

Algebraic Derivation of Quartic Fierz Identities in 11-Dimensional Supergravity

Abstract

This document provides a self-contained, step-by-step algebraic proof of two specific quartic ($4\text{-}\psi$) Fierz identities required in 11-dimensional supergravity. The methodology follows the iterative algebraic bootstrapping technique established by Naito, Osada, and Fukui [1]. The proof relies strictly on the properties of exterior differential forms and the Clifford algebra representation in 11 dimensions, ensuring complete mathematical rigor.

1 Preliminaries and Axiomatic Rules

Let $\psi = \psi_\mu dx^\mu$ be an 11-dimensional Majorana gravitino field. It carries both a Grassmann-odd spinor index (anti-commuting) and the structure of an exterior differential 1-form. All multiplications between such fields are implicitly exterior wedge products (\wedge). This algebraic structure necessitates three strict axiomatic rules:

1. **Commutativity of Spinor 1-Forms:** The exchange of two gravitino fields incurs a factor of (-1) from the Grassmann algebra (fermionic statistics) and a factor of (-1) from the exterior form algebra. Consequently, their tensor product is completely symmetric in spinor space:

$$\psi^\alpha \wedge \phi^\beta = \phi^\beta \wedge \psi^\alpha \quad \implies \quad \psi \wedge \bar{\psi} = \bar{\psi} \wedge \psi. \quad (1)$$

2. **Vanishing Majorana Bilinears:** A bilinear constructed as $(\bar{\psi}\Gamma^{(k)} \wedge \psi) = \psi^\alpha \wedge (C\Gamma^{(k)})_{\alpha\beta}\psi^\beta$ vanishes identically if the matrix $C\Gamma^{(k)}$ is antisymmetric. In 11 dimensions, the properties of the charge conjugation matrix C dictate that $(C\Gamma^{(k)})^T = -(-1)^{k(k+1)/2}C\Gamma^{(k)}$. Evaluating this parity establishes that these bilinears vanish identically for ranks $k \in \{0, 3, 4, 7, 8, 11\}$ (see [1], Equation 2.1).
3. **Commutativity of Scalar 2-Forms:** Any bilinear of the form $(\bar{\psi}\Gamma^{(k)} \wedge \psi)$ is a Lorentz scalar in spinor space (possessing no free spinor indices) and an exterior 2-form. In the exterior algebra, a 2-form commutes identically with any p -form, since the signature factor is $(-1)^{2 \times p} = +1$. Therefore, these bilinears commute past 1-forms (such as $\Gamma\psi$) and past each other without acquiring any negative signs.

2 Proof of the First Identity

Theorem 1. $(\bar{\psi}\Gamma_{da} \wedge \psi) \wedge (\bar{\psi}\Gamma^d \wedge \psi) = 0$.

Proof. By expanding the open product $\psi \wedge \bar{\psi}$ over the complete 11-dimensional Clifford basis, one can systematically invert the standard Fierz identities. As established by Naito, Osada, and Fukui, isolating the 1-index contraction yields the following fundamental matrix-valued operator equation:

Lemma 1 (NOF86, Equation 2.9).

$$\Gamma_{da}\psi \wedge \bar{\psi}\Gamma^d + \Gamma^d\psi \wedge \bar{\psi}\Gamma_{da} = -\frac{1}{2}(\bar{\psi}\Gamma^d \wedge \psi)\Gamma_{da} - \frac{1}{2}(\bar{\psi}\Gamma_{da} \wedge \psi)\Gamma^d. \quad (2)$$

We begin by wedging Lemma 1 from the right with the column-spinor 1-form ψ . Let the left-hand side (*LHS*) of the resulting equation be denoted as:

$$LHS = \Gamma_{da}\psi \wedge (\bar{\psi}\Gamma^d \wedge \psi) + \Gamma^d\psi \wedge (\bar{\psi}\Gamma_{da} \wedge \psi). \quad (3)$$

On the right-hand side (*RHS*), wedging with ψ attaches the spinor to the trailing Γ matrices. By the commutativity of scalar 2-forms (Rule 3), the parenthetical bilinears commute past the resulting $\Gamma\psi$ 1-forms. Reordering the terms gives:

$$RHS = -\frac{1}{2}\Gamma_{da}\psi \wedge (\bar{\psi}\Gamma^d \wedge \psi) - \frac{1}{2}\Gamma^d\psi \wedge (\bar{\psi}\Gamma_{da} \wedge \psi) = -\frac{1}{2}LHS. \quad (4)$$

Equating both sides, we obtain $LHS = -\frac{1}{2}LHS$. This strict algebraic relation demands that $LHS = 0$. We have thus established the intermediate 3- ψ identity (NOF86 Equation 2.21):

$$\Gamma_{da}\psi \wedge (\bar{\psi}\Gamma^d \wedge \psi) + \Gamma^d\psi \wedge (\bar{\psi}\Gamma_{da} \wedge \psi) = 0. \quad (5)$$

Next, we elevate this to a 4- ψ identity by wedging (5) from the left with the row-spinor $\bar{\psi}$:

$$(\bar{\psi}\Gamma_{da} \wedge \psi) \wedge (\bar{\psi}\Gamma^d \wedge \psi) + (\bar{\psi}\Gamma^d \wedge \psi) \wedge (\bar{\psi}\Gamma_{da} \wedge \psi) = 0. \quad (6)$$

Because both constructed bilinears are purely 2-forms, they commute with each other identically (Rule 3). The two terms are therefore identical and symmetric, allowing us to sum them:

$$2(\bar{\psi}\Gamma_{da} \wedge \psi) \wedge (\bar{\psi}\Gamma^d \wedge \psi) = 0. \quad (7)$$

Dividing by 2 yields the required result, concluding the proof. \square

3 Proof of the Second Identity

Theorem 2. $(\bar{\psi}\Gamma_{da_1\dots a_4} \wedge \psi) \wedge (\bar{\psi}\Gamma^d \wedge \psi) = 3(\bar{\psi}\Gamma_{[a_1a_2} \wedge \psi) \wedge (\bar{\psi}\Gamma_{a_3a_4]} \wedge \psi)$.

Proof. Applying the analogous Clifford basis inversion to an arbitrary rank- n contraction yields a generalized 3- ψ master equation. We select the branch applicable to even integers n , which corresponds to the lower set of coefficients derived in Equation 2.22 of [1]:

Lemma 2 (NOF86, Equation 2.22, Even n).

$$\begin{aligned} & (2-n)\Gamma_{da_1\dots a_n}\psi \wedge (\bar{\psi}\Gamma^d \wedge \psi) - 2\Gamma^d\psi \wedge (\bar{\psi}\Gamma_{da_1\dots a_n} \wedge \psi) \\ & = 2\psi \wedge (\bar{\psi}\Gamma_{a_1\dots a_n} \wedge \psi) + 2n\Gamma_{[a_1}\psi \wedge (\bar{\psi}\Gamma_{a_2\dots a_n]} \wedge \psi) \\ & + n(n-4)\Gamma_{[a_1\dots a_{n-1}}\psi \wedge (\bar{\psi}\Gamma_{a_n]} \wedge \psi) - n(n-1)\Gamma_{[a_1\dots a_{n-2}}\psi \wedge (\bar{\psi}\Gamma_{a_{n-1}a_n]} \wedge \psi). \end{aligned} \quad (8)$$

We evaluate Lemma 2 meticulously for the specific target rank of $n = 4$:

- **Left-Hand Side:** The leading coefficient evaluates to $(2 - 4) = -2$. Factoring out -2 , we obtain:

$$-2\left[\Gamma_{da_1\dots a_4}\psi \wedge (\bar{\psi}\Gamma^d \wedge \psi) + \Gamma^d\psi \wedge (\bar{\psi}\Gamma_{da_1\dots a_4} \wedge \psi)\right].$$

- **Right-Hand Side Term 1** ($k = 4$): Contains the rank-4 bilinear $(\bar{\psi}\Gamma_{a_1\dots a_4} \wedge \psi)$. By the vanishing Majorana rules (Rule 2), this evaluates identically to zero.

- **Right-Hand Side Term 2** ($k = 3$): Contains the rank-3 bilinear $(\bar{\psi}\Gamma_{a_2a_3a_4} \wedge \psi)$. By Rule 2, this also vanishes identically.
- **Right-Hand Side Term 3** ($k = 1$): The algebraic prefactor is $n(n - 4) = 4(4 - 4) = 0$. This term trivially vanishes.
- **Right-Hand Side Term 4** ($k = 2$): Contains the rank-2 bilinear $(\bar{\psi}\Gamma_{a_3a_4} \wedge \psi)$, which survives the vanishing theorems. The prefactor evaluates to $-n(n-1) = -4(3) = -12$. The surviving term is precisely:

$$-12\Gamma_{[a_1a_2}\psi \wedge (\bar{\psi}\Gamma_{a_3a_4]} \wedge \psi).$$

Equating the LHS to the single non-vanishing term on the RHS yields:

$$-2\left[\Gamma_{da_1\dots a_4}\psi \wedge (\bar{\psi}\Gamma^d \wedge \psi) + \Gamma^d\psi \wedge (\bar{\psi}\Gamma_{da_1\dots a_4} \wedge \psi)\right] = -12\Gamma_{[a_1a_2}\psi \wedge (\bar{\psi}\Gamma_{a_3a_4]} \wedge \psi). \quad (9)$$

Dividing the entire equation by -2 isolates the explicit 3- ψ relation (NOF86 Equation 2.23):

$$\Gamma_{da_1\dots a_4}\psi \wedge (\bar{\psi}\Gamma^d \wedge \psi) + \Gamma^d\psi \wedge (\bar{\psi}\Gamma_{da_1\dots a_4} \wedge \psi) = 6\Gamma_{[a_1a_2}\psi \wedge (\bar{\psi}\Gamma_{a_3a_4]} \wedge \psi). \quad (10)$$

Finally, we elevate (10) to the target 4- ψ identity by wedging from the left with the adjoint spinor $\bar{\psi}$:

$$(\bar{\psi}\Gamma_{da_1\dots a_4} \wedge \psi) \wedge (\bar{\psi}\Gamma^d \wedge \psi) + (\bar{\psi}\Gamma^d \wedge \psi) \wedge (\bar{\psi}\Gamma_{da_1\dots a_4} \wedge \psi) = 6(\bar{\psi}\Gamma_{[a_1a_2} \wedge \psi) \wedge (\bar{\psi}\Gamma_{a_3a_4]} \wedge \psi). \quad (11)$$

By the commutativity of scalar 2-forms (Rule 3), the two generated terms on the left-hand side are algebraically identical. Summing them together provides:

$$2(\bar{\psi}\Gamma_{da_1\dots a_4} \wedge \psi) \wedge (\bar{\psi}\Gamma^d \wedge \psi) = 6(\bar{\psi}\Gamma_{[a_1a_2} \wedge \psi) \wedge (\bar{\psi}\Gamma_{a_3a_4]} \wedge \psi). \quad (12)$$

Dividing the resultant equation by 2 yields the required target identity, thus concluding the formal proof. \square

References

- [1] Seichi Naito, Kazuo Osada, and Tetsuo Fukui. Fierz identities and invariance of 11-dimensional supergravity action. *Phys. Rev. D*, 34(2):536–552, 7 1986. DOI: 10.1103/PhysRevD.34.536.