On Exceptional Geometry and Supergravity

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Based on joint work with B. deWit, and H. and M. Godazgar:  
[dWN:NPB274(1986),1302.6219; GGN:1303.1013,1307.8295,1309.0266,1312.1061]  
as well as ongoing work with O. Hohm and H. Samtleben  
[GGNHS: hopefully to appear soon]
Motivation

There are many indications of exceptional geometrical structures in maximal supergravity and M theory:

- Ubiquity of exceptional groups: $E_6(6)$, $E_7(7)$, $E_8(8)$, ...
  [Cremmer,Julia(1979)]

- Presence of form fields beyond standard geometry

- Extra (central charge) coordinates beyond $D = 11$?

has led to several attempts to generalise geometry

- Double Field Theory [Siegel(1992); Hull(2005); Hohm, Hull, Zwiebach(2010), ...]

- Generalised geometry (and ‘non-geometry’) [Berman, Cederwall, Kleinschmidt, Thompson(2013); Coimbra, Strickland-Constable, Waldram(2014); ...]

- Exceptional geometry [dWN(1986,2001); HN(1987); KNS(2000); Hillmann(2009); Berman, Goadzgar, Perry, West(2011); Coimbra, Strickland-Constable, Waldram(2011); GGN(2013); Hohm, Samtleben(2013)]
Generalised Geometry

Idea: ‘lift’ exceptional structures found in lower dimensions back up to $D = 11$ (or $D = 10$).

- Extend tangent space in accordance with R symmetries \([dWN(1986);HN(1987)]\)
- Extend tangent space to include $p$-forms \([Hitchin(2003);Gualtieri(2004)]\)
- Include windings of M2, M5, and KK branes \([Hull(2007);Pacheco,Waldram(2008)]\)
- Extra (central charge) coordinates \([...,Siegel(1993);dWN(2001);West(2003);Hillmann(2009);Berman,Perry(2011)]\)

Exceptional duality symmetries necessitate new geometric structures (vielbeine, connections,...) and (perhaps) extra dimensions beyond $D = 11 \rightarrow \text{two options:}$

- *Postulate* new structures *ad hoc* (‘top-down approach’).
- *Derive* them by re-writing original theory (‘bottom-up’).
- In either case must ascertain full consistency, either intrinsically or by comparison with original theory.
Cartan’s Theorem (1909)

... states that the most general algebra of vector fields on a manifold consists (essentially) of the following three: diffeomorphisms, volume preserving diffeomorphisms, or symplectomorphisms. Or: there are no exceptional algebras of vector fields! Thus, if a generalised vielbein $V^M_A$ transforms according to

$$V^M_A(y) \rightarrow V'^M_A(y') = \frac{\partial y'^M}{\partial y^N} V^N_A(y)$$

we can never arrange things such that

$$\frac{\partial y'^M}{\partial y^N} \in E_7(7) \subset GL(56, \mathbb{R}) \quad \text{for all } y$$

$\Rightarrow$ extra coordinates are not for real!

... as was to be expected since there appear to exist no consistent supergravity theories beyond $D = 11$ dimensions (at least, no one has found any so far...)!
More Motivation

What is to be gained from re-writing a known theory (*D = 11* supergravity [*CJS(1978)*]) into a form that is (or is not??) on-shell equivalent to the original theory?

- **Derivation of non-linear Kaluza-Klein ansätze**
  - Consistency of $S^7$ compactification [*dWN(1987), Pilch, HN(2012), GGN(2013)*]
  - Scherk-Schwarz compactifications [*Samtleben(2008); GGN(2013)*]

- **Understanding origin of embedding tensor** from higher dimensions and compactification.

- **... and perhaps: new maximal supergravities?**
  [*Dall'Agata, Inverso, Trigiante(2012); dWN(2013)*]

Also, crucial new insights for (a long term project!)

- **Infinite dimensional extensions: E$_{10}$** [*Julia(1983); DHN(2002),...*]
  or E$_{11}$ [*West(2001)*] and emergent space-time?
Starting from $D = 11$ supergravity [Cremmer, Julia, Scherk (1978)]

split coordinates as $z^M = (x^\mu, y^m)$ and perform 4+7 split of bosonic fields $G_{MN}$ and $A_{MNP}$:

$$G_{MN} : \quad G_{mn}(28) \oplus G_{m\mu}(7) \oplus G_{\mu\nu}(1)$$

$$A_{MNP} : \quad A_{mnp}(35) \oplus A_{\mu mn}(21) \oplus A_{\mu \nu m}(7) \oplus A_{\mu \nu \rho}(1)$$

To get proper count of scalar degrees of freedom $\rightarrow$ dualize seven 2-form fields $A_{\mu \nu m}$ [Cremmer, Julia (1979)]

$$28 + 35 + 7 = 70 \rightarrow \mathcal{V}(x) \in E_7(7)/SU(8)$$

**Key Question:** is this structure peculiar to torus reduction, or can it be lifted back up to $D = 11$?

And: is there a way to reformulate $D = 11$ (or IIA, IIB,...) supergravity that makes these hidden symmetries manifest? [→ B. de Wit and HN, NPB274(1986)363; HN, PLB187(1987)316]
Dualities in eleven dimensions

3-form/6-form duality

\[ F_{M_1 \ldots M_7} = 7! D_{[M_1 A_{M_2 \ldots M_7]} + 7! \frac{\sqrt{2}}{2} A_{[M_1 M_2 M_3} D_{M_4} A_{M_5 M_6 M_7]} \]

\[-\frac{\sqrt{2}}{192} i \epsilon_{M_1 \ldots M_{11}} \left( \bar{\Psi}_R \tilde{\Gamma}^{M_8 \ldots M_{11}} R S \Psi_S + 12 \bar{\Psi}^{M_8} \tilde{\Gamma}^{M_9 M_{10}} \Psi^{M_{11}} \right)\]

defines dual 6-form \( A^{(6)} \equiv A_{MNPQRS} \), with

\[ \delta A_{MNPQRS} = -\frac{3}{6! \sqrt{2}} \varepsilon \Gamma_{MNPQR} \Psi_S + \frac{1}{8} \varepsilon \Gamma_{MN} \Psi_P A_{QRS} \]

Relations are valid on-shell and at full non-linear level.

By contrast, dualisation of gravity works only at linear level, and without matter sources:

\[ G_{MN} = \eta_{MN} + h_{MN} : \quad h_{MN} \leftrightarrow h_{M_1 \ldots M_8 | N} \]

In particular, ‘dual supergravity’ does not even exist at linear level. [Bergshoeff, de Roo, Kerstan, Kleinschmidt, Riccioni (2008)]
Existing no go theorems suggest that $D = 11$ Lorentz covariance must be abandoned if interactions are to be included consistently! \[\text{[Bekaert, Boulanger, Henneaux(2003)]}\]

⇒ more $4+7$ decompositions:

\[
\begin{align*}
A_{MNPQRS} : & \quad A_{mn p q r s}(7) \oplus A_{\mu m n p q r}(21) \oplus A_{\mu \nu m n p q}(35) \oplus A_{\mu \nu \rho m n p}(35) \oplus \cdots \\
h_{M_1 \cdots M_8 | N} : & \quad \emptyset \oplus h_{\mu m n p q r s t u}(7) \oplus h_{\mu \nu m n p q r s t}(49) \oplus h_{\mu \nu \rho m n p q r s}(147) \oplus \cdots
\end{align*}
\]

Now we see that also fields other than scalars can be re-packaged into $E_7(7)$ multiplets in eleven dimensions:

- **Vectors**: $7 \oplus 21 \oplus 21 \oplus \overline{7} = 56$ (electromagnetic duality)
- **2-forms**: $7 \oplus 35 \oplus 49 \oplus \cdots = 133$ ($E_7(7)$ Noether current)
- **3-forms**: $1 \oplus 35 \oplus 147 \oplus \cdots = 912$ (embedding tensor)

→ Beyond kinematics main challenge is to show that full $D = 11$ theory (supersymmetry variations and field equations) can be rewritten in an $E_7(7) \times SU(8)$ covariant way!
NPB274(1986)363 in short

4+7 decomposition of elfbein (in triangular gauge)

\[ E_M^A(x, y) = \begin{pmatrix} \Delta^{-1/2} e'_\mu^\alpha & B_\mu^m e_m^a \\ 0 & e_m^a \end{pmatrix}, \quad \Delta \equiv \det e_m^a \]

Similar redefinitions of fermions \( \rightarrow \) chiral \( SU(8) \)

\[ \varphi'_\mu = \Delta^{-1/4}(i\gamma_5)^{-1/2}e'_\mu^\alpha(\Psi_\alpha - \frac{1}{2}\gamma_5\gamma_a\Gamma^a\Psi_a) , \quad \varphi^A_\mu \text{ or } \varphi_\mu^A \equiv \frac{1}{2}(1 \pm \gamma_5)\varphi'_\mu^A \]

\[ \chi'_{ABC} = \frac{3}{4}\sqrt{2}i\Delta^{-1/4}(i\gamma_5)^{-1/2}\Psi_{[A}\Gamma^a_{BC]} , \quad \chi^{ABC} \text{ or } \chi_{ABC} \equiv (1 \pm \gamma_5)\chi'_{ABC} \]

\[ \Rightarrow \quad \delta B_\mu^m = \frac{\sqrt{2}}{8}e_m^{aB} \left[ 2\sqrt{2}e^{A}\varphi_\mu^B + \bar{\epsilon}_C\gamma'_\mu\chi^{ABC} \right] + \text{h.c.} \]

with generalised vielbein \( \equiv \) GV

\[ e_m^{AB} = i\Delta^{-1/2}(\Phi^T\Gamma^m\Phi)_{AB} , \quad \Phi(x, y) \in SU(8) \]

whence \( e_m^{AB} \) becomes an \( SU(8) \) tensor!

Tangent space symmetry: \( SO(1,10) \rightarrow SO(1,3) \times SU(8) \)
Generalization to remaining $21 + 21 + 7 = 49$ vectors: [dWN, GGN(2013)]

$$B^m_\mu = -\frac{1}{2} B^m_\mu; \quad B^{mn}_\mu = -3\sqrt{2} (A^{mn}_\mu - B^p_\mu A^{pn}_m),$$

$$B^{mn}_\mu = -3\sqrt{2} \eta^{mnp_1...p_5} \left( A^{\mu p_1...p_5} - B^q_\mu A^{qp_1...p_5} - \sqrt{2} \frac{(A^{\mu p_1 p_2} - B^q_\mu A^{qp_1 p_2}) A^{p_3 p_4 p_5}}{4} \right)$$

$$B^m_\mu = -18 \eta^{n_1...n_7} \left( A^{\mu n_1...n_7, m} + (3\tilde{c} - 1) (A^{\mu n_1...n_5} - B^p_\mu A^{pn_1...n_5}) A^{n_6 n_7 m} \right.$$\n
$$+ \tilde{c} A^{\mu n_1...n_6} (A^{n_7 m} - B^p_\mu A^{n_7 p m}) + \frac{\sqrt{2}}{12} \left( A^{\mu n_1 n_2} - B^p_\mu A^{pn_1 n_2} \right) A^{n_3 n_4 n_5} A^{n_6 n_7 m}) \right)$$

where $B^{m}_\mu = $ dual (magnetic) graviphoton. Requiring

$$\delta B^{mn}_\mu = \frac{\sqrt{2}}{8} e^{mn}_{AB} \left[ 2\sqrt{2} \varepsilon^A \phi^B + \varepsilon_C \gamma^C \mu \chi^{ABC} \right] + \text{h.c.}$$

leads to more generalised vielbein components $\Rightarrow$ extend $e^m_{AB}$ to full 56-plet ($e^m_{AB}, e_{mnAB}, e^{mn}_{AB}, e^m_{mAB}$) $\equiv$ 56-bein in eleven dimensions!
56-bein in eleven dimensions

\[
\nu^m_{AB} = \frac{\sqrt{2}i}{8} e^m_{AB} = -\frac{\sqrt{2}}{8} \Delta^{-1/2} \Gamma^m_{AB} \equiv \nu^m_{AB} \equiv -\nu^8_{AB},
\]

\[
\nu_{mnAB} = -\frac{\sqrt{2}}{8} \Delta^{-1/2} \left( \Gamma_{mnAB} + 6 \sqrt{2} A_{mnp} \Gamma^p_{AB} \right),
\]

\[
\nu^{mn}_{AB} = -\frac{\sqrt{2}}{8} \cdot \frac{1}{5!} \eta^{mnp_1\cdots p_5} \Delta^{-1/2} \left[ \Gamma_{p_1\cdots p_5AB} + 60 \sqrt{2} A_{p_1p_2p_3} \Gamma_{p_4p_5AB} 
\right.
\]
\[
- 6! \sqrt{2} \left( A_{q_1\cdots q_5} - \frac{\sqrt{2}}{4} A_{q_1q_2} A_{p_3p_4p_5} \right) \Gamma^q_{AB}
\left. \right],
\]

\[
\nu_{mAB} = -\frac{\sqrt{2}}{8} \cdot \frac{1}{7!} \eta^{p_1\cdots p_7} \Delta^{-1/2} \left[ \Gamma_{p_1\cdots p_7mAB} + 126 \sqrt{2} \ A_{mp_1p_2} \Gamma_{p_3\cdots p_7AB} 
\right.
\]
\[
+ 3 \sqrt{2} \times 7! \left( A_{mp_1\cdots p_5} + \frac{\sqrt{2}}{4} A_{mp_1p_2} A_{p_3p_4p_5} \right) \Gamma_{p_6p_7AB}
\]
\[
\left. + \frac{9!}{2} \left( A_{mp_1\cdots p_5} + \frac{\sqrt{2}}{12} A_{mp_1p_2} A_{p_3p_4p_5} \right) A_{p_6p_7q} \Gamma^q_{AB} \right]
\]
\( \mathcal{V}(e, A^{(3)}, A^{(6)}) \) has all the requisite properties of an \( E_{7(7)} \) matrix:

\[
\mathcal{V}_{MN}^{\ AB} \equiv (\mathcal{V}_{MN\ AB})^* , \quad \mathcal{V}^{MNAB} \equiv (\mathcal{V}^{MN\ AB})^* \\
\]

where we have combined the GL(7) indices into SL(8) indices

\[
\mathcal{V}_{mn}^{\ AB} , \quad \mathcal{V}^{m8}\equiv (\mathcal{V}^{mn\ AB})^* \\
\]

With proper \( E_{7(7)} \) indices \( M, N, \ldots \) in 56 representation

\[
\mathcal{V}_M \equiv (\mathcal{V}_{MN}, \mathcal{V}^{MN}) , \quad \mathcal{V}^M = \Omega^{MN} \mathcal{V}_N \equiv (\mathcal{V}^{MN}, -\mathcal{V}_{MN}) \\
\]

and symplectic form \( \Omega^{MN} \)

\[
\mathcal{V}_M^{\ AB} \mathcal{V}_N^{\ AB} - \mathcal{V}_M^{\ AB} \mathcal{V}_N^{\ AB} = i \Omega_{MN} , \\
\Omega^{MN} \mathcal{V}_M^{\ AB} \mathcal{V}_N^{\ CD} = i \delta^{AB}_{CD} , \\
\Omega^{MN} \mathcal{V}_M^{\ AB} \mathcal{V}_N^{\ CD} = 0 \quad \Rightarrow \quad \in Sp(56, \mathbb{R}) \\
\]

(for \( E_{7(7)} \) have to work a little harder...)

\[
\Rightarrow \quad E_{7(7)} \text{ covariant form of vector transformation in } D = 11: \\
\]

\[
\delta B^\mathcal{M}_\mu = i \mathcal{V}_M^{\ AB} \left( \bar{\epsilon}_C \gamma_\mu \chi^{ABC} + 2\sqrt{2} \bar{\epsilon}^A \psi^B_\mu \right) + \text{h.c.} \\
\]
Extending general covariance

Standard behaviour under internal diffeomorphisms $\xi^m = \xi^m(x,y)$:

$$
\delta \mathcal{V}^m_{AB} = \xi^p \partial_p \mathcal{V}^m_{AB} - \partial_p \xi^m \mathcal{V}^p_{AB} - \frac{1}{2} \partial_p \xi^p \mathcal{V}^m_{AB}
$$

$$
\delta \mathcal{V}_{mn \ AB} = \xi^p \partial_p \mathcal{V}_{mn \ AB} - 2 \partial_{[m} \xi^p \mathcal{V}_{n]p \ AB} - \frac{1}{2} \partial_p \xi^p \mathcal{V}_{mn \ AB}
$$

$$
\delta \mathcal{V}^{mn \ AB} = \xi^p \partial_p \mathcal{V}^{mn \ AB} + 2 \partial_p \xi^{[m} \mathcal{V}^{n]p \ AB} + \frac{1}{2} \partial_p \xi^p \mathcal{V}^{mn \ AB}
$$

$$
\delta \mathcal{V}_{m \ AB} = \xi^p \partial_p \mathcal{V}_{m \ AB} + \partial_m \xi^p \mathcal{V}_{p \ AB} + \frac{1}{2} \partial_p \xi^p \mathcal{V}_{m \ AB}
$$

Due to its explicit dependence on $A^{(3)}$ and $A^{(6)}$ $\mathcal{V}$ also transforms under 2-form gauge transformations with parameter $\xi_{mn}(x,y)$:

$$
\delta A_{mnp} = 3! \partial_{[m} \xi_{np]} \ , \quad \delta A_{mnpqrs} = 3 \sqrt{2} \partial_{[m} \xi_{np] A_{qrs]} \ \Rightarrow
$$

$$
\delta \mathcal{V}^m_{AB} = 0, \quad \delta \mathcal{V}_{mn \ AB} = 36 \sqrt{2} \partial_{[m} \xi_{np]} \mathcal{V}^p_{AB},
$$

$$
\delta \mathcal{V}^{mn \ AB} = 3 \sqrt{2} \eta^{mnqrs} \partial_p \xi_{qr} \mathcal{V}_{st \ AB}, \quad \delta \mathcal{V}_{m \ AB} = 18 \sqrt{2} \partial_{[m} \xi_{np]} \mathcal{V}^{np \ AB}
$$
Idem for 5-form gauge transformations

\[ \delta A_{mnp} = 0, \quad \delta A_{mnpqr} = 6! \partial_{[m} \xi_{npqr]} \Rightarrow \]

\[ \delta \mathcal{V}^m_{AB} = \delta \mathcal{V}_{mnAB} = 0, \quad \delta \mathcal{V}^{mn}_{AB} = 6 \cdot 6! \sqrt{2} \eta^{mnp1\ldots p5} \partial_{[q} \xi_{p1\ldots p5]} \mathcal{V}^q_{AB}; \]

\[ \delta \mathcal{V}_m_{AB} = 3 \cdot 6! \sqrt{2} \eta^{n1\ldots n7} \partial_{[m} \xi_{n1\ldots n5]} \mathcal{V}_{n6n7AB} \]

These formulas can be neatly summarised as

\[ \delta \Lambda \mathcal{V}_{MAB} = \hat{\mathcal{L}} \Lambda \mathcal{V}_{MAB} \]

with \( \Lambda^M \equiv (\xi^m, \xi_{mn}, \xi^{mn}, \xi_m) \) and generalised Lie derivative:

\[ \hat{\mathcal{L}} \Lambda X_M = \frac{1}{2} \Lambda^N \partial_N X_M + 6(t^\alpha)_M^N (t_\alpha)_P^Q \partial_Q \Lambda^P X_N + \frac{1}{2} \omega \partial_N \Lambda^N X_M \]

\[ \Rightarrow \text{unifies internal diffeomorphisms and tensor gauge transformations and suggests extra coordinates: 4+56 instead of 4+7?} \]

But only consistent with \textbf{Section Constraint}:

\[ t^M_N \partial_M \otimes \partial_N = \Omega^M_N \partial_M \otimes \partial_N = 0 \iff \partial_M = 0 \text{ for } M \neq m \]

[Coimbra, Strickland-Constable, Waldram(2012); Berman, Cederwall, Kleinschmidt, Thompson(2013)]

Back to seven (or six) internal coordinates!
Generalised Vielbein Postulate = GVP

56-bein obeys a generalisation of the usual GVP, both for external and internal dimensions. For external dimensions, we have

\[ \partial_\mu \mathcal{V}_{\mathcal{M}AB} + 2 \mathcal{L}_{B_\mu} \mathcal{V}_{\mathcal{M}AB} + \mathcal{Q}_\mu^C [A \mathcal{V}_{\mathcal{M}B]C} = \mathcal{P}_\mu ABCD \mathcal{V}_{\mathcal{M}CD} \]

where \( \mathcal{L}_A \) was defined above. To be compared with \( D = 4 \) relation

\[ \partial_\mu \mathcal{V}_{\mathcal{M}ij} - g \mathcal{B}_\mu^P X_{\mathcal{MP}}^N + \mathcal{Q}_\mu^k [i \mathcal{V}_{\mathcal{M}j]k} \mathcal{V}_{\mathcal{N}ij} = \mathcal{P}_\mu i j k l \mathcal{V}_{\mathcal{M}kl} \]

where \( X_\mathcal{M} \) generate the gauge algebra \( \Rightarrow \) furnishes higher dimensional origin of embedding tensor \( \Theta_{\mathcal{M}^\alpha} \) via

\[ X_{\mathcal{MN}^P} \equiv \Theta_{\mathcal{M}^\alpha(t_\alpha)}^\alpha N^P \]

This correspondence has been checked for \( S^7 \) compactification (where gauging is purely electric) \([GGN:1309.0266]\) and Scherk-Schwarz compactifications \([GGN:1312.1061]\) (where gauge fields are usually both electric and magnetic).

\( \rightarrow \) may thus explain new SO(8) gaugings \([Dall'Agata, Inverso, Trigiante, PRL109(2012)201301]\) via \( U(1) \) duality rotation in \( D = 11 \)!
Internal GVP à la dWN and GGN

\[ \partial_m \mathcal{V}_{MAB} - \mathbf{\Gamma}_{mMN} \mathcal{V}_{NAB} + Q^C_{m \{A \mathcal{V}_{MB} \} C} = P_{mABCD} \mathcal{V}_{MCD} \]

with SU(8) connection

\[ Q^B_{mA} = -\frac{1}{2} \omega_{mab} \mathbf{\Gamma}^{ab}_{AB} + \frac{\sqrt{2}}{48} F_{mabc} \mathbf{\Gamma}^{bc}_{AB} + \frac{\sqrt{2}}{14 \cdot 6!} F_{mabcdef} \Gamma^{abcef}_{AB} \]

and 'non-metricity'

\[ P_{mABCD} = \frac{\sqrt{2}}{32} F_{mabc} \mathbf{\Gamma}^{a}_{[AB} \mathbf{\Gamma}^{bc}_{CD]} - \frac{\sqrt{2}}{56 \cdot 5!} F_{mabcdef} \mathbf{\Gamma}^{a}_{[AB} \mathbf{\Gamma}^{bcdef}_{CD]} \]

\( \mathbb{E}_7(7) \)-valued generalised 'affine' connection \( \mathbf{\Gamma}_{mM}^N = \mathbf{\Gamma}_m^a (t_\alpha)_M^N \):

\( \mathbf{(\Gamma}_m)\n^p_m \equiv -\Gamma^p_{mn} + \frac{1}{4} \delta^p_n \Gamma^q_{mq} \), \( \mathbf{(\Gamma}_m)\n^8_m \equiv -\frac{3}{4} \Gamma^8_{mn} \)

\( \mathbf{(\Gamma}_m)\n^n_m = \sqrt{2} \eta^{np_1 \ldots p_6} \Xi_{m|p_1 \ldots p_6} \), \( \mathbf{(\Gamma}_m)\n^{n_1 \ldots n_4}_m = \frac{1}{\sqrt{2}} \eta^{n_1 \ldots n_4 p_1 p_2 p_3} \Xi_{m|p_1 p_2 p_3} \)

where

\( \Xi_{p|mnq} \equiv D_p A_{mnq} - \frac{1}{4!} F_{pmnq} \)

\( \Xi_{p|m_1 \ldots m_6} \equiv D_p A_{m_1 \ldots m_6} - \frac{1}{7!} F_{pm_1 \ldots m_6} + \ldots \)

\[ \Xi [m|npq] = 0 \]

\[ \Xi [p|m_1 \ldots m_6] = 0 \]
• These connections (as determined from $D = 11$ supergravity) satisfy all covariance properties!

• but have non-vanishing components only along seven dimensions, vanish along all other directions.

So what about connection coefficients for $\mathcal{M} \neq m$

$$\Rightarrow \partial_\mathcal{M} \mathcal{V}_{NAB} - \Gamma_{MN}^P \mathcal{V}_{PAB} + Q^C_{[A} \mathcal{V}_{NB]C} = P_{MABCD} \mathcal{V}_{N}^{CD} ??$$

Possible (and even required, see below), but:

• Connections become highly ambiguous, and are not fixed by requiring absence of (generalised) torsion.

• Full (generalised) covariance incompatible with expressibility in terms of $\mathcal{V}$ and $\partial \mathcal{V}$ only.

• Remarkably, supersymmetric theory is insensitive to these ambiguities and other difficulties!
**Torsion**

Definition from generalised geometry [CSW(2014);Cederwall,Edlund,Karlsson(2013)]

\[ \mathcal{T}_{NK}^M = \Gamma_{NK}^M - 12 \, \mathbb{P}^M_K P \, \Gamma_{PN}^Q + 4 \, \mathbb{P}^M_K P \, N \, \Gamma_{QP}^Q \]

This is the 912 representation in $56 \times 133 \rightarrow 56 \oplus 912 \oplus 6480$.

A simple component-wise calculation using the components of $\Gamma$ shows that the generalised torsion does indeed vanish, e.g.

\[ T_{m8n8}^{p8} = \Gamma_{m8n8}^{p8} - 48 \, \mathbb{P}^{p8}_{n8} q8 \, \Gamma_{q8m8}^{r8} + 16 \, \mathbb{P}^{p8}_{n8} q8 \, m8 \, \Gamma_{r8q8}^{r8} \]

\[ = \Gamma_{[mn]}^p - 2 \, \frac{1}{3} \, \Gamma_{r[m} \, \delta^n_{r]} = 0 \]

if ordinary torsion $\Gamma_{[mn]}^p = 0$. Similarly (using $\mathbb{P}^{pq}_{n8} \, r8 \, st = -\frac{1}{12} \, \delta^{pq}_{n[s} \, \delta^r_{t]}$)

\[ T_{m8n8}^{pq} = \Gamma_{m8n8}^{pq} + 2 \Gamma_{r8m8}^{[p} \delta^n_{q]} \]

\[ = 3\sqrt{2} \eta^{pq1...t5} (\Xi_{m|nt1...t5} - \Xi_{n|mt1...t5} + 5 \Xi_{t1|mnt2...t5}) \]

\[ = 21\sqrt{2} \eta^{pq1...t5} \Xi_{[m|nt1...t5]} = 0 \quad \text{etc.} \]

\[ \Rightarrow \quad \text{irreducibility properties of } \Gamma_{MN}^P \text{ are crucial for } \mathcal{T}_{MN}^P = 0! \]

[GGNHS, to appear]
Absorbing non-metricity

[Hehl,VonDerHeyde,Kerlick,Nester(1976); M.Perry, private communication]

Cf. GVP of ordinary differential geometry

\[ \partial_m e_n^a + \omega^a_{\ b} e_n^b - \Gamma^p_{\ mn} e_p^a = 0 \]

But there is a more general expression

\[ \partial_m e_n^a + \omega^a_{\ b} e_n^b - \Gamma^p_{\ mn} e_p^a = T^p_{\ mn} e_p^a + P_{\ mn}^a e_n^b \]

with torsion \( T^p_{\ mn} \) and non-metricity \( P_{\ mn}^a \equiv \frac{1}{2} D_{\ mn} g_{np} \), which can be absorbed by redefinitions

\[
\Gamma^p_{\ mn} \longrightarrow \Gamma^p_{\ mn} - P_{\ mn}^a e_n^d e_p^c, \\
T^p_{\ mn} \longrightarrow T^p_{\ mn} - P_{\ mn}^a e_n^d e_p^c
\]

Idem for exceptional geometry:

\[ \Gamma^p_{\ MN} \longrightarrow \tilde{\Gamma}^p_{\ MN} = \Gamma^p_{\ MN} - i \left( \mathcal{V}^A_{\ M} P_{\ ABCD} \mathcal{V}^C_{\ D} - \mathcal{V}^A_{\ N} P_{\ ABCD} \mathcal{V}^C_{\ D} \right) \]

so that the internal GVP becomes

\[ \partial_{\ M} \mathcal{V}^A_{\ NAB} - \tilde{\Gamma}^p_{\ MN} \mathcal{V}^p_{\ AB} + Q^C_{\ M[A} \mathcal{V}^C_{\ N|B]} = 0 \]
Supersymmetric theory

Supersymmetry variations of bosonic fields

\[ \delta e_\mu^\alpha = \bar{\epsilon}^A \gamma^\alpha \psi_{\mu A} + \bar{\epsilon}_A \gamma^\alpha \psi_\mu^a \]

\[ \delta B_\mu^M = i \mathcal{V}_M^A \left( \bar{\epsilon}_C \gamma_\mu \chi^{ABC} + 2\sqrt{2} \bar{\epsilon}^A \psi_\mu^B \right) + \text{h.c.} \]

\[ \delta \mathcal{V}_M^A B = 2\sqrt{2} \mathcal{V}_M^{MC} \left( \bar{\epsilon}_{[A} \chi_{BCD]} + \frac{1}{24} \epsilon_{ABCDEFGH} \bar{\epsilon}^E \chi^{FGH} \right) \]

are derived from \( D = 11 \) SUGRA in \([dWN,GGN]\), while postulated in recent approaches to exceptional geometry.

To establish agreement for the supersymmetry variations of fermions is more tricky! Recall \([dWN(1986)]\)

\[ \delta \psi_\mu^A \propto \cdots + e^{mAB} \partial_m (\gamma_\mu \epsilon_B) + \frac{1}{2} e^{mAB} Q_{mB}^C \gamma_\mu \epsilon_C - \frac{1}{2} e^{mCD} P_m^{ABCD} \gamma_\mu \epsilon_D \]

\[ \delta \chi^{ABC} \propto \cdots + e^{m[AB} \partial_m \epsilon_C] - \frac{1}{2} e^{m[AB} Q_{mD}^C \epsilon^D - \]

\[ - \frac{1}{2} e^{mDE} P_m^{DE[AB} \epsilon_C] - \frac{2}{3} e^{mDE} P_m^{ABCD} \epsilon^E \]
To absorb non-metricity $P_{m}^{ABCD}$ in these variations, must redefine SU(8) connection \cite{GGNHS, to appear}

$$Q_{mA}^{B} \rightarrow Q_{MA}^{B} \equiv Q_{MA}^{B} + Q_{MA}^{B}$$

where

$$Q_{MA}^{B} = R_{MA}^{B} + U_{MA}^{B}$$

with

$$R_{MA}^{B} = \frac{4i}{3} (\nu_{nBC} \nu_{M}^{DE} P_{nACDE} + \nu_{nAC} \nu_{MDE} P_{nBCDE})$$

$$+ \frac{20i}{27} (\nu_{nDE} \nu_{M}^{BC} P_{nACDE} + \nu_{nDE} \nu_{MAC} P_{nBCDE})$$

$$- \frac{7i}{27} \delta_{A}^{B} (\nu_{nCD} \nu_{M}^{EF} P_{nCDEF} + \nu_{nCD} \nu_{MEF} P_{nCDEF})$$

$$U_{MA}^{B} = \nu_{MCD} u^{CD,B}_{A} - \nu_{M}^{CD} u_{CD,A}^{B}$$

where $u^{[CD,B]}_{A} \equiv 0$, $u^{CA,B}_{C} \equiv 0$ in 1280 of SU(8).

Redefinition requires SU(8) connection components along $M \neq m$!
Leads to very compact expressions:

\[
\begin{align*}
\delta \psi^A_\mu &\propto \cdots + \mathcal{V}^{MAB} \mathcal{D}_M(\mathcal{Q})_B^C (\gamma_{\mu \varepsilon C}) \\
\delta \chi^{ABC} &\propto \cdots + \mathcal{V}^M[A \mathcal{D}_M(\mathcal{Q}) \varepsilon] 
\end{align*}
\]

Also: requires extra components \( Q_M \) for \( M \neq m \) and

\[
\Gamma_{MN}^P \rightarrow \hat{\Gamma}_{MN}^P \equiv \tilde{\Gamma}_{MN}^P + i (\mathcal{V}^P_{AB} Q_M^{AC} \mathcal{V}^B_{C} - \mathcal{V}^{PAB} Q_{MA}^C \mathcal{V}^{NBC})
\]

After all these operations we are left with fully covariant and torsion-free connections and a standard GVP

\[
\partial_M \mathcal{V}^{NAB} - \hat{\Gamma}_{MN}^P \mathcal{V}^{PAB} + Q^C_M[A \mathcal{V}^{N B}] = 0
\]

NB: absence of torsion does not fix affine connection uniquely, irremovable ambiguity is in 1280 of SU(8).

[Coimbra,Strickland-Constable,Waldram(2012);Cederwall,Edlund,Karlsson(2013);GGNHS(2014)]
Conclusions

• Starting from ‘old’ results \cite{dWN(1986);GGN(2013)} one can construct generalised SU(8) and affine connections that satisfy all required covariance properties.

• These cannot be written in terms of just \( V \) and \( \partial V \), unlike in General Relativity, even with zero torsion.

• SUSY theory smartly picks just the right combinations which are insensitive to ambiguities/difficulties encountered in generalised geometry constructions.

• Only in this supersymmetric context ‘old’ results agree with more recent constructions! \cite{GGNHS(2014)}

• New theories by \( \omega \)-deformations?