# Infinite Dimensional Symmetries

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**Abstract**

Lecture notes on Infinite Dimensional Symmetries given by Hermann Nicolai at the 15th Saalburg summer school in Wolfersdorf, Thuringia, in September 2009. Typed by Oliver Schlotterer.

# Contents



# 1 Motivation and introduction

Symmetry simplifies the description of phenomena in all areas of theoretical physics. Physicists usually distinguish the following types of symmetries:

- spacetime symmetry (Lorentz-, Poincare group, etc.)
- internal symmetry (isospin, flavour, etc.)
- duality symmetry (e.g. $\vec{E} + i\vec{B} \mapsto e^{i\alpha}(\vec{E} + i\vec{B})$ in Maxwell theory)

All of these examples have finite dimension. However, already the Standard Model and general relativity involve symmetries of infinite dimension such as the following two:

- Yang Mills $SU(3) \otimes SU(2) \otimes U(1)$ symmetry (local gauge transformations)
- general coordinate transformations $x^\mu \mapsto x^\mu + \xi^\mu(x)$ with vector fields $\xi^\mu(x)$

These lectures are mainly devoted to a different class of infinite dimensional symmetries, namely to symmetries of *Kac Moody* type. The latter for instance occur after dimensional reduction of Einstein's theory or of supergravity:

- $D = 4 \;\mapsto\; D = 3$ : Ehlers symmetry $SL_2(\mathbb{R})$ ($\sim$ 1957)
- $D = 4 \;\mapsto\; D = 2$ : Geroch symmetry $\widehat{SL_2(\mathbb{R})} \otimes$ central extension ($\sim$ 1972)
- $D = 4 \;\mapsto\; D = 1$ : even bigger symmetry of Kac Moody type

A similar approach to the dimensional reduction of gravity was taken by *Belinskii, Khalatnikof & Lifshitz* (BKL) around 1972 by examining light cones close to a singular surface in spacetime: spacelike separated points causally decouple and spatial derivatives become hierarchically small $\partial_x \ll \partial_t$. Therefore the dynamical evolution is asymptotically governed by ordinary differential equations in time. This scenario will be the topic of a later chapter.

## 2 Kac Moody Lie algebras

Lie algebras of Kac Moody type generalize the concepts of angular momentum in quantum mechanics which is described by a triplet of non-commuting operators subject to $\mathfrak{sl}_2 \equiv \mathfrak{su}_2 \otimes \mathbb{C}$ commutation relations. Let us adapt the following notation for the $\mathfrak{sl}_2$ generators $J^i \equiv \sigma^i$:

$$h \;\equiv\; J^3 \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \;, \qquad e \;\equiv\; J^+ \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \;, \qquad f \;\equiv\; J^- \;=\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.1}$$

Then the commutation relations $[J^i, J^j] = 2i\varepsilon^{ijk} J^k$ can be rewritten as

$$\big[h\,,\,e\big] \;=\; 2\,e\;, \qquad \big[h\,,\,f\big] \;=\; -\,2\,f\;, \qquad \big[e\,,\,f\big] \;=\; h\;. \tag{2.2}$$

Kac Moody Lie algebras are generated by several basic $sl_2$ building blocks. Consider $r$ sets of $sl_2$ generators:

$$\Big\{e_i,\ f_i,\ h_i:\ i = 1, 2, ..., r\Big\} \tag{2.3}$$

The information about their concatenation (i.e. the failure of $\{e_i,\ f_i,\ h_i\}$ to commute with another triplet $\{e_j,\ f_j,\ h_j\},\ i \neq j$) can be encoded in the so-called *Cartan matrix* $A_{ij},\ i, j = 1, 2, ..., r$ for which we state the following basic properties:

$$A_{ij} \in \mathbb{Z}\;, \qquad A_{ii} \;=\; 2\;, \qquad A_{ij} \;\leq\; 0 \quad \forall\, i \neq j \tag{2.4}$$

If the symmetry condition $A_{ij} = A_{ji}$ is additionally imposed, then the composed algebra will be simply-laced.

The Cartan matrix is defined in terms of the *Chevalley Serre* presentation of the algebra: First of all choose a *Cartan subalgebra* of mutually commuting generators $\{h_i : i = 1, ..., r\}$,

$$\big[h_i\,,\,h_j\big] \;=\; 0\;, \tag{2.5}$$

then the corresponding ladder operators $\{e_i, f_i : i = 1, ..., r\}$ obey commutation relations

$$\big[h_i\,,\,e_j\big] \;=\; A_{ij}\,e_j\;, \qquad \big[h_i\,,\,f_j\big] \;=\; -\,A_{ij}\,f_j\;, \qquad \big[e_i\,,\,f_j\big] \;=\; \delta_{ij}\,h_j \tag{2.6}$$

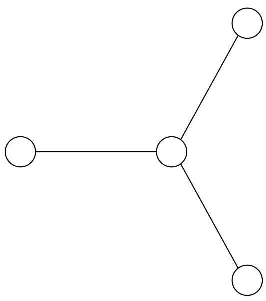
and the *Serre relations*

$$\begin{aligned} 0 \;&=\; \Big[\underbrace{e_i\,,\,\Big[e_i\,,\,...\big[e_i}_{1-A_{ij}\ \text{fold commutator}}\,,\,e_j\big]\,...\Big]\Big] \;=\; \Big[\underbrace{f_i\,,\,\Big[f_i\,,\,...\big[f_i}_{1-A_{ij}\ \text{fold commutator}}\,,\,f_j\big]\,...\Big]\Big] \\ &=\; \big(\mathrm{ad}\,e_i\big)^{1-A_{ij}}(e_j) \;=\; \big(\mathrm{ad}\,f_i\big)^{1-A_{ij}}(f_j)\;. \end{aligned} \tag{2.7}$$

Given a $r \times r$ Cartan matrix $A$, the corresponding *Kac Moody algebra* $\mathfrak{g}(A)$ is defined as a free Lie algebra generated by the $\{e_i, f_i, h_i\}$ modulo Serre relations.

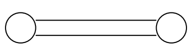
An alternative representation of the $r \times r$ Cartan matrix is the Dynkin diagram. One possible convention is the following:

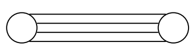
- draw $r$ nodes
- connect nodes $i, j$ by $A_{ij} \cdot A_{ji}$ lines



$$A_{ij} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

For infinite dimensional algebras with large offdiagonal entries, however, it might be preferable to conventionally draw $|A_{ij}|$ lines between nodes $i$ and $j$. Consider e.g. the Dynkin diagram associated with $A_{ij} = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ in both conventions:





## 2.1 Examples of Kac Moody algebras

**Example 1:** $A_2 \equiv \mathfrak{sl}_3$ This Lie algebra is characterized by the Cartan matrix $A_{ij} = \left( \begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix} \right)$. Its rank $r = 2$ implies that there are six basic generators $h_1, h_2, e_1, e_2, f_1, f_2$ which might generate further algebra elements via commutators. From $[e_i, f_j]$, one can only get $h_{1,2}$, also $[h_i, e_j]$ and $[h_i, f_j]$ involves combinations of $e$'s and $f$'s respectively, so the only possibility to construct further $\mathfrak{g}(A)$ elements are the commutators $[e_1, e_2]$ and $[f_1, f_2]$.
Serre relations ensure that the formation of new generator terminates. In this case, $1 - A_{12} = 2$ implies

$$\Big[ e_1 \left[ e_1 \, , \, e_2 \right] \Big] \;\;=\;\; \Big[ f_1 \left[ f_1 \, , \, f_2 \right] \Big] \;\;=\;\; 0 \; . \tag{2.8}$$

There is a matrix representation for the eight basis elements:

$$\begin{array}{llllll} h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} , & & & & h_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \\ e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , & & e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} , & & \left[ e_1 \, , \, e_2 \right] = \begin{pmatrix} 0 & 0 & +1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ f_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} , & & f_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} , & & \left[ f_1 \, , \, f_2 \right] = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \end{array} \tag{2.9}$$

**Example 2:** $A_1^{(1)}$ Let us slightly modify the off-diagonal entries of the $\mathfrak{sl}_3$ Cartan matrix and examine $A_{ij} = \left( \begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix} \right)$. Now the Serre relations give

$$0 \;\;\neq\;\; \Big[ e_1 \left[ e_1 \, , \, e_2 \right] \Big] \; , \qquad 0 \;\;=\;\; \Big[ e_1 \Big[ e_1 \left[ e_1 \, , \, e_2 \right] \Big] \Big] \; , \tag{2.10}$$

but one can still generate new elements by $[e_2, \cdot]$ action on the nonzero double commutator. It turns out that the $A_1^{(1)}$ is infinite dimensional.

**Example 3:** $\mathcal{H}(3)$ The Cartan matrix $A_{ij} = \left( \begin{smallmatrix} 2 & -3 \\ -3 & 2 \end{smallmatrix} \right)$ of the Fibonacci Algebra $\mathcal{H}(3)$ implies even less restrictive Serre relations

$$0 \;\;\neq\;\; \Big[ e_1 \Big[ e_1 \left[ e_1 \, , \, e_2 \right] \Big] \Big] \; , \qquad 0 \;\;=\;\; \Big[ e_1 \Big[ e_1 \Big[ e_1 \left[ e_1 \, , \, e_2 \right] \Big] \Big] \Big] \; . \tag{2.11}$$

This innocent looking modification of (2.10) leads to an explosion of the (nonzero commutators of the) algebra, we will go into more detail in the later subsection 2.6. The same is true for any $A_{ij} = \left( \begin{smallmatrix} 2 & -m \\ -m & 2 \end{smallmatrix} \right)$ with $m \geq 3$.

We will see that the existence of zero or negative eigenvalues of the Cartan matrix implies infinite dimension for the corresponding algebra. Nice textbook about these cases are [1], [2].

## 2.2 Basic properties of Kac Moody algebras

Kac Moody algebras share a number of features with finite dimensional matrix algebras:

- triangular decomposition

  Any Kac Moody algebra can be decomposed as

$$\mathfrak{g}(A) \quad = \quad \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \tag{2.12}$$

  where $\mathfrak{h}$ denotes the Cartan subalgebra (CSA) and

$$\begin{aligned} \mathfrak{n}^- \quad &\equiv \quad \text{lower triangular part: multiple } \{f_i\} \text{ commutators} \\ \mathfrak{n}^+ \quad &\equiv \quad \text{upper triangular part: multiple } \{e_i\} \text{ commutators .} \end{aligned} \tag{2.13}$$

  This property enables the development of representation theory by means of heighest weight states $|\cdot\rangle$ (analogous to quantum mechanics) such that $\mathfrak{n}^+|\cdot\rangle = 0$.

- root space decomposition

  The CSA acts diagonally on $\mathfrak{n}^\pm$, the eigenvalues are called *roots* $\alpha$ and can be represented as $r$ component vectors. To each root $\alpha$, one can associate a subspace of $\mathfrak{g}(A)$

$$\mathfrak{g}_\alpha \quad := \quad \Big\{ x \in \mathfrak{g}(A) : \ \big[h \, , \, x\big] \quad = \quad \alpha(h) \cdot x \Big\} \ , \tag{2.14}$$

  i.e. technically speaking $\alpha \in \mathfrak{h}^*$. Setting $x$ equal to a basic Chevalley Serre generator, one arrives at the simple root vectors $e_i \leftrightarrow \alpha_i$ and $f_i \leftrightarrow -\alpha_i$, so a multiple commutator is associated with the linear combination

$$\Big[ e_{i_1} \, , \, ... \, \big[ e_{i_{k-1}} \, , \, e_{i_k} \big] \, ... \Big] \quad \leftrightarrow \quad \sum_{\nu=1}^{k} \alpha_{i_\nu} \ . \tag{2.15}$$

  Note that the CSA $\mathfrak{h}$ formally belong to the subspace of the zero root.

- invariant bilinear form

  We define a bilinear form $\langle \cdot | \cdot \rangle : \ \mathfrak{g}(A) \times \mathfrak{g}(A) \rightarrow \mathbb{C}$ by nonzero entries

$$\langle e_i \, | \, f_j \rangle \quad = \quad \delta_{ij} \ , \qquad \langle h_i \, | \, h_j \rangle \quad = \quad A_{ij} \tag{2.16}$$

  whereas $\langle e|e \rangle = \langle e|h \rangle = ... = 0$. In addition, we require $\langle \cdot | \cdot \rangle$ to satisfy the *invariance property*

$$\big\langle \big[x \, , \, y\big] \, | \, z \big\rangle \quad = \quad \big\langle x \, | \, \big[y \, , \, z\big] \big\rangle \qquad \forall \ x, y, z \in \mathfrak{g}(A) \tag{2.17}$$

which helps to handle multiple commutators $\langle [e_i, ...[e_j, e_k]...] | [f_i, ...[f_j, f_k]...] \rangle$ by means of the following algorithm:

- use the invariance property (2.17) to shuffle one of the $e$'s to the right hand side
- apply the Jacobi identity to obtain commutators of type $[e, f] \equiv h$
- evaluate the diagonal action $[f_i, h_j] = A_{ji} f_j$ of the Cartan generators

This can be further illustrated by an explicit example:

$$\begin{aligned}
\langle [e_i\,,\, e_j] \,|\, [f_i\,,\, f_j] \rangle &= \langle e_i \,|\, \Big[ e_j\,,\, [f_i\,,\, f_j] \Big] \rangle \\
&= \langle e_i \,|\, \Big[ f_i\,,\, \underbrace{[e_j\,,\, f_j]}_{=h_j} \Big] \rangle \;-\; \langle e_i \,|\, \Big[ f_j\,,\, \underbrace{[e_j\,,\, f_i]}_{=0} \Big] \rangle \\
&= \langle e_i \,|\, [f_i\,,\, h_j] \rangle \;=\; \langle e_i \,|\, A_{ji}\, f_j \rangle \;. \qquad (2.18)
\end{aligned}$$

For finite dimensional $\mathfrak{g}(A)$, there is the explicit realization $\langle x|y \rangle = \mathrm{Tr}\{xy\}$ for the bilinear, the invariance property follows from cyclicity of the trace.

## 2.3 The Cartan Weyl basis

Sometimes it is convenient to use a so-called *Cartan-Weyl basis* for $\mathfrak{g}(A)$. Using Greek indices $\mu, \nu$ to label the root components corresponding to an arbitrary basis $H_\mu$ in the CSA, with the usual summation convention and metric $G_{\mu\nu}$, we have $h_i := \alpha_i^\mu H_\mu$ and define $e_i = E(\alpha_i)$. Therefore,

$$[H_\mu\,,\, E(\alpha)] = \alpha_\mu\, E(\alpha) \qquad (2.19)$$

$$\langle H_\mu \,|\, H_\nu \rangle = G_{\mu\nu} \qquad (2.20)$$

$$[E(\alpha)\,,\, E(\beta)] = \begin{cases} c_{\alpha\beta}\, E(\alpha+\beta) & : \ \alpha+\beta \text{ root} \\ 0 & : \text{otherwise} \end{cases} , \qquad (2.21)$$

the latter identity (with unfixed prefactor $c_{\alpha\beta}$) following from the Jacobi identity applied to the double commutator $\big[H_\mu, [E(\alpha), E(\beta)]\big] = (\alpha_\mu + \beta_\mu)[E(\alpha), E(\beta)]$.
One can also define the scalar product in the root space $\mathfrak{h}^*$ via

$$\begin{aligned}
\langle \alpha_i \,|\, \alpha_j \rangle &= \alpha_i^\mu\, \alpha_j^\mu\, G_{\mu\nu} \;=\; \alpha_i^\mu\, \alpha_j^\mu\, \langle H_\mu \,|\, H_\nu \rangle \\
&= \langle h_i \,|\, h_j \rangle \;=\; A_{ij} \;. \qquad (2.22)
\end{aligned}$$

In finite dimensional $\mathfrak{g}(A)$ with positive definite $A$, one can take $G_{\mu\nu} = \delta_{\mu\nu}$.

**Example:** $\mathfrak{sl}_3 \equiv A_2$ This algebra is governed by Cartan matrix $A_{ij} = \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$. Its positive definiteness admits the $G_{\mu\nu} = \delta_{\mu\nu}$ metric, and an explicit realization of the simple roots solving $\alpha_i^\mu \alpha_j^\mu G_{\mu\nu} = A_{ij}$ is

$$\alpha_1 \ = \ \left(\sqrt{2}\,,\,0\right)\,, \qquad \alpha_2 \ = \ \left(-\frac{1}{\sqrt{2}}\,,\,\sqrt{\frac{3}{2}}\right)\,. \tag{2.23}$$

The root lattice is a priori fixed to lie in $\subset \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2$, and application of the Serre relations shows that there is an additional positive root $\alpha_1 + \alpha_2$ associated with $[e_1, e_2]$.
Using relations $h_1 = \sqrt{2}H_1$ and $h_2 = -\frac{1}{\sqrt{2}}H_1 + \sqrt{\frac{3}{2}}H_2$ between Chevalley Serre- and Cartan Weyl generators in the CSA, one can translate the matrix realization (2.9) of $h_{1,2}$ into

$$H_1 \ = \ \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \qquad H_2 \ = \ \begin{pmatrix} \frac{1}{\sqrt{6}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{6}} & 0 \\ 0 & 0 & -\sqrt{\frac{2}{3}} \end{pmatrix}\,. \tag{2.24}$$

## 2.4 Classification according to Cartan matrix signature

Kac Moody algebras can be neatly classified according to their Cartan matrix $A$ and in particular to its signature. A necessary and sufficient criterion for $\mathfrak{g}(A)$ to have finite dimension is $A > 0$, i.e. $A$ being positive definite. Consider the most general ansatz $A_{ij} = \left(\begin{smallmatrix} 2 & -a \\ -b & 2 \end{smallmatrix}\right)$ for rank $r = 2$, then

$$A \ > \ 0 \quad \Leftrightarrow \quad a \cdot b \ \leq \ 3\,. \tag{2.25}$$

The integer solutions are the following:

$$\begin{array}{rcl@{\qquad}rcl} \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right) & \leftrightarrow & \mathfrak{sl}_2 \oplus \mathfrak{sl}_2\,, & \left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right) & \leftrightarrow & \mathfrak{sl}_3 \equiv A_2 \\ \left(\begin{smallmatrix} 2 & -2 \\ -1 & 2 \end{smallmatrix}\right) & \leftrightarrow & B_2,\, C_2\,, & \left(\begin{smallmatrix} 2 & -3 \\ -1 & 2 \end{smallmatrix}\right) & \leftrightarrow & G_2 \end{array} \tag{2.26}$$

In 1968, Kac and Moody dropped the restriction to $A > 0$ and considered positive *semi*definite Cartan matrices $A \geq 0$ known as *affine algebras*. Clearly, they are infinite dimensional, and there are many applications in physics, e.g. to 2 dimensional QFT (in particular to CFTs), to integrable systems and to dimensionally reduced Einstein gravity (the Geroch group upon reducing $D = 4 \mapsto D = 2$). The rank two examples satisfy $a \cdot b = 4$, these are $\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 & -4 \\ -1 & 2 \end{smallmatrix}\right)$.
For indefinite $A$ (i.e. $A$ with at least one negative eigenvalue), on the other hand, very little is known, and it remains an outstanding problem to find a manageable representation for $\mathfrak{g}(A)$. In particular, there is not a single example of an indefinite KM algebra for which the root multiplicities, i.e. the number of Lie algebra elements associated with a given root, are known in closed form.

One might think about further generalization of possibly indefinite Kac Moody algebras, for instance super Lie algebras, Borcherd's *generalized Kac Moody algebras* (with $A_{ii} \neq 2$) and toric algebras (defined by algebra valued functions on $S^1 \times S^1$).

## 2.5 Realization of affine algebras: current algebras

Consider a finite dimensional Lie algebra $\bar{\mathfrak{g}}$ with generators $J^A$ subject to commutation relations

$$\left[J^A\,,\,J^B\right] \quad = \quad f^{AB}{}_C\,J^C\ . \tag{2.27}$$

Its affinization is defined by corresponding Laurent modes $\{J^A_m :\ m \in \mathbb{Z}\}$ and a *central charge* $c$, in particular by the generalized commutation relations

$$\left[J^A_m\,,\,J^B_n\right] \quad = \quad f^{AB}{}_C\,J^C_{m+n} \ + \ m\,c\,\delta_{m+n,0}\,\delta^{AB}\ . \tag{2.28}$$

One way of giving a meaning to the extra label $m \in \mathbb{Z}$ of the $J^A_m$ generators is to define $J^A \equiv J^A(\theta)$ on $S^1 = \{z \in \mathbb{C} :\ |z| = 1\}$ parametrized by $\theta \in [0, 2\pi)$, then

$$J^A(\theta) \quad = \quad \sum_{m\in\mathbb{Z}} e^{im\theta}\,J^A_m\ . \tag{2.29}$$

Using the representations $2\pi\delta(\theta) = \sum_{m\in\mathbb{Z}} e^{im\theta}$ and $-2\pi i\delta(\theta) = \sum_{m\in\mathbb{Z}} me^{im\theta}$ of the $\delta$ function, one can show the commutator to be equivalent to

$$\left[J^A(\theta_1)\,,\,J^B(\theta_2)\right] \quad = \quad 2\pi\,f^{AB}{}_C\,J^C(\theta_1)\,\delta(\theta_1-\theta_2) \ - \ 2\pi i\,c\,\delta^{AB}\,\delta'(\theta_1-\theta_2)\ . \tag{2.30}$$

This is what Schwinger found in 1950 upon analyzing two dimensional quantum electrodynamics. Correspondingly, the piece $\sim c\delta'(\theta_1-\theta_2)$ is known as *Schwinger term*.

The set of maps $S^1 \to \bar{\mathfrak{g}}$ into some finite dimensional Lie algebra $\bar{\mathfrak{g}}$ is often referred to as the *loop algebra* $\hat{\mathfrak{g}}$. Loop groups $\hat{G}$ are analogously defined as maps from $S^1$ into finite dimensional groups.

One can explicitly construct the central element $c$ by means of the eigenvector $n_i$ with eigenvalue zero of the Cartan matrix, i.e. $\sum_j A_{ij}n_j = 0$. It is easy to check that

$$c \quad := \quad \sum_{j=1}^{r} n_j\,h_j \tag{2.31}$$

commutes with any $\mathfrak{g}(A)$ element.

**Example:** $A_1^{(1)} \equiv \widehat{\mathfrak{sl}_2} \oplus \mathbb{C}$ The Cartan matrix $A_{ij} = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ implies the commutation relations

$$\begin{aligned} \big[h_1 \, , \, e_{1,2}\big] &= \pm 2 \, e_{1,2} & \big[h_1 \, , \, f_{1,2}\big] &= \mp 2 \, f_{1,2} & (2.32) \\ \big[h_2 \, , \, e_{1,2}\big] &= \mp 2 \, e_{1,2} & \big[h_2 \, , \, f_{1,2}\big] &= \pm 2 \, f_{1,2} \ . & (2.33) \end{aligned}$$

These can be reproduced from the current algebra

$$\begin{aligned} \big[J^3_m \, , \, J^\pm_n\big] &= \pm 2 \, J^\pm_{m+n} & (2.34) \\ \big[J^+_m \, , \, J^-_n\big] &= J^3_{m+n} \ + \ m \, c \, \delta_{m+n,0} & (2.35) \end{aligned}$$

by identifying

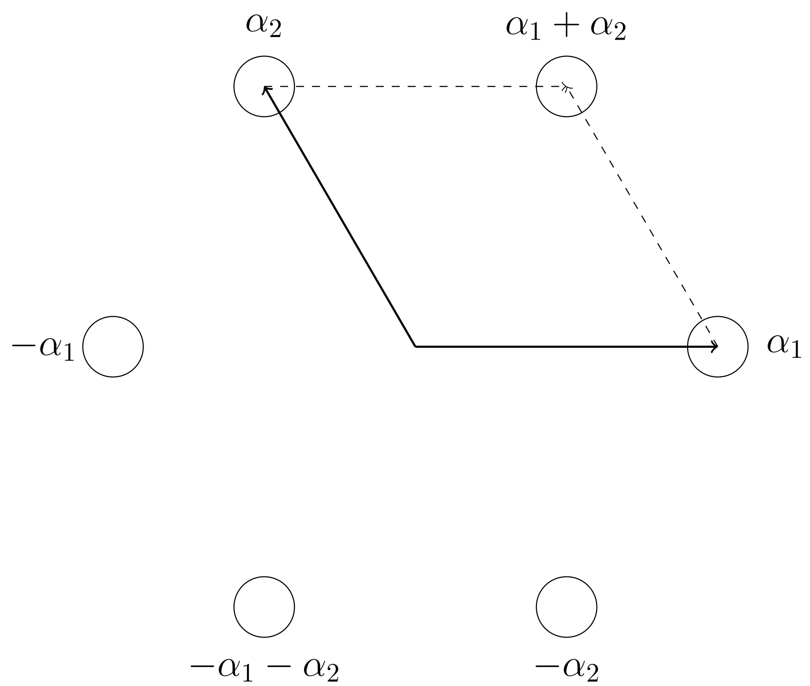
$$\begin{aligned} e_1 &\equiv J^+_0 \ , & e_2 &\equiv J^-_1 \ , & f_1 &\equiv J^-_0 \ , & f_2 &\equiv J^+_{-1} & (2.36) \\ h_1 &\equiv J^3_0 \ , & c &\equiv h_1 + h_2 \ . & & & & & (2.37) \end{aligned}$$

Further current modes $J^3_{\neq 0}$ and $J^+_{>0}, J^-_{<0}$ are obtained via commutators such as

$$\begin{aligned} \big[e_1 \, , \, e_2\big] &= + J^3_{+1} & \Big[e_1 \, , \, \big[e_1 \, , \, e_2\big]\Big] &= -2 \, J^+_{+1} & (2.38) \\ \big[f_1 \, , \, f_2\big] &= - J^3_{-1} & \Big[f_1 \, , \, \big[f_1 \, , \, f_2\big]\Big] &= -2 \, J^-_{-1} \ . & (2.39) \end{aligned}$$

## 2.6 Root systems

Recall the discussion of the algebra $\mathfrak{sl}_3$ in subsection (2.3). From the Cartan matrix and in particular from the Serre relations, we concluded the root lattice to be hexagonal with two simple roots $\alpha_{1,2}$:



Let us now focus on *simply-laced algebras* whose spacelike roots all have uniform length $\alpha^2 = 2$. In the finite dimensional sector, this is the case for the $A, D$- and $E$ algebras. Another way of expressing the limiting impact of the Serre relations on simply-laced algebras is

$$\forall \text{ roots } \alpha \text{ we have } \alpha^2 = 2 \text{ with respect to the metric } G_{\mu\nu} = \delta_{\mu\nu} \; . \tag{2.40}$$

However, for indefinite Cartan matrices $A_{ij} = \alpha_i^\mu \alpha_j^\nu G_{\mu\nu}$, the metric is of Lorentzian signature, e.g. $G_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, +1, ..., +1)$ in an appropriate basis. Then, the Serre relations are equivalent to $\alpha^2 \leq 2$, more precisely one has (assuming simply-laced algebras to exclude $0 < \alpha^2 < 2$)

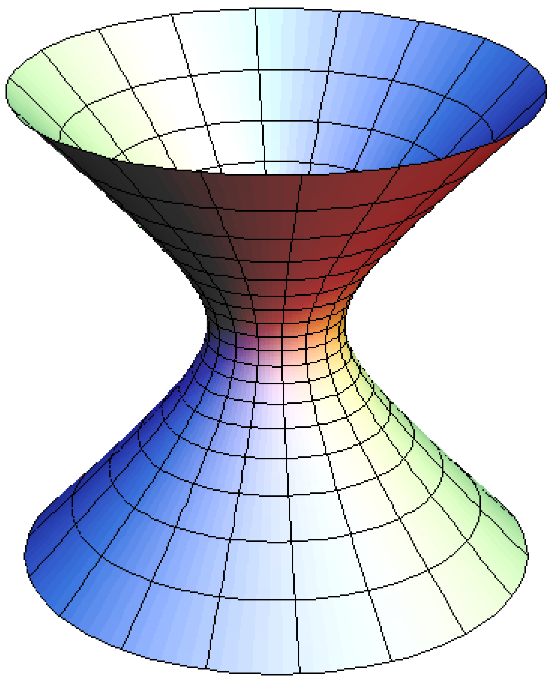


Figure 1: Roots with $\alpha^2 = 2$ lie on a hyperboloid (its axis pointing into the timelike direction)

- real roots $\alpha^2 = 2$ on the hyperboloid $-(\alpha^1)^2 + \sum_{j=2}^r (\alpha^j)^2 = 2$
- imaginary roots $\alpha^2 \leq 0$ (null or timelike)

In the affine case $A \geq 0$, all the roots $(\alpha, \delta)$ are either of *affine* type $\alpha^2 = 2$ or null $\delta^2 = 0$, whereas indefinite algebras certainly have timelike roots $\alpha^2 < 0$.

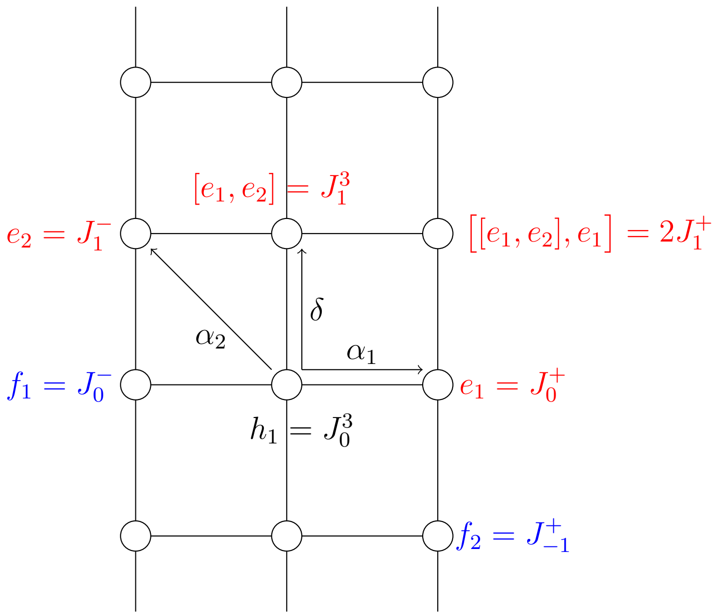
In string theory, one can assign masses $m^2 = -\alpha^2$ to roots $\alpha$, this gives the following dictionary:

$$\begin{aligned} \text{real root} &\quad\leftrightarrow\quad \text{tachyon} \\ \text{null root} &\quad\leftrightarrow\quad \text{photon} \\ \text{timelike root} &\quad\leftrightarrow\quad \text{massive Regge excitation} \end{aligned}$$

**Example 1:** $A_1^{(1)}$ The Cartan matrix $A_{ij} = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ obviously has a zero eigenvector $(1, 1)$ associated with the central element $c = h_1 + h_2$. The problem about $c$ is that all its commutators trivially vanish, so getting a two dimensional root lattice requires a *extension* of the Cartan subalgebra $\{J_0^3, c\} \oplus \{d\}$ by a further element $d$ to lift the degeneracy. The additional generator $d$ is defined by adjoint action

$$\left[d\, ,\, J_m^a\right] \;=\; m\, J_m^a\ . \tag{2.41}$$

In the following diagram, we will include the $d$ eigenvalue of the corresponding generator as a second component for the root. The simple roots read $\alpha_1 = \sqrt{2}\,(1, 0)$ and $\alpha_2 = \sqrt{2}\,(-1, 1)$ and reproduce the Cartan matrix entries $A_{ij} = \alpha_i^\mu \alpha_j^\nu G_{\mu\nu}$ in the degenerate metric $G_{\mu\nu} = \mathrm{diag}(1, 0)$. Hence, $\delta = \alpha_1 + \alpha_2$ is a null root!




The same ladder construction applies to any other affine algebra: Start with the root system of the zero modes $J_{n=0}^a$ of the corresponding current algebra, then the action of the ladder operators $J_{n\neq 0}^a$ gives rise to a copy of the zero modes' root system for each $d$ eigenvalue. In other words, each affine algebra can be constructed from a finite dimensional algebra whose root system is reproduced infinitely many times in a ladder-like structure.

We have already defined root spaces $\mathfrak{g}_\alpha$ by $x \in \mathfrak{g}_\alpha \Leftrightarrow [h, x] = \alpha(h)x$, let us now introduce *multiplicities* of roots $\alpha$ as

$$\mathrm{mult}(\alpha) \;:=\; \dim \mathfrak{g}_\alpha\ . \tag{2.42}$$

Real roots satisfy $\mathrm{mult}(\alpha) = 1$ whereas multiplicities of imaginary roots increase exponentially with $-\alpha^2$. Raising operators $E(\alpha)$ associated with a root $\alpha$ of nontrivial multiplicity require an extra label $E_s(\alpha) :\ s = 1, ..., \mathrm{mult}(\alpha)$.

**Example 2:** $\mathcal{H}(3)$ We already mentioned the example of the Fibonacci algebra $\mathcal{H}(3)$ algebra in section 2. For hyperbolic indefinite algebras, the multiplicities of raising operators increase in an uncontrolled way with the length of the associated root. In fact, there is not a single example of a indefinite hyperbolic algebra for which the mult$(\alpha)$ are known in closed form. The following diagram is a nice example for the proliferation of multiplicities:

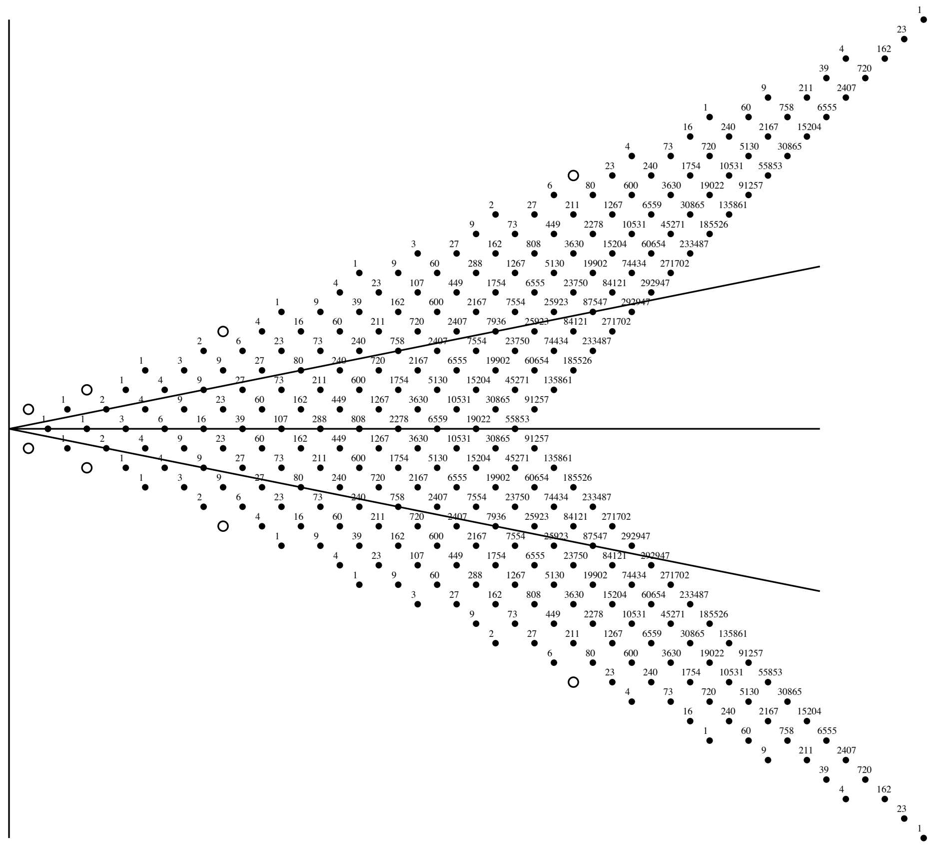


Figure 2: Elements of $\mathcal{H}(3)$ generated by multiple $e_1, e_2$ generators together with their multiplicities; figure taken from [3]

The numbers next to the $\mathcal{H}(3)$ elements denote the root multiplicities, computed by means of the *Peterson recursion formula* [1], [2]. Circles mark the real roots on the hyperbola $\alpha^2 = 2$; all the other roots are situated within the light cone. There are no null roots, i.e. $\mathcal{H}(3)$ does not have any affine subalgebra. (In this case the algebra is referred to as *strictly hyperbolic*).

## 2.7 Weyl group

An important tool for classifying root systems is the *Weyl group* $W(A)$. This discrete group is generated by reflections on the hyperplanes perpendicular to simple roots $\alpha_i : \; i = 1, ..., r$. An elementary reflection of a vector $v$ in root space is implemented as

$$w_i(v) \quad := \quad v \; - \; \langle \alpha_i \,|\, v \rangle \, \alpha_i \tag{2.43}$$

using $\alpha_i^2 = 2$ (which also enables to check $w_i^2 = 1$).

Each element of $W(A)$ can be written as a product $w_1 w_2 ... w_n$ of elementary reflections (2.43). The products with an even number $n \in 2\mathbb{N}$ of operations forms a subgroup $W^+(A)$, the *even Weyl group*.

If $A$ is finite, then $W(A) \subset O(r)$ with $|W(A)| < \infty$. For Lorentzian $A$, on the other hand, $W(A) \in O(1, r-1)$ and $|W(A)| = \infty$. Affine $A$ can be embedded into some Lorentzian $\tilde{A}$ such that $W(A)$ is the subgroup of $W(\tilde{A})$ leaving the null root $\delta$ invariant (e.g. $W(A_n) = S_{n+1}$).

Let us finally define the *fundamental Weyl chamber* $\mathcal{C}_W$ to be the wedge about the origin of root space delimited by the reflecting hyperplanes of $W(A)$ such that

$$\mathcal{C}_W \quad = \quad \Big\{ v \in \mathbb{R}^r : \; \langle v \,|\alpha_i \rangle \geq 0 \; \forall \; i = 1, ..., r \Big\} \; . \tag{2.44}$$

## 2.8 Hyperbolic Kac Moody algebras

In order to define the class of hyperbolic KM algebras, we first of all need the notion of a *regular subalgebra* which is the result of deleting a node from the original algebra's Dynkin diagram. An indefinite KM algebra $\mathfrak{g}(A)$ is called *hyperbolic* if all its regular subalgebras are finite or affine.

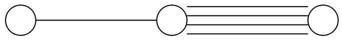
Hyperbolic algebras have been classified in [4]. Taking all examples into account, one arrives at the important lemma

$$\mathfrak{g}(A) \text{ hyperbolic} \quad \Rightarrow \quad \text{rank}(A) \leq 10 \; . \tag{2.45}$$

As an example for a hyperbolic KM algebra, consider the algebra $\mathcal{F} \equiv AE_3$ examined by Feingold & Frenkel in 1983 [5]. It is also relevant in the context of $D = 4$ gravity [9], see the later section 4. It is governed by the Cartan matrix

$$A \quad = \quad \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix} \tag{2.46}$$

or equivalently by the following Dynkin diagram



If one erases the right node, the surviving $\left(\begin{smallmatrix} 2 & -1 \\ -1 & 2 \end{smallmatrix}\right)$ block corresponds to an $\mathfrak{sl}_3$ acting on the spatial dreibein of the $D=4$ gravity model. On the other hand, after removing the left node, a $\left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$ block survives associated with the Geroch algebra $\widehat{\mathfrak{sl}_2}$.

The Weyl group $W(A)$ of the $\mathcal{F}$ algebra has the remarkable property that the even subgroup $W^+(A)$ coincides with the modular group

$$W(A) \ = \ PGL_2(\mathbb{Z}) \ \supset \ W^+(A) \ = \ PSL_2(\mathbb{Z}) \ . \tag{2.47}$$

# 3 Supergravity and $E_{10}$

In this subsection, we will demonstrate that one can extract the (bosonic) particle content of prominent supergravity theories from clever decompositions of the hyperbolic $E_{10}$ algebra. A nice reference about this topic is [6]. Let us first of all introduce the algebraic tool for this task.

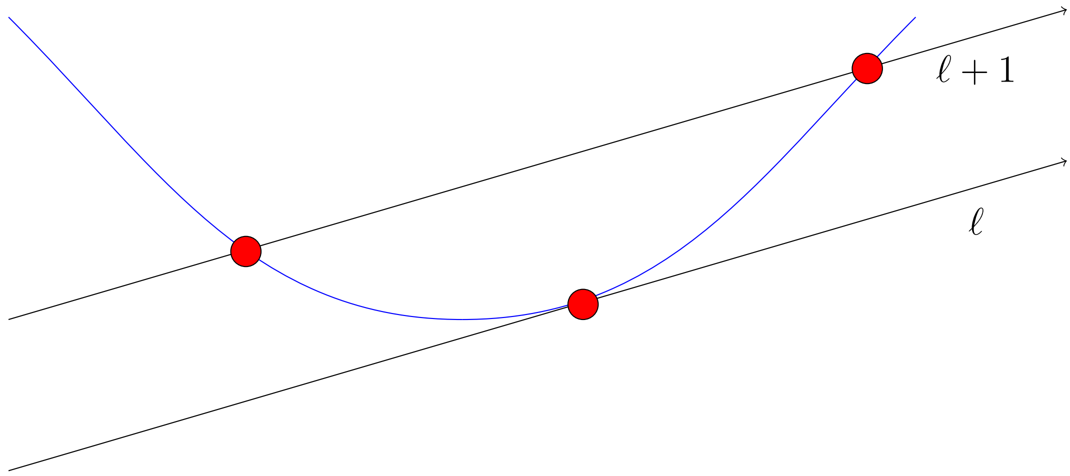
## 3.1 Level decomposition

The aim of *level decomposition* is the analysis of an unknown algebra in terms of well-understood subalgebras. Pick a distinguished node corresponding to simple root $\alpha_0$, then any root can be decomposed as

$$\alpha \ = \ \ell \cdot \alpha_0 \ + \ \sum_{j \neq 0} m_j \, \alpha_j \ , \tag{3.1}$$

the integer coefficient $\ell$ of the distinguished root $\alpha_0$ is referred to as *level*. The remaining $j$ sum in (3.1) is a root of the chosen subalgebra.

The geometric idea behind this procedure is a slicing of the root space, each slice being associated with a fixed value of the level $\ell$. Of course, there are various ways of slicing, and it is most convenient to pick a real root perpendicular to a spacelike hypersurface. Slicings by lightlike or timelike hyperplanes would produce gradings with respect to affine or indefinite KM subalgebras, with each slice containing infinitely many roots.




The algebra then obtains a graded structure

$$\mathfrak{g}(A) \quad = \quad \bigoplus_{\ell\in\mathbb{Z}} \mathfrak{g}^{(\ell)} \tag{3.2}$$

where the levels are additive under the commutator,

$$\left[\mathfrak{g}^{(\ell)}\,,\,\mathfrak{g}^{(\ell')}\right] \quad \subset \quad \mathfrak{g}^{(\ell+\ell')}\ . \tag{3.3}$$

## 3.2 $A_9$ decomposition of $E_{10}$

Now we apply this prescription of level decomposition to the hyperbolic $E_{10}$ algebra with root labelling as shown in the Dynkin diagram below. Singling out the branched node $\alpha_0$ leaves an

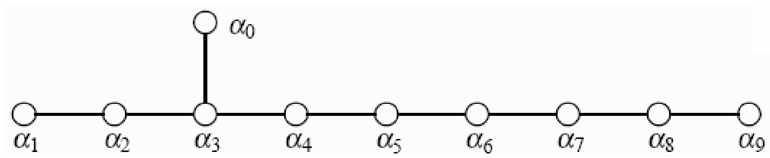



Figure 3: Dynkin diagram of $E_{10}$

$A_9 = sl_{10}$ subalgebra. Note the level $\ell(\alpha)$ stays invariant under the adjoint action of the $sl_{10}$ subalgebra. Hence, the set of $E_{10}$ elements corresponding to a given level $\ell$ can be decomposed into a (finite) number of irreducible representations of $\mathfrak{sl}_{10}$. We will next give an overview over the first levels $\ell = 0,1,2,3$:

- level $\ell = 0$

  At zero level, no $e_0, f_0$ generators are involved, the only contribution from the $\alpha_0$ node is the Cartan generator $h_0 \sim \mathbb{1}$, therefore

$$\mathfrak{g}^{(0)} \quad = \quad \mathfrak{sl}_{10} \oplus \mathrm{span}\{h_0\} \quad = \quad \mathfrak{gl}_{10}\ . \tag{3.4}$$

The $\mathfrak{gl}_{10}$ algebra is well-known to be spanned by generators $K^a{}_b$ with $a, b = 1, 2, ..., 10$ such that

$$\left[K^a{}_b\,,\,K^c{}_d\right] \;=\; \delta^c_b\,K^a_d \;-\; \delta^a_b\,K^c_d \tag{3.5}$$

$$\langle K^a{}_b\,|\,K^c{}_d\rangle \;=\; \delta^c_b\,\delta^a_d \;-\; \delta^a_b\,\delta^c_d\ . \tag{3.6}$$

- level $\ell = 1$

On the first level $\ell = 1$, one is faced with some representation of $\mathfrak{sl}_{10}$ which is determined by the presence of extra generators $\{e_0, f_0, h_0\}$. Take $f_0$ as a highest weight vector and examine the adjoint action $x(f_0) := [x, f_0]$ of $x \in \mathfrak{sl}_{10}$ on it via

$$e_i(f_0) \;=\; 0\ , \qquad f_i(f_0) \;\sim\; \delta_{i,3}\ , \qquad i = 1, 2, ..., 9 \tag{3.7}$$

and

$$h_i(f_0) \;=\; -\,A_{i0}\,f_0 \;=\; \delta_{i,3}\,f_0\ . \tag{3.8}$$

This defines the (001000000) representation of $\mathfrak{sl}_{10}$ spanned by antisymmetric three forms of dimension $120 = \binom{10}{3}$. We will label its members as $E^{abc} = E^{[abc]}$.

- level $\ell = 2$

By virtue of the level additivity (3.3), level two tensors can be obtained from commutators of $\ell = 1$ objects

$$\left[E^{abc}\,,\,E^{def}\right] \;=:\; E^{abcdef}\ . \tag{3.9}$$

The tensor product of the three form representations (e.g. by means of Young tableaus) yields Dynkin labels (000001000).

- level $\ell = 3$

Let us simply state the result at level three: The $[\ell = 1, \ell = 2]$ commutator yields objects

$$\left[E^{abc}\,,\,E^{d_1 d_2 \ldots d_6}\right] \;=:\; E^{[a|bc]d_1 d_2 \ldots d_6}\ . \tag{3.10}$$

in the (100000010) representation of $\mathfrak{sl}_{10}$ with the additional property that

$$E^{[a|bcd_1 \ldots d_6]} \;=\; 0\ . \tag{3.11}$$

## 3.3 $D = 11$ supergravity

In $D = 11$ spacetime dimensions [6], one can formulate the maximally supersymmetric theory of gravity. Its bosonic particle content is made of

$$G_{MN} \;\equiv\; \text{graviton}\ , \qquad A_{MNP} \;\equiv\; \text{three form}\ . \tag{3.12}$$

The $A_{MNP}$ field gives rise to a field strength four form

$$F_{MNPQ} \quad := \quad 4\,\partial_{[M}\,A_{NPQ]} \tag{3.13}$$

invariant under gauge transformation $\delta A_{MNP} = 3\partial_{[M}\Lambda_{NP]}$ with two form gauge parameter $\Lambda$. This can be rephrased in the language of differential forms as $F = \mathrm{d}A$ and $\delta A = \mathrm{d}\Lambda$.
The field strength can be dualized to a seven form field

$$\tilde{F} \quad := \quad *\,F\;, \qquad \tilde{F}_{M_1\ldots M_7} \quad = \quad \frac{1}{4!}\,\varepsilon_{M_1\ldots M_7 N_1\ldots N_4}\,F^{N_1\ldots N_4}\;. \tag{3.14}$$

In Maxwell theory, there are equations of motion and Bianci identities $\mathrm{d}F = \mathrm{d}*F = 0$ which allow to also derive the dual field strength $\tilde{F}$ from a potential $\tilde{A} \neq *A$ (at least locally). However, there is a little extra term in the supergravity equations of motion for the three form field $A$ due to a topological term $A \wedge F \wedge F$ in the Lagrangian:

$$\mathrm{d}\tilde{F} \;-\; 3\sqrt{2}\,F \wedge F \quad = \quad \mathrm{d}\left(\tilde{F} \;-\; 3\sqrt{2}\,A \wedge F\right) \quad = \quad 0 \tag{3.15}$$

We can thus locally express the closed form $\tilde{F} - 3\sqrt{2}A \wedge F$ in terms of a six form field $\tilde{A} \neq *A$,

$$\tilde{F} \;-\; 3\sqrt{2}\,A \wedge F \quad = \quad \mathrm{d}\tilde{A}\;. \tag{3.16}$$

There is a nice correspondence between the fields of this supergravity and the lowest $A_9$ levels of $E_{10}$. At $\ell = 0$, we have found a $\mathfrak{gl}_{10}$ representation under which the spatial *zehnbein* $e^M{}_A$ transforms. The antisymmetric $E^{[abc]}$ from the $\ell = 1$ level can be matched with the electric three form components $A_{MNP}$ whereas the $E^{[abcdef]}$ tensors at $\ell = 2$ correspond to the dual electric magnetic potential $\tilde{A}$. Also the $\ell = 3$ objects $E^{[a|bcd_1\ldots d_6]}$ can be found within the supergravity framework as some magnetic dual of the zehnbein.

## 3.4 $D = 10$ supergravities

There is no unique supergravity theory in $D = 10$ spacetime dimensions, the two possibilities are usually classified as type IIA and IIB according to the superstring theories from which they arise as the low energy limit. Also, the IIA supergravity can be obtained dimensionally reducing the (unique) $D = 11$ analogue. Both have a close connection to further level decompositions of $E_{10}$ (alternative to picking the $A_9$ subalgebra):

- type IIB supergravity from the $A_8 \oplus A_1$ subalgebra [7]

  Distinguishing the root $\alpha_0$ as shown in the figure leaves a disconnected Dynkin diagram associated with the direct sum $A_8 \oplus A_1$. This leads the following representations at lowest levels (see [7] for the details):

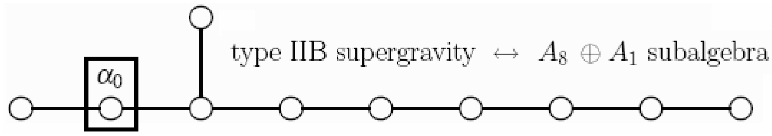



Figure 4: $E_{10}$ decomposition with respect to $A_8 \oplus A_1$

- $\ell = 0$: $\mathfrak{g}^{(0)} \;=\; \big(\mathfrak{sl}_9 \oplus \mathrm{span}\{h_0\}\big) \oplus \mathfrak{sl}_2 \;=\; \mathfrak{gl}_9 \oplus \mathfrak{sl}_2$
- $\ell = 1$: $E_\alpha{}^{pq} \;\leftrightarrow\; \mathfrak{sl}_2$ doublet of two form fields
- $\ell = 2$: $E^{pqrs} \;\leftrightarrow$ four form with self dual field strength $\mathrm{d}E = *\mathrm{d}E$
- $\ell = 3$: $E_\alpha{}^{pqrstu} \;\leftrightarrow$ magnetic dual of $\ell = 1$

This is precisely the massless (bosonic) particle content of type IIB supergravity.

- type IIA supergravity from the $D_9$ subalgebra [8]

  The choice of slicing as displayed in the next figure leads to a (non-compact) $D_9 = \mathfrak{so}(9,9)$ subalgebra.

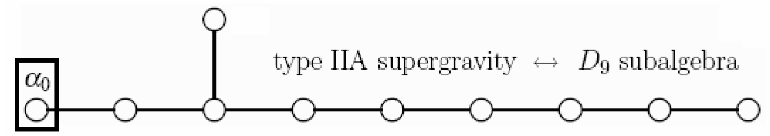



Figure 5: $E_{10}$ decomposition with respect to $D_9$

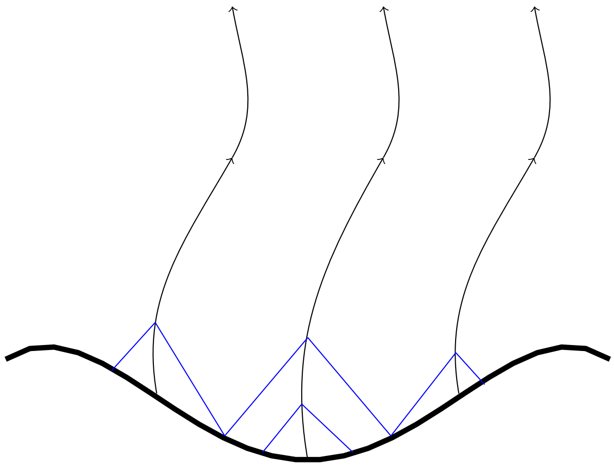
At level $\ell = 0$, the standard $\mathfrak{so}(9,9)$ generators $M^{[IJ]}$ arise and reflect the degrees of freedom in the metric $G_{\mu\nu}$ and a two form field $B_{\mu\nu}$. Things are more complicated at first level $\ell = 1$ where the standard procedure of the previous examples leads to a 256 component spinor $E_A$ of $SO(9,9)$. The maximal compact subgroup is $SO(9) \otimes SO(9) \subset SO(9,9)$ which on the other hand has diagonal subgroup $SO(9)_{\mathrm{diag}} \subset SO(9) \otimes SO(9)$. Decomposing the level one spinor representation with respect to the smaller groups,

$$\underbrace{(\mathbf{256})}_{\text{w.r.t } SO(9,9)} \;=\; \underbrace{(\mathbf{16}) \otimes (\mathbf{16})}_{\text{w.r.t } SO(9)\times SO(9)} \;=\; \underbrace{(\mathbf{9}) \oplus (\mathbf{84}) \oplus (\mathbf{126}) \oplus (\mathbf{36}) \oplus (\mathbf{1})}_{\text{w.r.t } SO(9)_{\mathrm{diag}}}\,, \tag{3.17}$$

we end up with all the odd $SO(9)_{\mathrm{diag}}$ forms, e.g. $A_\mu \in (\mathbf{9})$ and $A_{[\mu\nu\lambda]} \in (\mathbf{84})$ (and their magnetic five- and seven form duals). In string theorists' language, level $\ell = 0$ contains the degrees of freedom of the NS NS sector whereas $\ell = 1$ covers the R R sector. See [8] for further information on that topic.

# 4 Cosmological billard and Kac Moody algebras

In this section, we will discuss a cosmological problem concerning spacelike big bang like singularities. Consider causally decoupled events close to such a cosmological singulariy: It was conjectured by *Belinskii, Khalatnikof & Lifshitz* (BKL) that in this regime, spatial gradients become negligible $\partial_x \ll \partial_t$ such that the (generically partial and highly nonlinear) Einstein equations asymptotically reduce to ordinary differntial equations in time.




The material is mostly taken from [9].

## 4.1 The Kasner solution

The simplest realization of the BKL assumption is the *Kasner solution* (1926):

$$\mathrm{d}s^2 \;=\; -\,\mathrm{d}t^2 \;+\; t^{2p_1}\,\mathrm{d}x^2 \;+\; t^{2p_2}\,\mathrm{d}y^2 \;+\; t^{2p_3}\,\mathrm{d}z^2 \tag{4.1}$$

$$1 \;=\; p_1^2 + p_2^2 + p_3^2 \;=\; p_1 \,+\, p_2 \,+\, p_3 \tag{4.2}$$

This solves Einstein's equations and leaves a reparametrization freedom $t \mapsto t'(t)$. It is homogeneous but not isotropic, and singular at $t = 0$ (where we permute the $p_i$ such that $p_1 < 0 < p_2 < p_3 < 1$). Volumes at the singularity scale as

$$\sqrt{g} \;\sim\; t^{p_1+p_2+p_3} \;=\; t\ . \tag{4.3}$$

To actually test the BKL hypothesis, one has to introduce some curvature. Let us consider homogeneous spaces of constant curvature. It turns out that the dynamics is characterized by *Kasner bounces*

$$(p_1, p_2, p_3) \;\mapsto\; (p'_1, p'_2, p'_3)\ , \tag{4.4}$$

which can occur in two ways:

- infinite number of Kasner bounces as $t \to 0$, i.e. chaotic oscillations
- finite number of bounces, $\exists\, t = \varepsilon$ such that the solution is Kasner like for $0 < t < \varepsilon$ with $p_i = p_i(\vec{x})$ (which is referred to as the AVD regime for *asymptotically velocity dominated*)

BKL checked that the first (chaotic) case rather than the second one is realized in $D = 4$ gravity, in fact this holds in any dimension $D \leq 10$.

The Einstein Hilbert action for a metric in the gauge

$$\mathrm{d}s^2 \;=\; -\,N^2(t)\,\mathrm{d}t^2 \;+\; g_{mn}(t)\,\mathrm{d}x^m\,\mathrm{d}x^n \tag{4.5}$$

(with lapse function $N(t)$) is given by (see second exercise session)

$$\mathcal{S}_{\mathrm{EH}}[N,g] \;\sim\; \int \mathrm{d}t\,\frac{\sqrt{g}}{N}\,\Big(\mathrm{Tr}\{\mathrm{g}^{-1}\,\dot{\mathrm{g}}\,\mathrm{g}^{-1}\,\dot{\mathrm{g}}\} \;-\; \big(\mathrm{Tr}\{\mathrm{g}^{-1}\,\dot{\mathrm{g}}\}\big)^2\Big)\,. \tag{4.6}$$

Here, $g = \det g_{ij}$ denotes the determinant of the spatial metric, g its matrix form and $\dot{\mathrm{g}} \equiv \frac{\mathrm{dg}}{\mathrm{d}t}$. In particular, assuming the diagonal *Kasner type* metric $g^{(\mathrm{K})}_{mn} = \delta_{m,n}\exp(-2\beta^m(t))$, we can simplify the action to

$$\begin{aligned}\mathcal{S}_{\mathrm{EH}}[N,g^{(\mathrm{K})}] \;&\sim\; \int \mathrm{d}t\,\frac{\sqrt{g}}{N}\left[\sum_{a=1}^{D-1}\left(\frac{\mathrm{d}\beta^a}{\mathrm{d}t}\right)^2 \;-\; \left(\sum_{a=1}^{D-1}\frac{\mathrm{d}\beta^a}{\mathrm{d}t}\right)^2\right] \\ &=\; \int \mathrm{d}t\,\frac{1}{\tilde{N}}\,G_{ab}\,\dot{\beta}^a\,\dot{\beta}^b\,.\end{aligned} \tag{4.7}$$

This defines the rescaled lapse function $\tilde{N} = \frac{N}{\sqrt{g}}$ and the *DeWitt metric* $G_{ab}$ in $\beta^a$ parameter space, it is Lorentzian with signature $(-,+,+,...,+)$. A timelike direction in this space is $\mathrm{d}\beta^a \sim (1,1,...,1)$, this reflects the familiar fact that the gravitational action is not bounded from below (even with Euclidean signature). It is this characteristic feature of gravity which causes the Lorentzian nature of the emerging KM algebras.

It makes sense to introduce further time variables, first of all *proper time* $T$

$$\mathrm{d}T \;:=\; -N(t)\,\mathrm{d}t \quad\Rightarrow\quad \mathrm{d}s^2 \;=\; -\,\mathrm{d}T^2 \;+\; g_{mn}\big(t(T)\big)\,\mathrm{d}x^m\,\mathrm{d}x^n \tag{4.8}$$

such that the singularity occurs at $T \to 0^+$. Alternatively, the time coordinate $\tau$ defined by

$$\mathrm{d}\tau \;=\; -\,\frac{\mathrm{d}T}{T} \quad\Rightarrow\quad \tau \;\sim\; -\ln T \tag{4.9}$$

has the advantage to considerably simplify the equation of motion (from varying the action (4.7) with respect to the metric) to

$$\frac{\mathrm{d}^2\beta^a}{\mathrm{d}\tau^2} \;=\; 0 \quad\Rightarrow\quad \beta^a(\tau) \;=\; v^a\cdot\tau \;+\; \beta^a_0\,. \tag{4.10}$$

By varying $\mathcal{S}_{\rm EH}$ with respect to $\tilde{N}$, one obtains the Hamiltonian constraint

$$0 \;=\; G_{ab}\,\dot\beta^a\,\dot\beta^b \;=\; G_{ab}\,v^a\,v^b\ . \tag{4.11}$$

In this description, the Kasner solutions correspond to free relativistic particle motion in $\beta$ space. The singularity corresponds to $\sum_a \beta^a \to \infty$ (or in the hyperbolic polar parametrization $\beta^a = \rho\gamma^a$ with $G_{ab}\gamma^a\gamma^b = -1$ to $\rho^2 \to \infty$).

## 4.2 Iwasawa decomposition

Let us now drop the restriction on the metric to be diagonal. A convenient parametrization for such a more general g is the *Iwasawa decomposition* of the metric

$$\mathrm{g} \;=\; \mathcal{N}^{\rm T}\,\mathcal{A}^2\,\mathcal{N} \tag{4.12}$$

into a diagonal part $\mathcal{A}$ and an upper triangular factor (the $*$ denoting undetermined entries)

$$\mathcal{N} \;=\; \begin{pmatrix} 1 & * & \cdots & * \\ 0 & 1 & \cdots & * \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}\ . \tag{4.13}$$

Then the Einstein Hilbert Lagrangian associated with the general action (4.6) becomes

$$\mathcal{L}_{\rm EH} \;\sim\; \frac{1}{\tilde{N}}\left\{ G_{ab}\,\dot\beta^a\,\dot\beta^b \;+\; \sum_{a<b}^{D-1} e^{2(\beta^a-\beta^b)}\,(\dot{\mathcal{N}}^a{}_m\,\mathcal{N}^m{}_b)^2 \right\}\ . \tag{4.14}$$

A Hamiltonian description requires conjugate momenta

$$\pi_a \;:=\; \frac{\partial \mathcal{L}_{\rm EH}}{\partial\dot\beta^a}\ , \qquad p^a{}_m \;:=\; \frac{\partial \mathcal{L}_{\rm EH}}{\partial \dot{\mathcal{N}}^m{}_a} \tag{4.15}$$

such that the Einstein Hilbert Hamiltonian is given by

$$\mathcal{H}_{\rm EH} \;=\; \tilde{N}\left\{ G^{ab}\,\pi_a\,\pi_b \;+\; \sum_{a<b}^{D-1} e^{-2(\beta^a-\beta^b)}\,(p^m{}_a\,\mathcal{N}^b{}_m)^2 \right\} \tag{4.16}$$

$$G^{ab}\,\pi_a\,\pi_b \;=\; \sum_{a=1}^{D-1}\pi_a^2 \;-\; \frac{1}{d-1}\left(\sum_{a=1}^{D-1}\pi_a\right)^2\ . \tag{4.17}$$

According to a somewhat lengthy calculation in [9], this is a special case of a more general class of Hamiltonians

$$\mathcal{H} \;=\; \tilde{N}\left\{ G^{ab}\,\pi_a\,\pi_b \;+\; \sum_A c_A(Q,P,\partial\beta,\partial Q)\,e^{-2w_A(\beta)} \right\} \tag{4.18}$$

where $(Q, P)$ denote the remaining phase space variables other than $(\beta, \pi)$ and the $w_A$ are linear functions $w_A(\beta) = G_{ab} w_A^a \beta^b$. It is convenient to represent the $\beta^a$ in terms of hyperbolic polar coordinates (with radial variable $\rho$) as

$$\beta^a \;=\; \rho\,\gamma^a\;, \qquad G_{ab}\,\gamma^a\,\gamma^b \;=\; -1\;, \tag{4.19}$$

then the behaviour close to the singularity is given by the $\rho \to \infty$ limit:

$$\lim_{\rho\to\infty} e^{-2w_A(\beta)} \;=\; \lim_{\rho\to\infty} e^{-2\rho w_A(\gamma)} \;=:\; \Theta_\infty\big(-\,w_A(\gamma)\big) \tag{4.20}$$

The exponential suppression with increasing $\rho$ leads to an infinitely high step function

$$\Theta_\infty(x) \;=\; \begin{cases} \infty & : \; x > 0 \\ 0 & : \; x < 0 \end{cases} \tag{4.21}$$

in the limit $\rho \to \infty$, i.e. the Hamiltonian (4.20) contains a potential with infinitely high walls located at the zeros of $w_A(\gamma)$. These explain the Kasner bounces mentioned at the beginning of this section. At the $\rho \to \infty$ singularity, the whole dynamics of the system is confined to the $\gamma^a$ phase space region where $w_A(\gamma) > 0 \;\forall\; A$. This is a wedge delimited by the hyperplanes of vanishing $w_A(\gamma) = 0$, and if it lies within the light cone of parameter space, then we find chaotic bahviour.

## 4.3 The BKL assumption and Kac Moody algbras

In the previous subsections, we have explained consequences of the BKL assumption $\partial_x \ll \partial_t$: Except for a finite number of them, the infinite number of degrees of freedom encoded in the spatially inhomogeneous metric freeze when approaching the singularity, i.e. they tend to some finite limits as $T \to 0$. Equations (4.10) and (4.18), (4.20) show that the dynamics of the remaining "active" degrees of freedom (corresponding to the diagonal components of the metric) could be asymptotically described in terms of a simple "billard dynamics".
The "billard table" and its walls $\{\gamma \in \mathbb{R}^{D-1} : \; w_A(\gamma) = 0\}$ can be identified with the fundamental Weyl chamber and the roots of some hyperbolic KM algebra (which is $AE_3$ for pure gravity). In particular, the minimal set of constraints $w_A(\gamma) \geq 0$ which imply all the others correspond to the simple real roots. (The hyperplanes orthogonal to imaginary roots are spacelike, and the latter are of no relevance for the Weyl group.) Again, one should stress that the indefinite nature of the algebra is responsible for the action being unbounded from below.

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