}_X(D)$ for some non-negative divisor D. Then $\underline{L}^{-1}\simeq(-1_X)^*\underline{L}\simeq\mathcal{O}_X((-1_X)^*D)$, hence $\mathcal{O}_X\simeq\underline{L}\otimes\underline{L}^{-1}\simeq\mathcal{O}_X(D+(-1_X)^*D)$. Therefore $D+(-1_X)^*D=0$, hence $D=0$ and $\underline{L}\simeq\mathcal{O}_X$ which contradicts our assumption. This proves that $H^0(\underline{L})=(0)$. Let k be the smallest integer such that $H^k(\underline{L})\neq(0)$. Let $s_1\colon X\to X\times X$ be the map $s_1(x)=(x,0)$. Using the fact that $m^*L\simeq p_1^*L\otimes p_2^*L$ and the Künneth formula, we get the diagram:

$$\begin{tikzcd}
X \arrow[r, "s_1"] & X\times X \arrow[r, "m"] & X \\
H^k(X,\underline{L}) & H^k(X\times X,m^*\underline{L}) \arrow[l, "s_1^*"'] \arrow[d, phantom, "\wr\|"] & H^k(X,\underline{L}) \arrow[l, "m^*"'] \arrow[ll, dashed, bend right] \\
 & H^k(X\times X,p_1^*\underline{L}\otimes p_2^*\underline{L}) \arrow[d, phantom, "\wr\|"] & \\
 & \displaystyle\sum_{i+j=k} H^i(X,\underline{L})\otimes H^j(X,\underline{L}). &
\end{tikzcd}$$

Since $m\circ s_1=1_X$, the dotted arrow is the identity. But if $i+j=k>1$, then either $i<k$ or $j<k$, hence in all cases

$$H^i(X,\underline{L})\otimes H^j(X,\underline{L})=(0).$$

Therefore, the identity from $H^k(X,\underline{L})$ to $H^k(X,\underline{L})$ factors through a (0)-group. So $H^k(X,\underline{L})=(0)$ too.

We now come to the really key point in the theory of Pic^0.

THEOREM 1. *Let L be ample and $M\in\mathrm{Pic}^0(X)$. Then for some $x\in X$,*

$$M\simeq T_x^*L\otimes L^{-1},$$

i.e. the map $\phi_L\colon X\to\mathrm{Pic}^0(X)$ is surjective.

PROOF. The whole idea is to look at the cohomology on $X\times X$ of the line bundle

$$K=m^*L\otimes p_1^*L^{-1}\otimes p_2^*(L^{-1}\otimes M^{-1}).$$

This cohomology is the abutment of two Leray spectral sequences associated to the two projections of $X\times X$ onto X:

(1) $H^l(X,R^kp_{1,*}(\underline{K}))\Rightarrow H^{k+l}(X\times X,\underline{K})$,

(2) $H^l(X,R^kp_{2,*}(\underline{K}))\Rightarrow H^{k+l}(X\times X,\underline{K})$.

Notice that on the fibres $\{x\}\times X$ of p_1 and $X\times\{x\}$ of p_2, K restricts to the line bundles

$$K|_{\{x\}\times X}\simeq T_x^*L\otimes L^{-1}\otimes M^{-1},$$

$$K|_{X\times\{x\}}\simeq T_x^*L\otimes L^{-1}.$$

Therefore, if $M\not\simeq T_x^*L\otimes L^{-1}$ for any x, it follows that $K|_{\{x\}\times X}$ is a non-trivial line bundle in Pic^0 for every x. But by (vii), this means that all the cohomology groups of $K|_{\{x\}\times X}$ are (0). Therefore $R^kp_{1,*}(\underline{K})=(0)$ for all k by Corollary 2, §5. Therefore $H^k(X\times X,\underline{K})=(0)$ by spectral sequence (1).

Now use the other spectral sequence. Since $T_x^*L\otimes L^{-1}$ is non-trivial and in Pic^0 if $x\notin K(L)$, it follows that

$$\mathrm{supp}\,(R^kp_{2,*}(\underline{K}))\subset K(L).$$

Since $K(L)$ is a finite set, spectral sequence (2) degenerates to

$$\bigoplus_{x\in K(L)} R^kp_{2,*}(\underline{K})_x\simeq H^k(X\times X,\underline{K}).$$

But $H^k(X \times X, \underline{K}) = (0)$, so $R^k p_{2,*}(\underline{K}) = (0)$ too. Therefore $H^k(X, \underline{K}|_{X \times \{x\}}) = (0)$ for all x, by Cor. 4, § 5. But $K|_{X \times \{0\}}$ is the trivial line bundle, hence has a non-zero H^0! Thus we have a contradiction, so that the theorem must be true.

For a second proof of a slight weakening of the theorem, see Lang, p. 99. This important theorem shows that as an abstract group, $\mathrm{Pic}^0(X)$ is isomorphic to the abelian variety $X/K(L)$. If $\hat{X}$ is an abelian variety isomorphic as abstract group to $\mathrm{Pic}^0(X)$, what properties would we expect, which would characterize this "extra structure" on $\mathrm{Pic}^0(X)$?

(a) We want a line bundle P on $X \times \hat{X}$, the *Poincaré bundle* such that for all $\alpha \in \hat{X}$, the restriction P_α of P to $X \times \{\alpha\}$ represents the element of $\mathrm{Pic}^0(X)$ given by α under the isomorphism $\mathrm{Pic}^0(X) \simeq \hat{X}$. Moreover, we require that $P|_{\{0\} \times \hat{X}}$ is trivial. (These properties characterize P by the See-saw theorem.)

(b) For every normal variety S, and every line bundle K on $X \times S$ such that (i) $K_s = K|_{X \times \{s\}}$ is in $\mathrm{Pic}^0(X)$, for one and hence all $s \in S$, and (ii) $K|_{\{0\} \times S}$ is trivial, the unique set-theoretic map

$$f: S \to \hat{X}$$

such that $K_s \simeq P_{f(s)}$, is to be a morphism, and K is to be isomorphic to $(1_X \times f)^* P$.

It is easy to check that (a) and (b) uniquely characterize both $\hat{X}$ and P up to canonical isomorphisms. The problem is to construct such an $\hat{X}$ and P. So fix an ample L on X, and as suggested by the theorem, take $\hat{X}$ to be the quotient $X/K(L)$, constructed in §7. Let $\pi: X \to \hat{X}$ be the given morphism. To construct P, we shall use Prop. 2, §7. We want $(1_X \times \pi)^* P$ to be the line bundle $K = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$ on $X \times X$. [This clearly should be the case: apply (b) with $S = X$, $K = m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}$. Then $f = \pi$ so K should be isomorphic to $(1_X \times \pi)^* P$]. According to Prop. 2, we must lift the translation action of $\mathrm{Ker}(1_X \times \pi) = (0) \times K(L)$ on $X \times X$, to an action of the same group on the line bundle K. But for any $a \in K(L)$, compute the pull-back $T_{(0,a)}^* K$:

$$\begin{aligned} T_{(0,a)}^* K &\simeq T_{(0,a)}^* m^* L \otimes T_{(0,a)}^* p_1^* L^{-1} \otimes T_{(0,a)}^* p_2^* L^{-1} \\ &\simeq m^* T_a^* L \otimes p_1^* L^{-1} \otimes p_2^* T_a^* L^{-1} \\ &\simeq m^* L \otimes p_1^* L^{-1} \otimes p_2^* L^{-1}, \end{aligned}$$

since $a \in K(L)$. Therefore, there is an automorphism $\phi_a: K \to K$ covering the automorphism $T_{(0,a)}: X \times X \to X \times X$ on the base space. However, each ϕ_a could be changed by a scalar, so there is no reason why $\phi_a \circ \phi_b = \phi_{a+b}$ should hold if the ϕ_a's are chosen arbitrarily. However, if $L^{-1}(0)$ is the fibre of the line bundle L^{-1} over 0, then notice that there is a *canonical* isomorphism:

$$\begin{aligned} K|_{\{0\} \times X} &\simeq m^* L|_{\{0\} \times X} \otimes p_1^* L^{-1}|_{\{0\} \times X} \otimes p_2^* L^{-1}|_{\{0\} \times X} \\ &\simeq L \otimes \begin{pmatrix} \text{trivial bundle} \\ L^{-1}(0) \times X \end{pmatrix} \otimes L^{-1} \\ &\simeq L^{-1}(0) \times X. \end{aligned}$$

Suppose we require that the automorphism ϕ_a of K should restrict on $\{0\} \times X$ to the product automorphism:

$$(\lambda, x) \longmapsto (\lambda, x + a)$$

$$L^{-1}(0) \times X \to L^{-1}(0) \times X.$$

Clearly, there is a unique ϕ_a which has this restriction to $\{0\} \times X$. Since the restrictions then obey $\phi_a \circ \phi_b = \phi_{a+b}$, so do the ϕ_a's themselves. With this action of $\mathrm{Ker}(1_X \times \pi)$ on K, we construct a P on $X \times \hat{X}$ such that $(1_X \times \pi)^* P \simeq K$.

Notice first that for all $\alpha \in \hat{X}$, if $\alpha = \pi(x)$, then

$$\begin{aligned} P_\alpha &\underset{\text{def}}{=} P|_{X \times \{\alpha\}} \\ &\simeq \pi^*(P)|_{X \times \{x\}} \\ &\simeq T_x^* L \otimes L^{-1}, \end{aligned}$$

i.e. P_α represents the element $\phi_L(x) \in \mathrm{Pic}^0(X)$. Therefore, if $\hat{X}$ is identified with $\mathrm{Pic}^0(X)$ so as to make the diagram

$$\begin{tikzcd}
 & \mathrm{Pic}^0(X) \arrow[dd, phantom, "\wr\|"] \\
X \arrow[ur, "\phi_L"] \arrow[dr, "\pi"'] & \\
 & \tilde{X}
\end{tikzcd}$$

commute, the first part of (a) holds. Moreover, $P|_{\{0\}\times\hat{X}}$ is the quotient of $K|_{\{0\}\times X}$ by $\mathrm{Ker}(\pi)$, i.e. of $L^{-1}(0)\times X$ by the product action of $K(L)$. Therefore, $P|_{\{0\}\times\hat{X}} \simeq L^{-1}(0)\times\hat{X}$, a trivial bundle. Thus the second part of (a) holds.

To check that (b) holds, given S and K, consider the line bundle $E = p_{12}^*(K)\otimes p_{13}^*(P^{-1})$ on $X\times S\times\hat{X}$. Then $E|_{X\times\{\alpha,s\}} \simeq K_s\otimes P_\alpha^{-1}$, and the subset of $S\times X$:

$$\Gamma = \{\, (s,\alpha) \mid E|_{X\times\{s,\alpha\}} \text{ trivial}\}$$

is Zariski-closed in $S\times X$. But since $E|_{X\times\{s,\alpha\}}$ is trivial if and only if $K_s \simeq P_\alpha$, Γ is nothing but the graph of the set-theoretic map f. In particular, the projection $\Gamma\to S$ is a bijection. Now since the characteristic is 0†, this shows that Γ and S are birationally equivalent varieties, and since S is normal, $\Gamma\to S$ is an isomorphism of varieties by Zariski's Main Theorem. Therefore Γ is the graph of a morphism, i.e. f is a morphism. The last assertion in (b) follows from the See-saw theorem.

REMARKS. (1) For *every* line bundle L on X, the map $\phi_L: X\to\hat{X}$ is a morphism. This comes out of applying the universal mapping property (b) to the line bundle $m^*(L)\otimes p_1^*(L^{-1})\otimes p_2^*(L^{-1})$ on $X\times X$.

(2) If $X\xrightarrow{f} Y$ is a homomorphism of abelian varieties, the induced map $\mathrm{Pic}\,Y\to\mathrm{Pic}\,X$ maps $\mathrm{Pic}^0 Y$ into $\mathrm{Pic}^0 X$, and thus we get a natural map $\hat{f}: \hat{Y}\to\hat{X}$, and this is a morphism. In fact, if Q is the

Poincaré bundle on $Y\times\hat{Y}$, $(f\times 1)^*(Q)$ is a line bundle on $X\times\hat{Y}$ such that for $\hat{y}\in\hat{Y}$, $(f\times 1_{\hat{Y}})^*Q|_{X\times\{\hat{y}\}}$ represents $f^*(\hat{y})\in\mathrm{Pic}^0 X$, and by the universal mapping property, $\hat{f}: \hat{Y}\to\hat{X}$ is a morphism.

(3) If $f: X\to Y$ is an isogeny, so is $\hat{f}: \hat{Y}\to\hat{X}$, and there is a canonical duality (of finite abelian groups) between $\mathrm{Ker}\,f$ and $\mathrm{Ker}\,\hat{f}$.

PROOF. We have seen in §7 that $\mathrm{Ker}(f)$ and $\mathrm{Ker}(f^*\colon \mathrm{Pic}\,(Y)\to\mathrm{Pic}(X))$ are dual. If $\hat{y}\in\mathrm{Pic}(Y)$ is such that $f^*(\hat{y})=0$, then this shows that $\hat{y}$ has finite order, hence $\hat{y}\in\mathrm{Pic}^0(Y)$. Therefore, $\mathrm{Ker}(f^*\colon \mathrm{Pic}\,Y\to\mathrm{Pic}\,X) = \mathrm{Ker}\,\hat{f}$. Finally, since $\dim\hat{X} = \dim X = \dim Y = \dim\hat{Y}$, it follows that $\hat{f}$ is an isogeny too.

The final point we want to make is that the relationship between X and $\hat{X}$ is in reality *symmetric* like the relationship between two vector spaces set up by a bilinear pairing. We can see this as follows.

DEFINITION. *Let X and Y be abelian varieties. A divisorial correspondence between X and Y is a line bundle Q on $X\times Y$ whose restrictions to $\{0\}\times Y$ and $X\times\{0\}$ are trivial.*

PROPOSITION 2. *Let X and Y be two abelian varieties of the same dimension, and Q a divisorial correspondence between X and Y. The following are equivalent*

(1) *If $Q|_{\{x\}\times Y}$ is trivial, then $x=0$,*

(2) *If $Q|_{X\times\{y\}}$ is trivial, then $y=0$.*

If these hold, then $X\simeq\hat{Y}$ with Q isomorphic to the Poincaré bundle P_Y of Y, and $Y\simeq\hat{X}$ with Q isomorphic to the Poincaré bundle P_X of X.

PROOF. By symmetry, it suffices to deduce (2) from (1). If (1) holds, there is an injective morphism $\phi: X\to\hat{Y}$ such that $Q\simeq(\phi\times 1_Y)^*P_Y$, P_Y being the Poincaré bundle on $\hat{Y}\times Y$. Since $\dim X = \dim\hat{Y}$ and ϕ is injective, ϕ is also surjective; since the characteristic is 0, this implies that ϕ is an isomorphism, i.e. $X\simeq\hat{Y}$.

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†This is the only place where we use char. = 0! However, it is quite essential. The $\hat{X}$ we have constructed would definitely be "wrong" in char p.

Now, let $\psi: Y \to \hat{X}$ be the morphism such that if P_X is the Poincaré bundle on $X \times \hat{X}$, $(1_X \times \psi)^* P_X = Q$. To prove (2), we have to show that ψ is injective. If not, we can find a finite subgroup $K \subset \ker \psi$, $K \neq (0)$, and ψ factorizes as $Y \xrightarrow{\eta} Y/K \xrightarrow{\tilde{\psi}} \hat{X}$ where η is the natural homomorphism. Thus, if L is the line bundle $(1_X \times \tilde{\psi})^* P_X$ on $X \times Y/K$, we have that $Q \simeq (1_X \times \eta)^*(L)$. Now, L induces a homomorphism $\alpha: X \to (\widehat{Y/K})$, and the isomorphism $Q \simeq (1_X \times \eta)^*(L)$ means precisely that the composite $X \xrightarrow{\alpha} (\widehat{Y/K}) \xrightarrow{\hat{\eta}} \hat{Y}$ is the homomorphism defined by Q. Thus this composite is an isomorphism. Thus α is injective, and since $\dim X = \dim (\widehat{Y/K})$, α and $\hat{\eta}$ are both isomorphisms. But we know that $\hat{\eta}$ has a non-trivial kernel, viz. the dual abelian group of K. This is a contradiction, proving that ψ is injective.

9. The case $k = \mathbf{C}$. We want to link up the methods of Chapters 1 and 2 in this section. Therefore we assume that the ground field $k = \mathbf{C}$, and that $X = V/U$ (V a complex vector space, U a lattice) is an abelian variety over $\mathbf{C}$. Recall that every line bundle on X is isomorphic to $L(H, \alpha)$ for a unique Hermitian form H on V such that $E = \operatorname{Im} H$ is integral on $U \times U$, and a unique map $\alpha: U \to \mathbf{C}_1^*$ satisfying

$$\frac{\alpha(u_1 + u_2)}{\alpha(u_1)\,\alpha(u_2)} = e^{i\pi E(u_1, u_2)}, \quad u_1, u_2 \in U.$$

$L(H, \alpha)$ is, by definition, the quotient $(\mathbf{C} \times V)/U$ for the action

$$\phi_u(\lambda, z) = (\lambda.\,\alpha(u).\,e^{\pi H(z,u) + \frac{\pi}{2} H(u,u)},\ z + u).$$

We shall do two things: (A) compute $T^*_{\pi(x)}[L(H, \alpha)]$, $(x \in V)$, explicitly and hence interpret $\phi_{L(H,\alpha)}$ in the analytic case. (B) Describe divisorial correspondences as $L(H, \alpha)$'s and hence compute the dual $\hat{X}$ analytically.

(A) To keep the picture clear, it is convenient to generalize a little. Let a discrete group U act freely and discretely on V_1 and V_2 and let $T: V_1 \to V_2$ be a U-morphism. Let $X_i = V_i/U$, and let $\bar{T}: X_1 \to X_2$ be induced by T. Suppose U acts linearly on $\mathbf{C} \times V_2$ by

$$\phi_u(\lambda, z) = (\lambda.e_u(z),\ u(z)),\ z \in V_2,\ \lambda \in \mathbf{C},\ u \in U,$$

where $e_u(z)$ is a multiplicative co-cycle

$$e_{u_1+u_2}(z) = e_{u_1}(u_2(z)).e_{u_2}(z).$$

Let $L_2 = (\mathbf{C} \times V_2)/U$. Suppose we now want to describe the line bundle $L_1 = \bar{T}^*(L_2)$. Then

$$\begin{aligned} L_1 &= X_1 \times_{X_2} L_2 \\ &\simeq [V_1 \times_{V_2} (\mathbf{C} \times V_2)]/U, \text{ where } U \text{ acts on both factors.} \end{aligned}$$

Therefore, $L_1 = (\mathbf{C} \times V_1)/U$ with the action

$$\phi_u(\lambda, z) = (\lambda.e_u(Tz),\ u(z)),\ z \in V_1,\ \lambda \in \mathbf{C},\ u \in U.$$

Now the co-cycle $\{e_u\}$ might be normalized to have a special form which $\{e_u \circ T\}$ might not have. Then we might use an automorphism of $\mathbf{C} \times V_1$:

$$\begin{tikzcd} (\lambda, z) \arrow[r, mapsto] & (\lambda.g(z),\ z) \\ \mathbf{C} \times V_1 \arrow[r] & \mathbf{C} \times V_1 \end{tikzcd}$$

and carrying over the action of U, obtain a new description of L_1 as $(\mathbf{C} \times V_1)/U$ with action

$$\phi_u(\lambda, z) = (\lambda e_u(T z).g(u(z)).g(z)^{-1},\ u(z)).$$

Apply all this to the case $V_1 = V_2 = V$, $X_1 = X_2 = X$, T_a translation by $a \in V$, $\bar{T} = T_{\pi(a)}$ translation by $\pi(a)$, and $L_2 = L(H, \alpha)$. It follows that $T^*_{\pi(a)} L(H, \alpha)$ is $(\mathbf{C} \times V)/U$ with action

$$\begin{aligned} \phi_u(\lambda, z) &= (\lambda.\alpha(u).e^{\pi H(z+a,u) + \frac{1}{2}\pi H(u,u)},\ z + u) \\ &= (\lambda[\alpha(u).e^{\pi H(a,u)}].\ e^{\pi H(z,u) + \frac{1}{2}\pi H(u,u)},\ z + u). \end{aligned}$$

To simplify this co-cycle, take $g(z)$ to be the non-zero holomorphic function $e^{-\pi H(z,a)}$. We get the new action

$$\phi'_u(\lambda, z) = (\lambda.\alpha(u).e^{\pi[H(a,u) - H(u,a)]}.e^{\pi H(z,u) + \frac{1}{2}\pi H(u,u)},\ z + u)$$

and since $H(a, u) - H(u, a) = 2iE(a, u)$, we conclude:

Proposition. $T^*_{\pi(a)}[L(H, \alpha)] \simeq L(H, \alpha . \gamma_a)$ *where* $\gamma_a(u) = e^{2\pi i E(a,u)}$.

We get immediately lots of nice consequences.

(i) $\phi_{L(H,\alpha)}(\pi(a))$ is the point of $\mathrm{Pic}^0(X)$ represented by $L(0, \gamma_a)$.

(ii) In particular, since $\gamma_{a_1} \cdot \gamma_{a_2} = \gamma_{a_1+a_2}$, it follows that $\phi_{L(H,\alpha)}$ is a homomorphism; this is the theorem of the square.

(iii) We find

$$K(L(H, \alpha)) = U^{\perp}/U \subset V/U = X,$$

where $U^{\perp} = \{a | E(a, u) \in \mathbf{Z}, \text{ all } u \in U\}$.

(iv) Therefore $L(H, \alpha)$ is in $\mathrm{Pic}^0(X)$, as defined algebraically,

$\iff K(L(H, \alpha)) = X \iff U^{\perp} = V \iff E \equiv 0$

$\iff H \equiv 0$, i.e. $L(H, \alpha)$ is in $\mathrm{Pic}^0(X)$, as defined analytically.

(v) Moreover,

$K(L(H, \alpha))$ is finite $\iff U^{\perp}/U$ is finite

$\iff U^{\perp}$ is a lattice

$\iff E$ is non-degenerate

$\iff H$ is non-degenerate.

(vi) Notice that if H is non-degenerate, e.g. if $L(H, \alpha)$ is ample, then every homomorphism $U \to \mathbf{R}$ is given by $u \to E(a, u)$, for some $a \in V$. Therefore every homomorphism $\alpha : U \to \mathbf{C}_1^*$ is given by

$$u \to e^{2\pi i E(a,u)}$$

for some $a \in V$. Thus every element of $\mathrm{Pic}^0(X)$ is equal to $L(0, \gamma_a)$, some $a \in V$; this proves the main theorem of §8 when $k = \mathbf{C}$.

(B) Let $X_i = V_i/U_i$, $i = 1, 2$, be two abelian varieties. Then $Q = L(H, \alpha)$ on $X_1 \times X_2$ is a divisorial correspondence if Q is trivial on $\{0\} \times X_2$ and on $X_1 \times \{0\}$. This means that (a) the Hermitian form H is 0 on $\{0\} \times V_2$ and on $V_1 \times \{0\}$, and (b) $\alpha \equiv 1$ on $\{0\} \times U_2$ and on $U_1 \times \{0\}$. Define

$$B(x_1, x_2) = H((x_1, 0), (0, x_2)).$$

Then B is an $\mathbf{R}$-bilinear form on $V_1 \times V_2$, complex linear on V_1, and anti-linear on V_2. Let $\mathrm{Im}(B) = \beta$. Then β is integral on $U_1 \times U_2$, and we get

$$\begin{aligned} H((x_1, x_2), (y_1, y_2)) &= H((x_1, 0), (y_1, 0)) + H((x_1, 0), (0, y_2)) \\ &\quad + H((0, x_2), (y_1, 0)) + H((0, x_2), (0, y_2)) \\ &= B(x_1, y_2) + \overline{B(y_1, x_2)}. \\ \alpha((u_1, u_2)) &= \alpha((u_1, 0)) . \alpha((0, u_2)) . e^{\pi i E((u_1,0),(0,u_2))} \\ &= e^{\pi i \beta(u_1,u_2)}. \end{aligned}$$

Thus the divisorial correspondence Q is determined entirely by B.

In order to find the map from X_2 to $\widehat{X}_1$ induced by Q, we next calculate the restriction of Q to $X_1 \times \{\pi_2(a_2)\}$, $a_2 \in V_2$. Let $Q' = (\mathbf{C} \times V_1 \times V_2)/U_1$. If $1 \times \pi_2 : X_1 \times V_2 \to X_1 \times X_2$ is the natural map, then $Q' \simeq (1 \times \pi_2)^* Q$, and $Q'|_{X_1 \times \{a_2\}} \simeq Q|_{X_1 \times \pi_2(a_2)}$. The action of U_1 on $\mathbf{C} \times V_1 \times V_2$ is given by

$$\phi'_{u_1}(\lambda, x_1, x_2) = (\lambda . e^{\overline{\pi B(u_1,x_2)}}, x_1 + u_1, x_2).$$

Restricting to $V_1 \times \{a_2\}$, it follows that $Q'|_{X_1 \times \{a_2\}}$ equals $(\mathbf{C} \times V_1)/U_1$ with action

$$\phi''_{u_1}(\lambda, x_1) = (\lambda . e^{\overline{\pi B(u_1,a_2)}}, x_1 + u_1).$$

Modifying this group action by the automorphism of $\mathbf{C} \times V_1$, scalar multiplication by $e^{-\pi B(x_1,a_2)}$, we get the action

$$\phi'''_{u_1}(\lambda, x_1) = (\lambda . e^{-2\pi i \beta(u_1,a_2)}, x_1 + u_1).$$

Thus we have

Proposition. $Q|_{X_1 \times \{\pi_2(a_2)\}} \simeq L(0, \delta_{a_2})$ *where* $\delta_{a_2}(u_1) = e^{-2\pi i \beta(u_1,a_2)}$.

In particular, we see that in order that Q on $X_1 \times X_2$ satisfy the equivalent conditions of Proposition 2, §8, it is necessary and sufficient that $\dim X_1 = \dim X_2$ and that for all $x_2 \in V_2$, $x_2 \in U_2 \iff \beta(u_1, x_2) \in \mathbf{Z}$, all $u_1 \in U_1$. This implies that B is a non-degenerate pairing of V_1 and V_2. Hence, COROLLARY. *Via Q, $X_1 \simeq \widehat{X}_2$ and $X_2 \simeq \widehat{X}_1$ if and only if*

(i) *B is non-degenerate;*

(ii) *under β, U_1 and U_2 are dual lattices, i.e.*

$$U_2 = \{x_2 \in V_2 \mid \beta(u_1, x_2) \in \mathbf{Z}, \text{ all } u_1 \in U_1\} \text{ and vice versa.}$$

Explicitly, therefore, if $X = V/U$, then the dual abelian variety $\widehat{X}$ can be constructed as follows. Let

$$\overline{T} = \mathrm{Hom}_{\mathbf{C}\text{-antilinear}}(V, \mathbf{C}),$$

$$U' = \{l \in \overline{T} \mid \mathrm{Im}\, l(u) \in \mathbf{Z}, \text{ all } u \in U\}.$$

Then $X = \overline{T}/U'$. The form $B\colon V \times \overline{T} \to \mathbf{C}$ is simply $B(x, l) = \overline{l(x)}$, and the Poincaré bundle P_X on $X \times \widehat{X}$ is simply $L(H,\alpha)$ with

$$H((x_1, l_1), (x_2, l_2)) = \overline{l_2(x_1)} + l_1(x_2)$$

$$\alpha((u, l)) = e^{-\pi i\, \mathrm{Im}\, l(u)}.$$

Moreover the line bundle on X corresponding to a point $\pi(l) \in \widehat{X}$ $(l \in \overline{T})$ is then just $L(0, \alpha_l)$ where

$$\alpha_l(u) = e^{2\pi i\, \mathrm{Im}\, l(u)}, \quad u \in U.$$

There arises a small question of compatibility: we just constructed $\widehat{X} = \overline{T}/U'$ and showed that $\widehat{X} \simeq \mathrm{Pic}^0(X)$. But in §2, via the exact sequence

$$0 \longrightarrow \mathbf{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0,$$

we constructed an isomorphism

$$\mathrm{Pic}^0(X) \simeq H^1(X, \mathcal{O}_X)/H^1(X, \mathbf{Z})$$

and in §1, we found an isomorphism $H^1(X, \mathcal{O}_X) \simeq \overline{T}$. We would like to be sure that we have really found essentially the same description of $\mathrm{Pic}^0(X)$ twice. As we have just seen, our second description of $\mathrm{Pic}^0(X)$ rests on the map

$$\begin{aligned} \overline{T} &\longrightarrow \mathrm{Pic}^0(X) \\ l &\longmapsto L(0, \alpha_l). \end{aligned}$$

Let us compute the composite map $\overline{T} \xrightarrow{\sim} H^1(\mathcal{O}_X) \xrightarrow{e^{2\pi i}} \mathrm{Pic}(X)$ which gave us the first description. This map is given by

$$\overline{T} \subset T \oplus \overline{T} \xrightarrow[f]{\sim} H^1(X, \mathbf{C}) \longrightarrow H^1(X, \mathcal{O}_X) \xrightarrow{e^{2\pi i}} \mathrm{Pic}(X).$$

Using group cohomology of U, we get a diagram

$$\begin{tikzcd}
\mathrm{Hom}(U, \mathbf{C}) \arrow[r, equal, "\substack{\text{by def.}\\ \text{of gp.}\\ \text{coho.}}"] \arrow[d, equal, "\wr"'] & H^1(U, \mathbf{C}) \arrow[r, "e^{2\pi i}"'] \arrow[dd, "\wr"'] & H^1(U, H^*) \arrow[dd, "\wr"'] \\
\mathrm{Hom}_{\mathbf{R}}(V, \mathbf{C}) \arrow[d, "\wr"'] & & \\
\overline{T} \subset T \oplus \overline{T} \arrow[r, "\sim", "f"'] & H^1(X, \mathbf{C}) \arrow[r, "e^{2\pi i}"'] & \mathrm{Pic}(X)
\end{tikzcd}$$

where H^*=multiplicative group of non-zero holomorphic functions on V, and where the square on the left commutes according to the compactibilities verified in §1. Therefore, if $l \in \overline{T}$, the first description associates to l the U-co-cycle $u \to e^{2\pi i l(u)}$.

But

$$e^{2\pi i l(u)} = \frac{g(z+u)}{g(z)} \cdot e^{4\pi i \mathrm{Re}[l(u)]}$$

where $g(z) = e^{-2\pi i \overline{l(z)}}$ is holomorphic in z. In other words, the first description rests on the map

$$\begin{aligned} \overline{T} &\longrightarrow \mathrm{Pic}^0(X) \\ l &\longmapsto L(0, \alpha_l^*) \end{aligned}$$

$$\alpha_l^*(u) = e^{4\pi i \mathrm{Re}[l(u)]}.$$

So the two maps differ only by multiplication by $2i$ (experimental error!).