

# Excellent metrics and enhancements

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# Overview

- 1 Rickard's 1989 theorem
- 2 A bunch of definitions
- 3 The main 2018 theorem
- 4 Intrinsic equivalence classes of metrics
- 5 The metrics on  $\mathcal{L}(\mathcal{S})$  and  $\mathcal{G}(\mathcal{S})$

## Theorem

Let  $R$  and  $S$  be left-coherent rings. Then the following are equivalent:

- 1 There exists a triangle equivalence  $\mathbf{D}^b(R\text{-proj}) \cong \mathbf{D}^b(S\text{-proj})$ .
- 2 There exists a triangle equivalence  $\mathbf{D}^b(R\text{-mod}) \cong \mathbf{D}^b(S\text{-mod})$ .

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## Questions, Krause 2018:

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## Questions, Krause 2018:

- 3 Is it true that (2)  $\implies$  (1)? Challenge: **find a counterexample**.
- 4 Is there an algorithm to pass directly from the triangulated category  $\mathbf{D}^b(R\text{-proj})$  to the triangulated category  $\mathbf{D}^b(R\text{-mod})$ ?



Henning Krause, *Completing perfect complexes*, Math. Z. **296** (2020), no. 3-4, 1387–1427, With appendices by Tobias Barthel, Bernhard Keller and Krause.

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-  Amnon Neeman, *The categories  $\mathcal{T}^c$  and  $\mathcal{T}_c^b$  determine each other*, <https://arxiv.org/abs/1806.06471>.

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Following a 1974 article of Lawvere, a **metric** on a category is a function that assigns a positive real number (**length**) to every morphism, satisfying:

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- 1 For any identity map  $\text{id} : X \longrightarrow X$  we have

$$\text{Length}(\text{id}) = 0 ,$$

- 2 and if  $x \xrightarrow{f} y \xrightarrow{g} z$  are composable morphisms, then

$$\text{Length}(gf) \leq \text{Length}(f) + \text{Length}(g) .$$

# The classical literature on the topic







F. William Lawvere, *Metric spaces, generalized logic, and closed categories*, Rend. Sem. Mat. Fis. Milano **43** (1973), 135–166 (1974).



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-  G. Maxwell Kelly, *Basic concepts of enriched category theory*, London Mathematical Society Lecture Note Series, vol. 64, Cambridge University Press, Cambridge-New York, 1982.
-  G. Maxwell Kelly and Vincent Schmitt, *Notes on enriched categories with colimits of some class*, Theory Appl. Categ. **14** (2005), no. 17, 399–423.

## Definition (Equivalence of metrics)

We'd like to view two metrics on a category  $\mathcal{C}$  as **equivalent** if the identity functor  $\text{id} : \mathcal{C} \longrightarrow \mathcal{C}$  is uniformly continuous in both directions.

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More formally:

Let  $\mathcal{C}$  be a category. Two metrics

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$$\{\text{Length}_1(f) < \delta\} \implies \{\text{Length}_2(f) < \varepsilon\}$$

and

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## Definition (Cauchy sequences)

Let  $\mathcal{C}$  be a category with a metric. A **Cauchy sequence** in  $\mathcal{C}$  is a sequence  $E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow \dots$  of composable morphisms such that, for any  $\varepsilon > 0$ , there exists an  $M > 0$  such that the morphisms  $E_i \rightarrow E_j$  satisfy

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We will assume the category  $\mathcal{C}$  is  **$\mathbb{Z}$ -linear**. This means that  $\text{Hom}(a, b)$  is an abelian group for every pair of objects  $a, b \in \mathcal{C}$ , and that composition is bilinear.

## Definition (The categories $\mathcal{L}(\mathcal{C})$ and $\mathcal{S}(\mathcal{C})$ )

Let  $\mathcal{C}$  be a  $\mathbb{Z}$ -linear category with a metric. Let  $Y : \mathcal{C} \rightarrow \text{Mod-}\mathcal{C}$  be the Yoneda map, that is the map sending an object  $c \in \mathcal{C}$  to the functor  $Y(c) = \text{Hom}(-, c)$ , viewed as an additive functor  $\mathcal{C}^{\text{op}} \rightarrow \text{Ab}$ .

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- 1 Let  $\mathfrak{L}(\mathcal{C})$  be the **completion** of  $\mathcal{C}$ , meaning the full subcategory of  $\text{Mod-}\mathcal{C}$  whose objects are the colimits in  $\text{Mod-}\mathcal{C}$  of Cauchy sequences in  $\mathcal{C}$ .

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$F : \mathcal{C}^{\text{op}} \rightarrow \text{Ab}$  belongs to  $\mathfrak{S}(\mathcal{C})$  if there exists an  $\varepsilon > 0$  such that

$$\{\text{Length}(a \rightarrow b) < \varepsilon\} \implies$$

$$\{F(b) \rightarrow F(a) \text{ is an isomorphism}\}.$$

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Equivalent metrics lead to identical  $\mathfrak{L}(\mathcal{C})$  and  $\mathfrak{S}(\mathcal{C})$ .

## Heuristic

We want to specialize the above to a situation in which we can actually prove something.

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which means that for any homotopy cartesian square

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \downarrow \\ c & \xrightarrow{g} & d \end{array}$$

we must have

$$\text{Length}(f) = \text{Length}(g)$$

## Heuristic, continued

Given any  $f : a \longrightarrow b$  we may form the homotopy cartesian square

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \downarrow \\ 0 & \xrightarrow{g} & x \end{array}$$

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and our assumption tells us that

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Hence it suffices to know the lengths of the morphisms

$$0 \longrightarrow x .$$

## Heuristic, continued

We will soon be assuming that the metric is **non-archimedean**.  
Replacing the metric by an equivalent (if necessary), we may also assume our metric takes values in the set of rational numbers of the form

$$\{0, \infty\} \cup \{2^n \mid n \in \mathbb{Z}\} .$$

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To know everything about the metric it therefore suffices to specify the balls

$$B_n = \left\{ x \in \mathcal{S} \mid \text{the morphism } 0 \longrightarrow x \text{ has length } \leq \frac{1}{2^n} \right\}$$

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If  $f : x \longrightarrow y$  is any morphism, to compute its length you complete to a triangle  $x \xrightarrow{f} y \longrightarrow z$ , and then

$$\text{Length}(f) = \inf \left\{ \frac{1}{2^n} \mid z \in B_n \right\}$$

## Definition (good metric)

Let  $\mathcal{S}$  be a triangulated category. A **good metric** on  $\mathcal{S}$  is a sequence of full subcategories  $\{B_n, n \in \mathbb{Z}\}$ , containing 0 and satisfying

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- 2  $B_{n+1} \subset B_n$ .

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## Example

Suppose  $\mathcal{S}$  has a t-structure. Then  $B_n = \mathcal{S}^{\leq -n}$  works.

## Theorem (1)

Let  $\mathcal{S}$  be a category with a metric. Some slides ago we defined categories

$$\mathfrak{G}(\mathcal{S}) \subset \mathfrak{L}(\mathcal{S}) .$$

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Let  $S$  be a *triangulated* category with a *metric*. Some slides ago we defined categories

$$\mathfrak{S}(S) \subset \mathfrak{L}(S).$$

Now define the distinguished triangles in  $\mathfrak{S}(S)$  to be the colimits in  $\mathfrak{S}(S) \subset \text{Mod-}S$  of Cauchy sequences of distinguished triangles in  $S$ .

## Theorem (1)

Let  $\mathcal{S}$  be a *triangulated* category with a *good* metric. Some slides ago we defined categories

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Now define the distinguished triangles in  $\mathfrak{G}(\mathcal{S})$  to be the colimits in  $\mathfrak{G}(\mathcal{S}) \subset \text{Mod-}\mathcal{S}$  of Cauchy sequences of distinguished triangles in  $\mathcal{S}$ .

*With this definition of distinguished triangles, the category  $\mathfrak{G}(\mathcal{S})$  is triangulated.*

## Example (the six triangulated categories to keep in mind)

Let  $R$  be an associative ring.

- 1  $\mathbf{D}(R\text{-Mod})$  has for objects all cochain complexes of  $R$ -modules, no conditions.
- 2  $\mathbf{D}^b(R\text{-proj})$  is the derived category of bounded complexes of finitely generated, projective  $R$ -modules.
- 3 Suppose the ring  $R$  is coherent. Then  $\mathbf{D}^b(R\text{-mod})$  is the bounded derived category of finitely presented  $R$ -modules.

## Example (the six triangulated categories to keep in mind, continued)

Let  $X$  be a quasicompact, quasiseparated scheme, and let  $Z \subset X$  be a closed subset with quasicompact complement.

- ④  $\mathbf{D}_{\text{qc},Z}(X)$  will be our shorthand for  $\mathbf{D}_{\text{qc},Z}(\mathcal{O}_X\text{-Mod})$ . The objects are the complexes of  $\mathcal{O}_X$ -modules, and the conditions are that (1) the cohomology must be quasicoherent, and (2) the restriction to  $X - Z$  is acyclic.
- ⑤ The objects of  $\mathbf{D}_Z^{\text{perf}}(X) \subset \mathbf{D}_{\text{qc},Z}(X)$  are the perfect complexes. A complex  $F \in \mathbf{D}_{\text{qc}}(X)$  is *perfect* if there exists an open cover  $X = \cup_i U_i$  such that, for each  $U_i$ , the restriction map  $u_i^* : \mathbf{D}_{\text{qc}}(X) \rightarrow \mathbf{D}_{\text{qc}}(U_i)$  takes  $F$  to an object  $u_i^*(F)$  isomorphic in  $\mathbf{D}_{\text{qc}}(U_i)$  to a bounded complex of vector bundles.
- ⑥ Assume  $X$  is noetherian. The objects of  $\mathbf{D}_{\text{coh},Z}^b(X) \subset \mathbf{D}_{\text{qc},Z}(X)$  are the complexes with coherent cohomology which vanishes in all but finitely many degrees.

## Theorem (1, continued)

Now let  $R$  be an associative ring. Then the category  $\mathbf{D}^b(R\text{-proj})$  admits an **intrinsic metric** [up to equivalence], so that

$$\mathfrak{S}[\mathbf{D}^b(R\text{-proj})] = \mathbf{D}^b(R\text{-mod}).$$

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If we further assume that  $R$  is *left-coherent* then there is on  $[\mathbf{D}^b(R\text{-mod})]^{\text{op}}$  an **intrinsic metric** [again up to equivalence], such that

$$\mathfrak{S}\left([\mathbf{D}^b(R\text{-mod})]^{\text{op}}\right) = [\mathbf{D}^b(R\text{-proj})]^{\text{op}}.$$

## Theorem (1, continued)

Let  $X$  be a quasicompact, quasiseparated scheme, and let  $Z \subset X$  be a closed subset with quasicompact complement. There is an **intrinsic equivalence class of metrics** on  $\mathbf{D}_Z^{\text{perf}}(X)$  for which

$$\mathfrak{G}[\mathbf{D}_Z^{\text{perf}}(X)] = \mathbf{D}_{\text{coh},Z}^b(X) .$$

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Now assume that  $X$  is a **coherent scheme**. Then the category  $[\mathbf{D}_{\text{coh},Z}^b(X)]^{\text{op}}$  can be given an **intrinsic metric** [up to equivalence], so that

$$\mathfrak{S}\left([\mathbf{D}_{\text{coh},Z}^b(X)]^{\text{op}}\right) = [\mathbf{D}_Z^{\text{perf}}(X)]^{\text{op}}.$$

# Intrinsic equivalence classes of metrics

Recall Rickard's 1989 theorem:

## Theorem

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**The theorem makes no mention of metrics.** Until now, what we have honestly explained is that

- 1 The category  $\mathbf{D}^b(R\text{-proj})$  can be given **some metric**  $\{B_i, i \in \mathbb{N}\}$  for which

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The category  $\mathbf{D}^b(R\text{-mod})^{\text{op}}$  can be given **some metric**  $\{\tilde{B}_i, i \in \mathbb{N}\}$

$$\mathfrak{S}\left([\mathbf{D}^b(R\text{-mod})]^{\text{op}}\right) = [\mathbf{D}^b(R\text{-proj})]^{\text{op}}.$$

## The article



Amnon Neeman, *The categories  $\mathcal{T}^c$  and  $\mathcal{T}_c^b$  determine each other*,  
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provides recipes, constructing metrics on triangulated categories  $\mathcal{S}$ .

## The article



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<https://arxiv.org/abs/1806.06471>.

provides **recipes**, **constructing metrics** on triangulated categories  $\mathcal{S}$ .

### Example

Let  $\mathcal{S}$  be a triangulated category, and let  $G \in \mathcal{S}$  be an object. For any integer  $n > 0$ , the full subcategory  $\langle G \rangle^{(-\infty, -n]}$  is the smallest  $\mathcal{L} \subset \mathcal{S}$  subject to

$$G[i] \in \mathcal{L} \quad \forall i \geq n, \quad \mathcal{L} * \mathcal{L} \subset \mathcal{L}, \quad \text{smd}(\mathcal{L}) \subset \mathcal{L}.$$

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With this notation, the recipe

$$B_n(G) = \langle G \rangle^{(-\infty, -n]}$$

**provides a good metric** on the category  $\mathcal{S}$ , for any choice of object  $G \in \mathcal{S}$ .

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**provides a good metric** on the category  $\mathcal{S}$ , for any choice of object  $G \in \mathcal{S}$ .  
And **if we stipulate that  $G \in \mathcal{S}$  is a classical generator**, then the metrics  $\{B_n(G), n \in \mathbb{N}\}$  **are all equivalent**.

## Theorem

The category  $\mathbf{D}(R\text{-proj})$  has a classical generator. And with the metric being any member of the equivalence class  $\{B_n(G), n \in \mathbb{N}\}$  in the example above, we obtain

$$\mathfrak{S}[\mathbf{D}^b(R\text{-proj})] = \mathbf{D}^b(R\text{-mod}).$$

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## Corollary

Any autoequivalence of the category  $\mathbf{D}(R\text{-proj})$  takes a metric  $\{B_n(G), n \in \mathbb{N}\}$  to an equivalent one  $\{B_n(H), n \in \mathbb{N}\}$ , and hence induces an autoequivalence on

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The category  $\mathbf{D}^b(R\text{-mod})$  does not in general have a classical generator. But there is a (more complicated) recipe, providing an equivalence class of metrics that works.

**Summarizing:** in the article



Amnon Neeman, *The categories  $\mathcal{T}^c$  and  $\mathcal{T}_c^b$  determine each other*,  
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And in the recent work which I will discuss today, **we reverse this**. The question we want to ask is: **what hypotheses do we have to impose on the metric, for the passage from  $\mathcal{S}$  to  $\mathfrak{S}(\mathcal{S})^{\text{op}}$  to be an involution?**

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Note that this **really is a question about the metric**. For any triangulated category  $\mathcal{S}$ , we can define a good metric  $\{B_n, n \in \mathbb{N}\}$  by the formula  $B_n = \mathcal{S}$ . And it is easy to show that, for this metric,  $\mathfrak{S}(\mathcal{S}) = \{0\}$ . Hence this metric will only be involutive if  $\mathcal{S} = \{0\}$ .

# The metrics on $\mathcal{L}(\mathcal{S})$ and $\mathcal{G}(\mathcal{S})$

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- 2 Consider the diagram below.

$$\begin{array}{ccc} & & \mathcal{L}_n \\ & & \downarrow \\ \mathcal{G}(\mathcal{S}) \subset & \longrightarrow & \mathcal{L}(\mathcal{S}) \end{array}$$

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- 2 Consider the diagram below. In the category  $\mathcal{G}(\mathcal{S})$ , we define full subcategories  $\mathcal{N}_n \subset \mathcal{G}(\mathcal{S})$  to be the pullback

$$\begin{array}{ccc} \mathcal{N}_n & \hookrightarrow & \mathcal{L}_n \\ \downarrow & & \downarrow \\ \mathcal{G}(\mathcal{S}) & \hookrightarrow & \mathcal{L}(\mathcal{S}) \end{array}$$

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It can be proved that  $\{\mathcal{N}_i, i \in \mathbb{N}\}$  is a good metric on  $\mathcal{G}(\mathcal{S})$ .

The category  $\mathfrak{S}(\mathcal{S})^{\text{op}}$  has a good metric  $\mathcal{N}_i^{\text{op}}$ , and we can perform on  $\mathfrak{S}(\mathcal{S})^{\text{op}}$  the constructions

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But all of this data came from  $\mathcal{S}$  and its metric, and there is a Yoneda map

$$\widehat{Y} : (\text{Mod-}\mathcal{S})^{\text{op}} \longrightarrow \text{Mod-}\mathfrak{S}(\mathcal{S})^{\text{op}}$$

$\{\mathcal{M}_i, i \in \mathbb{N}\}$

$\cap$   
 $\mathcal{S}$

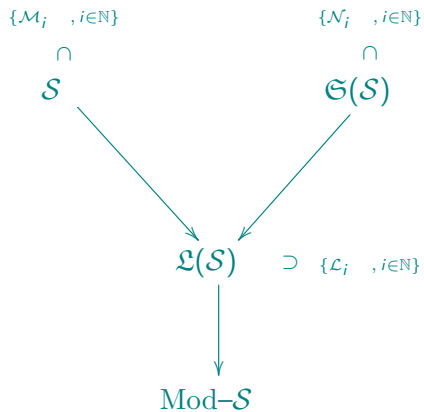
$\{\mathcal{M}_i, i \in \mathbb{N}\}$

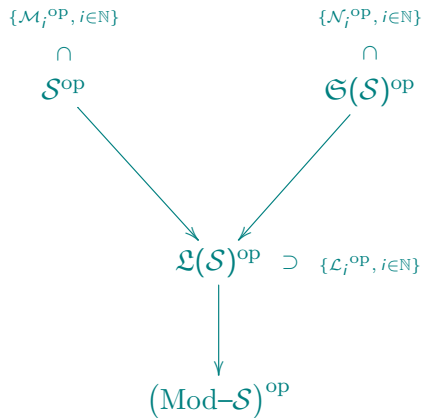
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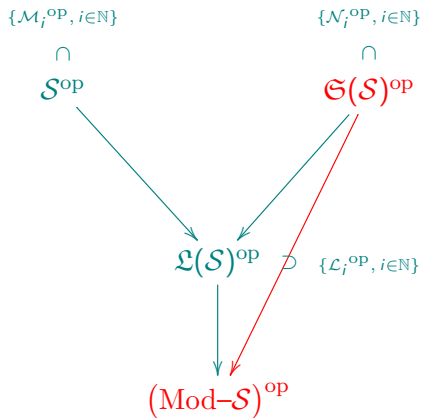
$\gamma$

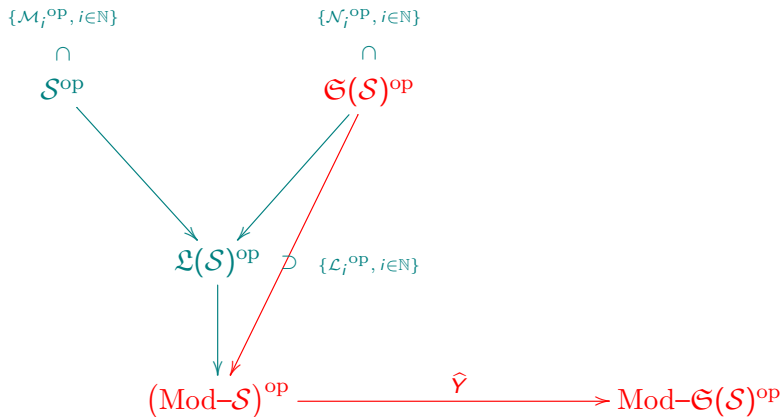
$\text{Mod-}\mathcal{S}$

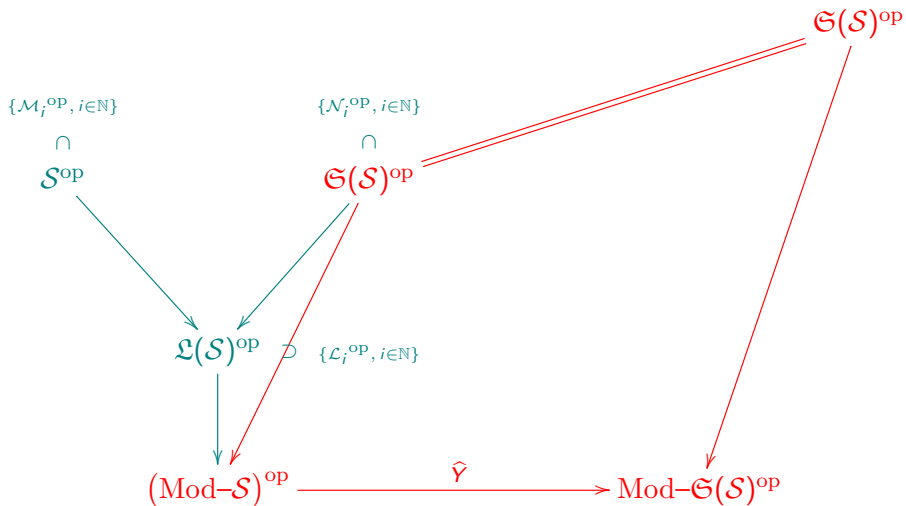
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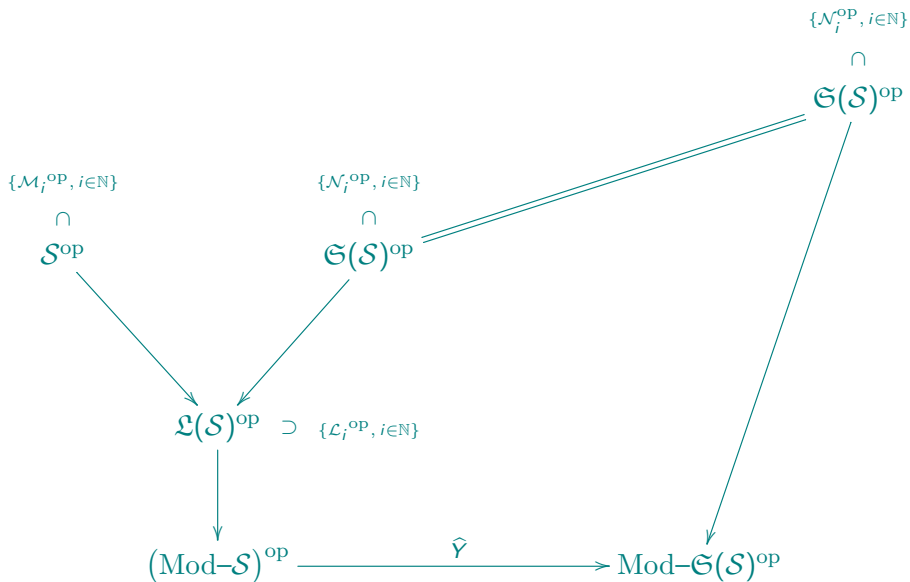


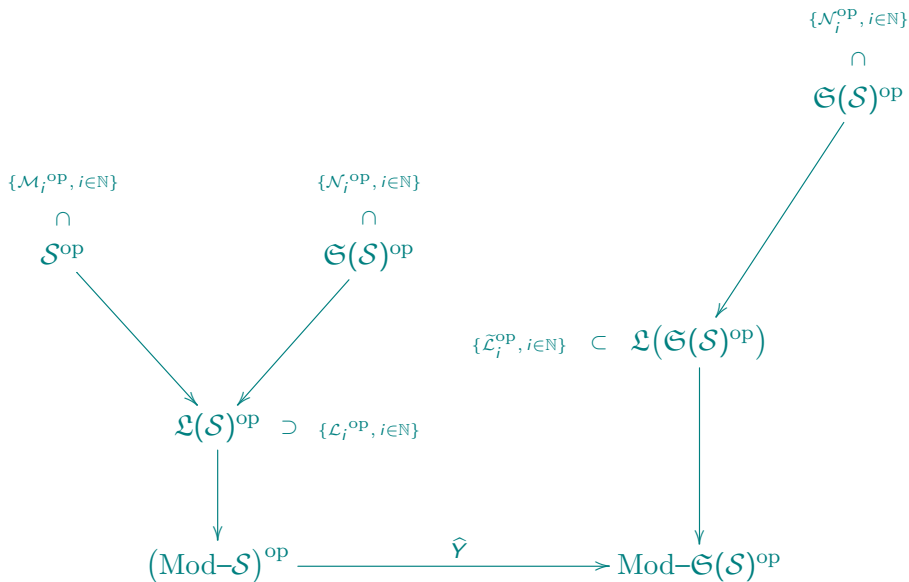


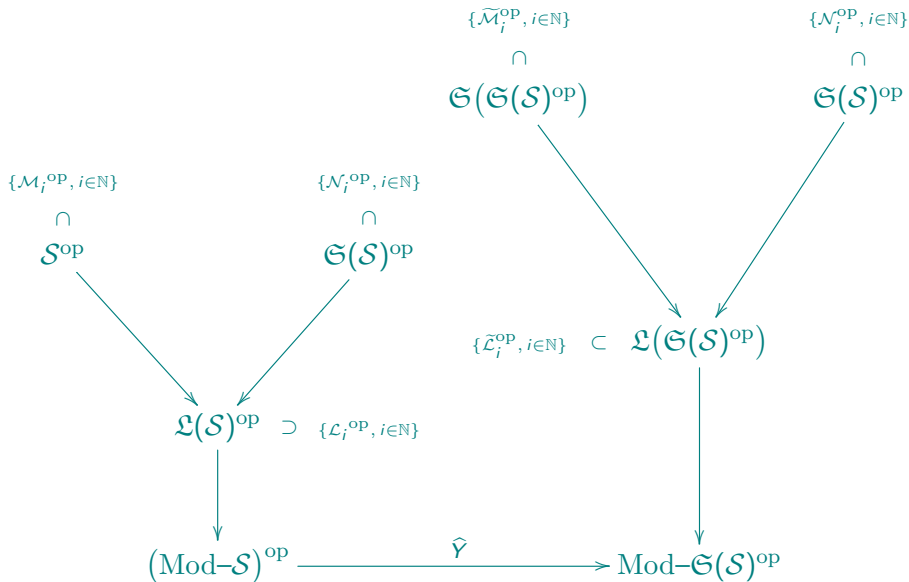


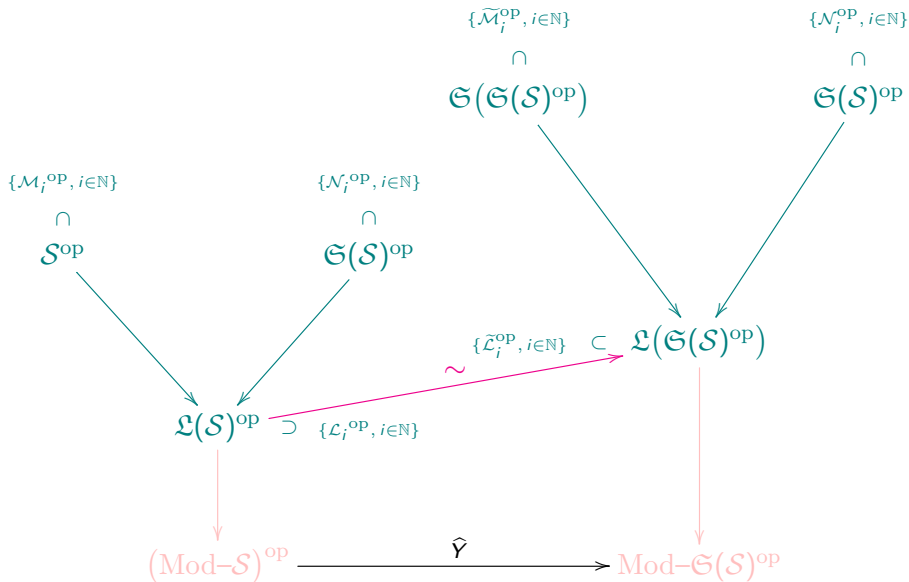












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We would like to view  $F : A \rightarrow B$  as a **short morphism** if  $C \in \mathcal{L}_n$  for  $n \gg 0$ .

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The category  $\mathcal{L}(\mathcal{S})$  isn't triangulated, and hence a morphism

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$$A = \underset{\rightarrow}{\text{colim}} Y(a'_*) , \quad B = \underset{\rightarrow}{\text{colim}} Y(b'_*) .$$

## Lemma

Let  $\mathcal{S}$  be a triangulated category, and let  $\{\mathcal{M}_i, i \in \mathbb{N}\}$  be a good metric on  $\mathcal{S}$ . Suppose we are given a **length data**, meaning a morphism  $F : A \rightarrow B$  in the category  $\mathcal{L}(\mathcal{S})$ , as well as a pair of Cauchy sequences  $a'_*$  and  $b'_*$  in the category  $\mathcal{S}$ , satisfying the requirements.

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Consider the set  $\Lambda$ , of **all possible** Cauchy sequence of exact triangles  $a_* \rightarrow b_* \rightarrow c_* \rightarrow a_*[1]$  in the category  $\mathcal{S}$ , with

- 1  $a_*$  is a subsequence of  $a'_*$ , and  $b_*$  is a subsequence of  $b'_*$ .
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If one of the Cauchy sequences in  $\Lambda$  is such that  $c_k \in \mathcal{M}_n$  for all  $k \gg 0$ , **the same is true for all of them.**

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- 3 The length data given by (i) and (ii) above is such that, for any Cauchy sequence of exact triangles  $a_* \longrightarrow b_* \longrightarrow c_* \longrightarrow a_*[1]$  in the category  $\mathcal{S}$ , belonging to the set  $\mathbf{\Lambda}$  of the previous slide, we have  $c_k \in \mathcal{M}_n$  for all  $k \gg 0$ .

## Remark

Informally: we could consider the category  $\mathbf{LD}(\mathcal{S})$ , where the **objects** are pairs  $(A, a'_*)$ , with

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With this definition, **type- $n$**  morphisms should be viewed as morphisms in  $\mathbf{LD}(\mathcal{S})$  of length  $\leq 2^{-n}$ , and this defines a Lawvere metric on  $\mathbf{LD}(\mathcal{S})$ .

## Definition

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In the notation  $(F, d_*)$ , the Cauchy sequences  $F$  is taken to be the constant sequence  $F \xrightarrow{\text{id}} F \xrightarrow{\text{id}} F \xrightarrow{\text{id}} \dots$

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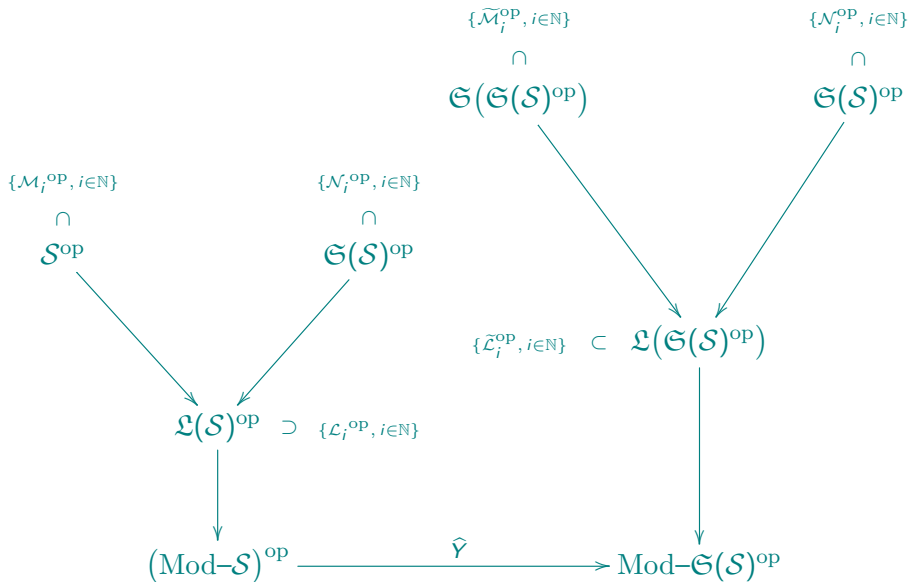
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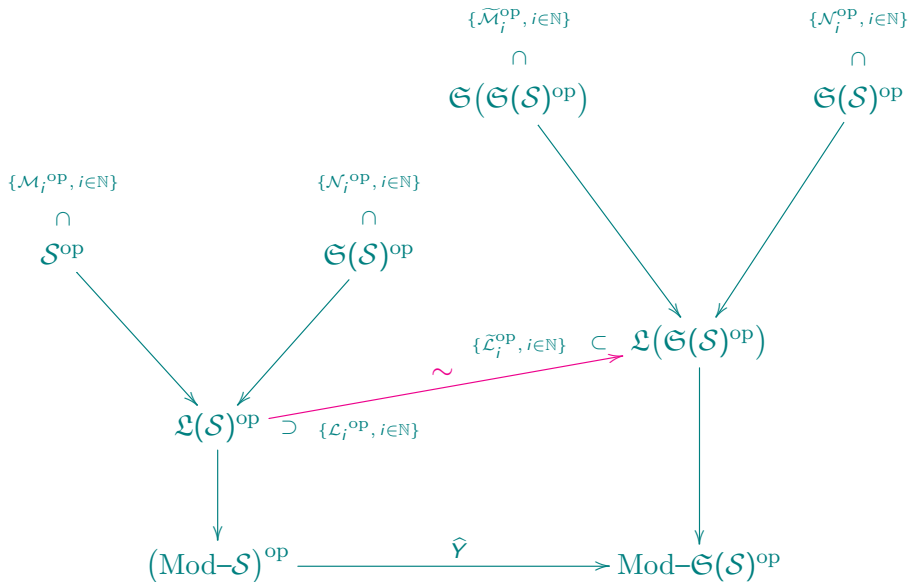
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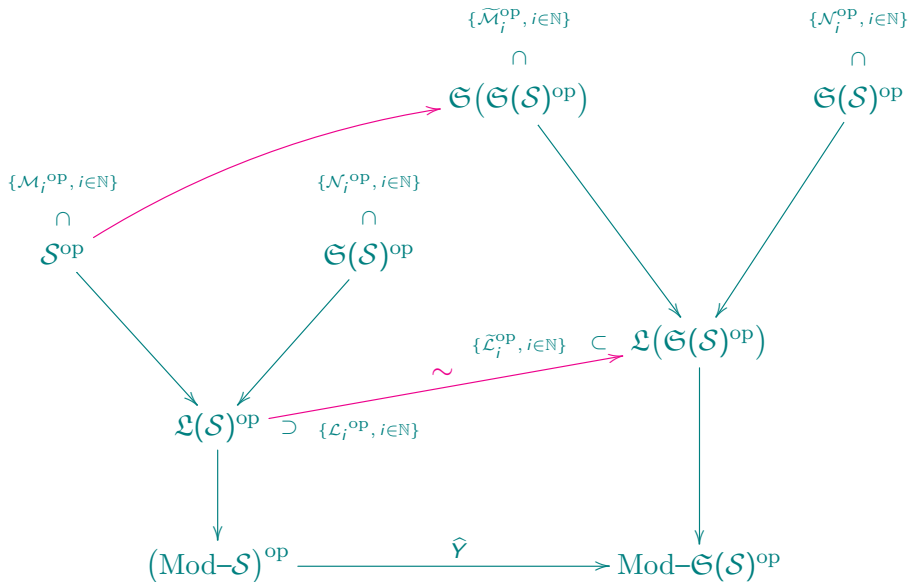
## Corollary

Let  $\mathcal{S}$  be a triangulated category, let  $\{\mathcal{M}_i, i \in \mathbb{N}\}$  be an **excellent** metric on  $\mathcal{S}$ .

Then the induced metric  $\{\mathcal{N}_i^{\text{op}}, i \in \mathbb{N}\}$  on the triangulated category  $\mathcal{S}(\mathcal{S})^{\text{op}}$  is **also excellent**.







$$\begin{array}{ccc}
 & & \{\widetilde{\mathcal{M}}_i^{\text{op}}, i \in \mathbb{N}\} \\
 & & \cap \\
 & & \mathfrak{S}(\mathfrak{S}(\mathcal{S})^{\text{op}}) \\
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## Example (old)

Let  $\mathcal{T}$  be a **coherent, weakly approximable** triangulated category, let  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  be a  **$t$ -structure** in the **preferred equivalence class**, and let the subcategories  $\mathcal{T}^c$  and  $\mathcal{T}_c^b$  be given the usual meaning.

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If  $\mathcal{T} = \mathbf{D}(R\text{-Mod})$  and the  $t$ -structure is the standard one, then  $\mathcal{T}^c = \mathbf{D}^b(R\text{-proj})$  and  $\mathcal{T}_c^b = \mathbf{D}^b(R\text{-mod})$ .

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$$\begin{aligned} \mathfrak{S}\left(\mathbf{D}^b(R\text{-proj})\right) &= \mathbf{D}^b(R\text{-mod}), \\ \mathfrak{S}\left(\mathbf{D}^b(R\text{-mod})^{\text{op}}\right) &= \mathbf{D}^b(R\text{-proj})^{\text{op}}. \end{aligned}$$

## Example (new)

Let  $R$  be a ring, let  $\mathcal{S} = \mathbf{D}^b(R\text{-Proj})$ , and let  $\mathcal{M}_i = \mathbf{D}^b(R\text{-Proj})^{\leq -i}$ . Then the metric  $\{\mathcal{M}_i, i \in \mathbb{N}\}$  is excellent on the triangulated category  $\mathcal{S}$ .

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It can be computed that  $\mathfrak{S}(\mathcal{S}) = \mathbf{D}^b(R\text{-Mod})$ , and that the metric  $\{\mathcal{N}_i, i \in \mathbb{N}\}$  on  $\mathfrak{S}(\mathcal{S})$  is given by the formula  $\mathcal{N}_i = \mathbf{D}^b(R\text{-Mod})^{\leq -i}$ .

## Example (new, in gorgeous generality)

More generally: let  $\mathcal{T}$  a weakly approximable triangulated category, and let  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  be a  $t$ -structure in the preferred equivalence class.

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## Example (old and new, schemes)

Let  $X$  be a quasicompact, quasiseparated scheme, and let  $\mathcal{T}$  be either one of the the pair of triangulated categories below

$$\mathbf{D}_{\text{coh}}^b(X) \subset \mathbf{D}_{\text{qc}}^b(X) .$$

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For  $X$  arbitrary, the category


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seems new, although it is easy enough to describe explicitly.



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# Thank you!