Elements of Topological M-Theory

(with R. Dijkgraaf, S. Gukov, C. Vafa)

Andrew Neitzke

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Preface

The topological string on a Calabi-Yau threefold X is (loosely speaking) an "integrable spine" of the Type II string theory on $X \times \mathbb{R}^{3,1}$. Calabi-Yau spaces are the natural target space because they preserve some supersymmetry.

The full Type II theory in 10 dimensions is known to develop an 11-dimensional Poincare invariance at strong coupling — leading to the conjecture that there is an M-theory in 11 dimensions whose low energy limit is 11-dimensional supergravity.

Could something similar happen for the topological string — could the Calabi-Yau space X grow an extra dimension? The natural target spaces in this case would be G_2 holonomy manifolds. But what is the appropriate low energy action?

Outline

Stable forms and Hitchin's functionals

Relation to the topological string

Topological M-theory?

Geometric structures and forms

Hitchin introduced new action functionals for which the critical points are geometric structures on a manifold X — e.g. symplectic structure, complex structure, G_2 holonomy metric.

The construction is based on the idea that geometric structures are often characterized by the existence of particular differential forms on X — e.g. presymplectic form ω, holomorphic volume form Ω, associative 3-form Φ — obeying some integrability conditions — e.g. $d\omega = 0$, $d\Omega = 0$, $d\Phi = d * \Phi = 0$.

Stable forms

A form ω on an n-dimensional manifold X can give rise to a geometric structure because it defines a reduction of the group $GL(n, \mathbb{R})$ of coordinate changes (structure group of TX) to the subgroup that preserves ω.

In order to get the same structure at every point of X, independent of small perturbations of ω, want ω to be nondegenerate and generic in an appropriate sense. What does this mean for a general p-form?

Stable 2-forms

If $p = 2$, we know how to define a nondegenerate form: it is $\omega = M^{ij} dx_i \wedge dx_j$ with $\det M \neq 0$.

A nondegenerate real 2-form in dimension $n = 2m$ can always locally be written

$$\omega = e_1 \wedge f_1 + \cdots + e_m \wedge f_m,$$

for some choice of basis $\{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ for T^*X, varying over X ("vielbein"). So ω defines a presymplectic structure: reduces $GL(2m, \mathbb{R}) \to Sp(2m, \mathbb{R})$ at each point.

If $d\omega = 0$, then there exist local coordinates $(p_1, \ldots, p_m, q_1, \ldots, q_m)$ such that

$$\omega = dp_1 \wedge dq_1 + \cdots + dp_m \wedge dq_m.$$

Then ω defines a symplectic structure.

Stable forms

Another way of expressing the statement that a 2-form ω is nondegenerate is to say that any small perturbation $\omega \to \omega + \delta\omega$ can be undone by a local $GL(n, \mathbb{R})$ transformation. In this sense ω is stable.

This formulation can be generalized to other p-forms: we say $\omega \in \Omega^p(X, \mathbb{R})$ is stable if it lies in an open orbit of the local $GL(n, \mathbb{R})$ action, i.e. if any small perturbation can be undone by a local $GL(n, \mathbb{R})$ action.

So e.g. there are no stable 0-forms; any 1-form that is everywhere nonvanishing is stable; for 2-forms stability is equivalent to nondegeneracy (when n is even!)

Stable 3-forms

What about $p = 3$? The dimension of $\wedge^3(\mathbb{R}^n)$ grows like n^3, but the dimension of $GL(n, \mathbb{R})$ grows like $n^2 \Rightarrow$ for large enough n, there cannot be stable 3-forms!

In large enough dimensions, every 3-form is different.

Stable 3-forms in dimension 6

But some exceptional examples exist — e.g. $n = 6$.

$$\dim \wedge^3(\mathbb{R}^6) = 20$$

$$\dim GL(6, \mathbb{R}) = 36$$

So consider a stable real 3-form ρ in dimension 6. The stabilizer of ρ inside $GL(6, \mathbb{R})$ has real dimension $36 - 20 = 16$; in fact it is either $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$ or $SL(3, \mathbb{C}) \cup SL(3, \mathbb{C})$.

We're interested in the case of $SL(3, \mathbb{C}) \cup SL(3, \mathbb{C})$.

Stable 3-forms in dimension 6

If ρ has stabilizer $SL(3, \mathbb{C}) \cup SL(3, \mathbb{C})$ it can be written locally in the form

$$\rho = \frac{1}{2} \left(\zeta_1 \wedge \zeta_2 \wedge \zeta_3 + \bar{\zeta}_1 \wedge \bar{\zeta}_2 \wedge \bar{\zeta}_3 \right)$$

where $\zeta_1 = e_1 + ie_2$, $\zeta_2 = e_3 + ie_4$, $\zeta_3 = e_5 + ie_6$, and the e_i are a basis for T^*X, varying over X. The ζ_i determine an almost complex structure on X. If we are lucky, there exist local complex coordinates (z_1, z_2, z_3) such that $\zeta_i = dz_i$; in that case we say the almost complex structure is integrable, i.e. it is an honest complex structure.

Stable 3-forms in $d = 6$

Given the stable $\rho \in \Omega^3(X, \mathbb{R})$

$$\rho = \frac{1}{2}\left(\zeta_1 \wedge \zeta_2 \wedge \zeta_3 + \bar{\zeta}_1 \wedge \bar{\zeta}_2 \wedge \bar{\zeta}_3\right)$$

we can define another real 3-form

$$\hat{\rho}(\rho) = \frac{i}{2}\left(\zeta_1 \wedge \zeta_2 \wedge \zeta_3 - \bar{\zeta}_1 \wedge \bar{\zeta}_2 \wedge \bar{\zeta}_3\right)$$

This $\hat{\rho}$ is algebraically determined by ρ. Then $\Omega = \rho + i\hat{\rho}$ is a holomorphic 3-form in the almost complex structure determined by ρ.

The complex structure is integrable just if $d\Omega = 0$, i.e. $d\rho = 0$, $d\hat{\rho}(\rho) = 0$.

Hitchin's holomorphic volume functional

The integrability condition $d\rho = 0$, $d\hat{\rho}(\rho) = 0$ can be obtained by extremization of a volume functional:

$$V_H(\rho) = \frac{1}{2}\int_X \hat{\rho}(\rho) \wedge \rho = \frac{-i}{4}\int \Omega \wedge \overline{\Omega}.$$

Here ρ varies within a cohomology class, $[\rho] \in H^3(X, \mathbb{R})$ — i.e. $\rho = \rho_0 + d\beta$ for some fixed closed ρ_0. So $d\rho = 0$ of course; and the effect of variation of β is

$$\delta V_H(\rho) = \int_X \hat{\rho}(\rho) \wedge d(\delta\beta),$$

so $\delta V_H(\rho) = 0$ for all $\delta\beta \Rightarrow d\hat{\rho}(\rho) = 0$.

Intermezzo

One can write $V_H(\rho)$ more explicitly in terms of ρ:

$$V_H(\rho) = \frac{1}{2}\int d^6x \sqrt{\rho_{a_1a_2a_3}\rho_{a_4a_5a_6}\rho_{a_7a_8a_9}\rho_{a_{10}a_{11}a_{12}}\epsilon^{a_2a_3a_4a_5a_6a_7}\epsilon^{a_8a_9a_{10}a_{11}a_{12}a_1}}$$

Hitchin's holomorphic volume functional

So extremization of V_H, with a fixed $[\rho] \in H^3(X, \mathbb{R})$, leads to integrable complex structures on X, equipped with holomorphic 3-forms Ω, such that the real parts of the periods are fixed by $[\mathrm{Re}\ \Omega] = [\rho]$. (Almost "Calabi-Yau structures" on X, except that we didn't say X was Kähler. No Ricci flat metrics here!)

Complex geometry emerges from real 3-forms! A peculiarity of $d = 6$.

Hitchin vs. the B model

So altogether we have

$$V_H(\rho) = \frac{-i}{4} \int \Omega \wedge \overline{\Omega},$$

the action of a "2-form abelian gauge theory" in 6 dimensions, for which the classical solutions are roughly Calabi-Yau structures. (A stripped-down gravity theory.)

We already know a theory in 6 dimensions with these classical solutions — the B model topological string, or "Kodaira-Spencer gravity." So could V_H be a target space action for the B model?

Define formally the partition function,

$$Z_H([\rho]) = \int_{\rho \in [\rho]} D\rho \exp\left(V_H(\rho)\right).$$

This is a real function of $[\rho] \in H^3(X, \mathbb{R})$. We want to compare it to the B model partition function.

The B model and background dependence

The B model partition function is naively a holomorphic function of the complex moduli of X, $Z_B(t)$. But more precisely, it has a background dependence: depends on choice of base-point $\Omega_0 \in H^3(X, \mathbb{R})$, so it should be written $Z_B^{\Omega_0}(t)$. Here t parameterizes tangent vectors to the extended Teichmuller space (complex structures together with a choice of holomorphic 3-form): $t \in H^{3,0}(X_{\Omega_0}, \mathbb{C}) \oplus H^{2,1}(X_{\Omega_0}, \mathbb{C})$. [Bershadsky-Cecotti-Ooguri-Vafa]

The various $Z_B^{\Omega_0}$ are related by a holomorphic anomaly equation which gives the parallel transport from one Ω_0 to another Ω_0'. This equation has an elegant interpretation. [Witten]

Background dependence as the wavefunction property

Consider $H^3(X, \mathbb{R})$ as a symplectic vector space; then we can quantize it. (Think of $\mathbb{R}^2$ with $\omega = dp \wedge dq$.) The Hilbert space consists of functions ψ which depend on "half the coordinates," e.g. functions on a Lagrangian subspace (choice of polarization). Different polarizations are related by Fourier-like transforms.

So we can have real polarizations (like $\psi(q)$ or $\psi(p) = \int dq\, e^{ipq} \psi(q)$), obtained by splitting $H_3(X, \mathbb{Z})$ into "A and B cycles" — symplectic marking, $H_3(X, \mathbb{Z}) = H_3(X, \mathbb{Z})_A \oplus H_3(X, \mathbb{Z})_B$.

Can also have holomorphic polarizations (like $\psi(q + ip)$ or $\psi(q + \tau p)$) obtained by splitting $H^3(X, \mathbb{C})$, e.g. Hodge splitting $H^3(X, \mathbb{C}) = \left(H^{3,0} \oplus H^{2,1}\right) \oplus \left(H^{0,3} \oplus H^{1,2}\right)$.

The various $Z_B^{\Omega_0}$ are expressions of the same wavefunction in different polarizations of $H^3(X, \mathbb{R})$ — polarization given by the Hodge splitting determined by Ω_0.

Hitchin vs. the B model

So we've seen that Z_B depends on only half the coordinates of $H^3(X, \mathbb{R})$ (Lagrangian subspace) and requires a choice of polarization. These features are visible already classically (genus zero).

On the other hand, Z_H is a function on all of $H^3(X, \mathbb{R})$, and doesn't seem to require any choice.

So we can't say $Z_H = Z_B$. Instead, propose that $Z_H = Z_B \otimes \overline{Z_B}$, or more precisely, Z_H is the Wigner function associated to Z_B: this is the phase-space density,

$$(Z_B \otimes \overline{Z_B})([\rho]) = \int d\Phi \; e^{-\langle \Phi, \rho_B \rangle} \; |Z_B(\rho_A + i\Phi)|^2.$$

Here we wrote Z_B in the real polarization determined by a choice of A and B cycles (symplectic marking): $\rho_A = \rho|_{H_3(X,\mathbb{R})_A}$, $\rho_B = \rho|_{H_3(X,\mathbb{R})_B}$.

Hitchin vs. the B model

The relation $Z_H = Z_B \otimes \overline{Z_B}$ holds at least classically — i.e. leading order asymptotics in large $[\rho]$ expansion. In that limit the saddle-point evaluation of

$$\int d\Phi \; e^{-\langle \Phi, \rho_B \rangle} \; |Z_B(\rho_A + i\Phi)|^2$$

indeed gives

$$(Z_B \otimes \overline{Z_B})([\rho]) \sim e^{i \int \Omega \wedge \overline{\Omega}},$$

where Ω is the complex structure with $[\mathrm{Re}\ \Omega] = [\rho]$.
This agrees with the saddle-point evaluation of $Z_H([\rho])$, almost by definition.

Hitchin vs. the B model

At one-loop the situation is subtler: careful BV quantization shows that, in order to agree with the known one-loop $Z_B \otimes \overline{Z_B}$, one needs to replace Z_H by an extended Z_H. [Pestun-Witten]

The extended Z_H includes fields describing variations of generalized complex structures. [Hitchin]

Hitchin B model and black holes

The Wigner function of the B model (which we have now identified as $Z_H([\rho]))$ has appeared recently in another context: it was conjectured that $Z_H([\rho])$ computes the number of states of a black hole.

Motivation from the attractor mechanism: suppose we consider Type IIB superstring on $X \times \mathbb{R}^{3,1}$. Then we can construct a charged black hole by wrapping a D3-brane on a 3-cycle $Q \in H_3(X, \mathbb{Z})$. The complex moduli of X near the horizon then get fixed to an Ω satisfying $[\mathrm{Re}\ \Omega] = Q^*$.

This is exactly the Ω that Hitchin's gauge theory constructs if we fix the class $[\rho] = Q^*$. And $Z_H([\rho])$ is exactly the number of states of the black hole! [Ooguri-Strominger-Vafa]

So Z_H reformulates the B model in a way naturally adapted to the counting of black hole states.

Hitchin's symplectic volume functional

What about the A model? This has to do with variations of symplectic structures.

Hitchin introduced a functional which produces at its critical points symplectic structures in $d = 6$.

A stable 4-form σ in $d = 6$ may be written $\sigma = \frac{1}{2} k \wedge k$. Then k gives a presymplectic structure. Define

$$V_S(\sigma) = \frac{1}{6} \int k \wedge k \wedge k = \frac{1}{3} \int k \wedge \sigma.$$

Varying $V_S(\sigma)$ with $[\sigma] \in H^4(X, \mathbb{R})$ fixed, i.e. $\sigma = \sigma_0 + d\gamma$ for some closed σ_0, get

$$\delta V_S = \frac{1}{2} \int k \wedge d\delta\gamma,$$

so $\delta V_S = 0 \Rightarrow dk = 0$, i.e. k defines a symplectic structure.

So V_S has the same classical solutions as the A model.

Hitchin A model and black holes

Another attractor mechanism: consider M-theory on $X \times \mathbb{R}^{4,1}$. Then we can construct a charged black hole by wrapping an M2-brane on a 2-cycle $Q \in H_2(X, \mathbb{Z})$. The Kähler moduli of X near the horizon then get fixed to an k satisfying $\frac{1}{2}[k \wedge k] = Q^*$.

This is exactly the k that Hitchin's gauge theory constructs if we fix the class $[\sigma] = Q^*$. So the weird fact that V_S involves the 4-form $k \wedge k$ instead of the 2-form k gets naturally related to the fact that we want to fix a 2-cycle charge.

The leading number of states of this black hole is $\sim \exp \int k \wedge k \wedge k$ at large k, which agrees with the classical value of the Hitchin A model partition function,

$$Z_S([\sigma]) = \int_{\sigma \in [\sigma]} D\sigma \exp \left(V_S(\sigma) \right).$$

Hitchin A model and black holes

It was known before that one can extract counts of BPS states of wrapped M2-branes in five dimensions from the perturbative A model. [Gopakumar-Vafa]

Here we are finding that the Hitchin version of the A model is organized differently — its partition function seems to be directly counting the number of states.

Hitchin and topological strings

So Hitchin's functionals V_H and V_S in 6 dimensions seem to give reformulations of the target space dynamics of the B model and A model topological string theories, naturally adapted to the problem of counting black holes.

We only argued for this classically; it remains to be seen how much V_H and V_S can capture about the quantum theories.

These reformulations may be of interest in their own right. They are also naturally related to Hitchin's functional in 7 dimensions.

Stable 3-forms in dimension 7

Another exceptional example — $n = 7$.

$$\dim \wedge^3(\mathbb{R}^7) = 35$$

$$\dim GL(7, \mathbb{R}) = 49$$

So consider a stable real 3-form Φ in dimension 7. The stabilizer of Φ inside $GL(7, \mathbb{R})$ has dimension $49 - 35 = 14$; in one open subset it is the compact form of G_2.

Stable 3-forms in dimension 7 and G_2 structures

If $\Phi \in \Omega^3(Y, \mathbb{R})$ is stable of the appropriate sort, it determines a "G_2 structure" on Y (reduction of the structure group to G_2).

Concretely, Φ can be written in the form

$$\Phi = \sum_{i,j,k=1}^{7} \Psi_{ijk} e_i \wedge e_j \wedge e_k,$$

where Ψ_{ijk} are the structure constants of the imaginary octonions, and the e_i are a basis for T^*Y, varying over Y. G_2 occurs as the automorphism group of the imaginary octonions.

We can construct a metric from Φ, namely

$$g_\Phi = \sum_{i=1}^{7} e_i \otimes e_i.$$

This metric has G_2 holonomy just if $d\Phi = 0$, $d *_\Phi \Phi = 0$.

Hitchin's G_2 volume functional

The integrability condition $d\Phi = 0$, $d *_\Phi \Phi = 0$ can – again – be obtained by extremization of the volume functional:

$$V_7(\Phi) = \int_Y \Phi \wedge *_\Phi \Phi.$$

Again Φ varies within a cohomology class, $[\Phi] \in H^3(Y, \mathbb{R})$ — i.e. $\Phi = \Phi_0 + d\Gamma$ for some fixed closed Φ_0. So $d\Phi = 0$ of course; and the effect of variation of Γ is

$$\delta V_7(\Phi) = \frac{7}{3} \int_Y *_\Phi \Phi \wedge d(\delta\Gamma),$$

so $\delta V_7(\Phi) = 0 \Rightarrow d *_\Phi \Phi = 0$.

Hitchin's G_2 volume functional and topological M-theory

So

$$V_7(\Phi) = \int_Y \Phi \wedge *_\Phi \Phi$$

generates G_2 holonomy metrics at its critical points. In this sense it is a candidate action for topological M-theory.

By analogy with physical M-theory, one would expect that topological M-theory on $X \times S^1$ should be related to topological string theory on X. Indeed, this V_7 can be connected to the A and B models.

Hitchin's G_2 volume functional and topological M-theory

Namely, letting t be the coordinate along S^1 in $Y = X \times S^1$, and splitting

$$\Phi = kdt + \rho$$

one gets (assuming the constraints $k \wedge \rho = 0$, $2V_S(\sigma) - V_H(\rho) = 0$)

$$*_\Phi \Phi = \hat{\rho} dt + \sigma$$

which implies

$$V_7(\Phi) = 2V_H(\rho) + 3V_S(\sigma)$$

So topological M-theory seems to reduce to the sum of the A and B models at least in this formal sense.

Hamiltonian reduction

Another perspective on this relation: consider canonical quantization of topological M-theory on $X \times \mathbb{R}$. The phase space is then $\Omega^3_{exact}(X, \mathbb{R}) \times \Omega^4_{exact}(X, \mathbb{R})$ (omitting zero modes), with the symplectic pairing determined by

$$\langle \delta\rho, \delta\sigma \rangle = \int \delta\rho \wedge \frac{1}{d} \delta\sigma \tag{1}$$

and the Hamiltonian

$$H = 2V_S(\sigma) - V_H(\rho). \tag{2}$$

The conditions $k \wedge \rho = 0$, $2V_S(\sigma) - V_H(\rho) = 0$ then (should) show up as the diffeomorphism and Hamiltonian constraints, as usual for quantization of a diffeomorphism invariant theory.

The A model and B model appear as conjugate degrees of freedom!

Open questions

There are many open questions:

- How does topological M-theory embed into the physical string/M-theory (what quantities does it compute?)
- Can it be used to give a nonperturbative definition of the topological string? The splitting between A and B models is not covariant in 7 dimensions — does this mean the A and B models have to be mixed together nonperturbatively? How should the 6-dimensional couplings g_A, g_B be identified?
- What is the meaning of the fact that the A and B model appear as conjugate variables? (S-duality?) [Nekrasov-Ooguri-Vafa]
- Should the theory be augmented to one which describes generalized G_2 structures? [Pestun-Witten, Hitchin, Witt]
- Is there a lift to 8 dimensions? [Anguelova-de Medeiros-Sinkovics]