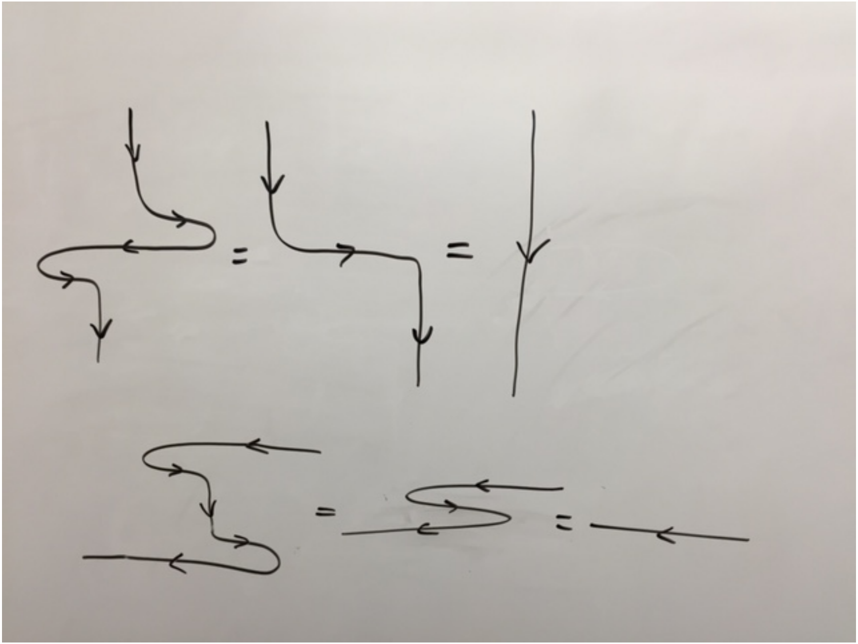
String Diagrams for (Virtual) Proarrow Equipments

David Jaz Myers

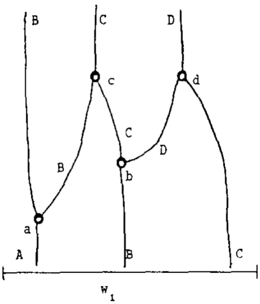
July 22, 2017



Theorem (Joyal and Street)

The graphical calculus for monoidal categories is sound.

For any deformation $h : \Gamma \times [0,1] \to [a,b] \times [c,d]$ *of diagrams, the value of* $h(-,0)$ *equals that of* $h(-,1)$*.*



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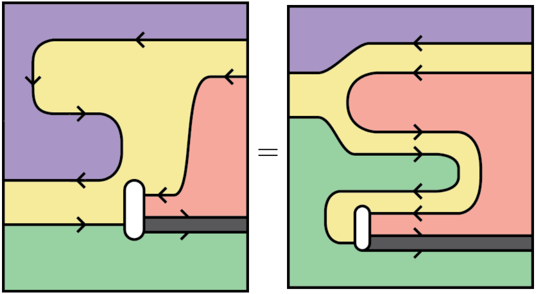

Theorem (M.)

There is a canonical (Yoneda-style) embedding $|\cdot| : \mathcal{E} \to \mathcal{E}$-**Cat** *of a virtual equipment into the virtual equipment of categories enriched in it, which is full on arrows and coreflective on proarrows.*

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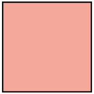
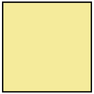
- Objects A, B, . . . , , , . . .

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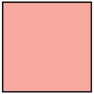
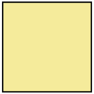
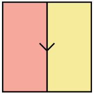
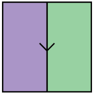
- Objects A, B,. . ., , ,. . .
- Arrows $f : A \to B$,. . . , , ,. . .

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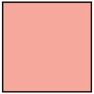
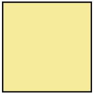
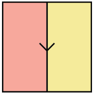
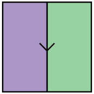
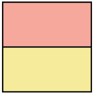
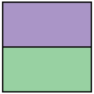
- Objects A, B,. . ., , ,. . .
- Arrows $f : A \to B$,. . . , , ,. . .
- Proarrows $J : A \nrightarrow B$, . . . , , ,. . .

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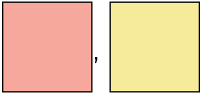
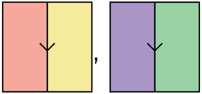
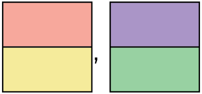
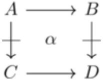
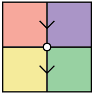
- Objects A, B,...,  ,...
- Arrows $f : A \to B$,...,  ,...
- Proarrows $J : A \nrightarrow B$, ...,  ,...
- 2-cells 

,...,  ,...

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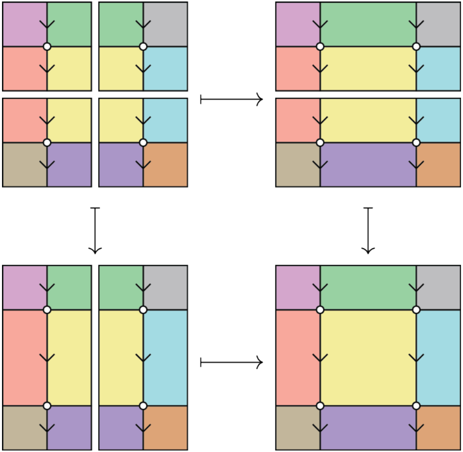

Companions and Conjoints

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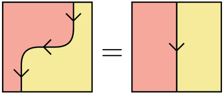

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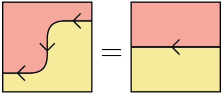.

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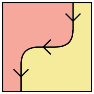
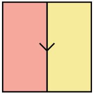
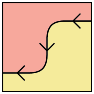
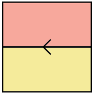
 =  and  = .

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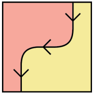
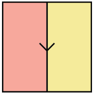
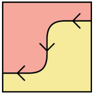
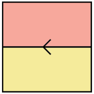
 =  and  = .

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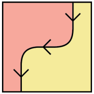
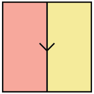
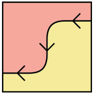
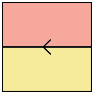
 $=$  and  $=$ .

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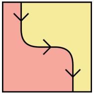
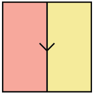
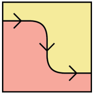
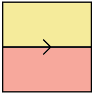
 $=$  and  $=$ .

Proarrow Equipments

Definition

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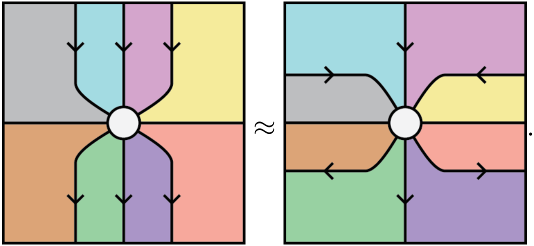
Examples

- Sets, Functions, Relations.
- Rings, Homomorphisms, Bimodules.
- Categories, Functors, Profunctors.
- Enriched Categories, Enriched Functors, Enriched Profunctors, etc.

Spider Lemma

Lemma (Spider Lemma)

In an equipment, we can bend arrows. More formally, there is a bijective correspondence between diagrams of form of the left, and diagrams of the form of the right:



Hom-Set to Zig-Zag Adjunctions

Given arrows  and ,

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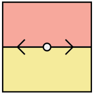
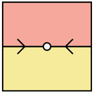
Given arrows  and , with an isomorphism  with inverse :

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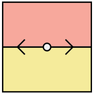
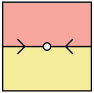
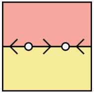
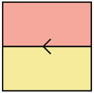
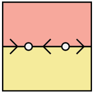
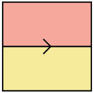
Given arrows  and , with an isomorphism  with inverse :  $=$  and  $=$ .

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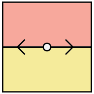
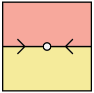
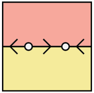
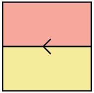
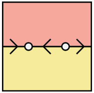
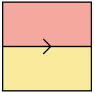
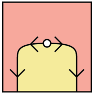
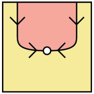
Given arrows  and , with an isomorphism  with

inverse :  =  and  = .

Bend to  and ,

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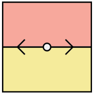
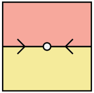
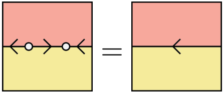
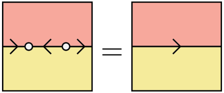
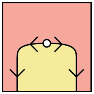
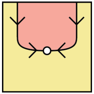
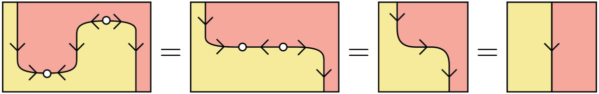
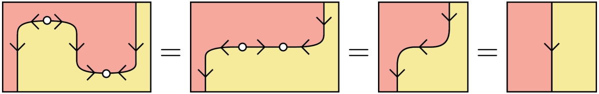
Given arrows  and , with an isomorphism  with

inverse :  and .

Bend to  and , then

,

.

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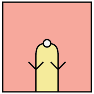
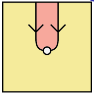
Given  and ,

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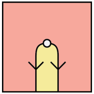
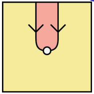
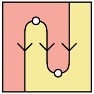
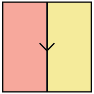
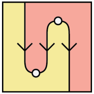
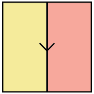
Given  and , satisfying

 $=$  and  $=$ .

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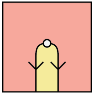
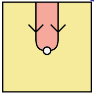
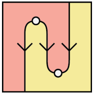
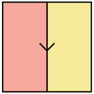
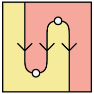
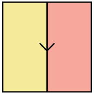
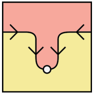
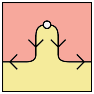
Given  and , satisfying

 =  and  = .

Bend to  and ,

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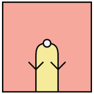
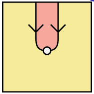
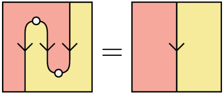
Given  and , satisfying



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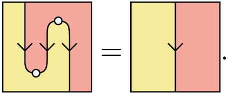
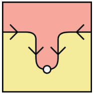
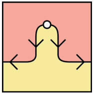
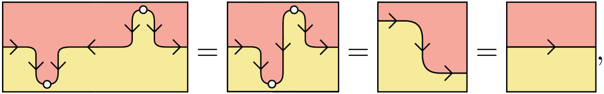
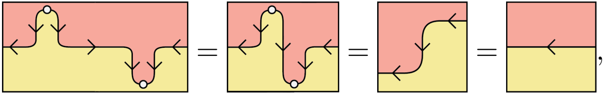


Bend to  and , then





Enriching in a Virtual Equipment

- Lawvere ('73):
  - Not only are the most fundamental structures of mathematics organized in categories,
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  - Not only are the most fundamental structures of mathematics organized in categories,
  - They are in many cases (enriched) categories themselves.
- With the graphical calculus, we can show that so long as our objects form a virtual equipment, then they are enriched categories of a sort.

Theorem (M.)

There is a Yoneda-style embedding $|\cdot| : \mathcal{E} \to \mathcal{E}\text{-}\mathbf{Cat}$ *of a virtual equipment into the virtual equipment of categories enriched in it.*

Enrichment and Virtual Equipments

- Composing proarrows requires taking a colimit in the base category.
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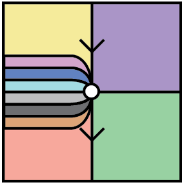
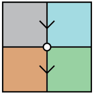
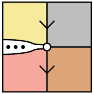
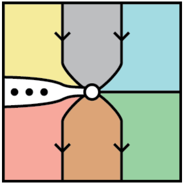
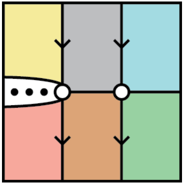
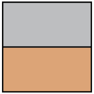
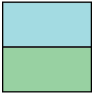
- But what if the base category is not suitably cocomplete?
- Then we use "virtual equipments" instead.



Restrictions

Definition

A cell  is called *cartesian* if for any , there exists a unique  so that  $=$ .

We call  the **restriction** of  along  and .

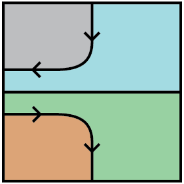
Restrictions

Definition

A **Virtual Equipment** is a virtual double category with all restrictions (and a unit condition).

Lemma (Cruttwell and Shulman)

In a virtual equipment, every restriction is of the form



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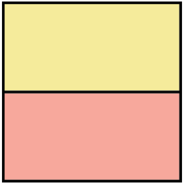

Examples of Enrichment in a Virtual Equipment

With a single object:

- In Sets and Spans: Categories.
- In Rings and Bimodules: Algebras.
- In Enriched Cats and Profunctors: Arrows.
- Multicategories, Many-sorted Lawvere theories, Virtual double categories, etc.

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Conjecture (M.)

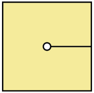
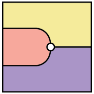
There is a full and faithful functor

$$\mathbf{Kleisli}(\mathbf{Jet}) \hookrightarrow \mathbf{Span\text{-}Cat}.$$

sending a smooth manifold to its category of smooth paths.

Enriching in a Virtual Equipment

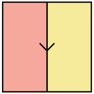
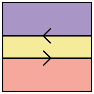
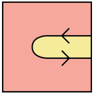
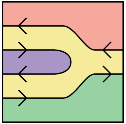
A category $\mathcal{C}$ enriched in a virtual equipment $\mathcal{E}$ consists of the following data:

- A class of objects $\mathcal{C}_0$, with each object $A \in \mathcal{C}_0$ associated with an object $\mathcal{C}(A) =$  in $\mathcal{E}$ called its *extent*.
- For each pair of objects  and  in $\mathcal{C}_0$, a proarrow $\mathcal{C}($, $) =$  in $\mathcal{E}$.
- For each object  in $\mathcal{C}_0$, a 2-cell **id** $=$  called the *identity*.
- For each triple of objects , , , a 2-cell  called *composition*.

Defining the “Yoneda” Embedding

For an object  of $\mathcal{E}$, we define its representative to be

|| :=

- Objects are vertical arrows , with each object’s extent being its domain.
- Between objects  and , a hom-object .
- For object , an identity arrow .
- For each composable triple, a composition arrow .

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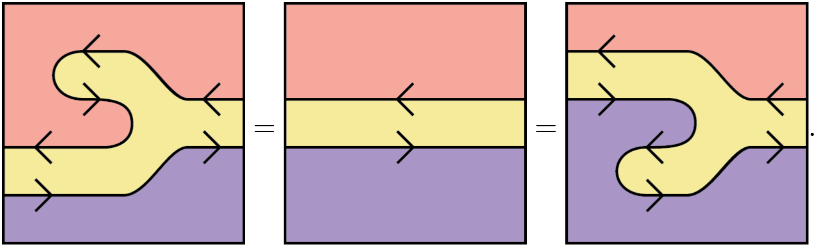

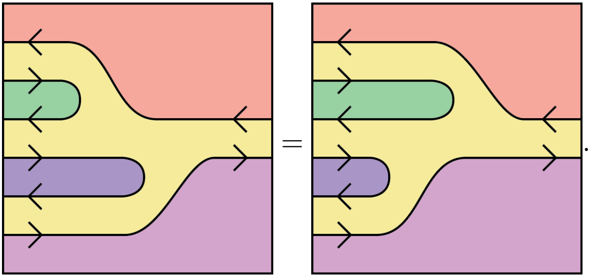

Properties of "Yoneda" Embedding

Proposition (M.)

The "Yoneda" embedding $|\cdot| : \mathcal{E} \to \mathcal{E}\text{-}\mathbf{Cat}$

- *is full on 2-cells (and therefore faithful on arrows and proarrows);*

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Conjecture (M.)

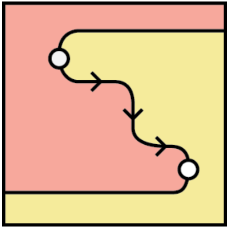
For a "fibrantly enriched" $\mathcal{E}$*-category* $\mathcal{C}$*, denote by* $\mathcal{C}[A]$ *the full subcategory of* $\mathcal{C}$ *whose objects have extent* A*. Then*

$$\mathcal{E}\text{-}\mathbf{Cat}(|A|, \mathcal{C}) \simeq \mathcal{C}[A].$$

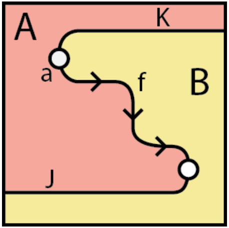
Soundness of Graphical Calculi

- Similar to the proof of Joyal and Street for monoidal categories.
- But using the tile-order machinery of Dawson and Paré to handle the two sorts of composition.

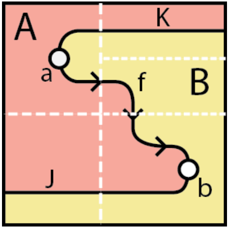
Take a Diagram,



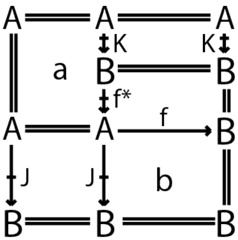
Label it,



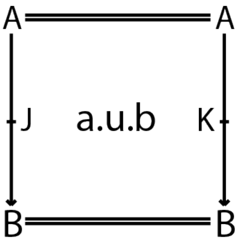
Tile it,



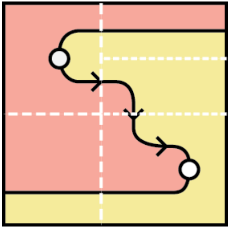
Turn it into the usual notation,

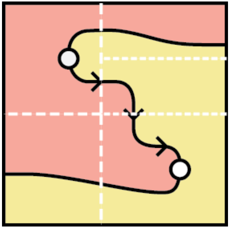


Compose it.



Tilings are stable under small deformations.





In Conclusion

Equipments are fundamental and useful objects

1. for combining “scalar” arrows and “linear” proarrows, and
2. as a setting for formal (enriched, internal, higher) category theory.

I hope that the string diagrams can make working with them easier!

References

Acknowledgements: Many thanks to Emily Riehl and Mike Shulman for reading drafts and giving very helpful comments.

1. D. J. M., *String diagrams for double categories and equipments*. arXiv:1612.02762
2. Joyal and Street, *Geometry of tensor calculus, I*.
3. Dawson and Paré, *General associativity and general composition for double categories*
4. Dawson, *A forbidden-suborder characterization of binarily-composable diagrams in double categories.*
5. Cruttwell and Shulman, *A unified framework for generalized multicategories*
6. Leinster, *Generalized enrichment of categories*