Lectures on Supersymmetry and Supergravity in 2+1 Dimensions and Regularization of Suspersymmetric Gauge Theories

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Abstract

These notes are intended to provide an introduction in both x-space and superspace to $N = 1$, $d = 2{+}1$ rigid supersymmetry and supergravity.

We give a detailed discussion at the classical level of various supersymmetric models, namely the Wess-Zumino model, Yang-Mills theory, Chern-Simons theory and supergravity. We also consider rigidly supersymmetric Yang-Mill-Chern-Simons theory at the quantum level and prove that the theory is ultraviolet finite to all loops. At the one, two and three-loop level in x-space, and at the one and two-loop level in superspace, certain diagrams are power-counting divergent. This raises the possibility that different regularization schemes may give finite but different results for the effective action. We consider the two most used schemes: ordinary dimensional regularization with $d > 3$ in x-space, and dimensional reduction with $d < 3$ in superspace. The well-known inconsistency of dimensional reduction (an ambiguity in the evaluation of the product of three epsilon tensors) is multiplied by $d - 3$, so that it vanishes at $d = 3$. Using BRST Ward identities, supersymmetry Ward identities and general theorems of quantum field theory, we show that both schemes yield the same effective action. Hence, for this model at least, **the superspace approach respects gauge invariance**.

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1 Classical rigid supersymmetry

1.1 When and why supersymmetry?

The assumption that field theories have a Fermi-Bose symmetry leads to predictions which will be tested in the next decade, certainly at the LHC at CERN, and possibly earlier at the Tevatron at Fermilab. For example, in the minimal supersymmetric extension of the standard model, one needs two (instead of one) Higgs doublets, with one of the Higgs scalars classically lighter than the Z boson and quantum corrections being able to lift its mass at most to about 150 GeV. For some of the predicted suspersymmetric partners, upper and lower limits on their masses can be given, so that not finding these suspersymmetric particles will be a serious problem for SUSY. On the other hand, discovery of supersymmetric particles will rank, with quantum mechanics, special and general relativity and gauge theories, among the most important physical discoveries of our century.

It is sometimes stated that there is not the slightest indication that nature is supersymmetric. This is not the whole story, though. The standard model becomes probably inconsistent at very high energies, of the order of 10^{15} GeV, due to what is called "triviality". This means that some extension or modification of the Standard Model is needed. SUSY is one such modification, perhaps the most consistent one available today. When combined with string theory, SUSY produces a theory of quantum gravity without infinities. In addition, supersymmetric quantum gauge field theories have duality symmetries which give detailed information on the nonperturbative sector of the corresponding effective actions, the hope being that also nonsupersymmetric gauge theories have similar features.

In these notes we first give an introduction to SUSY, both in x-space and in superspace. Then we discuss a fundamental problem with SUSY that has been around since its beginning: a regularization scheme that respects both SUSY and gauge invariance. In 2+1 dimensions we have found such a scheme, but we make no claims concerning 3+1 dimensions. We conclude with a detailed discussion of $2+1$ classical supergravity. Rather than giving a full account of the subject and the literature on it, something which would require much more space, we have opted for a more direct presentation which hardly requires any background on supersymmetry.

1.2 General properties of supersymmetric field theories

SUSY is a symmetry of certain actions with an anticommuting spinorial parameter, such that in the (anti)commutator of two supersymmetries one finds a translation. There are other symmetries with anticommuting parameters, BRST symmetry for example, which has an anticommuting constant parameter, but this parameter is a scalar instead of a spinor under Lorentz transformations.

Suppose one has an action with some bosonic fields $b(x)$ and fermionic fields $f(x)$ which satisfy the usual spin-statistics connection. A crucial property from which SUSY springs is their **mass dimension**. Actions for bosonic fields contain two derivatives. Examples are the Maxwell and the Klein-Gordon actions, and also the spin 2 action of Fierz and Pauli which is the linearized limit of the Einstein-Hilbert action of general relativity. On the other hand, actions for fermions contain only one derivative. Quite familiar is the spin 1/2 Dirac action; but also the spin 3/2 Rarita-Schwinger action for "gravitinos", the supersymmetric partners of gravitons, has only one derivative. It follows that the sum of the mass dimensions of two bosonic fields and two derivatives is equal to the sum of the dimensions of two fermionic fields and one derivative. Equivalently, the dimensions of a Fermi and a Bose field differ by one half the mass dimension of a derivative. If we define the latter to be unity, $[\partial_\mu] = 1$, we find

$$[f] - [b] = 1/2\,. \tag{1}$$

This leads to SUSY as we now show.

Suppose that there are SUSY transformation rules which transform a boson into a fermion and vice-versa and which leave the action invariant. Then[1] $\delta b \sim f\epsilon$ and $\delta f \sim b\epsilon$, with ϵ a Fermi field. However, if $\delta b \sim f\epsilon$ contains no derivative, it follows from eq. (1) that $[\epsilon] = -1/2$, so that in $\delta f \sim b\epsilon$ there is a gap of one unit of mass dimension. If there are no masses in the theory and we there are no dimensionful coupling constants, the only object that can fill this gap is a derivative. Hence

$$\delta b \sim f\epsilon \qquad \delta f \sim \partial b\epsilon\,. \tag{2}$$

(1) It is clear from the statistics of Bose and Fermi fields that ϵ must be anticommuting. Usually one chooses Grassmann variables which anticommute, so $\epsilon_1\epsilon_2 + \epsilon_2\epsilon_1 = 0$. Recently, another choice for ϵ has been studied, namely Clifford variables Bouwknegt, McCarthy, and Nieuwenhuizen (1997). This seems to lead to quantum groups, and could lead to a completely different quantum superspace.

(2) From the conservation of angular momenta and the integer/half-integer spin of bosons/fermions it follows that ϵ has half-integer spin. The simplest case is clearly spin 1/2. Spin 3/2 for ϵ leads to field theories which have no positive definite Hilbert space in flat spacetime. Thus, by angular momentum theory, bosons transform into fermions and fermions into bosons whose spins differ by 1/2. This means that **the basic building blocks are Fermi-Bose doublets.**

(3) The commutator of SUSY transformations of $b(x)$ leads indeed to a translation:

$$\delta_1\,(\,\delta_2 b\,) \sim \delta_1(f\epsilon_2) \sim (\partial b\epsilon_1)\,\epsilon_2 \sim \epsilon_1\epsilon_2\,\partial b \tag{3}$$

[1] To keep the notation simple, we denote SUSY variations by the letter δ without any subscript.

The translation ∂_μ of $b(x)$ is over a distance $\epsilon_1\epsilon_2$ which is the same everywhere, since in rigid SUSY the ϵ's are x-independent. For the fermion $f(x)$ one finds a similar result

$$\delta_1\,(\,\delta_2 f\,) \sim \delta_1\,(\partial b\epsilon_2) \sim \partial\,(\epsilon_1 f)\,\epsilon_2 \sim \epsilon_1\epsilon_2\,\partial f\,. \tag{4}$$

(4) We shall see that matters are a bit more complicated than in point 3) above. In general, there are also "auxiliary fields" F which do not correspond to physical particles. Often they appear in the action as F^2 and have field equations $F = 0$. In 2+1 dimensions they have mass dimension $[F] = 3/2$, so that now one can fill the mass dimension gap also with an auxiliary field

$$\delta f \sim \partial_\mu b\,\epsilon + F\epsilon\,. \tag{5}$$

With auxiliary fields, the SUSY algebra "closes", meaning that the commutator of two supersymmetries leads to a sum of symmetries, namely a translation and sometimes a gauge transformation. Without auxiliary fields, the SUSY commutator of a fermionic field contains in general fermionic field equations. Their origin is clear: because $F = 0$ is a bosonic field equation and, in general, field equations rotate into field equations under SUSY, the SUSY variation of $F(x)$ is a fermionic field equation. Hence, if the SUSY commutator in the theory with F closes, omitting F from the theory will introduce fermionic field equations.

1.3 Spinors and Dirac matrices in three dimensions

Before discussing the WZ model in 2+1 dimensions, we recall first some simple facts about spinors and Dirac matrices in 3 dimensions.

We shall exclusively discuss Minkowski spacetime and, in Section 1.7, its superspace. In Euclidean space, the treatment of spinors is different; for example, no real spinors exist in three-dimensional Euclidean space while in Minkowski spacetime they exist. One can go from Minkowski spacetime to Euclidean space by a Wick rotation, and construct in this way supersymmetric theories in Euclidean space from supersymmetric theories in Minkowski spacetime Nieuwenhuizen and Waldron (1996).

We take the metric in flat (2+1)-dimensional Minkowski spacetime to be

$$\eta_{\mu\nu} = \mathrm{diag}\,(-1,1,1) \qquad \mu = 0,1,2\,, \tag{6}$$

while for the epsilon symbol $\epsilon^{\mu\nu\rho}$ we take $\epsilon^{012} = +1$. A spinor $\psi^a(x)$ has two components ($\alpha = 1,2$) since the Dirac matrices

$$\{\gamma^\mu,\gamma^\nu\} = 2\,\eta^{\mu\nu} \tag{7}$$

are 2×2 matrices. If we choose the real representation[2]

$$(\gamma^0)^\alpha{}_\beta = -i\sigma^2 \qquad (\gamma^1)^\alpha{}_\beta = \sigma^1 \qquad (\gamma^2)^a{}_\beta = \sigma^3 \ , \tag{8}$$

where

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{9}$$

are the Pauli matrices, the Dirac operator $\not\partial$ is real and we can take $\psi^\alpha(x)$ to be real. Then $\psi^\alpha(x)$ describes **one** physical state. Compare with four dimensions, where one can also choose a real representation of the Dirac matrices, and where a real 4-component spinor describes 2 physical states, namely particles with helicities $\pm 1/2$ which are their own antiparticles. An important difference with four dimensions is that in three dimensions, and more generally in any odd number of dimensions, there is no γ^5 and hence no chiral spinors exist. This is due to the fact that in an odd number $2n+1$ of dimensions the product of all the Dirac matrices $\gamma^0\gamma^1\ldots\gamma^{2n+1}$ is proportional to the unit matrix. In three dimensions, the matrices $\{\mathbb{1}, \gamma^\mu\}$ form a basis of the Clifford algebra, so that any 2×2 matrix will be a linear combination of them. These observations will be important when considering supersymmetric extensions of purely bosonic models.

The real spinors we consider are a special case of Majorana spinors. In an arbitrary representation of the Dirac matrices, a Majorana spinor is a spinor for which its Dirac conjugate $\bar\psi = \psi^\dagger i\gamma^0$ is equal to its Majorana conjugate $\bar\psi = \psi^T C$, where C is the charge conjugation matrix, $C\gamma^\mu C^{-1} = -(\gamma^\mu)^T$. For our real spinors, C equals $i\gamma^0$.

By definition, we raise and lower spinor indices with the epsilon symbols $\epsilon^{\alpha\beta}$ and $\epsilon_{\alpha\beta}$ and the northwest-southeast convention: $\lambda^\alpha = \epsilon^{\alpha\beta}\lambda_\beta$ and $\lambda_\alpha = \lambda^\beta\epsilon_{\beta\alpha}$. W= e define $\epsilon^{12} = +1$. Raising the indices of $\epsilon_{\alpha\beta}$ as stated, $\epsilon^{\alpha\beta} = \epsilon^{\alpha\alpha'}\epsilon^{\beta\beta'}\epsilon_{\alpha'\beta'}$, shows that also $\epsilon_{12} = 1$. The Dirac matrices with both spinor indices down are given by $\gamma^\mu{}_{\alpha\beta} = (\gamma^\mu)^{\alpha'}{}_\beta\,\epsilon_{\alpha'\alpha}$. It is straightforward to check that $\epsilon_{\alpha\beta} = -(\gamma^0)^\alpha{}_\beta$, so that $(\gamma^\mu)_{\alpha\beta} = (\gamma^0\gamma^\mu)^\alpha{}_\beta$, which leads to

$$(\gamma^\mu)_{\alpha\beta} = \{-\mathbb{1}, -\sigma^3, \sigma^1\} \ . \tag{10}$$

The Dirac matrices with both indices up are obtained analogously

$$(\gamma_\mu)^{\alpha\beta} = \{\mathbb{1}, \sigma^3, -\sigma^1\} \ . \tag{11}$$

From these equations it follows that

$$\begin{aligned} (\gamma^\mu)_{\alpha\beta}\,(\gamma_\nu)^{\alpha\beta} &= -2\,\delta^\mu{}_\nu \\ (\gamma^\mu)_{\alpha\beta}\,(\gamma_\mu)^{\gamma\delta} &= -\,(\,\delta_\alpha{}^\gamma\delta_\beta{}^\delta + \delta_\alpha{}^\delta\delta_\beta{}^\gamma\,) \ . \end{aligned} \tag{12}$$

[2]In Euclidean space, one of the Dirac matrices is necessarily complex and no real spinors can be defined.

Lowering spinor indices with $\epsilon_{\alpha\beta}$ in the last equation, we obtain

$$(\gamma^\mu)_{\alpha\beta}(\gamma_\mu)_{\gamma\delta} = -\left(\,\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma}\,\right)\,. \tag{13}$$

It is useful to write vectors as bispinors

$$\begin{aligned} v_{\alpha\beta} &= (\gamma^\mu)_{\alpha\beta}\, v_\mu \\ v_\mu &= -\frac{1}{2}\,(\gamma_\mu)^{\alpha\beta}\, v_{\alpha\beta} \\ v^\alpha{}_\alpha &= 0\,. \end{aligned} \tag{14}$$

In this way both spinors and vectors can be described with one formalism (spinor formalism). In 4 dimensions one can distinguish between “dotted” and “undotted” indices, for spinors can be decomposed into left- and right-handed parts, but (again) in odd dimensions there exists no matrix γ_5 and hence no chiral spinors exist.

The normalization of the action for real spinors is as usual

$$\mathcal{L} = -\frac{1}{2}\,\bar{\lambda}\,\partial\!\!\!/\lambda\ , \tag{15}$$

and is chosen such that the hamiltonian is positive definite

$$\begin{aligned} H &= \int dx^1 dx^2\, \mathcal{H} \\ &= \frac{1}{2}\int dx^1 dx^2\, \bar{\lambda}\gamma^k\partial_k\lambda \\ &= -\frac{1}{2}\int dx^1 dx^2\, \bar{\lambda}\gamma^0\partial_0\lambda \\ &= \frac{i}{2}\int dx^1 dx^2 \lambda^T\dot{\lambda} \\ &= \sum_{\vec{k}} E(\vec{k})\left[c^\dagger(\vec{k})\,c(\vec{k}) - \frac{1}{2}\right]. \end{aligned} \tag{16}$$

Two identities that the Dirac matrices satisfy in three dimensions and that will be used in the sequel are

$$\epsilon^{\mu\nu\rho}\gamma_\rho = -\gamma^{[\mu}\gamma^{\nu]} \tag{17}$$

and

$$\gamma^\mu\gamma^\rho\gamma^\sigma = \gamma^{[\mu}\gamma^\rho\gamma^{\sigma]} + \eta^{\mu\rho}\gamma^\sigma + \eta^{\rho\sigma}\gamma^\mu - \eta^{\mu\sigma}\gamma^\rho. \tag{18}$$

It is instructive to check them by taking particular values for μ, ρ and σ. Another two identities for real spinors ψ and χ that we will repeatedly use are

$$\bar{\psi}\chi = \bar{\chi}\psi \qquad \bar{\psi}\gamma^\mu\chi = -\bar{\chi}\gamma^\mu\psi\ . \tag{19}$$

To prove them, it is enough to use the definition of $\bar{\psi}$: $\bar{\psi} = \psi^T\sigma^2$. Note that the second equation in (19) ensures that the lagrangian $\mathcal{L}$ in (15) is not a total derivative.

1.4 The simplest case: the Wess-Zumino model in x-space

Suppose that we begin with a real scalar field $\varphi(x)$ in (2+1)-dimensional Minkowski spacetime, and choose as its action the Klein-Gordon action

$$S = \int d^3x \, \left(-\frac{1}{2} \eta^{\mu\nu} \, \partial_\mu \varphi \, \partial_\nu \varphi \right) \, . \tag{20}$$

To make the system supersymmetric, we introduce a spin 1/2 fermion. We have already said that in 2+1 dimensions a spinor $\psi^\alpha(x)$ has two components and describes **one** physical state. This gives equal numbers of states but not yet equal numbers of bosonic and fermionic field components, since we have one bosonic field component $\varphi(x)$ and two fermionic field components $\psi^\alpha(x)$. Thus we expect that we must add a bosonic auxiliary field $F(x)$. This suggests the free field action

$$S_{\mathrm{WZ}} = \int d^3x \, \left[-\frac{1}{2} \, \partial^\mu \varphi \, \partial_\mu \varphi - \frac{1}{2} \, \bar{\psi} \partial\!\!\!/ \psi + \frac{\alpha}{2} \, F^2 \right] \, . \tag{21}$$

At this point it is not clear what the sign of the F^2 term is going to be, so we have introduced a coefficient α. From the previous section we are motivated to consider

$$\begin{aligned} \delta\varphi &= \bar{\epsilon}\psi \\ \delta F &= \bar{\epsilon} \partial\!\!\!/ \psi \\ \delta\psi &= \beta \, \partial\!\!\!/ \varphi \, \epsilon + \gamma \, F \epsilon \, , \end{aligned} \tag{22}$$

with β and γ constant coefficients to be determined. We have scaled ϵ and F such that $\delta\varphi$ and δF are normalized to $\bar{\epsilon}\psi$ and $\bar{\epsilon}\partial\!\!\!/\psi$, respectively. Note that, since $\bar{\epsilon}\psi$ and $\bar{\epsilon}\partial\!\!\!/\psi$ are real, the coefficients β and γ are real. We recall that in three dimensions there is no γ^5 and that the matrices $\{\mathbb{1}, \gamma^\mu\}$ form a basis of the Clifford algebra. This accounts for the two terms in the SUSY transformation law for $\delta\psi$ above, Lorentz covariance requiring the γ^μ matrices in the first term to be contracted with ∂_μ. Invariance of the action under the transformations (22) fixes $\beta = 1$ and $\alpha = \gamma$. Indeed, the variation of the Klein Gordon action gives $\bar{\epsilon}\psi \, (\Box\varphi)$, while the Dirac action varies into $-\beta \, \bar{\psi}\epsilon \, (\Box\varphi) - \gamma \, \bar{\psi} \partial\!\!\!/ F \epsilon$ and the F^2 term into $\alpha \, F \bar{\epsilon} \partial\!\!\!/ \psi$, the identities (19) implying then the thesis.

Exercise 1: Show that the mass term

$$S_m = \int d^3x \, m \, \left(F\varphi - \frac{1}{2} \bar{\psi}\psi \right) \tag{23}$$

is supersymmetric provided $\beta = 1$ and $\gamma = 1$. Hence, invariance of both S_{WZ} and S_m requires that the F^2 term in (21) has positive sign, *i.e.* $\alpha = 1$. This has important consequences for SUSY breaking that we do not discuss in these notes.

Exercise 2: Show that the self-interaction term

$$S_g = \int d^3x \; g\,(F\varphi^2 - \bar{\psi}\psi\varphi) \tag{24}$$

is supersymmetric provided $\beta = 1$ and $\gamma = 1$. To do this, note that the $(\bar{\psi}\psi)\,(\bar{\epsilon}\psi)$ vanishes by itself, since $(\bar{\psi}\psi)\psi$ is completely antisymmetric in all 3 ψ's while there are only two independent ψ's. Thus, one can also use SUSY invariance of $S_{\rm WZ}$ and S_g to fix $\alpha = \beta = \gamma = 1$, so that

$$\begin{aligned} \delta\varphi &= \bar{\epsilon}\psi \\ \delta F &= \bar{\epsilon}\partial\!\!\!/\psi & (25) \\ \delta\psi &= \partial\!\!\!/\varphi\,\epsilon + F\epsilon\;. & (26) \end{aligned}$$

Exercise 3: Using eq. (22), derive the SUSY commutator

$$\begin{aligned} [\delta_1,\delta_2]\,\varphi &= \beta\,\bar{\epsilon}_2\partial\!\!\!/\varphi\,\epsilon_1 + \gamma\,\bar{\epsilon}_2\epsilon_1 F - (1\leftrightarrow 2) \\ &= 2\,\beta\,(\bar{\epsilon}_2\gamma^\mu\epsilon_1)\,\partial_\mu\varphi & (27) \end{aligned}$$

and show that exactly the same result holds for $\psi^\alpha(x)$ and $F(x)$ provided $\gamma = \beta$.

The fields $\varphi(x)$, $\psi^\alpha(x)$ and $F(x)$ fill up a real scalar superfield $\phi(x,\theta)$ living in superspace,

$$\phi(x,\theta) = \varphi(x) + i\,\theta^\alpha\psi_\alpha(x) + \frac{i}{2}\,\theta^\alpha\theta_\beta\epsilon_{\beta\alpha}F(x)\;, \tag{28}$$

as we shall discuss in more detail in Section 1.6.

1.5 Supersymmetric Yang-Mills theory in x-space

Next we consider Yang-Mills fields A^a_μ, with a a gauge, Lie algebra index. Since A^a_μ describes one degree of freedom for fixed index a, we add a real spinor field λ^a_α. Counting field components shows that there are 2 fermionic field components and 2 (and not 3) bosonic field components. The reason is that gauge invariance can be used to gauge away one of the components of A^a_μ, for example A^a_0 or any other combination. Hence, we do not need an auxiliary field this time, and the action reads

$$S_{\rm YM} = \frac{1}{m\,g^2}\int d^3x\,\left[\,-\,\frac{1}{4}\,F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2}\,\bar{\lambda}^a(D\!\!\!\!/\,\lambda)^a\right]\;, \tag{29}$$

where m is a parameter with dimensions of mass, g is the dimensionless coupling constant, $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc}A^b_\mu A^c_\nu$ is the field strength and $D_\mu{}^a{}_c = \partial_\mu\delta^a{}_c + f^{abc}A^b_\mu$ is the covariant derivative. The gauge transformations are

$$\delta_{\rm g}\,A^a_\mu = \,(D_\mu\zeta)^a \qquad \delta_{\rm g}\,\lambda^a = f^{abc}\,\lambda^b\zeta^c\;. \tag{30}$$

It is straightforward to check that the gauge transforms of $F^a_{\mu\nu}$ and $(D_\mu\lambda)^a$ are $\delta_{\rm g} F^a_{\mu\nu} = f^{abc}\, F^b_{\mu\nu}\,\zeta^c$ and $\delta_{\rm g}(D_\mu\lambda)^a = f^{abc}\,(D_\mu\lambda)^b\,\zeta^c$, from which the gauge invariance of the action follows.

By writing in eq. (29) the overall factor $1/m$, we have taken the coupling constant g to be dimensionless. In the literature one also finds the action (29) written without the factor $1/m$. This corresponds to taking for g mass dimension $[g] = 1/2$. In either case, there is always a dimensionful parameter, which in our conventions is m.

The parameter m can be used to fill the mass dimension gap between δf and $f\epsilon$ discussed in Section 1.2, so that the SUSY transformation law for λ^a may in principle have more terms than those given in eq. (2) for δf. The most general SUSY transformation rules which are Lorentz covariant read

$$\delta A^a_\mu = \alpha\,\bar\epsilon\,\gamma_\mu\lambda^a \qquad \delta\lambda^a = \beta\, F^a_{\mu\nu}\gamma^\mu\gamma^\nu\epsilon + \gamma\,\partial^\mu A^a_\mu\,\epsilon + \delta\, m \not{A}^a\epsilon\ . \tag{31}$$

Note that a term $\epsilon^{\mu\nu\rho}F^a_{\mu\nu}\gamma_\rho\epsilon$ in $\delta\lambda^a$ is not independent because of the identity (17). The variation of the action (29) under (31) is given by

$$\delta S_{\rm YM} = \frac{1}{m\,g^2}\int d^3x\ \left[-F^{a\,\mu\nu}\,(D_\mu\delta A_\nu)^a - \bar\lambda\,(\not{D}\delta\lambda)^a - \frac{1}{2}\,f^{abc}\,\bar\lambda^a(\delta\not{A}^b)\lambda^c\right]\ . \tag{32}$$

We have used that $(\delta\bar\lambda^a)(\not{D}\lambda)^a$ is equal to $\bar\lambda^a(\not{D}\delta\lambda)^a$, as can easily be shown by partially integrating. The first two terms in (32) are linear in λ and must cancel each other, while the last one is cubic in λ and must separately cancel. For the first two, one finds after partial integration

$$\frac{\alpha}{m}\,(D_\mu F^{\mu\nu})^a\,\bar\epsilon\,\gamma_\nu\lambda^a - \frac{1}{m}\,\bar\lambda^a\,[\,\beta\,(\not{D}F_{\rho\sigma})^a\,\gamma^\rho\gamma^\sigma\epsilon - \gamma\,(\not{D}\partial\cdot A)^a\epsilon - \delta\, m\,(\not{D}A_\nu)^a\,\gamma^\nu\epsilon\,]\ . \tag{33}$$

From this we already see that $\gamma = \delta = 0$. We now use the identity (18). The term that arises with $\gamma^{[\mu}\gamma^\rho\gamma^{\sigma]}$ does not contribute due to the Bianchi identity

$$\left(D_{[\mu}F_{\rho\sigma]}\right)^a = 0\ , \tag{34}$$

the term with $\eta^{\rho\sigma}$ does not contribute since $F_{\rho\sigma}$ is antisymmetric, and the two remaining terms give each the same contribution $-\beta\,(D^\mu F_{\mu\sigma})\,\bar\lambda\gamma^\sigma\epsilon$. Recalling that $\bar\epsilon\,\gamma_\nu\lambda = -\bar\lambda\gamma_\nu\epsilon$, we find $\alpha + 2\beta = 0$. Hence,

$$\delta A^a_\mu = \alpha\,\bar\epsilon\,\gamma_\mu\lambda \qquad \delta\lambda^a = -\frac{\alpha}{2}\,F_{\mu\nu}\gamma^\mu\gamma^\nu\epsilon\ . \tag{35}$$

We still have to show that the third term in (14) cancels by itself. The same term appears in 4-, 6- and 10-dimensional supersymmetric theories, and always vanishes. In our case we must show that

$$f^{abc}\,(\bar\lambda^a\gamma^\mu\lambda^b)\,(\bar\epsilon\gamma_\mu\lambda^c) = 0\ . \tag{36}$$

Using that $\bar{\lambda}^a = \lambda^T i\gamma^0$, that $(\gamma^0\gamma^\mu)^\alpha{}_\beta = (\gamma^\mu)\alpha\beta$ and eq. (13), we have for the left-hand side

$$-f^{abc}\ \lambda^a_\alpha\,\lambda^b_\beta\,\epsilon^\gamma\,\lambda^c_\delta\ (\,\epsilon_{\alpha\gamma}\epsilon_{\beta\delta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma}\,)\ , \tag{37}$$

which vanishes by (anti)symmetrization.

For the SUSY commutator acting on A^a_μ we find

$$[\delta_1,\delta_2]\,A^a_\mu = -\frac{\alpha^2}{2}\,\bar{\epsilon}_2\gamma_\mu\gamma^\rho\gamma^\sigma\epsilon_1\,F^a_{\rho\sigma} - (1\leftrightarrow 2)\ . \tag{38}$$

We now use the identities (18) and (19), and obtain for the right-hand side $-2\alpha^2\bar{\epsilon}_2\gamma^\sigma\epsilon_1F^a_{\mu\sigma}$. Hence

$$[\delta_1,\delta_2]\,A^a_\mu = -2\alpha^2\,(\bar{\epsilon}_2\gamma^\sigma\epsilon_1)\,\left(\partial_\mu A^a_\sigma - \partial_\sigma A^a_\mu + f^{abc}A^b_\mu A^c_\sigma\right)\ . \tag{39}$$

The SUSY commutator acting on the gauge field gives thus a **covariant** translation. To interpret its meaning we split off the ordinary translation term with $\partial_\sigma A_\mu{}^a$ and write the two remaining terms as a covariant derivative

$$[\delta_1,\delta_2]\,A^a_\mu = 2\alpha^2\,(\bar{\epsilon}_2\gamma^\sigma\epsilon_1)\,\partial_\sigma A^a_\mu - 2\alpha^2\,(\bar{\epsilon}_2\gamma^\sigma\epsilon_1)\,(D_\mu A_\sigma\,)^a\ . \tag{40}$$

We have found a translation and a gauge transformation, the latter with parameter $\zeta^a = -2\alpha^2(\bar{\epsilon}_2\gamma^\sigma\epsilon_1)A^a_\sigma$. Thus, the algebra closes, but not only on translations: it also produces gauge transformations. To find the same translation as for the WZ model we need $\alpha^2 = 1$, which we assume from now on. Thus, $\alpha = \pm 1$. We choose the sign of λ^a such that $\alpha = 1$. In other words, if $\alpha = -1$, we redefine λ^a as $-\lambda^a$ and get the same action and SUSY transformation laws as for $\alpha = 1$. Hence

$$\delta A^a_\mu = \bar{\epsilon}\gamma_\mu\lambda \qquad \delta\lambda^a = -\frac{1}{2}\,F_{\mu\nu}\gamma^\mu\gamma^\nu\epsilon\ . \tag{41}$$

Note that one can not set $\alpha = 1$ by rescaling ϵ, since one could put together the WZ model and supersymmetric Yang-Mills theory and ϵ was already rescaled to normalize the SUSY transformation $\delta\varphi$ of the scalar filed φ in the WZ model to $\bar{\epsilon}\psi$.

Coming back to eq. (40), in superspace one finds only a translation. However, if one chooses a so-called Wess-Zumino gauge, one needs to add compensating gauge transformations to the ordinary SUSY transformations in order to stay in this gauge, and these produce then the terms with gauge transformations in the SUSY commutator.

Exercise 4: Check that for λ^a one finds the same result as in (25), namely a covariant translation with $(D_\mu\lambda)^a$.

Exercise 5: Suppose one were to add a mass term

$$S_m = \int d^3x\ \left[\,c_1 m\,(A^a_\mu)^2 + c_2\bar{\lambda}^a\lambda^a\,\right] \tag{42}$$

to the action $S_{\rm YM}$. Show by counting states that one would need another real physical spinor field. Counting field components, show that one would need one real auxiliary bosonic field. All these fields fill up a real spinor superfield

$$A^a_\alpha(x,\theta) = \chi^a_\alpha(x) + \theta_\alpha H^a(x) + \theta^\beta V^a_{\beta\alpha}(x) + i\,\theta^2 \left[\frac{1}{2}\,\partial_{\alpha\beta}\chi^{a\beta}(x) - \Lambda^a_\alpha(x) \right] , \tag{43}$$

where Λ_α is essentially λ_α and V^a_μ is essentially A^a_μ, as we shall see in Sections 1.9 and 1.10.

The mass term (42) breaks gauge invariance, since $(A^a_\mu)^2$ is not gauge invariant. In three dimensions, however, it is possible to give a mass to Yang-Mills fields without breaking invariance under infinitesimal gauge transformations by adding to the Yang-Mills action a Chern-Simons term Jackiw and Templeton (1981), Schonfeld (1981). We see this in the next section.

1.6 Supersymmetric Chern-Simons theory in x-space

In three dimensions, out of the gauge field A^a_μ, the derivatives ∂_μ and a dimensionless coupling constant g, one can construct the following local action invariant under gauge transformations (30)

$$S_{\rm CS} = \frac{1}{g^2} \int d^3x\; \epsilon^{\mu\nu\rho} \left(\frac{1}{2}\, A^a_\mu \partial_\nu A^a_\rho + \frac{1}{6}\, f^{abc}\, A^a_\mu A^b_\nu A^c_\rho \right) \ . \tag{44}$$

Here, as is usual in quantum field theory, **local** means polynomial in the field A^a_μ and its derivatives. This action is known as the Chern-Simons action and has field equation

$$F^a_{\mu\nu} = 0 \ . \tag{45}$$

As opposed to the Yang-Mills action, the Chern-Simons action is only invariant under infinitesimal gauge transformations (30). Suppose that the gauge group is $SU(N)$ and consider finite gauge transformations $A_\mu \to A^h_\mu = h^{-1}\partial_\mu h + g\, h^{-1} A_\mu h$, where $A_\mu = A^a_\mu T^a$, with T^a antihermitean generators of the gauge Lie algebra. Then $S_{\rm CS}$ is not invariant under large gauge transformations. Only the quantity $e^{S_{\rm CS}}$ is invariant, provided $4\pi/g^2$ is an integer Deser, Jackiw, and Templeton (1982). In these notes, however, we are concerned with perturbative quantization, so that we are only interested in infinitesimal gauge transformations, for which there is no restriction on g. In what follows, unless stated otherwise, we will refer to infinitesimal gauge transformations as gauge transformations.

The same counting of states and components as for Yang-Mills theory implies that the supersymmetric extension of the Chern-Simons action will involve the field A_μ^a and the spinor field λ^a. In this case, it is straightforward to verify that the action

$$S_{\rm CS} = \frac{1}{g^2}\int d^3x\,\left[\epsilon^{\mu\nu\rho}\left(\frac{1}{2}\,A^a_\mu\partial_\nu A^a_\rho + \frac{1}{6}\,f^{abc}\,A^a_\mu A^b_\nu A^c_\rho\right) - \frac{1}{2}\,\bar\lambda^a\lambda^a\right]\,, \tag{46}$$

is gauge invariant and supersymmetric. Absence of dimensionful parameters and locality imply that the fermion λ^a can only enter the action as a term $\bar\lambda^a\lambda^a$, but does not fix the coefficient. Invariance under the SUSY transformations (41) fixes the value of the coefficient.

> **Exercise 6:** Show that indeed SUSY requires the coefficient of the term $\bar\lambda^a\lambda^a$ to be -1/2.

One can combine the Yang-Mills and Chern-Simons actions into one single action

$$S_{\rm YMCS} = S_{\rm YM} + S_{\rm CS}\ . \tag{47}$$

The resulting theory is called Yang-Mills-Chern-Simons theory or topologically massive Yang-Mills theory. The action $S_{\rm YMCS}$ is gauge invariant and gives, after gauge fixing, a massive propagator for the field A^a_μ. To see this, let us consider the nonsupersymmetric theory and work in the ordinary Landau gauge $\partial^\mu A^a_\mu = 0$. The Faddeev-Popov procedure gives then for the gauge-fixed classical action

$$S = S_{\rm YM} + S_{\rm CS} + S_{\rm GF}\ , \tag{48}$$

where the gauge-fixing term $S_{\rm GF}$ reads

$$S_{\rm GF} = \int d^3x\ \left[-b^a\partial_\mu A^{a\mu} - (\partial^\mu\hat c^a)\,(\,D_\mu c)^a\right]\ , \tag{49}$$

b^a is a Lagrange multiplier field imposing the condition $\partial^\mu A^a_\mu = 0$, and $\hat c^a$ and c^a are the Faddeev-Popov antighost and ghost fields. The part of the gauge-fixed action quadratic in A^a_μ and b^a has in momentum space the form

$$-\,\frac{1}{2}\int\frac{d^3p}{(2\pi)^3}\ \Big[\,A^a_\rho(p)\,K^{\rho\mu}(p)\,A^a_\mu(-p) + \,b^a(p)\,p^\rho\ A^a_\rho(-p)\Big]\ , \tag{50}$$

where

$$K^{\rho\mu}(p) = -\epsilon^{\rho\sigma\mu}\,p_\sigma + \frac{i}{m}\,\left(\,p^2\eta^{\rho\mu} - p^\rho p^\mu\,\right)\ . \tag{51}$$

This defines the kinetic matrix of A^a_μ and b^a as

$$T(p)\ =\ \begin{pmatrix} K^{\rho\mu}(p) & -p^\rho \\ p^\mu & 0 \end{pmatrix}\ . \tag{52}$$

The propagator matrix

$$\Delta(p) \;=\; \begin{pmatrix} \Delta_{\mu\nu}(p) & \Delta_\mu(p) \\ \Delta_\nu(-p) & 0 \end{pmatrix} \tag{53}$$

is the result of inverting $T(p)$:

$$T(p)\,\Delta(p) \;=\; \begin{pmatrix} \delta^\rho{}_\nu & 0 \\ 0 & 1 \end{pmatrix} \;. \tag{54}$$

To find $\Delta(p)$, we write for $\Delta_{\mu\nu}(p)$ and $\Delta_\mu(p)$ the most general expressions compatible with Lorentz covariance,

$$\Delta_{\mu\nu}(p) = f_1\,\epsilon_{\mu\sigma\nu}p^\sigma + f_2\,\eta_{\mu\nu} + f_3\,p_\mu p_\nu \qquad \Delta_\mu(p) = f_4\,p_\mu \;, \tag{55}$$

with $f_1, \ldots, f_4$ functions of p^2 and m to be determined, and impose eq. (54). One thus finds

$$\Delta_{\mu\nu}(p) = -\,\frac{mg^2}{p^2\,(p^2\!+\!m^2\!-\!io)}\,\Big(m\,\epsilon_{\mu\rho\nu}\,p^\rho + i\,p^2\eta_{\mu\nu} - i\,p_\mu p_\nu\Big) \tag{56}$$

and

$$\Delta_\mu(p) = \frac{p_\mu}{p^2} \;. \tag{57}$$

We see that the propagator (56) has a pole at $p^2 = -m^2$, which shows that the gauge field has a mass. In Section 2.1 we will consider a supersymmetric gauge in which the propagator of the gauge field is the same as in (56).

The propagator (56) has been obtained in three dimensions. When we define dimensional regularization, we will use the original 't Hooft-Veltman prescription, Hooft and Veltman (1972) and Breitenlohner and Maison (1977), for $\epsilon^{\mu\nu\rho}$ in n dimensions, which is the only algebraically consistent one known to date. We will see that this prescription introduces in the n-dimensional propagator extra terms which vanish for $n\!=\!3$ and which loosely speaking can be regarded as proportional to $n\!-\!3$.

1.7 Three-dimensional rigid superspace

Having a symmetry between bosonic and fermionic fields suggests also to consider a symmetry between bosonic coordinates x^μ and new fermionic coordinates. The simplest choice are "spin 1/2 coordinates" θ^α, with $\alpha = 1, 2$. Since x^μ are real, we take θ^α also real. According to eqs. (1) and (2), under SUSY, x^μ should vary into $\bar\epsilon\gamma^\mu\theta$. Hence $[\theta] = -1/2$, just as $[\epsilon] = -1/2$. The reverse law would be $\delta\theta \sim \partial_\mu x_\nu(q_1\eta^{\mu\nu} + q_2\gamma^\mu\gamma^\nu)\epsilon$, with q_1 and q_2 constants, but since $\partial_\mu x^\nu = \delta_\mu{}^\nu$, this simplifies to $\delta\theta^\alpha \sim \epsilon^\alpha$. Hence

$$\delta x^\mu = p\,\bar\epsilon\gamma^\mu\theta \qquad \delta\theta^\alpha = q\,\epsilon^\alpha \;, \tag{58}$$

with p and q real constants.

We denote the derivative with respect to θ^α by ∂_α : $\partial_\alpha \equiv \partial/\partial^\alpha_\theta$. Noting that ∂_α satisfies $\{\partial_\alpha, \theta^\beta\} = \delta_\alpha{}^\beta$, it is clear that $(\partial_\alpha)^\dagger = \partial_\alpha$. Similarly, from $[\partial_\mu, x^\nu] = \delta_\mu{}^\nu$ it follows that $(\partial_\mu)^\dagger = -\partial_\mu$. Since both x^μ and ∂_μ can be written as bispinors by means of eq. (14), $x^{\alpha\beta} = (\gamma_\mu)^{\alpha\beta} x^\mu$ and $\partial_{\alpha\beta} = (\gamma^\mu)_{\alpha\beta}\partial_\mu$, and since the matrices $(\gamma^\mu)_{\alpha\beta}$ are real, we also have $(\partial_{\alpha\beta})^\dagger = -\partial_{\alpha\beta}$.

Fields $\Phi(x,\theta)$ defined on superspace are called superfields and are functions of both coordinates x^μ and θ^α. A superfield will have an expansion in powers of θ^α, with terms of order 0, 1 and 2 in θ^α. This is so since the coordinates θ^α anticommute and there are two such coordinates ($\alpha = 1, 2$), so that one can have at most products $\theta^\alpha\theta^\beta\epsilon_{\beta\alpha} = -2\theta^1\theta^2$ of two θ's. For example, for a scalar superfield $\phi(x,\theta)$, one has

$$\phi(x,\theta) = \varphi(x) + i\,\theta^\alpha\psi_\alpha(x) + \frac{i}{2}\,\theta^\alpha\theta^\beta\epsilon_{\beta\alpha}\,F(x)\ , \tag{59}$$

where the coefficients φ, χ_α and F are fields defined on x-space, usually called component fields.

The SUSY transformations (58) can be viewed as a translation in superspace. Superfields $\Phi(x,\theta)$ will then transform with respect to SUSY as scalars, *i.e.* only with orbital parts but not with spin parts. In other words, $\Phi'(x',\theta') = \Phi(x,\theta)$, where $x'^\mu = x^\mu + p\,\bar\epsilon\gamma^\mu\theta$ and $\theta'^\alpha = \theta^\alpha + q\epsilon^\alpha$. The SUSY generator Q_α, called supercharge, will therefore be such that

$$\delta\Phi(x,\theta) = \epsilon^\alpha Q_\alpha \Phi(x,\theta)\ . \tag{60}$$

Note that Q_α must be a spinor operator, for SUSY transformations are linear in ϵ^α. In order that the commutator of two SUSY transformation gives a translation, we claim that we need

$$Q_\alpha = \partial_\alpha - i\,\theta^\beta\partial_{\beta\alpha}\ . \tag{61}$$

To see this, let us take a scalar superfield $\phi(x,\theta)$. Using the expansion (59) and acting with δ on it, we get on the one hand

$$\begin{aligned}\delta\phi(x,\theta) &= \epsilon^\alpha\,(\,\partial_\alpha - i\,\theta^\beta\partial_{\beta\alpha}\,)\,\phi(x,\theta)\\ &= i\,\epsilon^\alpha\psi_\alpha - i\,\epsilon^\alpha\theta^\beta\partial_{\beta\alpha}\varphi(x) + i\,\epsilon^\alpha\theta_\alpha\,F(x) - \frac{1}{2}\,\theta^2\epsilon^\alpha\partial_{\alpha\beta}\psi^\beta\ ,\end{aligned} \tag{62}$$

and on the other

$$\delta\phi(x,\theta) = \delta\varphi(x) + i\,\theta^\alpha\delta\psi_\alpha(x) + \frac{i}{2}\,\theta^2\delta F(x)\ . \tag{63}$$

In deriving (62), we have used that $\theta^\alpha\theta^\beta = -\frac{1}{2}\,\epsilon^{\alpha\beta}\theta^2$, where θ^2 denotes $\theta^2 \equiv \theta^\alpha\theta_\alpha$. Comparing eqs. (62) and (63), we have

$$\delta\varphi = i\,\epsilon^\alpha\psi_\alpha = \epsilon^\alpha(-i\epsilon_{\alpha\beta})\,\psi^\beta = \bar\epsilon\psi \tag{64}$$

$$\delta\psi_\beta = \partial_{\beta\alpha}\varphi\,\epsilon^\alpha + F\epsilon_\beta \quad \Leftrightarrow \quad \delta\psi = \partial\!\!\!/\varphi\epsilon + F\epsilon \tag{65}$$

$$\delta F = -i\epsilon^a\partial_{\alpha\beta}\psi^\beta \quad = \bar\epsilon\partial\!\!\!/\psi = \ , \tag{66}$$

in accordance with eq. (25). From these transformation laws, and using eqs. (19), the SUSY commutators

$$[\delta_1, \delta_2] \left\{ \begin{array}{c} \varphi \\ \psi \\ F \end{array} \right\} = 2\,(\bar{\epsilon}_2 \gamma^\mu \epsilon_1)\,\partial_\mu \left\{ \begin{array}{c} \varphi \\ \psi \\ F \end{array} \right\} \tag{67}$$

follow. The observant reader may notice that

$$[\,\epsilon_1^\alpha Q_\alpha\,,\,\epsilon_2^\beta Q_\beta\,] = \epsilon_2^\beta\,\epsilon_1^\alpha\,\{Q_\alpha, Q_\beta\} = \epsilon_2^\beta\,\epsilon_1^\alpha(-2i\partial_{\alpha\beta}) = -2\,\bar{\epsilon}_2\gamma^\mu\epsilon_1\partial_\mu \tag{68}$$

has opposite sign. The reason is that eq. (61) gives a representation for the supercharge Q_α as a Lie derivative and the generator P_μ of translations is represented also by the Lie derivative ∂_μ, and **minus** the Lie derivatives form (on general coset manifolds) a representation of the algebra. For example,

$$\{\partial_\alpha - i\,\theta^\delta\partial_{\delta\alpha}\,,\,\partial_\beta - i\,\theta^\gamma\partial_{\gamma\beta}\} = 2i\,(\gamma^\mu)_{\alpha\beta}\,(-\partial_\mu). \tag{69}$$

From either (68) or (69) it follows that in superspace the commutator of two SUSY transformations yields only a translation and no gauge transformation.

As always in field theory, it is useful to introduce the notion of covariant derivatives. Here this means derivatives, denoted by D_α and D_μ, which (anti)commute with the Lie derivatives Q_α and ∂_μ. It is very easy to find that they are given by

$$D_\alpha = \partial_\alpha + i\,\theta^\beta\partial_{\beta\alpha} \qquad D_\mu = \partial_\mu\ . \tag{70}$$

[The theory of coset manifolds can be applied to the coset $(P+Q+M)/M$, where M is the Lorentz subalgebra, finding that the Lorentz connections on Q_α, P_μ, D_α and D_μ all vanish].

Summarizing so far: Superspace is parameterized by coordinates x^μ and θ^α, superfields $\Phi(x,\theta)$ transform as $\delta\Phi(x,\theta) = \epsilon^\alpha Q_\alpha\phi(x,\theta)$, where $Q_\alpha = \partial_\alpha - i\,\theta^\beta\partial_{\beta\alpha}$ is the supercharge, and there exist covariant derivatives $D_\alpha = \partial_\alpha + i\,\theta^\beta\partial_{\beta\alpha}$ and D_μ such that

$$\{D_\alpha, Q_\beta\} = 0\ . \tag{71}$$

Hence $\delta D_\alpha\Phi = \epsilon^\beta Q_\beta\,(D_\alpha\Phi) = D_\alpha(\epsilon^\beta Q_\beta\Phi)$. Furthermore, since $(\theta^a)^\dagger = \theta^\alpha$, $(\partial_\alpha)^\dagger = \partial_\alpha$ and $(\partial_{\alpha\beta})^\dagger = -\partial_{\alpha\beta}$, one has $(D_\alpha)^\dagger = D_\alpha$. It is clear that

$$\{D_\alpha, D_\beta\} = 2i\partial_{\alpha\beta} \qquad [D_\alpha, D_\beta] = -\epsilon_{\alpha\beta}D^2 \qquad [D_\alpha, \partial_{\beta\gamma}] = 0\ , \tag{72}$$

where $D^2 \equiv D^\alpha D_\alpha$.

Three-dimensional $N=1$ superspace is much simpler than four-dimensional $N=1$ superspace. There are no chiral superfields, and hence no representation "preserving constraints". We recall that, as already mentioned, the notion of chirality does not exist in an odd number of dimensions. Imagine one were nevertheless to define a chiral superfield ϕ by the condition $D_1\phi = 0$. Then

$D_1D_1\phi = i\partial_{11}\phi = 0$, where $\partial_{11} = \gamma^{\mu}_{11}\partial_\mu = -(\partial_0 + \partial_1)\phi$. This restricts the x-dependence of ϕ, which is inadmissible. Another simplification in three dimensions is due to the simple fact that objects with three spinor indices which are totally antisymmetric vanish. Namely, for any object $O_{\alpha\beta\gamma}$ one has the identity

$$O_{\alpha\beta\gamma} + O_{\beta\gamma\alpha} + O_{\gamma\alpha\beta} - O_{\gamma\beta\alpha} - O_{\beta\alpha\gamma} - O_{\alpha\gamma\beta} = 0 \ . \tag{73}$$

Although this follows trivially from the observation that spinors in three dimension have only two indices, it leads to many simplifications. For example, taking $O_{\alpha\beta\gamma} = D_\alpha D_\beta D_\gamma$ and contracting with $\epsilon^{\beta\gamma}$, we find

$$D_\alpha D_\beta D^\beta + D_\beta D^\beta D_\alpha + D^\beta D_\alpha D_\beta = 0 \ . \tag{74}$$

If one next writes $D_\alpha D_\beta = -D_\beta D_\alpha + \{D_\alpha, D_\beta\}$ in the first term and $D^\beta D_\alpha = -D_\alpha D^\beta + \{D^\beta, D_\alpha\}$ in the second term, the two terms with an anticommutator cancel each other, $\{D_\alpha, D_\beta\}D^\beta + D_\beta\{D^\beta, D_\alpha\} = [\{D_\alpha, D_\beta\}, D^\beta] = 2i\,[\partial_{\alpha\beta}, D^\beta] = 0$, and one is left with $-D_\beta D_\alpha D^\beta - D_\beta D_\alpha D^\beta + D^\beta D_\alpha D_\beta = 3D^\beta D_\alpha D_\beta = 0$. Hence

$$D^\beta D_\alpha D_\beta = 0\,. \tag{75}$$

From this fundamental identity, others follow; *e.g.*

$$D_\alpha D^2 + D^2 D_\alpha = 0 \ . \tag{76}$$

The measure in superspace is $d^3x d^2\theta$, where d^3x is real and has mass dimension $[d^3x] = -3$ while $d^2\theta \equiv -2d\theta^1 d\theta^2$ is imaginary and has mass dimension $[d^2\theta] = 1$. The normalization factor -2 in the definition of $d^2\theta$ has been introduced for convenience (see below). Integration over Grassmann variables is defined by

$$\int d\theta = 0 \qquad \int d\theta\, \theta = 1 \ . \tag{77}$$

In the case we are considering here of two Grassmann coordinates, we have

$$\int d\theta^a = 0 \quad \int d\theta^a\, \theta^\beta = \delta^{\alpha\beta} \quad \Rightarrow \quad \int d^2\theta\, \theta^\alpha\theta^\beta = 2\epsilon^{\alpha\beta} \ . \tag{78}$$

Thus, in an integral $\int d^3x\, d^2\theta\, F(x,\theta)$, integration over $d^2\theta$ picks the term in $F(x,\theta)$ quadratic in θ's. This coincides precisely with the result of acting with D^2 on $F(x,\theta)$ and taking afterwards $\theta^\alpha = 0$, the reason for this being that $D^2(\theta^\alpha\theta^\beta) = 2\epsilon^{\alpha\beta}$. Hence one has

$$\int d^3x\, d^2\theta\, F(x,\theta) = \int d^3x\ D^2 F(x,\theta)\Big| \ , \tag{79}$$

where the vertical bar denotes restriction to $\theta^\alpha = 0$. With another choice of normalization for $d^2\theta$, this identity would have to be modified accordingly.

Let us consider the action

$$S = \int d^3x\, d^2\theta\, \mathcal{L}(\Phi, D_\alpha\Phi, D_\alpha D_\beta\Phi, \dots)\ , \tag{80}$$

with $\mathcal{L}$ a lagrangian that does not depend explicitly on coordinates. Under a SUSY transformation, the variation of $\mathcal{L}$ is $\delta\mathcal{L} = \epsilon^\alpha Q_\alpha \mathcal{L}$. The term $\epsilon^\alpha \partial_\alpha \mathcal{L}$ that arises from taking ∂_α in Q_α is made of terms which are order zero and one in θ^α and which, therefore, vanish upon integration over $d^2\theta$. Similarly, the term $i\epsilon^\alpha\theta^\beta\partial_{\beta\alpha}\mathcal{L}$ that arises from taking $i\theta^\beta\partial_{\beta\alpha}$ in Q_α gives rise to a total spacetime derivative which can be ignored. Having $\delta\mathcal{L} = 0$, one concludes that the action S is supersymmetric: $\delta S = 0$.

1.8 The Wess-Zumino model in superspace

Since actions are dimensionless (we set $\hbar = 1$) and $d^3x\, d^2\theta$ has mass dimension -2, to obtain the superspace action for the WZ multiplet, we need a lagrangian $\mathcal{L}_{\rm WZ}$ with mass dimension 2. The scalar superfield $\phi(x,\theta)$ in eq. (59) has two scalars, φ and F, and one spinor, ψ^α. In three dimensions, and assuming that there are no dimensionful parameters, a scalar field has mass dimension 1/2, and a spinor field has mass dimension 1. This and the fact that $[\theta^\alpha] = 1/2$ forces us to take φ as the scalar with mass dimension 1/2, since only then ψ^a has mass dimension 1. Thus $[\phi] = [\varphi] = 1/2$. Recalling tha= t $[D_\alpha] = 1/2$, we see that $\mathcal{L}_{\rm WZ} = (D^\alpha\phi)(D_\alpha\phi)$ has the correct mass dimension. Furthermore, because $\mathcal{L}_{\rm WZ}$ is a function of ϕ and $D_\alpha\phi$, the argument given at the end of the last section implies that

$$S_{\rm WZ} = \frac{1}{8}\int d^3x\, d^2\theta\, (D^\alpha\phi)\, (D_\alpha\phi) \tag{81}$$

is supersymmetric, where the factor 1/8 has been introduced for convenience. We can also add a mass term $\mathcal{L}_m = m\phi^2$ and a self-coupling $\mathcal{L}_g = g\phi^3$. Note that $[m] = 1$ but $[g] = \frac{1}{2}$.

To obtain the component action from the superfield action (81), we use eq. (79):

$$S_{\rm WZ} = \frac{1}{4}\int d^3x\, \left[(D^\alpha\phi)\, (D^2 D_\alpha\phi) - (D^\beta D^\alpha\phi)\, (D_\beta D_\alpha\phi)\right]\ . \tag{82}$$

If we write $D^2 D_\alpha$ in the first term as $D^2 D_\alpha = D^\beta(-D_\alpha D_\beta + \{D_\beta, D_\alpha\}) = 2iD^\beta\partial_{\beta\alpha}$, recast $D_\beta D_\alpha$ in the second term as $D_\beta D_\alpha = \frac{1}{2}\{D_\beta, D_\alpha\} + \frac{1}{2}[D_\beta, D_\alpha] = i\,\partial_{\beta\alpha} - \frac{1}{2}\,\epsilon_{\beta\alpha}D^2$ and note that

$$\varphi = \phi\Big| \qquad \psi_\alpha = -iD_\alpha\phi\Big| \qquad F = \frac{i}{2}\, D^2\phi\,\Big|\ , \tag{83}$$

we obtain

$$S_{\rm WZ} = \int d^3x \, \left[-\frac{1}{4} \, (\partial_\alpha{}^\beta \, \varphi) \, (\partial_\beta{}^\alpha \, \varphi) - \frac{i}{2} \, \psi^a \partial_{\alpha\beta} \psi^\beta + \frac{1}{2} \, F^2 \right] \, . \tag{84}$$

This is precisely the WZ action (21). Note that the F^2 term comes out indeed with a positive sign.

1.9 The covariant approach to Yang-Mills theory

To describe Yang-Mills theory in superspace, we need a superfield with a spin 1 field. The real scalar superfield $\phi(x,\theta)$ in (59) can therefore not be used. The spinor superfield

$$\mathcal{A}(x,\theta) = \chi_\alpha(x) + \theta_\alpha H(x) + \theta^\beta V_{\beta\alpha}(x) + i\,\theta^2 \left[\frac{1}{2} \, \partial_{\alpha\beta} \chi^\beta(x) - \Lambda_\alpha(x) \right] \, , \tag{85}$$

contains a vector $V^a_{\beta\alpha}$ and hence can be taken as starting point. Because we want to construct covariant derivatives for ∂_μ and D_α, we consider $\mathcal{A}_\alpha$ as the spinor part of a vector superconnection

$$\mathcal{A}_M = \{\mathcal{A}_\alpha, \mathcal{A}_{\alpha\beta}\} \qquad \mathcal{A}_{\alpha\beta} = (\gamma^\mu)_{\alpha\beta} \mathcal{A}_\mu \, . \tag{86}$$

The connections are Lie algebra valued

$$\mathcal{A}_M = \mathcal{A}^a_M \, T^a \, , \tag{87}$$

with T^a the antihermitean generators of the gauge Lie algebra

$$T^{c\dagger} = -T^c \qquad [T_a, T_b] = f^{abc} T^c \, . \tag{88}$$

$$A_\mu = \mathcal{A}_\mu \Big| \, . \tag{89}$$

In order that the vector field $V^a_{\beta\alpha}$ be real, $\mathcal{A}^a_\alpha$ must be real. Then also χ^a_α, H^a and Λ^a_α are real fields. Once we have a superconnection, we define a gauge covariant superderivative and use it to construct gauge transformations. Since D_α is real, as we already saw, we define the spinor part ∇_α of the gauge covariant superderivative ∇_M by

$$\nabla_\alpha \equiv D_\alpha + i\mathcal{A}_\alpha \, . \tag{90}$$

Note that the i in front of $\mathcal{A}_\alpha$ is needed because $\mathcal{A}_\alpha$ is Lie algebra valued and the generators T_a are antihermitean. We define the vector part of the gauge covariant superderivative by

$$\nabla_{\alpha\beta} \equiv \partial_{\alpha\beta} + \mathcal{A}_{\alpha\beta} \, . \tag{91}$$

Because $\partial_{\alpha\beta}$ is imaginary, $\mathcal{A}^a_{\alpha\beta}$ imaginary; note that there is no i in front of $\mathcal{A}_{\alpha\beta}$. Gauge transformations are defined by

$$\delta_{\rm g}\,(i\mathcal{A}_\alpha) = \nabla_\alpha\Omega = D_\alpha\Omega + i\,[\mathcal{A}_\alpha, \Omega] \tag{92}$$

$$\delta_{\rm g}\,\mathcal{A}_{\alpha\beta} = \nabla_{\alpha\beta}\Omega = \partial_{\alpha\beta}\Omega + [\mathcal{A}_{\alpha\beta}, \Omega]\ , \tag{93}$$

where $\Omega = \Omega^a T^a$ with Ω^a real. The covariant derivatives themselves transform covariantly

$$\delta_{\rm g}\,\nabla_\alpha = [\nabla_\alpha, \Omega] \qquad \delta_{\rm g}\,\nabla_{\alpha\beta} = [\nabla_{\alpha\beta}, \Omega]\ . \tag{94}$$

In general, given a covariant derivative ∇_M, the supertorsion $\mathcal{T}_{MN}{}^P$ and the group supercurvature $\mathcal{F}_{MN}$ are **defined** by

$$[\nabla_M, \nabla_N\} = \mathcal{T}_{MN}{}^P\,\nabla_P + \mathcal{F}_{MN}\ , \tag{95}$$

where $[a, b\}$ is the graded commutator, equal to $\{a, b\}$ if both a and b are fermionic, and equal to $[a, b]$ otherwise. Explicit evaluation gives that only $\mathcal{T}_{\alpha,\beta}{}^{\gamma\delta}$ is nonvanishing and yields

$$\{\nabla_\alpha, \nabla_\beta\} = 2i\nabla_{\alpha\beta} + \mathcal{F}_{\alpha\beta}\ , \tag{96}$$

with

$$\mathcal{F}_{\alpha\beta} = iD_\alpha\mathcal{A}_\beta + iD_\beta\mathcal{A}_\alpha - \{i\mathcal{A}_\alpha, \mathcal{A}_\beta\} - 2i\mathcal{A}_{\alpha\beta}\ . \tag{97}$$

The unusual term $-2i\mathcal{A}_{\alpha\beta}$ ensures that $\mathcal{F}_{\alpha\beta}$ transforms covariantly under gauge transformations. Indeed, under a gauge transformation (92)-(93), some straightforward algebra shows that $\delta_{\rm g}\,\mathcal{F}_{\alpha\beta} = [\mathcal{F}_{\alpha\beta}, \Omega]$. The presence of $\mathcal{A}_{\alpha\beta}$ in $\mathcal{A}_{\alpha\beta}$ can be understood by noting that rigid superspace, though flat, has a nontrivial spin connection. The inverse rigid vielbeins $E_{(0)\,\alpha}{}^M$ and $E_{(0)\,\mu}{}^M$ follow from $D_\alpha = E_{(0)\,\alpha}{}^M\,\partial_M$, $D_\mu = E_{(0)\,\mu}{}^M\,\partial_M$, and read

$$E_{(0)\,\alpha}{}^M = \left\{\delta_\alpha{}^\beta\,,\ i\,\theta^{\alpha'}(\gamma^\mu)_{\alpha'\alpha}\right\} \qquad E_{(0)\,\mu}{}^M = \{0\ ,\ \delta_\mu{}^\nu\}\ . \tag{98}$$

If one changes the basis from $\{\mathcal{A}_M\}$ to $\{\tilde{\mathcal{A}}_M\}$, with $\mathcal{A}_\alpha = E_{(0)\,\alpha}{}^M\,\tilde{\mathcal{A}}_M$ and $\mathcal{A}_{\alpha\beta} = E_{(0)\,\alpha\beta}{}^M\,\tilde{\mathcal{A}}_M$, the curvature takes on the usual Yang-Mills form, as one may check.

The connection in $\nabla_{\alpha\beta}$ is $\mathcal{A}_{\alpha\beta}$, but one may always add a tensor $\mathcal{O}_{\alpha\beta}$ that transforms covariantly under gauge transformations, since the new connection $\mathcal{A}'_{\alpha\beta} = \mathcal{A}_{\alpha\beta} + \mathcal{O}_{\alpha\beta}$ will also transform as $\delta_{\rm g}\,\mathcal{A}'_{\alpha\beta} = \nabla'_{\alpha\beta}\Omega$. If we go back to the beginning and start with the modified connection $\mathcal{A}'_{\alpha\beta} = \mathcal{A}_{\alpha\beta} + \frac{1}{2i}\mathcal{F}_{\alpha\beta}$, we then end up with $\{\nabla_\alpha, \nabla_\beta\} = 2 = i\nabla'_{\alpha\beta}$. Thus, by a redefinition of the vector connection we have obtained $\mathcal{F}_{\alpha\beta} = 0$, and from $\mathcal{F}_{\alpha\beta} = 0$ we have th= at

$$\mathcal{A}'_{\alpha\beta} = \frac{1}{2}\left[\,D_\alpha\mathcal{A}_\beta + D_\beta\mathcal{A}_\alpha + i\,\{\mathcal{A}_\alpha, \mathcal{A}_\beta\}\,\right]\ . \tag{99}$$

Hence we have imposed the conventional constraint $\mathcal{F}_{\alpha\beta} = 0$, which is simply an allowed redefinition of $\mathcal{A}_{\alpha\beta}$. From now on we drop primes. Only $\mathcal{A}_\alpha$ is left as an independent field, while $\mathcal{A}_{\alpha\beta}$ is expressed in terms of $\mathcal{A}_\alpha$ by

$$\mathcal{A}_{\alpha\beta} = \frac{1}{2}\left[D_\alpha \mathcal{A}_\beta + D_\beta \mathcal{A}_\alpha + i\left\{\mathcal{A}_\alpha, \mathcal{A}_\beta\right\}\right] . \tag{100}$$

Clearly, $\mathcal{A}^a_{\alpha\beta}$ is real.

Next we study the Bianchi identities

$$[\nabla_M, [\nabla_N, \nabla_L\}\} + \text{cyclic} = 0 . \tag{101}$$

We first look at the identity

$$[\nabla_\alpha, \{\nabla_\beta, \nabla_\gamma\}] + \text{cyclic} = 0 . \tag{102}$$

From this equation, the anticommutator $\{\nabla_\alpha, \nabla_\beta\} = 2i\nabla_{\alpha\beta}$ and $[\nabla_\alpha, \nabla_{\beta\gamma}] = \mathcal{F}_{\alpha,\beta\gamma}$, we get

$$\mathcal{F}_{\alpha,\beta\gamma} + \mathcal{F}_{\gamma,\alpha\beta} + \mathcal{F}_{\beta,\gamma\alpha} = 0 . \tag{103}$$

This, the decomposition

$$\mathcal{F}_{\alpha,\beta\gamma} = \frac{1}{3}\left[(\mathcal{F}_{\alpha,\beta\gamma} + \mathcal{F}_{\gamma,\alpha\beta} + \mathcal{F}_{\beta,\gamma\alpha}) + (\mathcal{F}_{\alpha,\beta\gamma} - \mathcal{F}_{\beta,\gamma\alpha}) + (\mathcal{F}_{\alpha,\beta\gamma} - \mathcal{F}_{\gamma,\alpha\beta})\right] \tag{104}$$

and

$$\mathcal{F}_{\alpha,\beta\gamma} - \mathcal{F}_{\beta,\gamma\alpha} = \epsilon_{\alpha\beta}\,\epsilon^{\sigma\tau}\mathcal{F}_{\sigma,\tau\gamma} \tag{105}$$

implies that

$$\mathcal{F}_{\alpha,\beta\gamma} = \frac{1}{3}\left[\epsilon_{\alpha\beta}\,(-\mathcal{F}^\tau{}_{,\tau\gamma}) + \epsilon_{\alpha\gamma}\,(-\mathcal{F}^\tau{}_{,\tau\beta})\right] . \tag{106}$$

The object $\mathcal{F}^\tau{}_{,\tau\gamma}$ is the basic superfield strength in the theory. For reasons to become clear, we normalize it as

$$[\nabla^\alpha, \nabla_{\alpha\beta}] = \mathcal{F}^\alpha{}_{,\alpha\beta} = -\frac{3}{2}\,W_\beta . \tag{107}$$

Then we have

$$[\nabla_\alpha, \nabla_{\beta\gamma}] = \mathcal{F}_{\alpha,\beta\gamma} = \frac{1}{2}\,\epsilon_{\alpha\beta}W_\gamma + \frac{1}{2}\,\epsilon_{\alpha\gamma}W_\beta . \tag{108}$$

The field strength is thus given by a (graded) commutator of two covariant derivatives, as in ordinary Yang-Mills theory, but not by $\{\nabla_\alpha, \nabla_\beta\}$, which only yields a torsion term, but rather by $[\nabla_\alpha, \nabla_{\beta\gamma}]$. The third commutator that can be formed with the covariant derivatives, namely $[\nabla_{\alpha\beta}, \nabla_{\gamma\delta}]$, gives the derivative of the field strength, as we show below. Note that W_β is real because ∇^α is real and $\nabla_{\alpha\beta}$ is imaginary. Hence W^a_β is imaginary. Explicit evaluation using (107) and the relation in eq. (100) yields

$$W_\beta = -D^\alpha D_\beta \mathcal{A}_\alpha - i\,[\mathcal{A}^\alpha, D_\alpha \mathcal{A}_\beta] + \frac{1}{3}\,[\mathcal{A}^\alpha, \{\mathcal{A}_\alpha, \mathcal{A}_\beta\}]\ . \tag{109}$$

Another expression for W_β is

$$W_\beta = i\,\nabla^\alpha \nabla_\beta \nabla_\alpha\ . \tag{110}$$

To derive it, apply eq. (73) to $O_{\alpha\beta\gamma} = \nabla_\alpha \nabla_\beta \nabla_\gamma$ to find

$$\nabla^\alpha \nabla_\alpha \nabla_\beta + \nabla_\alpha \nabla_\beta \nabla^\alpha + \nabla_\beta \nabla^\alpha \nabla_\alpha = 0\ , \tag{111}$$

and use this in the definition of W_β

$$W_\beta = \frac{i}{3}\,[\nabla^\alpha, \{\nabla_\alpha, \nabla_\beta\}]\ . \tag{112}$$

Note that the right-hand side in eq. (110) defines a function in superspace, not an operator. This is so since, as a result of the basic identity $D^\alpha D_\beta D_\alpha = 0$, no free derivatives are left in $\nabla^\alpha \nabla_\beta \nabla_\alpha$. The easiest way to check this is to first act with $\nabla^\alpha \nabla_\beta \nabla_\alpha$ on a function Ω, and then show that the expression $\nabla^\alpha \nabla_\beta \nabla_\alpha \Omega$ contains no derivatives of Ω. From eq. (94) it follows that W_β is covariant since it transforms covariantly under gauge transformations

$$\delta_{\rm g}\, W_\beta = [W_\beta, \Omega]\ . \tag{113}$$

In x-space, the variation (any variation, not necessarily a gauge variation) of a curvature is the covariant derivative of the variation: $\delta F_{\mu\nu} = D_\mu(\delta A_\nu) - D_\nu(\delta A_\mu)$. The same holds in superspace: $\delta W^a_\beta = -\nabla^\alpha \nabla_\beta (\delta \mathcal{A}_\alpha)$. This follows easily from (107) if one uses that $\delta \mathcal{A}_{\alpha\beta} = \frac{1}{2}\,(\nabla_\alpha\, \delta\mathcal{A}_\beta + \nabla_\beta\, \delta \mathcal{A}_\alpha)$, which in turn arises from (100).

The next Bianchi identity we study is

$$\{\nabla_\alpha, [\nabla_\beta, \nabla_{\gamma\delta}]\} + [\nabla_{\gamma\delta}, \{\nabla_\alpha, \nabla_\beta\}] - \{\nabla_\beta, [\nabla_{\gamma\delta}, \nabla_\alpha]\} = 0\ . \tag{114}$$

It can be used to express

$$\begin{aligned}\mathcal{F}_{\alpha\beta,\gamma\delta} &\equiv \frac{1}{2i}\,[\{\nabla_\alpha, \nabla_\beta\}, \nabla_{\gamma\delta}] \\ &= (\gamma^\mu)_{\alpha\beta}\,(\gamma^\nu)_{\gamma\delta}\,(\,\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]\,)\end{aligned} \tag{115}$$

in terms of W_α. We begin by decomposing the curvature $\mathcal{F}_{\alpha\beta,\gamma\delta}$ into the sum of terms symmetric in β,γ and terms antisymmetric in β,γ. From the definition of $\mathcal{F}_{\alpha\beta,\gamma\delta}$ in eq. (115) it follows that $\mathcal{F}_{\alpha\beta,\gamma\delta} = cf_{\beta\alpha,\gamma\delta} = -\mathcal{F}_{\gamma\delta,\alpha\beta}$, which in turn implies that the terms in $\mathcal{F}_{\alpha\beta,\gamma\delta}$ symmetric (respectively antisymmetric) in β,γ are antisymmetric (respectively symmetric) in α,δ. This allows us to write without loss of generality

$$\mathcal{F}_{\alpha\beta,\gamma\delta} = \epsilon_{\beta\gamma} f_{\alpha\delta} + \epsilon_{\alpha\delta} f_{\beta\gamma}\ , \tag{116}$$

with $f_{\alpha\delta}$ symmetric in its indices. We could have decomposed $\mathcal{F}_{\alpha\beta,\gamma\delta}$ using other pairs of indices, with the first index in $\{\alpha,\beta\}$ and the second index in $\{\gamma,\delta\}$. For example, we could have decomposed in β,δ and written

$$\mathcal{F}_{\alpha\beta,\gamma\delta} = \epsilon_{\beta\delta} f_{\alpha\gamma} + \epsilon_{\alpha\gamma} f_{\beta\delta} \ . \tag{117}$$

Tracing eq. (114) with $\epsilon^{\gamma\beta}$ and using eq. (116) yields

$$\{\nabla_\alpha, [\nabla^\gamma, \nabla_{\gamma\delta}]\} - \{\nabla^\gamma, [\nabla_{\gamma\delta}, \nabla_\alpha]\} = -4i f_{\alpha\delta} \ . \tag{118}$$

Recalling now eq. (108) we find

$$-2\,\nabla_\alpha W_\delta + \frac{1}{2}\,\epsilon_{\alpha\delta}\nabla^\gamma W_\gamma = -4i f_{\alpha\delta} \ . \tag{119}$$

Hence

$$\nabla^\alpha W_\alpha = 0 \qquad f_{\alpha\beta} = \frac{1}{2i}\,\nabla_{(\alpha} W_{\beta)} \ . \tag{120}$$

Exercise 7: Verify that the remaining Bianchi identities

$$\begin{aligned} &[\nabla_\alpha, [\nabla_{\beta\gamma}, \nabla_{\delta\epsilon}]\,] + \text{cyclic} = 0 \\ &[\nabla_{\alpha\beta}, [\nabla_{\gamma\delta}, \nabla_{\epsilon\zeta}]\,] + \text{cyclic} = 0 \end{aligned} \tag{121}$$

give no further information. Hint: substitute (116) and (108) and then use that $\nabla_\alpha W_\beta = 2i f_{\alpha\beta}$.

Let us now obtain the gauge action. Recalling that g is in our conventions dimensionless and that W^a_α has mass dimension 1/2, an action which is gauge and super Poincaré invariant and has the correct mass dimension is given by Gates Jr., Grisaru, Roček, and Siegel (1983) and Siegel (1979b)

$$S_{\rm YM} = \frac{c}{g^2}\int d^3x\, d^2\theta\, W^{a\alpha} W^a_\alpha \ , \tag{122}$$

with c a constant. Using $\delta W_\alpha = -\nabla^\beta\nabla_\alpha \delta A_\beta$ and integrating by parts, the field equations are found to be given by

$$\nabla^\alpha\nabla^\beta W_\alpha = 2i\nabla^{\alpha\beta} W_\alpha = 0 \ . \tag{123}$$

To find the component content of $S_{\rm YM}$, we use again $d^2\theta = D^2$, but we may replace D^2 by $\nabla^2 = \nabla^\alpha\nabla_\alpha$ since the action is gauge invariant. In other words, for a gauge invariant action,

$$\int d^3x\, d^2\theta\, \mathcal{L} = \int d^3x\, D^2\,\mathcal{L}\Big| = \int d^3x\ \nabla^2\mathcal{L}\,\Big| \ . \tag{124}$$

We obtain then

$$S_{\rm YM} = 2c\int d^3x\ \left[W^{a\alpha}\nabla^2 W^a_\alpha - (\nabla^\beta W^{a\alpha})\,(\nabla_\beta W^a_\alpha)\right]\,\Big| \ . \tag{125}$$

Using $\nabla^\alpha W_\alpha = 0$ and the identity (73), one gets for $\nabla^2 W^a_\alpha$ in the first term

$$\nabla^2 W_\alpha + 2i \nabla_{\alpha\beta} W^\beta = 0 \ . \tag{126}$$

From this and eq. (120) it follows

$$S_{\rm YM} = 2c \int d^3x \, \left(-2i \, W^{a\alpha} \nabla_{\alpha\beta} W^{a\beta} + 4 f^{a\,\alpha\beta} f^a_{\alpha\beta} \right) \Big| \ . \tag{127}$$

Noting the relations in eq. (13) for $(\gamma^\mu)_{\alpha\beta}$, we obtain

$$\begin{aligned} \mathcal{F}_{\alpha\beta,\gamma\delta} \, \mathcal{F}^{\alpha\beta,\gamma\delta} &= (\gamma^\mu)_{\alpha\beta} \, (\gamma^\nu)_{\gamma\delta} \, \mathcal{F}_{\mu\nu} \, (\gamma_\rho)^{\alpha\beta} \, (\gamma_\sigma)^{\gamma\delta} \, \mathcal{F}^{\rho\sigma} = 4 \, \mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu} \\ &= (\epsilon_{\beta\gamma} f_{\alpha\delta} + \epsilon_{\alpha\delta} f_{\beta\gamma}) \, (\epsilon^{\beta\gamma} f^{\alpha\delta} + \epsilon^{\alpha\delta} f^{\beta\gamma}) = 4 \, f_{\alpha\beta} f^{\alpha\beta} \ . \end{aligned} \tag{128}$$

Recalling that $W^{a\alpha}$ is imaginary and noting eq. (115), we define

$$\lambda^a_\alpha = \frac{i}{2} \, W^a_\alpha \Big| \qquad A^a_\mu = \mathcal{A}^a_\mu \Big| \ . \tag{129}$$

Finally, using that $i\lambda^{a\alpha} \, (\gamma^\mu)_{\alpha\beta} \lambda^{b\beta} = \bar{\lambda}^a \gamma^\mu \lambda^b$ and making the choice $c = -1/32 \, g^2$, we obtain

$$S_{\rm YM} = \frac{1}{mg^2} \int d^3x \left[-\frac{1}{4} \, F^a_{\mu\nu} F^{a\mu\nu} - \frac{1}{2} \, \bar{\lambda}^a (\not{D} \lambda)^a \right] \ . \tag{130}$$

This is indeed the component action of eq. (29).

The last subject we wish to study in Yang-Mills theory are the SUSY transformation laws. The fact that $f^a_{\alpha\beta}$ and W^a_α transform covariantly under gauge transformations suggests to use covariant derivatives ∇_α for the SUSY transformations. Thus we write $\delta' = \epsilon^\alpha \nabla_\alpha$. The result of acting with $\delta_{\rm SUSY}$ on any gauge covariant quantity consists of the sum of the usual SUSY transformation δ going with D_α (whose commutator yields an ordinary translation) plus a gauge transformation (which leads to terms quadratic in superfields). The invariance of the action $S_{\rm YM}$ in eq. (122) under δ' follows from its gauge invariance and the fact that $(\delta' - \delta) W^a_\alpha = \delta_{\rm g} W^a_\alpha$. Using δ' we have

$$\delta' \lambda^a_\alpha = \frac{i}{2} \, \epsilon^\beta \nabla_\beta W^a_\alpha \Big| = -i \epsilon^\beta f^a_{\alpha\beta} \Big| \ , \tag{131}$$

or in vector notation

$$\delta' \lambda^a = -\frac{1}{2} F^a_{\mu\nu} \, \gamma^\mu \gamma^\nu \epsilon \ . \tag{132}$$

To find the SUSY transformation law for the field $A^a_{\alpha\beta}$ we note that the action of $\delta \mathcal{A}_{\alpha\beta}$ on any superfield ϕ, with δ an arbitrary variation, is given by $[\delta \mathcal{A}_{\alpha\beta}, \phi]$. This and the identities

$$\begin{gathered} [\delta \mathcal{A}_{\alpha\beta}, \phi] = \delta(\nabla_{\alpha\beta} \phi) - \nabla_{\alpha\beta}(\delta\phi) \\ \epsilon^\gamma \nabla_\gamma \nabla_{\alpha\beta} \phi - \nabla_{\alpha\beta} \epsilon^\gamma \nabla_\gamma \phi = \epsilon^\gamma [\nabla_\gamma, \nabla_{\alpha\beta}] \phi = \frac{1}{2} \, (\epsilon_\alpha W_\beta + \epsilon_\beta W_\alpha) \, \phi \end{gathered} \tag{133}$$

implies that

$$\delta' A^a_\mu = -\frac{1}{2}\,(\gamma_\mu)^{\alpha\beta}\,\delta'\mathcal{A}^a_{\alpha\beta}\Big| = i\epsilon_\alpha\,(\gamma_\mu)^{\alpha\beta}\lambda^a_\beta = \bar{\epsilon}\gamma_\mu\lambda^a\ . \tag{134}$$

Eqs. (132) and (134) are the x-space transformation rules of (41).

1.10 The covariant approach to Chern-Simons theory

The non-supersymmetric Chern-Simons action (44) can also be written as

$$S_{\rm CS} = \frac{1}{4}\int d^3x\,\epsilon^{\mu\nu\rho}A^a_\mu\left(F^a_{\nu\rho} - \frac{1}{3}\,f^{abc}\,A^b_\nu A^c_\rho\right)\ . \tag{135}$$

In superspace we therefore expect an action of the form

$$S_{\rm CS} = \frac{i}{g^2}\int d^3x\,d^2\theta\,\mathcal{A}^{a\alpha}\left[c_1 W^a_\alpha + ic_2 f^{abc}\,\mathcal{A}^{b\beta}(D_\beta\mathcal{A}^c_\alpha) + c_3 f^{abc}f^{cde}\mathcal{A}^{b\beta}\mathcal{A}^d_\alpha\mathcal{A}^e_\beta\right]\ , \tag{136}$$

with c_1, c_2 and c_3 real coefficients. Invariance under gauge transformations $\delta_{\rm g}(i\mathcal{A}^a_\alpha) = (\nabla_\alpha\Omega)^a$ requires $c_2 = c_1/3$ and $c_3 = -c_1/6$, which gives Gates Jr. et al. (1983), Siegel (1979b)

$$\begin{aligned} S_{\rm CS} = \frac{ic_1}{g^2}\int d^3x\,d^2\theta\Big[&(D^\alpha\mathcal{A}^{a\beta})(D_\beta\mathcal{A}^a_\alpha) + \frac{2i}{3}\,f^{abc}\,\mathcal{A}^{a\alpha}\mathcal{A}^{b\beta}(D_\beta\mathcal{A}^c_\alpha) \\ &- \frac{1}{6}\,f^{abc}f^{cde}\mathcal{A}^{a\alpha}\mathcal{A}^{b\beta}\mathcal{A}^d_\alpha\mathcal{A}^e_\beta\Big]\ . \end{aligned} \tag{137}$$

Another way to obtain this expression is the following. We expect the field equation $F^a_{\mu\nu} = 0$ for the nonsupersymmetric theory to generalize to $W^a_\alpha = 0$. Any action which under an arbitrary variation yields

$$\delta S_{\rm CS} \sim \int d^3x\,d^2\theta\ W^{a\alpha}\,\delta\mathcal{A}^a_\alpha \tag{138}$$

will be gauge invariant, since $\delta_{\rm g}(i\mathcal{A}^a_\alpha) = (\nabla_\alpha\Omega)^a$ and $\nabla^\alpha W^a_\alpha = 0$. Hence it is enough to construct an action of the form (136) whose variation is (138). The answer is eq. (137).

To find the component expression for the action (137),we make use of the fact that the action is gauge invariant to set

$$\mathcal{A}_\alpha| = 0 \qquad D^\alpha\mathcal{A}^a_\alpha| = 0\ , \tag{139}$$

which defines a Wess-Zumino gauge. The point is that these two conditions can be imposed by suitably choosing the components $D_\alpha\Omega^a|$ and $D^2\Omega^a|$ of the superfield Ω^a in $\delta_{\rm g}(i\mathcal{A}_\alpha) = \nabla_a\Omega$, while leaving the component $\Omega^a|$ arbitrary, which is the only one that enters the gauge transformation laws of the physical fields A^a_μ and λ^a_α. Indeed, from $\delta_{\rm g}(i\mathcal{A}^a_\alpha)| = D_\alpha\Omega^a| + if^{abc}\mathcal{A}^b_\alpha\Omega^c|$ it follows that it is enough to take $D_\alpha\Omega^a| = 0$ in order to have $\mathcal{A}^a_\alpha| = 0$. Similarly, onc= e we have $\mathcal{A}^a_\alpha| = 0$ and $D_\alpha\Omega^a| = 0$, it follows from $\delta_{\rm g}(iD^\alpha\mathcal{A}^a_\alpha)| = D^2\Omega^a| + if^{abc}D^\alpha(\mathcal{A}^b_\alpha\Omega^c)|$ that, to have $D^\alpha\mathcal{A}^a_\alpha| = 0$, it is enough to take $D^2\Omega^a| = 0$. Note, however, that no restriction has been imposed on $\Omega^a|$, which according to eqs. (93) and (113) is the only component of Ω^a that enters in $\delta_{\rm g}A^a_\mu$ and $\delta_{\rm g}\lambda^a$. Then, in the Wess-Zumino gauge (139),

$$A^a_{\alpha\beta} = \mathcal{A}^a_{\alpha\beta}\Big| = D_{(\alpha}\mathcal{A}^a_{\beta)}\Big| \qquad \lambda^a_\alpha = \frac{i}{2}\,W^a_\alpha\Big| = -\frac{i}{2}\,D^\beta D_\alpha\mathcal{A}^a_\beta\Big| \tag{140}$$

and the action becomes

$$\begin{aligned} S_{\rm CS} = \frac{2ic_1}{g^2}\int d^3x\,\Big[&(D^2D^\alpha\mathcal{A}^{a\beta})\,(D_\beta\mathcal{A}^a_\alpha) + (D^\gamma D^\alpha\mathcal{A}^{a\beta})\,(D_\gamma D_\beta\mathcal{A}^a_\alpha) \\ &- \frac{2i}{3}\,f^{abc}(D^\gamma\mathcal{A}^{a\alpha})\,(D_\gamma\mathcal{A}^{b\beta})\,(D_\beta\mathcal{A}^a_\alpha)\Big]\Big| \,. \end{aligned} \tag{141}$$

Furthermore, using eqs. (75) and (76) to derive

$$D^2D_\alpha\mathcal{A}_\beta = 2i\partial_\alpha{}^\gamma D_\gamma\mathcal{A}_\beta \qquad D^2\mathcal{A}_\alpha = 2D^\gamma D_\alpha\mathcal{A}_\gamma - 2i\,\partial_\alpha{}^\gamma\mathcal{A}_\gamma \;, \tag{142}$$

and noting

$$D_\gamma D_\beta\mathcal{A}_\alpha = i\partial_{\gamma\beta}\mathcal{A}_\alpha - \frac{1}{2}\,\epsilon_{\gamma\beta}\,D^2\mathcal{A}_\alpha \qquad D_\alpha\mathcal{A}_\beta = D_{(\alpha}\mathcal{A}_{\beta)} - \frac{1}{2}\,\epsilon_{\alpha\beta}\,D^\gamma\mathcal{A}_\gamma \;, \tag{143}$$

we have

$$\begin{aligned} S_{\rm CS} = \frac{2ic_1}{g^2}\int d^3x\,\Big[&2i\,(\partial_\alpha{}^\gamma\mathcal{A}^a{}_\gamma{}^\beta)\,\mathcal{A}^a{}_\beta{}^\alpha + (D^\gamma D^\alpha\mathcal{A}^a_\gamma)\,(D^\beta D_\alpha\mathcal{A}^a_\beta) \\ &+ \frac{2i}{3}\,f^{abc}\mathcal{A}^a{}_\alpha{}^\gamma\,\mathcal{A}^b{}_\gamma{}^\beta\,\mathcal{A}^c{}_\beta{}^\alpha\Big]\Big| \,. \end{aligned} \tag{144}$$

Finally, recalling eq. (140) and taking $c_1 = 1/16$, we arrive at

$$S_{\rm CS} = \frac{1}{g^2}\int d^3x\,\left[\epsilon^{\mu\nu\rho}\left(\frac{1}{2}\,A^a_\mu\partial_\nu A^a_\rho + \frac{1}{6}\,f^{abc}\,A^a_\mu A^b_\nu A^c_\rho\right) - \frac{1}{2}\,\bar\lambda^a\lambda^a\right] \,. \tag{145}$$

This is the component action of eq. (46). Here we have used the Wess-Zumino gauge (139) to derive the component form of action from the superfield form (137). We must emphasize, though, that the same component action is obtained if one does not make assumptions about the components of the superfield $\mathcal{A}^a_\alpha$. To prove this, one directly integrates (137) over $d^2\theta$ using $d^2\theta = D^2|$ and expresses everything in terms of $\mathcal{A}^{\alpha\beta}$ and W^a_α. It is very important to keep this in mind since in Section 2.2 we will work in a supersymmetric gauge which imposes different conditions on $\mathcal{A}^a_\alpha|$ and $D^a\mathcal{A}^a_\alpha|$.

One may define the components of the superfield $\mathcal{A}^a_\alpha$ by

$$\chi^a_\alpha = \mathcal{A}^a_\alpha\Big| \qquad V^a_{\alpha\beta} = D_{(\alpha}\mathcal{A}^a_{\beta)}\Big| \qquad H^a = \frac{1}{2}\,D^\alpha\mathcal{A}^a_\alpha\Big| \qquad \Lambda^a = \frac{i}{2}\,D_\alpha D^\beta\mathcal{A}^a_\beta\Big| \,. \tag{146}$$

Note that up to this moment we have not used any explicit form for the spinor superfield $\mathcal{A}^a_\alpha$ as an expansion in powers of θ^a, but only the fact that it contains a vector $V^a_{\alpha\beta}$. The definition of the components given here reproduces the θ-expansion in eq. (85). Using the expressions of $\mathcal{A}^a_{\alpha\beta}$ and W^a_α in eqs. (100) and (109), the physical fields A^a_μ and λ^a_α can be written in terms of χ^a_α, H^a, V^a_μ and Λ^a_α as

$$A^a_\mu = V^a_\mu + \frac{1}{4} f^{abc} \bar{\chi}^b \gamma_\mu \chi^c \tag{147}$$

$$\lambda^a = \Lambda^a - \partial\!\!\!/ \chi^a + \frac{1}{2} f^{abc} H^b \chi^c - \frac{1}{2} f^{abc} A\!\!\!/^b \chi^c + \frac{1}{24} f^{abc} f^{cde} \gamma^\mu \chi^b (\bar{\chi}^d \gamma_\mu \chi^e) \tag{148}$$

The transformation laws of χ^a, H^a, $V^a\mu$ and $\Lambda^a\alpha$ under SUSY as given by $\delta = \epsilon^\beta Q_\beta$ are linear in fields and read

$$\begin{aligned} \delta\chi^a &= V\!\!\!/^a \epsilon - H^a \epsilon \\ \delta H^a &= -\bar{\epsilon}\Lambda^a \\ \delta V^a_\mu &= \bar{\epsilon}\gamma_\mu(\Lambda^a - \partial\!\!\!/ \chi^a) + \bar{\epsilon}\partial_\mu \chi^a \\ \delta\Lambda^a &= \partial_\mu V^{a\mu} \epsilon - \partial\!\!\!/ H^a \epsilon \ . \end{aligned} \tag{149}$$

From these and the expressions in eqs. (147) and (148), we get the transformations rules

$$\begin{aligned} \delta A^a_\mu &= \bar{\epsilon}\gamma_\mu \lambda^a + \bar{\epsilon}(D_\mu \chi)^a \\ \delta\lambda^a &= -\frac{1}{2}\gamma^\mu \gamma^\nu F^a_{\mu\nu} \epsilon + f^{abc} \lambda^b (\bar{\epsilon}\chi^c) \end{aligned} \tag{150}$$

By subtracting a gauge transformation with parameter $\bar{\epsilon}\chi^a$, we obtain the usual x-space rules (41) for A^a_μ and λ^a. The same result is obtained in superspace if one adds a compensating gauge transformation which keeps one in the Wess-Zumino gauge $\chi^a = H^a = 0$.

1.11 Higher N models and gauge couplings to matter

One can construct rigidly supersymmetric models with $N \leq 8$ SUSY. One way to obtain them is by dimensional reduction from the $d = 3 + 1$ models where rigid SUSY exists for $N \leq 4$. For example, the $N = 2$ Wess-Zumino model in $d = 2 + 1$ corresponds to the $N = 1$ model in $d = 3 + 1$ and contains two real spinors, two real scalars and two auxiliary fields. It can clearly be written in complex notation as a model with one complex scalar, one complex spinor and one complex auxiliary field. The reader may check that

$$S\mathrm{WZ} = \int d^3x \left[-(\partial_\mu \varphi^\dagger)(\partial^\mu \varphi) - \bar{\psi}\partial\!\!\!/ \psi + F^\dagger F \right] \tag{151}$$

is invariant under

$$\delta\varphi = \bar{\epsilon}\psi \qquad \delta\psi = \partial\!\!\!/ \varphi \epsilon + F\epsilon \qquad \delta F = \bar{\epsilon}\partial\!\!\!/ \psi \ . \tag{152}$$

One may consider ϵ and $\bar{\epsilon}$ as independent parameters, and consider separately the variations of the action with ϵ and $\bar{\epsilon}$. For example, for the variation with ϵ, one finds

$$(\Box\varphi)\,\bar{\psi}\epsilon - \bar{\psi}\not{\partial}(\not{\partial}\varphi\epsilon + F\epsilon) - F(\partial_\mu\bar{\psi})\gamma^\mu\epsilon \ , \tag{153}$$

which clearly cancels after partial integration.

Similarly, one can write the action for the $N = 2$ Yang-Mills and Chern-Simons models. In this case, the $N=2$ multiplet consists of the gauge field A^a_μ, two real spinors λ^a_i $(i = 1, 2)$ and two real auxiliary fields C^a and D^a, and the actions have the form

$$\begin{aligned} S_{\rm YM} = \frac{1}{mg^2}\int d^3x \Big[& -\frac{1}{4}\,F^a_{\mu\nu}F^{a\mu\nu} - \frac{1}{2}\,\bar{\lambda}^a_1(\not{D}\lambda_i)^a \\ & -\frac{1}{2}\,(D_\mu C)^a(D^\mu C)^a + \frac{1}{2}\,(D^a)^2 - \frac{1}{2}f^{abc}\epsilon_{ij}\bar{\lambda}^a_i\lambda^b_j C^c\Big] \end{aligned} \tag{154}$$

and

$$S_{\rm CS} = \frac{1}{g^2}\int d^3x \left[\epsilon^{\mu\nu\rho}\left(\frac{1}{2}\,A^a_\mu\partial_\nu A^a_\rho + \frac{1}{6}\,f^{abc}\,A^a_\mu A^b_\nu A^c_\rho\right) - \frac{1}{2}\,\bar{\lambda}^a\lambda^a + C^a D^a\right] \ , \tag{155}$$

where $\epsilon_{12} = 1$. The SUSY transformation rules that leave these action invariant are

$$\begin{aligned} \delta A^a_\mu &= \bar{\epsilon}_i\gamma_\mu\lambda^a_i \\ \delta\lambda^a_i &= -\frac{1}{2}\,\gamma^\mu\gamma^\nu F^a_{\mu\nu}\epsilon_i + \epsilon_{ij}D^a\epsilon_j + \epsilon_{ij}\not{D}C^a\epsilon_j \\ \delta C^a &= -\epsilon_{ij}\bar{\epsilon}_i\lambda^a_j \\ \delta D^a &= -\epsilon_{ij}\bar{\epsilon}_i\not{D}\lambda^a_j + f^{abc}\bar{\epsilon}_i\lambda^b_i C^c \ . \end{aligned} \tag{156}$$

It is also possible to set a superfield formalism for $N=2$ SUSY, Aragone (1983) and Ivanov (1991), but we will not discuss this here. The $N=2$ actions $S_{\rm YM}$ and $S_{\rm CS}$ can also be obtained from a truncation of corresponding $N=3$ actions in 2+1 dimensions Kao, Lee, and Lee (1996).

> **Exercise 8:** Verify that S_{YM} and $S_{\rm CS}$ in (154) and (155) are invariant under the transformations (156).

So far we have discussed supersymmetric models for scalar fields and for gauge fields. It possible to construct supersymmetric models for matter fields coupled to gauge fields. Although this subject lies outside the scope of these notes, let us briefly mention how to couple the $N=1$ Wess-Zumino model for scalars to gauge fields while preserving SUSY. To do this, one puts the scalars in a particular representation R with generators $(T^a)^i{}_j$ and replaces in x-space the ordinary derivatives in the action and the SUSY transformation rules by gauge-covariant derivatives. This takes care of minimal coupling to A_μ^a, but since A_μ^a and λ^a form a SUSY multiplet, one also needs a "minimal" coupling of λ^a to the scalar multiplet. This leads to additional Yukawa couplings of the form $\bar{\lambda}\psi\varphi$. The complete model reads

$$S = \int d^3x \left[-\frac{1}{2} (D_\mu \varphi)^i (D^\mu \varphi)^i - \frac{1}{2} \bar{\psi}^i \not{D} \psi^i + \frac{1}{2} F^2 - \bar{\lambda}^a \varphi^i (Ta)^i{}_j \psi^j \right] \tag{157}$$

where $D_\mu \varphi^i = \partial_\mu \varphi^i + g A_\mu^a (T^a)^i{}_j \varphi^j$. For simplicity we consider the case that the representation $(T_a)^i{}_j$ is real; otherwise one must consider complex fields, *i.e.* an $N=2$ model. Then one finds from the variations of first three terms in the action the following extra terms

$$-\frac{1}{2} (\bar{\psi}^i \gamma^\mu \gamma^\nu \epsilon) F_{\mu\nu}^a (T^a)^i{}_j \varphi^j - (\delta A_\mu^b) \left[g (D_\mu \varphi^i) (T_a)^i{}_j \varphi^j + \frac{1}{2} g \bar{\psi}^i (T_a)^i{}_j \psi^j \right] . \tag{158}$$

The Yukawa coupling, in turn, yields the following variations

$$\frac{1}{2} \bar{\psi}^j (T^a)^j{}_k \gamma^\rho \gamma^\sigma \epsilon F_{\rho\sigma}^a \varphi^k + \dots \tag{159}$$

Exercise 9: Check that all variations indeed cancel.

In superspace this coupling is given by Gates Jr. et al. (1983), Siegel (1979b)

$$S = \frac{1}{8} \int d^3x \, d^2\theta \, (\nabla^\alpha \phi^i) (\nabla_\alpha \phi^i) \tag{160}$$

To find the component action, one may again use $d^2\theta = D^2|$. However, since we already have covariant derivatives ∇_α, it is more convenient to use $d^2\theta = \nabla^2|$. As we have discussed earlier, this gives the same result because the action is gauge invariant. The SUSY rules $\delta\phi = \epsilon^\alpha Q_\alpha \phi$ leave the action invariant if one also uses $\delta \mathcal{A}_\alpha^a = \epsilon^\beta Q_\beta \mathcal{A}_\alpha^a$, but we already saw that it was simpler to use $\delta' \mathcal{A}_\alpha^a = \epsilon^\beta \nabla_\beta \mathcal{A}_\alpha^a$. Hence, also for matter we use $\delta_{\mathrm{SUSY}} \phi = \epsilon^\alpha \nabla_\alpha \phi$, which contains now also gauge transformations, and which becomes now nonlinear in fields, namely it contains Yang-Mills covariant derivatives.

As a last topic, we discuss Euclidean SUSY. Since the Dirac operator $\gamma_E^\mu \partial_\mu$ in Euclidean space is complex, we need complex spinors. Clearly we also need then a complex scalar and a complex auxiliary field. The action

$$S = \int d^3x \left[-\delta^{\mu\nu} (\partial_\mu \varphi^\dagger) (\partial_\nu \varphi) - i \psi^\dagger \not{\partial}_E \psi + F^\dagger F \right]$$

is then hermitean and invariant under

$$\delta\varphi = -i\epsilon^\dagger \psi \qquad \delta\psi = \not{\partial}_{ssE} \varphi \epsilon + F\epsilon \qquad \delta F = i\epsilon^\dagger \not{\partial} \psi \ . \tag{161}$$

The variation of the action contains the terms

$$i\,(\Box\varphi)\,(\psi^\dagger\epsilon) - i\,\psi^\dagger \not{\partial}_E(\not{\partial}_E\psi\epsilon + F\epsilon) - i\,F(\partial_\mu\psi^\dagger)\,\gamma^\mu\epsilon + \text{h.c.} \tag{162}$$

The construction of a hermitian action in Euclidean space is due to Schwinger, and hermitian SUSY actions in Euclidean space were studied by Zumino. One can also abandon hermiticity and introduce two independent complex spinors. This is the approach of Osterwalder-Schrader. Let us consider, extending the work of Nieuwenhuizen and Waldron (1996), the following **continuous** Wick rotation for complex spinors in $d = 2 + 1$ dimensions (or any odd dimension):

$$\begin{aligned} \psi(\vec{x},t) \;\to\; \psi_\theta(\vec{x},t_\theta) &\equiv S(\theta)\,\psi(\vec{x},t_\theta) \\ (\theta) = e^{\pm\frac{1}{2}\gamma^0\theta} \qquad & t_\theta = e^{-i\theta}t \end{aligned} \tag{163}$$

Performing this substitution in Minkowski theory yields the Euclidean theory.

2 Quantum rigid supersymmetry: Yang-Mills-Chern-Simons theory

2.1 Supersymmetric regularization of gauge theories

Supersymmetric gauge theories contain two symmetries: rigid SUSY and Yang-Mills gauge invariance. If one covariantly quantizes these theories, one must add to the classical action gauge fixing terms and a ghost terms. Then gauge invariance is replaced with a rigid BRST symmetry. In principle, one can use any gauge fixing term to fix the gauge invariance. The usual Lorentz gauge fixing term $-\frac{1}{2\alpha}\,(\partial^\mu A^a_\mu)^2$ for supersymmetric gauge theories in four dimensions has been used in Capper, Jones, and Nieuwenhuizen (1980). It breaks SUSY but one can still derive Ward identities and study whether they are satisfied at the quantum level. Here, we shall use gauge fixing terms which are themselves invariant under rigid SUSY, so that the corresponding ghost terms are also SUSY invariant. The resulting gauge-fixed classical action will then have two rigid symmetries: SUSY and BRST symmetry. To compute the effective action perturbatively, one must evaluate Feynman graphs, and one must regulate the divergences which many of these graphs possess.

We shall study the two most used regularization schemes: ordinary 't Hooft-Veltman dimensional regularization (DReG), Hooft and Veltman (1972), and Siegel's regularization by dimensional reduction (DReD), Siegel (1979a).

The DReG scheme is formulated in $d > 3$ dimensions and **treats** ϵ-tensors $\epsilon^{\mu\nu\rho}$ as essentially three-dimensional objects. This leads to two kinds of indices, three-dimensional and $(d-3)$-dimensional, and the $SO(d-1,1)$ symmetry of the action is broken down to $SO(2,1)\times SO(d-3)$. Since in $d > 3$ the number of bosons and fermions is no longer equal, one may violate SUSY and one cannot use superfields. We must then use a component action in x-space. We show below that this prescription for $\epsilon^{\mu\nu\rho}$ yields a consistent regularization which manifestly preserves BRST invariance. The DReD scheme is formulated for $d < 3$. One works at all times with superfields and one performs first all θ-integrals. The final momentum integrals are then treated as in DReG. Because the algebra of the Feynman superdiagrams is performed with superfields, DReD manifestly preserves SUSY. However, as we shall see later, it may violate BRST.

Thus DReG may violate SUSY but it preserves BRST. On the other hand, DReD preserves SUSY but it may violate BRST. Our main result is that for supersymmetric Yang-Mills-Chern-Simons theory, both schemes give the same effective action, hence each scheme preserves both SUSY and BRST. In other words, for this model at least, the superfield approach preserves "gauge invariance", rather BRST symmetry, at the quantum level.

Let us describe in more detail 't Hooft-Veltman's prescription for $\epsilon^{\mu\nu\rho}$. Following Breitenlohner and Maison (1977), we consider n-dimensional Minkowski spacetime $\mathbb{M}^n$ with metric $\eta_{\mu\nu}$ and decompose it as $\mathbb{M}^n = \mathbb{M}^3 \otimes \mathbb{R}^{n-3}$. Here $n \leq 3$. We call $\tilde{\eta}_{\mu\nu}$ and $\hat{\eta}_{\mu\nu}$ to the metric on $\mathbb{M}^3$ and $\mathbb{R}^{n-3}$, respectively. Any vector v^μ will have a projection $\tilde{v}^\mu = \tilde{\eta}^{\mu\nu} v_\nu$ onto $\mathbb{M}^3$ and a projection $\hat{v}^\mu = \hat{\eta}^{\mu\nu} v_\nu$ onto $\mathbb{R}^{n-3}$. One may define $\epsilon^{\mu\nu\rho}$ in n dimensions as a completely antisymmetric object in its indices which satisfies

$$\epsilon^{\mu\nu\rho}\epsilon_{\alpha\beta\gamma} = f(n)\,(\delta^\mu{}_\alpha\,\delta^\nu{}_\beta\,\delta^\rho{}_\gamma + 5 \text{ terms}) \ , \tag{164}$$

where $f(n)$ is a function of n such that $f(3) = -1$ and δ^μ_ν is n-dimensional. Consider now three ϵ-tensors contracted in two different ways:

$$(\epsilon^{\mu\nu\rho}\epsilon_{\alpha\nu\rho})\,\epsilon_{\mu\beta\gamma} \qquad \text{and} \qquad \epsilon^{\mu\nu\rho}\,(\epsilon_{\alpha\nu\rho}\epsilon_{\mu\beta\gamma}) \ . \tag{165}$$

The result should be the same for consistency. It is easy to check that for the first contraction eq. (164) yields $f(n)(n-1)(n-2)\epsilon_{\alpha\beta\gamma}$, while for the second contraction it gives $2f(n)\epsilon_{\alpha\beta\gamma}$. Clearly, for $n \neq 3$, both contractions disagree. This shows that the definition of $\epsilon^{\mu\nu\rho}$ provided by eq. (164) is not algebraically consistent, and suggests to treat $\epsilon^{\mu\nu\rho}$ as a three-dimensional object. Thus we replace eq. (164) with

$$\epsilon^{\mu\nu\rho}\epsilon_{\alpha\beta\gamma} = \left(\tilde{\delta}^\mu{}_\alpha\,\tilde{\delta}^\nu{}_\beta\,\tilde{\delta}^\rho{}_\gamma + 5 \text{ terms}\right) \qquad \epsilon_{\mu\nu\rho}\hat{v}^\rho = 0 \ . \tag{166}$$

Then the inconsistency above is no longer present. This is the 't Hooft-Veltman prescription, Hooft and Veltman (1972) and Breitenlohner and Maison (1977), and amounts to treating $\epsilon^{\mu\nu\rho}$ as three-dimensional. Quantities with a caret vanish at $n=3$ and are called **evanescent**.

A similar result holds for γ_5 in four dimensions. Suppose one were to consider in n dimensions the existence of n Dirac matrices γ_μ satisfying $\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}$ and $\{\gamma_5, \gamma_\mu\} = 0$, with $\eta_{\mu\nu}$ the n-dimensional Minkowski metric. Then one finds for $n \neq 4$ again an inconsistency. The proof proceeds by evaluating traces of one γ_5 and a set of Dirac matrices in two ways: one by commuting a Dirac matrix through the others, and one by using cyclicity of the trace. For $\mathrm{tr}\,(\gamma_5\gamma_\lambda\gamma^\lambda)$ and

$\mathrm{tr}\,(\gamma_5\gamma_\lambda\gamma_\mu\gamma^\lambda)$, these two ways to compute traces give

$$
\begin{aligned}
\mathrm{tr}\,(\gamma_5\gamma_\lambda\gamma^\lambda) &= n\,\mathrm{tr}\,\gamma_5 \\
&= \mathrm{tr}\,(\gamma^\lambda\gamma_5\gamma_\lambda) = -\mathrm{tr}\,(\gamma_5\gamma^\lambda\gamma_\lambda) = -n\,\mathrm{tr}\,\gamma_5
\end{aligned}
\tag{167}
$$

and

$$
\begin{aligned}
\mathrm{tr}\,(\gamma_5\gamma_\lambda\gamma_\mu\gamma^\lambda) &= -\mathrm{tr}\,(\gamma_5\gamma_\mu\gamma_\lambda\gamma^\lambda) + 2\,\mathrm{tr}\,(\gamma_5\gamma_\mu) = (2-n)\,\mathrm{tr}\,(\gamma_5\gamma_\mu) \\
&= \mathrm{tr}\,(\gamma^\lambda\gamma_5\gamma_\lambda\gamma_\mu) = -n\,\mathrm{tr}\,(\gamma_5\gamma_\mu)\ .
\end{aligned}
\tag{168}
$$

Hence $\mathrm{tr}\,\gamma_5 = \mathrm{tr}\,(\gamma_5\gamma_\mu) = 0$ in n dimensions. Proceeding in the same way for $\mathrm{tr}\,(\gamma_5\gamma_\lambda\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}\gamma^\lambda)$ and using $\mathrm{tr}\,\gamma_5 = \mathrm{tr}\,(\gamma_5\gamma_\mu) = 0$, one finds that

$$
(2n-8)\,\mathrm{tr},(\gamma_5\gamma_{\mu_1}\gamma_{\mu_2}\gamma_{\mu_3}\gamma_{\mu_4}) = 0\ . \tag{169}
$$

This implies that $\mathrm{tr}\,(\gamma_5\gamma_{\mu_1}\dots\gamma_{\mu_4}) = 0$ for $n \neq 4$, but at $n = 4$ the result is nonzero, so the limit $n \to 4$ would be discontinuous. Since the only two assumptions made are $\{\gamma_\mu,\gamma_\nu\} = 2\eta_{\mu\nu}$ and $\{\gamma_5,\gamma_\mu\} = 0$, and one wants to keep $\{\gamma_\mu,\gamma_\nu\} = 2\eta_{\mu\nu}$, one is led to give up a fully anticommuting γ_5. The prescription of 't Hooft and Veltman, Hooft and Veltman (1972), studied in detail by Breitenlohner and Maison Breitenlohner and Maison (1977), takes $\{\gamma_5,\tilde{\gamma}_\mu\} = 0$ but $[\gamma_5,\tilde{\gamma}_\mu] = 0$, where $\tilde{\gamma}_\mu$ denotes the first four Dirac matrices, while $\hat{\gamma}_\mu$ denotes the extra $n-4$ Dirac matrices. So γ_5 is the usual product of the first four Dirac matrices, even in n dimensions.

2.2 Supersymmetric Yang-Mills-Chern-Simons theory

We consider now the following gauge-fixed action in three dimensions Ruiz Ruiz and Nieuwenhuizen (1997):

$$
?_0 = S_{\mathrm{CS}} + S_{\mathrm{YM}} + S_{\mathrm{GF}} + S_{\mathrm{ES}}\ . \tag{170}
$$

where S_{GF} is the gauge fixing term in the action and S_{ES} contains the nonlinear BRST transforms. We work in the Landau gauge, characterized by the condition $D^\alpha\mathcal{A}^a_\alpha = 0$. In this gauge the gauge-fixing action is

$$
S_{\mathrm{GF}} = \int d^3x\, d^2\theta\, \left[B^a(D^\alpha\mathcal{A}^a_\alpha) - i\,\hat{C}_a(D^\alpha\nabla_\alpha C)^a\right]\ , \tag{171}
$$

where $B^a(x,\theta)$ is a real superfield Lagrange multiplier imposing the Landau condition, and $\hat{C}^a(x,\theta)$ and $C^a(x,\theta)$ are real antighost and ghost superfields. We have already said that after gauge fixing gauge invariance is replaced by BRST invariance. The BRST transformation laws are given by

$$
s\,(iA^a_\alpha) = (\nabla_\alpha C)^a \qquad sB^a = 0 \qquad s\hat{C}_a = B_a \qquad sC^a = \frac{1}{2}\,f^{abc}C^bC^c\ . \tag{172}
$$

The expression for sA^a_α is as usual obtained by replacing the gauge parameter Ω^a in a gauge transformation by the corresponding ghost field

$$\delta_{\rm g}(i\mathcal{A}^a_{\mathcal{A}}) = (\nabla_\alpha\Omega)^a \;\to\; s(i\mathcal{A}^a_\alpha) = (\nabla_\alpha C)^a \ . \tag{173}$$

The result for sC^a follows from using the expression for $s\mathcal{A}^a_\alpha$ and requiring that $s^2\mathcal{A}^a_\alpha$ vanishes. As a check of the expression above, one may verify that nilpotency on the ghosts, $s^2C^a = 0$, follows from the Jacobi identities for the structure constants. We may define the components of the superfields B^a, $\hat{C}^a$ and C^a by

$$\begin{array}{ccc} b^a = B^a\Big| & c^a = C^a\Big| & \hat{c}^a = \hat{C}^a\Big| \\ \zeta^a_\alpha = iD_\alpha B^a\Big| & \varphi^a_\alpha = D_\alpha C^a\Big| & \hat{\varphi}^a_\alpha = D_\alpha \hat{C}^a\Big| \\ h^a = -\dfrac{i}{2}D^2B^a\Big| & \omega^a = -\dfrac{i}{2}D^2C^a\Big| & \hat{\omega}^a = -\dfrac{i}{2}D^2\hat{C}^a\Big| \ . \end{array} \tag{174}$$

After using $d^2\theta = D^2|$ to integrate over $d^2\theta$, the gauge-fixing action takes then the form

$$\begin{aligned} S_{\rm GF} \;=\; \int d^3x \Big\{ &- b^a\partial_\mu V^{a\mu} - (\partial^\mu\hat{c}^a)\left(\partial_\mu c^a + f^{abc}\, V^b_\mu c^c - \frac{i}{2}\, f^{abc}\,\bar{\chi}^b\gamma_\mu\varphi^c\right) \\ &-\bar{\zeta}^a\Lambda^a - \bar{\hat{\varphi}}^a\Big[\partial\!\!\!/\varphi^a + f^{abc}\Big(i\Lambda^b c^c + \frac{i}{2}\,\gamma^\mu\chi^b\,\partial_\mu c^c \\ &\qquad + \frac{1}{2}\,V\!\!\!\!/^{\,b}\varphi^c - \frac{1}{2}\,H^b\varphi^c + \frac{i}{2}\,\chi^b\omega^c\Big)\Big] \\ &-h^aH^a + \hat{\omega}^a\left(\omega^a + f^{abc}\,H^bc^c - \frac{i}{2}\,f^{abc}\bar{\chi}^b\varphi^c\right)\Big\} \ . \end{aligned} \tag{175}$$

The BRST transformation laws for the components are obtained from those for the superfields and the definition of the components as projections. They read

$$\begin{array}{ll} s\chi^a = -i\varphi^a + f^{abc}\chi^bc^c & sb^a = 0 \\ sA^a_\mu = -(D_\mu c)^a & s\zeta^a = 0 \\ sH^a = -\omega^a - f^{abc}H^bc^c + \dfrac{i}{2}\,f^{abc}\bar{\chi}^b\varphi^c & sh^a = 0 \\ s\lambda^a = f^{abc}\lambda^bc^c & \\ s\hat{c}^a = b^a & sc^a = \dfrac{1}{2}\,f^{abc}c^bc^c \\ s\hat{\varphi}^a = i\zeta^a & s\varphi^a = -f^{abc}\varphi^bc^c \\ s\hat{\omega}^a = h^a & s\omega^a = f^{abc}\omega^bc^c + \dfrac{1}{2}\,f^{abc}\,\bar{\varphi}^b\varphi^c \ . \end{array} \tag{176}$$

The supersymmetry transformations for the components are obtained similarly from $\delta = \epsilon^\alpha Q_\alpha$ and the definition of components as projections. For the components of the gauge supermultiplet they are given in eqs. (149) and (150); for the components of B^a, $\hat{C}^a$ and C^a they take the form

$$\begin{aligned} \delta b^a &= -\bar{\zeta}^a \epsilon \\ \delta \zeta^a &= h^a \epsilon - \partial\!\!\!/\, b^a \epsilon \\ \delta h^a &= \bar{\epsilon}\, \partial\!\!\!/\, \zeta^a \end{aligned} \tag{177}$$

and

$$\begin{aligned} \delta \hat{c}^a &= i\, \bar{\hat{\varphi}}^a \epsilon & \delta c^a &= i\, \bar{\varphi}^a \epsilon \\ \delta \hat{\varphi}^a &= -i\, \partial\!\!\!/\, \hat{c}^a \epsilon + i\, \hat{\omega}^a \epsilon & \delta \varphi^a &= -i\, \partial\!\!\!/\, c^a \epsilon + i\, \omega^a \epsilon \\ \delta \hat{\omega}^a &= i\, \bar{\epsilon}\, \partial\!\!\!/\, \hat{\varphi}^a & \delta \omega^a &= i\, \bar{\epsilon}\, \partial\!\!\!/\, \varphi^a \ . \end{aligned} \tag{178}$$

Exercise 10: Since our starting point was the superfield gauge-fixing term (171) and this is supersymmetric, the component action (175) is supersymmetric. Verify that indeed the SUSY transformations for the components leave S_{GF} invariant.

Exercise 11: Check that the operators s and δ generating the BRST and SUSY transformations in above commute:

$$[s, \delta] = 0. \tag{179}$$

Exercise 12: Verify that S_{GF} can also be written as

$$S_{\mathrm{GF}} = \int d^3x\ s\left(- \hat{c}^a\, \partial_\mu V^{a\mu} + i\, \bar{\hat{\varphi}}^a \Lambda^a - \hat{\omega}^a H^a \right) . \tag{180}$$

We thus see that the gauge-fixing condition $D^\alpha \mathcal{A}^a_\alpha = 0$ is equivalent to the the conditions $H^a = 0$, $\partial^\mu V^a_\mu = 0$ and $\Lambda^a = 0$. Check that these conditions remain invariant under SUSY transformations.

The term S_{ES}

$$S_{\mathrm{ES}} = \frac{i}{2} \int d^3x\, d^2\theta\, \left(\frac{1}{2}\, K^{a\alpha}_{\mathcal{A}}\, s\mathcal{A}^a_\alpha + K^a_C\, sC^a \right) . \tag{181}$$

couples the nonlinear BRST variations $s\mathcal{A}^a_\alpha$ and sC^a to external sources $K^{a\alpha}_{\mathcal{A}}$ and K^a_C. We may define the components of the latter by

$$\begin{aligned} \kappa^a_\alpha &= K^a_{\alpha\,\mathcal{S}} \Big| & \ell^a &= K^a_C \Big| \\ G^a &= -\frac{i}{2}\, D^\alpha K^a_{\alpha\,\mathcal{S}} \Big| & \tau^a_\alpha &= i\, D_\alpha K^a_C \Big| \\ K^a_{\alpha\beta} &= i\, D_{(\alpha} K^a_{\beta)\,\mathcal{A}} \Big| & L^a &= -\frac{i}{2}\, D^2 K^a_C \Big| \ . \\ \sigma^a_\alpha &= -\frac{i}{2}\, D^\beta D_\alpha K^a_{\beta\,\mathcal{A}} \Big| & & \end{aligned} \tag{182}$$

The component expression of $S_{\rm ES}$ is then

$$S_{\rm ES} = \int d^3x \, \left[i\bar\kappa^a s\Lambda^a + K^{a\mu} sV_\mu^a + G^a sH^a + i\bar\sigma^a s\chi^a + \ell^a s\omega^a + i\bar\tau^a s\varphi^a + L^a sc^a \right] . \tag{183}$$

The SUSY transformations for the components of $K_A^{a\alpha}$ and K_C^a are obtained again from the definition of components as projections and the action of $\delta = \epsilon^\alpha Q_\alpha$ on the supersources. They have the form

$$\begin{aligned} &\delta\kappa^a = i \not{K}^a \epsilon + i\, G^a \epsilon && \delta\ell^a = \bar\epsilon\tau^a \\ &dG^a = i\,\bar\epsilon \not{\partial}\, \kappa^a + i\,\bar\epsilon\sigma^a && \delta\tau^a = i \not{\partial}\, \ell^a \epsilon - i\, L^a \epsilon \\ &\delta K_\mu^a = i\,\bar\epsilon\partial_\mu\kappa^a + i\,\bar\epsilon\gamma_\mu\sigma^a && \delta L^a = \bar\epsilon \not{\partial}\, \tau^a \ . \\ &\delta\sigma^a = \frac{i}{2}\gamma^\mu\gamma^\nu\epsilon\,(\,\partial_\mu K_\nu^a - \partial_\nu K_\mu^a\,) && \end{aligned} \tag{184}$$

Power counting for $?_0$ shows that there is only a finite number of superficially divergent diagrams, thus proving that the theory is superrenormalizable. At one loop there are quadratic, linear and logarithmic divergences; at two loops there are linear and logarithmic divergences; and at three loops only logarithmic divergences survive. Furthermore, quadratically divergent one-loop diagrams do not have internal gauge lines and the only primitively divergent one-, two- and three-loop 1PI diagrams are those in Table 1, where $\bar\omega$ denotes the superficial UV degree of divergence of the diagram.

external lines						1 loop	2 loops	3 loops
$\chi\bar\chi$						$\bar\omega = 2$	$\bar\omega = 1$	$\bar\omega = 0$
$\lambda\bar\chi$	A^2	AH	H^2			$\bar\omega = 1$	$\bar\omega = 0$	
$\chi\bar\chi A$	$\chi\bar\chi H$	$(\chi\bar\chi)^2$						
$\lambda\bar\lambda$	$\hat c c$	$\hat\varphi\bar\varphi$	$\zeta\bar\chi$			$\bar\omega = 0$		
$\chi\bar\lambda A$	$\chi\bar\lambda H$	A^3	A^2H	AH^2	H^3			
$(\chi\bar\chi)\,(\chi\bar\lambda)$	$\chi\bar\chi A^2$	$\chi\bar\chi AH$	$\chi\bar\chi H^2$					
$(\chi\bar\chi)^2 A$	$(\chi\bar\chi)^2 H$	$(\chi\bar\chi)^3$						

Table 1: Power counting for component fields

2.3 Ward identities, dimensional regularization and regularization by dimensional reduction

The BRST identity for the full renormalized effective action ? takes the form

$$\int d^3x \, \left(\sum_\phi \frac{\delta ?}{\delta\phi}\frac{\delta ?}{\delta K_\phi} + b\,\frac{\delta ?}{\delta\hat c} + i\bar\zeta\,\frac{\delta ?}{\delta\bar{\hat\varphi}} + h\,\frac{\delta ?}{\delta\hat\omega} \right) = 0 \ , \tag{185}$$

where the sum is extended over $\phi^a = \chi^a,\, V_\mu^a,\, H^a,\ \ \Lambda^a,\, c^a,\, \varphi^a,\, \omega^a$. In what follows, we will write this equation as

$$(\,?\,,?\,) = 0 \tag{186}$$

and use the notation Θ for the Slavnov-Taylor operator:

$$\Theta = (?_0, \ ) \ . \tag{187}$$

It is very important to note that the operator Θ commutes with the supersymmetry generator δ:

$$[\Theta, \delta] = 0 \ . \tag{188}$$

If we write for ? a loop expansion

$$? = \sum_{k=0}^{\infty} \hbar^k \, ?_k \tag{189}$$

and substitute it into eq. (186), we obtain at one, two and three loops

$$\begin{aligned} \Theta ?_1 &= 0 \\ \Theta ?_2 + (?_1, ?_1) &= 0 \\ \Theta ?_3 + (?_1, ?_2) + (?_2, ?_1) &= 0 \ . \end{aligned} \tag{190}$$

The SUSY Ward identity for the effective action is

$$\delta ? = 0 \, . \tag{191}$$

We remark that ? generates 1PI Green functions for the fields V_μ^a and Λ^a and not for the elementary fields A_μ^a and λ^a. This is due to the fact that $S_{\rm ES}$ introduces external sources for the BRST variations of V_μ^a and Λ^a, and not for those of A_μ^a and λ^a. To compute ?, we use the Feynman rules for A_μ^a and λ^a and treat V_μ^a and Λ as composite fields defined by (146). It is not difficult to see that, given a 1PI diagram with superficial degree of divergence $\bar{\omega}$, all the diagrams that result from replacing one or more of the external A_μ^a and/or λ^a-lines with any of the composite fields have superficial degree of divergence strictly less than $\bar{\omega}$. Regarding then V_μ^a and Λ^a as composite fields does not worsen power counting.

To define DReG, we follow Giavarini, Martin, and Ruiz Ruiz (1992) and use for $\epsilon^{\mu\nu\rho}$ the definition in eq. (166). Since the terms in the action $?_0$ which are linear and quadratic in the gauge field A_μ^a are the same as in the nonsupersymmetric Landau gauge of Section 1.6, the kinetic matrix is the same as in eq. (52). The propagator matrix is then given by its inverse and has the structure in (53), but now $\Delta_{\mu\nu}(p)$ and $\Delta_\mu(p)$ have the form

$$\begin{aligned} \Delta_{\mu\nu}(p) &= f_1 \, \epsilon_{\mu\sigma\nu} p^\sigma + f_2 \, \eta_{\mu\nu} + f_3 \hat{\eta}_{\mu\nu} + f_4 p_\mu p_\nu + f_5 p_\mu \hat{p}_\nu + f_6 \hat{p}_\mu p_\nu + f_7 \hat{p}_\mu \hat{p}_\nu \\ \Delta_\mu(p) &= f_8 \, p_\mu + f_9 \, \tilde{p}_\mu \ . \end{aligned} \tag{192}$$

The distinction between n-dimensional objects and $(n-3)$-dimensional objects, or equivalently between 3-dimensional and $(n-3)$-dimensional arises from the fact that the definition of $\epsilon^{\mu\nu\rho}$ in eq. (166) is not $SO(n-1,1)$-covariant but rather $SO(2,1)\times SO(n-1)$. After substituting (192) in (53) and imposing (54), we find

$$\Delta_{\mu\nu}(p) = -\frac{g^2\, m}{(p^2-io)^2+m^2\tilde{p}^2}\left[m\,\epsilon_{\mu\rho\nu}\,p^\rho + i\,p^2\eta_{\mu\nu} - i\,p_\mu p_\nu \right.$$
$$\left. + \frac{i\,m^2}{p^2-io}\left(\tilde{p}^2\hat{\eta}_{\mu\nu} + \frac{\hat{p}^2}{p^2}\,p_\mu p_\nu - p_\mu\hat{p}_\nu - \hat{p}_\mu p_\nu + \hat{p}_\mu\hat{p}_\nu\right)\right]. \tag{193}$$

Because, by construction, the propagator is the inverse of the kinetic term in the $n\leq 3$ dimensions and the BRST transformation for the gauge field is the same as in the unregularized theory, DReG preserves BRST invariance, Giavarini et al. (1992) and Breitenlohner and Maison (1977). Hence, the DReG regularized effective action $?^{\rm DReG}$ satisfies the BRST identity

$$(?^{\rm DReG}, ?^{\rm DReG}) = 0\,. \tag{194}$$

The complicated propagator for the gauge field is the price for having a consistent treatment of $\epsilon^{\mu\nu\rho}$ while manifestly preserving BRST invariance. As regards supersymmetry, we have already explained that DReG does not manifestly preserve it. The propagator $\Delta_{\mu\nu}(p)$ can be decomposed into the sum

$$\Delta_{\mu\nu}(p) = D_{\mu\nu}(p) + R_{\mu\nu}(p)\ , \tag{195}$$

of the naive covariant generalization

$$D_{\mu\nu}(p) = -\,g^2 m\ \frac{m\,\epsilon_{\mu\rho\nu}\,p^\rho + i\,p^2\eta_{\mu\nu} - i\,p_\mu p_\nu}{p^2\,(p^2+m^2-io)} \tag{196}$$

to n dimensions of the three-dimensional propagator plus an evanescent term

$$R_{\mu\nu}(p) = -\frac{g^2\,m^3}{(p^2-io)^2+m^2\,\tilde{p}^2}\left[\frac{1}{p^2+m^2-io}\right.$$
$$\times\,\frac{\hat{p}^2}{p^2}\left(m\,\epsilon_{\mu\rho\nu}\,p^\rho + i\,p^2\eta_{\mu\nu} + \frac{i\,m^2}{p^2-io}\,p_\mu p_\nu\right)$$
$$\left. + \frac{i}{p^2-io}\left(\tilde{p}^2\hat{\eta}_{\mu\nu} - p_\mu\hat{p}_\nu - \hat{p}_\mu p_\nu + \hat{p}_\mu\hat{p}_\nu\right)\right]. \tag{197}$$

Note that $R_{\mu\nu}(p)$ is more UV convergent than $\Delta_{\mu\nu}(p)$, but less IR convergent:

$$\begin{array}{lll} \Delta_{\mu\nu}(p)\sim\dfrac{1}{p^2} & R_{\mu\nu}(p)\sim\dfrac{1}{p^4} & \text{for large } p \\ \Delta_{\mu\nu}(p)\sim\dfrac{1}{p} & R_{\mu\nu}(p)\sim\dfrac{1}{p^2} & \text{for small } p\,. \end{array} \tag{198}$$

This will be important in the sequel.

DReD can also be formulated in terms of components. In DReD, all the fields and matrices are kept three-dimensional and the momenta are continued in the sense of ordinary DReG to $d<3$. Because the Dirac algebra is performed in three dimensions, DReD manifestly preserves supersymmetry. The regularized action $c?^{\,\mathrm{DReD}}$ computed with DReD satisfies then

$$\delta ?^{\,\mathrm{DReD}} = 0\,. \tag{199}$$

The BRST transformation for the gauge field in DReD, however, is not the same as in the unregularized theory. Indeed, whereas the first $d<3$ components of the gauge field have the same BRST transformation law as the gauge field in the unregularized theory, the last $3-d$ components transform as $sA_\mu^a = f^{abc}A_\mu^b c^c$. Hence one has to introduce two external sources, one for the first d components of V_μ^a and one for the last $3-d$, which in turns yields a regularized BRST identity different from that in eq. (186). It may happen that at the end of all calculations, once the limit $d\to 3$ has been taken, all effects due to the splitting of the gauge field into d and $3-d$ components go away, but this is not what is meant by manifest BRST invariance. Concerning the well known algebraic inconsistency Siegel (1980) that occurs in products of three or more epsilons in DReD, we mention that it disappears in the limit $d\to 3$, since contributions with three or more epsilons are finite by power counting at $d=3$.

Our goal is to prove that DReG and DReD preserve both supersymmetry and BRST invariance and define the same Green functions Ruiz Ruiz and Nieuwenhuizen (1997). Our strategy is to first prove that the theory is finite to all loop orders, so that the regularized effective actions $?^{\,\mathrm{DReG}}$ and $?^{\,\mathrm{DReD}}$ are also renormalized effective actions and the difference $\Delta ? = ?^{\,\mathrm{DReG}} - ?^{\,\mathrm{DReD}}$ is the difference of two renormalized effective actions. Next we show that this difference vanishes. This, together with the observations that DReG preserves at all stages the BRST identities of local gauge invariance and that DReD preserves supersymmetry, implies the thesis.

One may try to define DReG for pure Chern-Simons theory in a way analogous to the way defined here for Yang-Mills-Chern-Simons theory. In this case, the kinetic matrix has the same form as in eq. (52) with $K^{\rho\mu}(p)$ now given by

$$K^{\rho\mu}_{\mathrm{CS}}(p) = \epsilon^{\rho\sigma\mu}p_\sigma\ . \tag{200}$$

To invert the kinetic matrix in $n\leq 3$ dimensions, one has to use the same $\Delta_{\mu\nu}(p)$ and Δ_μ as in eq. (192), since these are the most general expressions for $\Delta_{\mu\nu}(p)$ and Δ_μ in the propagator matrix allowed by $SO(2,1)\times SO(2-3)$ covariance. It happens, however, that the equation (54) has then as only solution $n=3,\ f_1 = -g^2/p^2$ and $f_8 = 1/p^2$. = In other words, the propagator only exists in three dimensions Martin (1990). This can be understood by noting that, since $\epsilon^{\mu\nu\rho}$ is essentially three-dimensional, in $n>3$ dimensions the kinetic matrix has rows and a columns with all zeros and hence does not have an inverse. To dimensionally regularize Chern-Simons theory, one then has to add a term to the action such that in $n>3$ dimensions it has nontrivial projection onto the $(n-3)$-sector. If, in addition one want to preserve BRST invariance, the added term must be BRST invariant. An obvious candidate is a Yang-Mills term S_{YM} Giavarini et al. (1992), but other higher covariant derivative terms can also be considered, Giavarini, Martin, and Ruiz Ruiz (1993a), Giavarini, Martin, and Ruiz Ruiz (1993b), and Giavarini, Martin, and Ruiz Ruiz (1994). In case one adds a Yang-Mills term, the complete regularization method consists of two regulators: the mass m in $S_{\rm YM}$ and the ϵ from DReG. They should be removed in a very specific order: one must first take the limit $\epsilon \to 0$ and then take $m \to \infty$. Because the limit $\epsilon \to 0$ is finite to all orders in perturbation theory (see next section), it then makes sense to take $m \to \infty$ in the result.

2.4 Perturbative finiteness

To prove perturbative finiteness at one loop, we consider a one-loop 1PI diagram and denote by $\mathcal{D}(d)$ its value in DReG. According to eq. (195), if the diagram has an internal gauge line, $\mathcal{D}(d)$ is the sum of two contributions: $\mathcal{D}(d) = \mathcal{D}_D(d)+\mathcal{D}_R(d)$. The contribution $\mathcal{D}_D(d)$ contains the $SO(d)$ covariant part $D_{\mu\nu}$ of all the gauge propagators[3]. The contribution $\mathcal{D}_R(d)$ contains at least one $R_{\mu\nu}$ and can be easily seen to be both UV and IR finite at $d{=}3$ by power counting. Recall that diagrams with an internal gauge field are at most linearly UV divergent. Being finite at $d{=}3$ and being at least linear in $\hat{\eta}_{\mu\nu}$, $\mathcal{D}_R(d)$ vanishes as $d \to 3$. We are thus left with only the $SO(d)$ covariant contribution $\mathcal{D}_D(d)$. If the diagram has no internal gauge line, $\mathcal{D}(d)$ is already $SO(d)$ covariant. The one-loop $SO(d)$-covariant dimensionally integrals we have are of the form

$$I_{\mu_1\dots\mu_N}(p_e,m,d) = \int \frac{d^dq}{(2\pi)^d}\ \frac{q_{\mu_1}\cdots q_{\mu_N}}{\prod_{r,s}(Q_r^2)^{n_r}\,(Q_s^2{+}m^2)^{n_s}}\ , \tag{201}$$

where Q_r^μ and Q_s^μ are linear combinations of the loop momentum q^μ and the external momenta p_e^μ, and n_r and n_s are nonnegative integers. These integrals do not produce poles when d is analytically continued to a an odd integer, Speer (1974) and Speer (1975). This completes the proof of perturbative finiteness at one loop and shows the result is independent of the number of dimensions in which the Lorentz algebra of the diagrams is performed, which in turn implies that in the limit $d \to 3$ 1PI Green functions at one loop are identical in DReG and in DReD.

At two loops we proceed differently, since two-loop $SO(d)$-covariant integrals have poles in an odd number of dimensions. Let us assume that the two-loop correction $?_2^{\rm DReG}$ to the effective action consists in the limit $d \to 3$ of a divergent part $?_{2,\rm div}^{\rm DReG}$ and a finite part $?_{2,\rm fin}^{\rm DReG}$. Since DReG manifestly preserves BRST invariance, $?_2^{\rm DReG}$ satisfies the BRST identity at two loops

$$\Theta ?_2^{\rm DReG} + (?_1^{\rm DReG}, ?_1^{\rm DReG}) = 0\ . \tag{202}$$

Recalling that $?_1^{\rm DReG}$ is finite, we have that the divergent part $?_{2,\rm div}^{\rm DReG}$ satisfies $\Theta ?_{2,\rm div}^{\rm DReG} = 0$. Because 1PI Feynman diagrams with external sources as external lines are finite by power counting and there are no one-loop subdivergences,

[3]In DReD, and also in DReD, to compute dimensionally regularized integrals, a Wick rotation is performed. This transforms $SO(d-1,1)$ covariance in $SO(d)$ covariance.

$?^{\rm DReG}_{2,\rm div}$ does not depend on the external sources and $\Theta ?^{\rm DReG}_{2,\rm div} = 0$ reduces to $s ?^{\rm DReG}_{2,\rm div} = 0$. Using the power counting in Table 1 and that contributions to two-loop 1PI diagrams from $R_{\mu\nu}$ are finite, we have that the most general form of $?^{\rm DReG}_{2,\rm div}$ is $?^{\rm DReG}_{2,\rm div} = \frac{1}{d-3} P_{\bar\omega_2}$, where

$$P_{\bar\omega_2} = m \int d^3x \, \Big[\alpha_1 \, m \, \bar\chi^a \chi^a + \alpha_2 \, \bar\chi^a \partial\!\!\!/ \chi^a + \alpha_3 \, \bar\chi^a \lambda^a + \alpha_4 \, A^a A^a + \alpha_5 \, H^a H^a + \alpha_6 \, f^{abc} \bar\chi^a A\!\!\!/^b \chi^c + \alpha_7 \, f^{abc} f^{cde} (\bar\chi^a \gamma^\mu \chi^b) \, (\bar\chi^d \gamma_\mu \chi^e) \Big] \tag{203}$$

and $\alpha_1, \ldots, \alpha_7$ are numerical coefficients. The terms in $P_{\bar\omega_2}$ correspond to all two-loop Lorentz invariant divergences that can be constructed from Table 1 with $\bar\omega_2$ derivatives. The equation $s ?^{\rm DReG}_{2,\rm div} = 0$ is an equation in the coefficients α_i whose only solution is $\alpha_i = 0$. This completes the proof at two loops.

The proof at three loops is analogous. Now the only three-loop Lorentz invariant divergence is $?^{\rm DReG}_{3,div} = \frac{1}{d-3} P_{\bar\omega_3}$, with

$$P_{\bar\omega_3} = \alpha m^2 \int d^3x \, \bar\chi^a \chi^a \; , \tag{204}$$

but $P_{\bar\omega_3}$ is not BRST invariant. At higher loops, finiteness follows from power counting and from absence of subdivergences.

2.5 A BRST invariant and supersymmetric effective action

Since the theory is finite, every regularization method defines a renormalization scheme. We consider two renormalization schemes: scheme one uses DReG as regulator and performs no subtractions, scheme two uses DReD and performs no subtractions. We want to prove that the difference $\Delta ? = ?^{\rm DReG} - ?^{\rm DReD}$ between the corresponding renormalized effective actions is zero. We have seen in Section 2.4 that this is the case at one loop. So let us consider the two-loop case.

There is a general theorem in quantum field theory, Hepp (1971) and Epstein and Glasser (1973), that states that if two different renormalization (not regularization) schemes yield the same Green functions up to $k-1$ loops, then at k loops they give Green functions that can differ at most by a local finite polynomial in the external momenta of degree equal to the superficial overall UV degree of divergence $\bar\omega_k$ at k loops. This, and the power counting in Table 1, implies that the difference $\Delta ?_2$ at two loops can at most be of the form

$$?^{\rm DReG}_2 - ?^{\rm DReD}_2 = P_{\bar\omega_2} \; , \tag{205}$$

with $P_{\bar\omega_2}$ as in eq. (203). We recall that $?^{\rm DReG}_2$ satisfies eq. (202) and observe that, since DReD preserves supersymmetry, $?^{\rm DReD}_2$ satisfies

$$\delta ?^{\rm DReD}_2 = 0 \; . \tag{206}$$

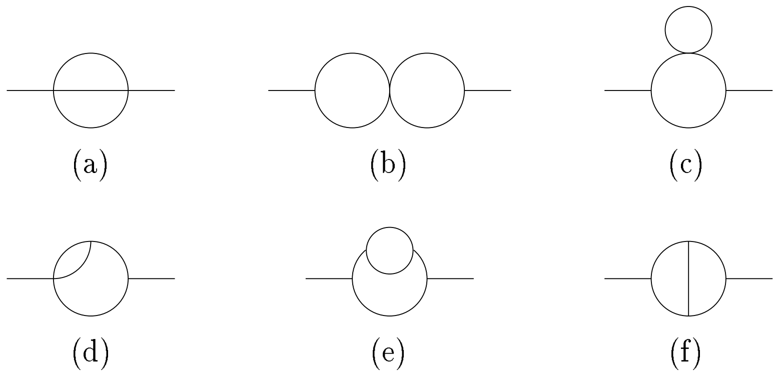

Figure 1: Two-loop topologies for $\langle HH \rangle_{\rm 1PI}$

Acting with δ on eq. (202), using eqs. (205) and (206), and recalling that $[\Theta, \delta] = 0$ and that $\Delta ?_1 = 0$, we obtain that $\Theta \delta P_{\bar{\omega}_2} = 0$. Since $P_{\bar{\omega}_2}$ does not depend on the external sources, $\delta P_{\bar{\omega}_2}$ is independent of the external sources and $\Theta \delta P_{\bar{\omega}_2} = 0$ reduces to $s \delta P_{\bar{\omega}_2} = 0$, which is an equation in the coefficients $\alpha_{=}86, \ldots, \alpha_7$ in $P_{\bar{\omega}_2}$. Because $\delta P_{\bar{\omega}_2}$ depends polynomially on the components of the gauge multiplet and their derivatives and has an overall factor of m, any nontrivial $\delta P_{\bar{\omega}_2}$ satisfying $s \delta P_{\bar{\omega}_2} = 0$ must be m times a BRST invariant of mass dimension two. However, there are no such invariants. Hence, $\delta P_{\bar{\omega}_2} = 0$. The only supersymmetry invariant that can be formed from $P_{\bar{\omega}_2}$ is

$$
\begin{aligned}
P_{\bar{\omega}_2}^{\rm susy} \;\; = \;\; \alpha\, m \int d^3x \, \Big[& \frac{1}{2}\, \bar{\chi}^a \partial\!\!\!/ \chi^a + \bar{\chi}^a \lambda^a + A^a A^a - H^a H^a \\
& - \frac{1}{48}\, f^{abc} f^{cde} (\bar{\chi}^a \gamma^\mu \chi^b)\, (\bar{\chi}^d \gamma_\mu \chi^e) \Big] \, . \qquad (207)
\end{aligned}
$$

At this point we have exhausted all the information given by BRST symmetry and supersymmetry. We determine the value of the coefficient α in $P_{\bar{\omega}_2}^{\rm susy}$ by means of an explicit calculation (see below) and find $\alpha = 0$.

At three loops, the difference is $\Delta ?_3 = \alpha P_{\bar{\omega}_3}$. Since $\Delta ?_3$ is not BRST invariant, nor supersymmetric, the same arguments as used at the two-loop level are now powerful enough to conclude that $\alpha = 0$ without the need of any explicit computation. At higher loops, the difference $\Delta ?$ vanishes since at one, two and three loops it vanishes and there are no overall divergences by power counting.

We now compute α in $P_{\bar{\omega}_2}^{\rm susy}$. To do this, we evaluate the difference between the contributions from DReG and DReD to the selfenergy of the field H^a. The vertices with an H are $H \zeta \chi$, $H \hat{\varphi} \varphi$, $H \hat{\omega} c$ and $H \hat{\varphi} \chi c$ [see eq. (175)]. Using them, one can construct two-loop 1PI diagrams with the six topologies in Fig.1.

In fact, since $\hat{\varphi}$ only propagates in φ and c into $\hat{c}$, and there is no four-point vertex containing the fields H, φ and $\hat{c}$, no graphs with the topology of Fig.1a can be constructed. The topologies in Figs.1b and 1c, being products of one-loop topologies, give the same contributions in DReG as in DReD, hence they do not contribute to α. We are thus left with the topologies in Figs.1d, 1e and 1f. Because one-loop subdiagrams give the same contributions in DReG as in DReD, only the overall divergent part of the corresponding two-loop diagrams contribute to α. Since the two-loop diagrams are logarithmically divergent, contributions to α come from setting in the numerators the external momentum p^μ and the mass m equal to zero, except, of course, for the overall factor m. The overall divergent part of every diagram then reads

$$m \int \frac{d^d k}{(2\pi)^d} \; \frac{d^d q}{(2\pi)^d} \; \frac{N(k,q)}{D(k,q,p,m)} \; . \tag{208}$$

Due to the different propagators that DReG and DReD use for the gauge field, the diagrams with internal gauge lines may give different contributions to α. It happens, however, that such diagrams only occur in topology 1e and that their contributions separately cancel, so that their net contribution in both DReG and DReD is zero.

The other source for different results is the different way in which DReG and DReD treat the Dirac matrices. In fact, the numerator $N(k,q)$ always contains a trace over a fermion loop. This is obvious for those diagrams in which H couples to fermions. The only vertex where H does not couple to fermions is the vertex $H\hat{\omega}c$, but in this case $\hat{\omega}$ propagates into ω and now ω couples to fermions; in fact, closer inspection reveals that no two-loop diagram with this structure can be constructed. It then follows that the overall divergence in DReG and DReD is the same except for the trace over the fermions. Now, the trace of a sum of products of $\not{q}$ and $\not{k}$ can always be written as d-dimensional scalar products k^2, kq and q^2 times an overall trace of the unit matrix. So, after summing over diagrams, α can be written as

$$\alpha = (\,\mathrm{tr}_{\,\mathrm{DReG}}\, 1\!\!1 - \mathrm{tr}_{\,\mathrm{DReD}}\, 1\!\!1\,) \int \frac{d^d k}{(2\pi)^d} \; \frac{d^d q}{(2\pi)^d} \; \frac{f(k^2,kq,q^2)}{D(k,q,p,m)} \; , \tag{209}$$

where $f(k^2,kq,q^2)$ is a polynomial of its arguments. Because the theory is finite, the integral is finite and therefore the difference due to the trace vanishes in the limit $d \to 3$. Hence $\alpha = 0$.

The equality of $?^{\,\mathrm{DReG}}$ and $?^{\,\mathrm{DReD}}$ is not explained by local quantum field theory. One possible explanation might be that there exists a third, as yet unknown, symmetry of the model. Another explanation might be that the existing theorems of local quantum field theory, Hepp (1971) and Epstein and Glasser (1973), concerning the difference between the renormalized expressions for the same Green function computed in two different renormalization schemes can be sharpened for finite models which are superrenormalizable by power counting and which have symmetries.

Our analysis relies on the fact that our model is superrenormalizable by power counting and finite. There exist several all-loop finite supersymmetric models in four dimensions, Ermushev, Kazakov, and Tarasov (1987), Kazakov (1986), Lucchesi, Piguet, and Sibold (1988b), and Lucchesi, Piguet, and Sibold (1988a), and $N=4$ Yang-Mills theory is also all-loop finite. It would be interesting to apply the methods developed here to these models. See Capper et al. (1980) for a partial comparison of DReG and DReD in 4-dimensional $N=1$ Yang-Mills theory in a non-supersymmetric gauge.

3 Classical supergravity

3.1 Supergravity in $(2+1)$-dimensional x-space

We discuss $N=1$ (simple) supergravity in 2+1 dimensions, first in x-space and then in superspace. Euclidean supergravity differs at some essential points from Minkowski supergravity, having to do with the different way in which real spinors are described in Euclidean space. Superspace supergravity in 2+1 dimensions is perhaps a bit too easy as compared with the (3+1)-dimensional case, since there are no $N=1$ chiral superfields and as a consequence there are no representation preserving constraints and no prepotentials, but it is an excellent introduction to the subject, and the student who has understood it, can always afterwards tackle the (3+1)-dimensional case.

The gravitational field is described by the vielbein[4] field $e_\mu{}^m$, with $\mu = 0,1,2$ and $m = 0,1,2$, which satisfies $e_\mu{}^m e_\nu{}^n \eta_{mn} = \eta_{\mu\nu}$, with $\eta_{mn} = \mathrm{diag}\,(-1,+1,+1)$ the Minkowski metric. In flat spacetime $e_\mu{}^m = \delta_\mu{}^m$. One can use a local Lorentz transformation to make $e_{\mu m}$ symmetric, and then it is transversal and traceless on-shell. This shows that on-shell $e_\mu{}^m$ contains no degrees of freedom. The same argument shows that in 3+1 dimensions there are two graviton states.

As the fermionic partner of $e_\mu{}^m$ we choose the gravitino field $\psi_\mu{}^\alpha$, with $\alpha = 1,2$. It is the gauge field for local supersymmetry, so it transforms as $\delta\psi_\mu = \partial_\mu \epsilon + \ldots$ This, in fact, is the best reason for choosing $\psi_\mu{}^\alpha$ and not, for example, $\psi_{\alpha\beta\gamma}$ which also contains a spin 3/2 part.

As gravitational action we take the Hilbert action

$$I_2 = -\frac{1}{8\kappa^2} \int d^3x \, eR \ , \tag{210}$$

where κ^2 is the gravitational constant, with mass dimension $[\kappa^2] = -1$ in 3 dimensions, $e = \det\,(e_\mu{}^m) = \sqrt{-g}$,

$$R = R_{\mu\nu}{}^{mn}(\omega)\, e_m{}^\nu e_n{}^\mu \tag{211}$$

is the the Ricci scalar and

$$R_{\mu\nu}{}^{mn}(\omega) = \partial_\mu\, \omega_\nu{}^{mn} + \omega_\mu{}^m{}_k\, \omega_\nu{}^k{}_n - (\mu \leftrightarrow \nu) \ , \tag{212}$$

[4]In three dimensions the name “dreibein” is also used, drei means three in German.

is the Riemann tensor. We choose the normalization factor 1/8 in order to obtain, for constant ϵ's, the same SUSY algebra as for the matter fields. In units $\hbar = 1$, the action is dimensionless. It is also Einstein, or general coordinate, and local Lorentz invariant, the corresponding transformation being

$$\begin{aligned}
\delta_{\rm E} e_\mu{}^m &= \xi^\nu \partial_\nu e_\mu{}^m + (\partial_\mu \xi^\nu)\, e_\nu{}^m \\
\delta_{\rm L} e_\mu{}^m &= \lambda^m{}_n\, e_\mu{}^n \\
\delta_{\rm E} \omega_\mu{}^{mn} &= \xi^\nu\, \partial_\nu \omega_\mu{}^{mn} + (\partial_\mu \xi^\nu)\, \omega_\nu{}^{mn} \\
\delta_{\rm L} \omega_\mu{}^{mn} &= -\partial_\mu \lambda^{mn} - \omega_\mu{}^m{}_k\, \lambda^{kn} - \omega_\mu{}^n{}_k\, \lambda^{mk} \equiv -(D_\mu \lambda)^{mn}\ .
\end{aligned} \tag{213}$$

Exercise 13: Check that

$$D_\mu e_\nu{}^m = \partial_\mu e_\nu{}^m + \omega_\mu{}^m{}_k\, e_\nu{}^k - ?^{\,\rho}_{\mu\nu} e_\rho{}^m \tag{214}$$

is Einstein and local Lorentz covariant. Therefore one may set it equal to zero ("the vielbein postulate"). As a result, one can always express $\omega_\mu{}^{mn}$ in terms of $?_{\mu\nu}{}^\rho$, and vice-versa. We shall exclusively work with $\omega_\mu{}^{mn}$ and never use $?_{\mu\nu}{}^\rho$.

As action for the gravitino in curved space we take

$$I_{3/2} = -\frac{1}{2} \int d^3x\, e\, \bar\psi_\mu \gamma^{[\mu} \gamma^\rho \gamma^{\sigma]} D_\rho(\omega)\, \psi_\sigma\ , \tag{215}$$

with

$$D_\rho(\omega)\psi_\sigma = \partial_\rho \psi_\sigma + \frac{1}{4}\, \omega_\rho{}^{mn}\, \gamma_m \gamma_n \psi_\sigma \tag{216}$$

and

$$\gamma^\mu = \gamma^m e_m{}^\mu \qquad\qquad \gamma^m = \gamma^\nu e_\nu{}^m\ . \tag{217}$$

A term with $?_{\rho\sigma}{}^\tau$ in (215) cancels due to (anti)symmetry. For the Dirac matrices γ^m we choose the same real representation as in eq. (8). Then the operator $\gamma^\mu\gamma^\rho\gamma^\sigma D_\rho$ is also real and hence we can take ψ_σ to be real. The action $I_{3/2}$ is hermitian. It is also the unique action without ghosts in any dimension. By the latter statement we mean the following: if one adds a source term $\bar J^\mu \psi_\mu = \bar\psi_\mu J^\mu$ to the most general flat-space free-field expression for $I_{3/2}$,

$$I_{3/2} = -\frac{1}{2} \int d^3x\, \bar\psi_\mu O^{\mu\rho\sigma} \partial_\rho \psi_\sigma\ , \tag{218}$$

where $O^{\mu\rho\sigma}$ depends on Dirac matrices, Minkowski metrics and $\epsilon^{\mu\nu\rho}$ tensors, to couple the couple the gravitino to an external real vector-spinor source $J^{\mu\alpha}$, and one completes squares, one finds the propagator term

$$\int d^3x\, \bar J^\mu_\psi P_{\mu\nu} J^\nu_\psi \tag{219}$$

The source $J^{\mu\alpha}$ is supposed to satisfy those constraints (and only those) which follow from the linearized field equations

$$O^{\mu\rho\sigma}\partial_\rho\psi_\sigma - J^\mu = 0 \tag{220}$$

In general $O^{\mu\rho\sigma}$ is singular, so it has no inverse. However, the parts which are ambiguous, due to the singularity of $O^{\mu\rho\sigma}$, cancel in the propagator term (219) due to the constraints on J_μ. Requiring then that at the poles $k^2 = 0$ the residue is non-negative definite, singles out $O^{\mu\rho\sigma} = \gamma^{[\mu}\gamma^\rho\gamma^{\sigma]}$ in any dimension. The physical meaning of this requirement is tree unitarity, a necessary but not sufficient condition for unitarity. Since the free field action with

$$\int d^3x\, \bar\psi_\mu\, \gamma^{[\mu}\gamma^\rho\gamma^{\sigma]}\,\partial_\rho\psi_\sigma \tag{221}$$

has the gauge invariance $\delta\psi_\sigma = \partial_\sigma\epsilon$, which will later become local SUSY, we have deduced the interesting result that gauge invariance follows from unitarity. The same holds for the actions with spin 1 and spin 2. We can in 2+1 dimensions simplify the gravitino action by using

$$\epsilon^{\mu\nu\rho}\gamma_\rho = -\gamma^{[\mu}\gamma^{\nu]}\ , \tag{222}$$

where the indices are curved indices. Then

$$I_{3/2} = \frac{1}{2}\int d^3x\, \epsilon^{\mu\rho\sigma}\,\bar\psi_\mu\, D_\rho(\omega)\,\psi_\sigma\ . \tag{223}$$

Note that, since $\epsilon^{\mu\rho\sigma}$ is a density, we do not need a factor e in the integrand in (223)). The spin 3/2 action is also Einstein and local Lorentz invariant,

$$\begin{aligned} \delta_{\mathrm{E}}\psi_\mu{}^\alpha &= \xi^\nu\,\partial_\nu\psi_\mu{}^\alpha + (\partial_\mu\xi^\nu)\,\psi_\nu{}^\alpha \\ \delta_{\mathrm{L}}\psi_\mu{}^\alpha &= \frac{1}{4}\,\lambda_{mn}\,\gamma^m\gamma^n\,\psi_\mu{}^\alpha\ . \end{aligned} \tag{224}$$

Let us do the usual counting of states and field components. We already saw that $e_\mu{}^m$ contains no states. The field ψ_μ satisfies the linearized field equation $\epsilon^{\mu\rho\sigma}\partial_\rho\psi_\sigma = 0$, so locally $\psi_\sigma = \partial_\sigma\psi$, with ψ a spin 1/2 field. Local SUSY, $\delta\psi_\mu = \partial_\mu\epsilon + \ldots$, can then be used to gauge away ψ, so also ψ_μ^α contains no states on-shell. However, the number of bosonic and fermionic field components does not match. The dreibein $e_\mu{}^m$ has 3×3 components, but 3 components can be gauged away by local Lorentz symmetry (for example, by making $e_{\mu m}$ symmetric), and another 3 components can be gauged away by Einstein symmetry (for example, by setting $e_{01} = e_{02} = e_{00} = 0$). Thus, in total, there are 3 bosonic field components left in $e_{\mu m}$. The gravitino field has 3×2 components, but local SUSY can be used to gauge away 2 components (for example, by setting $\psi_0{}^\alpha = 0$). Hence there are 4 fermionic components left. Because the operator P_μ is nonsingular and $\{Q_\alpha, Q_\beta\} \sim (\gamma^\mu)_{\alpha\beta}P_\mu$, Q_α is nonsingular. Thus the number of bosonic and fermionic field components must be the same and we need at least one bosonic auxiliary field S. This suggests the following action for $N=1$ supergravity

$$I_{\rm sugra} = \int d^3x \left[-\frac{1}{8\kappa^2}\, eR(e,\omega) + \frac{1}{2}\,\epsilon^{\mu\rho\sigma}\,\bar{\psi}_\mu\, D_\rho(\omega)\,\psi_\sigma - \frac{1}{2}\, eS^2 \right] \ . \qquad (225)$$

It is at this point not clear that the sign of the term with S^2 is negative, and thus opposite to the sign of the auxiliary field in the Wess-Zumino model. We shall see that local SUSY requires this sign. This has important consequences for the super-Higgs effect: the auxiliary fields yield the cosmological term and in supergravity one can obtain vanishing cosmological constant due to cancelations between the matter and gauge sectors. Taking

$$\begin{aligned} \delta_{\rm E} S &= \xi^\nu \partial_\nu S \\ \delta_{\rm L} S &= 0 \ , \end{aligned} \qquad (226)$$

the action (225) is still Einstein and local Lorentz invariant. We must now show that it is locally supersymmetric. First we must specify whether we take the spin connection ${\omega_\mu}^{mn}$ as an independent field or as a composite expression. In some sense we shall do both, as we next explain.

The spin connection ${\omega_\mu}^{mn}$ we do not take as an independent field, but we assume that it is expressed in terms of ${e_\mu}^m$ and ${\psi_\mu}^\alpha$ by solving its own nonpropagating field equation:

$${\omega_\mu}^{mn} = {\omega_\mu}^{mn}(e,\psi) \ . \qquad (227)$$

However, we do not expand ${\omega_\mu}^{mn}(e,\psi)$ in terms of ψ_μ; rather, we keep it as a composite object in the action. The reason is that whenever we **vary** ${\omega_\mu}^{mn}$, it is multiplied by its own field equation $\delta I/\delta{\omega_\mu}^{mn}(x)$, which vanishes identically when we substitute ${\omega_\mu}^{mn}(e,\psi)$ for ${\omega_\mu}^{mn}$. So **we need not vary** ${\omega_\mu}^{mn}$ **at all!** Taking ${\omega_\mu}^{mn}$ as an independent field is known as the first-order formalism, expressing ${\omega_\mu}^{mn}$ in terms of ${e_\mu}^m$ and $\psi_\mu\alpha$ receives the names of second-order formalism or Palatini formalism, and doing the latter but not expanding ${\omega_\mu}^{mn}$ in terms of ψ_μ goes under the name of 1.5 order formalism.

To study the invariance of the action under local SUSY, we recall that we only need to vary the explicit ${e_\mu}^m$ and ψ_μ, but not those ${e_\mu}^m$ and ψ_μ which are contained in ${\omega_\mu}^{mn}(e,\psi)$. For the transformation law of ψ_μ, we take

$$\delta\psi_\mu = \frac{1}{\kappa}\, D_\mu(\omega)\,\epsilon = \frac{1}{\kappa}\left(\partial_\mu\epsilon + \frac{1}{4}\,{\omega_\mu}^{mn}\,\gamma_m\gamma_n\epsilon \right) , \qquad (228)$$

with ${\omega_\mu}^{mn} = {\omega_\mu}^{mn}(e,\psi)$. This is the gravitational covariantization of $\delta\psi_\mu = \frac{1}{\kappa}\,\partial_\mu\epsilon$. The factor $1/\kappa$ is needed in order that the dimensions match: $[\psi_\mu] = [\partial_\mu] = 1$ and $[\epsilon] = -1/2$, so we need $1/\kappa$, since $[1/\kappa] = 1/2$. Of course, $[{\omega_\mu}^{mn}] = 1$, since it contains one derivative [see eq. (214)]. A further term $\sim S\gamma_\mu\epsilon$ will be added later to (228). We shall not postulate $\delta{e_\mu}^m$, but rather derive it by requiring SUSY invariance of the action. Noting that $\delta e = e_m{}^\nu \delta e_\nu{}^m$, we have for the variation of the first term in the action (225)

$$\delta I_2 = -\frac{1}{8\kappa^2} \int d^3x\, R_{\mu\nu}{}^{mn}(\omega)\, \delta[e e_m{}^\nu e_n{}^\mu] - \frac{1}{4\kappa^2} \int d^3x\, e \left(R_\nu{}^m - \frac{1}{2}\, e_\nu{}^m R \right) \delta e_m{}^\nu \ . \tag{229}$$

Using that the two gravitinos in $I_{3/2}$ give the same variation (see exercise below) and

$$[D_\rho(\omega), D_\sigma(\omega)]\, \epsilon = \frac{1}{4}\, R_{\rho\sigma}{}^{mn}(\omega)\, \gamma_m \gamma_n\, \epsilon \ , \tag{230}$$

we find

$$\delta I_{3/2} = \frac{1}{\kappa} \int d^3x\, \epsilon^{\mu\rho\sigma} \bar{\psi}_\mu D_\rho(\omega) D_\sigma(\omega)\, \epsilon = \frac{1}{8\kappa} \int d^3x\, \epsilon^{\mu\rho\sigma} \bar{\psi}_\mu\, R_{\rho\sigma}{}^{mn}(\omega)\, \gamma_m \gamma_n\, \epsilon \ . \tag{231}$$

We simplify $\delta I_{3/2}$ by using the identities

$$\gamma_{[m}\gamma_{n]} = -\epsilon_{mnr}\gamma^r \tag{232}$$

$$\epsilon^{\mu\rho\sigma}\epsilon_{mnr} = -6\, e\, e_{[m}{}^\mu e_n{}^\rho e_{r]}{}^\sigma \ . \tag{233}$$

Then, using that the scalar curvature R is defined by $R_\sigma{}^m\, e_m{}^\sigma$ and introducing the notation $\psi_m = e_m{}^\mu\, \bar{\psi}_\mu$, we get

$$\delta I_{3/2} = \frac{1}{2\kappa} \int d^3x\, e \left(R_\sigma{}^m - \frac{1}{2}\, e_\sigma{}^m R \right) \bar{\psi}_m \gamma^\sigma \epsilon \tag{234}$$

From eqs. (229) and (234) we see that if we choose

$$\delta e_m{}^\nu = 2\kappa\, \bar{\psi}_m \gamma^\nu \epsilon \ , \tag{235}$$

the variations of I_2 and $I_{3/2}$ under local SUSY cancel each other. We can obtain $\delta e_\nu{}^m$ from this result by using that

$$\delta\left(e_\nu{}^m e_m{}^\nu\right) = \left(\delta e_\nu{}^m\right) e_m{}^\nu + e_\nu{}^m \left(\delta e_m{}^\nu\right) = 0 \ . \tag{236}$$

Using also that $\bar{\psi}_m \gamma^\nu \epsilon = -\bar{\epsilon}\gamma^\nu \psi_m$, we find

$$\delta e_\nu{}^m = 2\kappa\, \bar{\epsilon}\gamma^m \psi_\nu \tag{237}$$

One may check that the factor κ is again needed for dimensions: $[e_\nu{}^m] = 0$, $[\epsilon] = -1/2$, $[\psi_\nu] = 1$ and $[\kappa] = -1/2$.

Exercise 14: Show that for any $\delta\psi_\rho$,

$$\int d^3x\, \epsilon^{\mu\rho\sigma}\, \bar{\psi}_\mu D_\rho(\omega)\, \delta\psi_\sigma = \int d^3x\, \epsilon^{\mu\rho\sigma}\, \delta\bar{\psi}_\mu\, D_\rho(\omega)\, \psi_\sigma \ . \tag{238}$$

Before moving on to the auxiliary field S, let us evaluate the local SUSY commutator on the dreibein:

$$\begin{aligned}[\delta_1, \delta_2]\, e_\mu{}^m &= 2\, \bar\epsilon_2 \gamma^m D_\mu \epsilon_1 - (1 \leftrightarrow 2) \\ &= 2\, \partial_\mu (\bar\epsilon_2 \gamma^m \epsilon_1) + 2\, \left[\frac{1}{4}\, \omega_\mu{}^{rs}\, \bar\epsilon_2 \gamma^m \gamma_r \gamma_s \epsilon_1 - (1 \leftrightarrow 2) \right] \,. \end{aligned} \tag{239}$$

Setting

$$\xi^\nu \equiv 2 \bar\epsilon_2 \gamma^\nu \epsilon_1 \ , \tag{240}$$

we can rewrite this as

$$[\delta_1, \delta_2]\, e_\mu{}^m = (\partial_\mu \xi^\nu)\, e_\nu{}^m + \xi^\nu\, (\partial_\mu e_\nu{}^m) + 2\, [\ldots] \,. \tag{241}$$

Recasting $[\delta_1, \delta_2]\, e_\mu{}^m$ as the expected Einstein transformation of $e_\mu{}^m$ plus other terms, we find

$$[\delta_1, \delta_2]\, e_\mu{}^m = (\partial_\mu \xi^\nu)\, e_\nu{}^m + \xi^\nu\, (\partial_\nu e_\mu{}^m) + \xi^\nu (\partial_\mu e_\nu{}^m - \partial_\nu e_\mu{}^m) + 2\, [\ldots] \tag{242}$$

Because, in ordinary general relativity with Riemannian connection $\omega_\mu{}^{mn}(e)$, the vielbein satisfies the vielbein postulate

$$\partial_\mu e_\nu{}^m + \omega_\mu{}^m{}_n(e)\, e_\nu{}^n - ?\,_{\mu\nu}{}^\rho(g)\, e_\rho{}^m = 0 \ , \tag{243}$$

we can replace the curl $\partial_\mu e_\nu{}^m - \partial_\nu e_\mu{}^m$ by

$$\partial_\mu e_\nu{}^m - \partial_\nu e_\mu{}^m = -\omega_\mu{}^m{}_n(e)\, e_\nu{}^n + \omega_\nu{}^m{}_n(e)\, e_\mu{}^n \tag{244}$$

Hence, the extra terms in the SUSY commutator are

$$-\omega_\mu{}^m{}_n(e)\, \xi^n + \xi^\nu \omega_\nu{}^m{}_n(e)\, e_\mu{}^n + 2\, [\ldots] \ , \tag{245}$$

where we have used the notation $\xi^n = \xi^\nu e_\nu{}^n$. The terms in $2\,[\ldots]$, defined in (239), depend on $\omega_\mu{}^{mn}(e, \psi)$, **not** on $\omega_\mu{}^{mn}(e)$. We can simplify them by using the identity (18) for "flat" Dirac matrices γ^m, γ^r, γ^s. Since $\gamma_{[m}\gamma_r\gamma_{s]}$ is proportional to the unit matrix, and $\bar\epsilon_2 \epsilon_1 = \bar\epsilon_1 \epsilon_2$, the terms $\bar\epsilon_2 \gamma_{[m}\gamma_r\gamma_{s]}\epsilon_1$ and $-\bar\epsilon_1 \gamma_{[m}\gamma_r\gamma_{s]}\epsilon_2$ that arise from using this identity cancel each other, and we find

$$2\, [\ldots] = 2\, \omega_\mu{}^{ms}(e, \psi)\, \bar\epsilon_2 \gamma_s \epsilon_1 = \omega_\mu{}^{ms}(e, \psi) \xi_s \ . \tag{246}$$

Thus we have

$$[\delta_1, \delta_2]\, e_\mu{}^m = \delta_{\rm gc}(\xi^\nu)\, e_\mu{}^m + [\xi^\nu\, \omega_\nu{}^m{}_n(e)]\, e_\mu{}^n + [\omega_\mu{}^{ms}(e, \psi) - \omega_\mu{}^{ms}(e)]\, \xi_s \ . \tag{247}$$

The first term is the Einstein or general coordinate transformation of $e_\mu{}^m$, the local equivalent of the usual translation in $\{Q, Q\} = P$, while the second term is a local Lorentz transformation with composite parameter $\xi^\nu\, \omega_\nu{}^m{}_n(e)$. The last term is quadratic in ψ_μ and is a sum of a local Lorentz transformation, which can be added to the local Lorentz transformation with $\xi^\nu \, \omega_\nu{}^m{}_n(e)$ to yield a local Lorentz transformation with $\xi^\nu \, \omega_\nu{}^m{}_n(e,\psi)$, and a local SUSY transformation with composite parameter $\epsilon \sim \xi^\nu \psi_\nu$, as we now show. The super-uninterested reader may skip the details and go directly to the next section.

In first–order formalism with an independent spin connection $\omega_\nu{}^{mn}$, the variation of the sum $I_2 + I_{3/2}$ with respect to the spin connection reads

$$\begin{aligned} \delta_\omega(I_2 + I_{3/2}) \quad = \quad & \int d^3x \Bigg\{ -\frac{e}{4\kappa^2} \left[D_\mu(\omega) \, \delta\omega_\nu{}^{mn} \right] e_m{}^\nu e_n{}^\mu \\ & \qquad + \frac{1}{8} \, \epsilon^{\mu\nu\rho} \, \bar{\psi}_\mu \, \delta\omega_\nu{}^{mn} \, \gamma_m \gamma_n \, \psi_\rho \Bigg\} \, . \end{aligned} \tag{248}$$

Using eqs. (232) and (233), we obtain, after partial integration, the field equation for $\omega_\mu{}^{mn}$

$$\frac{1}{\kappa^2} \, D_\mu(\omega) \, (e \, e_{[m}{}^\nu e_{n]}{}^\mu) + 3 \, (\bar{\psi}_\mu \gamma^r \psi_\rho) \, (e \, e_{[m}{}^\mu e_n{}^\nu e_{r]}{}^\rho) = 0 \tag{249}$$

From this equation we must find the solution $\omega_\mu{}^{mn}(e,\psi)$. To do this, we split $\omega_\mu{}^{mn}(e,\psi)$ into the torsionless part $\omega_\mu{}^{mn}(e)$ and a torsion piece $\omega_\mu{}^{mn}(\psi)$:

$$\omega_\mu{}^{mn}(e,\psi) = \omega_\mu{}^{mn}(e) + \omega_\mu{}^{mn}(\psi) \ . \tag{250}$$

The tensionless part

$$\begin{aligned} \omega_{\mu\nu\rho}(e) \quad = \quad & \frac{1}{2} \Big[- e_{m\mu}(\partial_\nu e_\rho{}^m - \partial_\rho e_\nu{}^m) \\ & + e_{m\nu}(\partial_\rho e_\mu{}^m - \partial_\mu e_\rho{}^m) - e_{m\rho}(\partial_\nu e_\mu{}^m - \partial_\mu e_\nu{}^m) \Big] \end{aligned} \tag{251}$$

follows from the vielbein postulate (243) and is computed in many textbooks on general relativity. As a check, one may verify that the transformation law for $\omega_\mu{}^{mn}$, as given by (251), under a local Lorentz transformation $\delta e_\mu{}^m = \lambda^m{}_n e_\mu{}^n$ agrees with that in eq. (213). Substituting eqs. (250) and (251) in (249), we find

$$\frac{1}{\kappa^2} \, \omega_{\mu m}{}^k(\psi) \, e_k{}^{[\nu} e_n{}^{\mu]} - (m \leftrightarrow n) = - \Big(\bar{\psi}_m \, \gamma \cdot \psi \, e_n{}^\nu + \bar{\psi}_n \gamma^\nu \psi_m + \bar{\psi} \cdot \gamma \, \psi_n \, e_m{}^\nu \Big) \ . \tag{252}$$

The left-hand side yields four terms

$$\frac{1}{2\kappa^2} \left[\omega_{nm}{}^\nu(\psi) - \omega_{km}{}^k(\psi) \, e_n{}^\nu - \omega_{mn}{}^\nu(\psi) + \omega_{kn}{}^k(\psi) \, e_m{}^\nu \right] \tag{253}$$

Tracing with $e_\nu{}^n$ shows that $\omega_{km}{}^k(\psi) = 2 \, \kappa^2 \bar{\psi}_m \gamma \cdot \psi$ and then the terms with $e_n{}^\nu$ and $e_m{}^\mu$ match. The solution is

$$\omega_{\mu m}{}^n(\psi) = \kappa^2 \left(\bar{\psi}_\mu \gamma_m \psi^n - \bar{\psi}_\mu \gamma^n \psi_m + \bar{\psi}_m \gamma_\mu \psi^n \right) \, . \tag{254}$$

So indeed the last term in eq. (247) is quadratic in ψ_μ. Adding a subtracting a term $\xi^\nu\, \omega_\nu{}^m{}_n(\psi)\, e_\mu{}^n$, we find

$$[\delta_1, \delta_2]\, e_\mu{}^m = \delta_{\rm gc}(\xi^\nu)\, e_\mu{}^m + \delta_{\rm L}[\xi^\nu \omega_\nu{}^{mn}(e,\psi)] e_\mu{}^m + \Big[\omega_\mu{}^m{}_s(\psi) - \omega_s{}^m{}_\mu(\psi)\Big]\xi^s\ . \tag{255}$$

The second term is, as anticipated, a local Lorentz transformation with parameter $\xi^\nu \Omega_\nu{}^m{}_n(e,\psi)$, while the last term is equal to $2\kappa^2 \xi^s \bar\psi_\mu \gamma^m \psi_s$ and is a local SUSY transformation with composite parameter $\epsilon = -\kappa\, \xi^s \psi_s$. The reason for the negative sign is that we defined $\delta\psi_\mu = \frac{1}{\kappa}\,\partial_\mu \epsilon + \dots$ in eq. (228). We can write the result in a uniform way as

$$[\delta_1, \delta_2] = \delta_{\rm gc}(\xi^\nu) + \delta_{\rm L}[\xi^\nu \omega_\nu{}^{mn}(e,\psi)] + \delta_{\rm susy}(-\kappa \xi^\nu \psi_\nu)\ . \tag{256}$$

Thus we have shown that the local SUSY commutator on $e_\mu{}^m$ closes: it is equal to a sum of local gauge transformations of the dreibein.

> **Exercise 15:** Show that in the limit of rigid SUSY, constant ξ^s, the term linear in the fields $h_\mu{}^m \equiv e_\mu{}^m - \delta_\mu{}^m$ and ψ_μ reduce to
>
> $$[\delta_1, \delta_2]\, h_\mu{}^m = \frac{1}{2}\, \xi^\nu \partial_\nu (h_\mu{}^m + h^m{}_\mu) + \frac{1}{2}\, \xi^\nu \partial_\mu (h^m{}_\nu + h_\nu{}^m) - \frac{1}{2}\, \xi^\nu \partial^m (h_{\mu\nu} + h_{\nu\mu})\ . \tag{257}$$
>
> Interpret this result.

3.2 Closure on the gravitino, the auxiliary field S

Let us now add the auxiliary field S to our considerations. Since the variation of the last term in the action (225) yields

$$\delta \int d^3x\, \left(-\frac{1}{2}\, e S^2\right) = \int d^3x\, \left(-\kappa \bar\epsilon \gamma{\cdot}\psi S^2 - e S \delta S\right)\ , \tag{258}$$

we add a term $\delta^{(S)}\psi_\mu = c S \gamma_\mu \epsilon$ to the gravitino law with c a constant to be determined. This yields the following new contribution to the variation of the gravitino action

$$\delta^{(S)} I_{3/2} = c \int d^3x\, S\, \epsilon^{\mu\rho\sigma}\, [\bar\epsilon \gamma_\mu D_\rho(\omega)\, \psi_\sigma]\ . \tag{259}$$

Clearly, the variation of S must be chosen such that the sum of the three S-dependent variations cancels. This leads to

$$\delta S = -\kappa \bar\epsilon \gamma{\cdot}\psi S - \frac{c}{e}\, \epsilon^{\mu\rho\sigma}\, \bar\epsilon\, \gamma_\mu D_\rho(\omega) \psi_\sigma\ . \tag{260}$$

The constant c is still free at this point; the two variations proportional to c constitute an "equation of motion symmetry". The equation of motion for S is $S = 0$).

The extra term $\delta\psi_\mu = cS\gamma_\mu\epsilon$ in the local SUSY law leads to an extra local Lorentz transformation of parameter $2c\kappa S\bar{\epsilon}_2(\gamma^m\gamma_n - \gamma_n\gamma^m)\epsilon_1$ in the local SUSY commutator on $e_\mu{}^m$

$$\begin{aligned}[\delta_1,,\delta_2]\; e_\mu{}^m &= \text{as before} + 2c\kappa S\bar{\epsilon}_2\gamma^m\gamma_\mu\epsilon_1 - (1\leftrightarrow 2) \\ &= \text{as before} + 2c\kappa S\,\bar{\epsilon}_2(\gamma^m\gamma_n - \gamma_n\gamma^m)\epsilon_1\, e_\mu{}^n\ .\end{aligned} \tag{261}$$

Consider now the local SUSY commutator on ψ_μ. We obtain

$$[\delta_1,\delta_2]\psi_\mu = \frac{1}{4\kappa}\left[\delta_1\omega_\mu{}^{mn}(e,\psi)\right]\}\gamma_m\gamma_n\epsilon_2 + c\,(\delta_1 S)\,\gamma_\mu\epsilon_2 - (1\leftrightarrow 2)\ . \tag{262}$$

After a long and tedious calculation, we obtain $\delta\omega_\mu{}^{mn}(e,\psi)$, and using this we find that the local SUSY commutator also closes on the gravitino. In $\delta\omega_\mu{}^{mn}(e,\psi)$ we only find undifferentiated local SUSY parameters, and no terms with $\partial_\mu\epsilon$. This shows that the terms $\bar{\psi}\psi$ in $\omega_\mu{}^{mn}(e,\psi)$ are "super covariantizations" of $\omega_\mu{}^{mn}(e)$.

Note that δS is proportional to the field equations of both the gravitino and the auxiliary field, and that the extra term $\delta\psi_\mu = cS\gamma_\mu\epsilon$ in the local SUSY law for the gravitino is linear in S. Not having an auxiliary field in the theory will therefore lead to terms proportional to the ψ_μ field equation in the local SUSY commutator on the gravitino.

Exercise 16: Show that in $\delta\omega_\mu{}^{mn}(e,\psi)$ all terms with $\partial_\rho\epsilon$ cancel. This is one way of fixing the relative sign of the transformation rules of of the vielbein and the gravitino, and hence of the Einstein and Rarita-Schwinger actions. Closure of the algebra is another way. Note that SUSY of the action does not fix this sign, since one can always adjust the sign of $\delta e_\mu{}^m$.

Exercise 17: Consider the local SUSY commutator $[\delta_1,\delta_2]\,S$ on S. Locate where the Einstein transformation comes from. Show that in order that the algebra closes on S, there should be no terms with $\partial_\mu\epsilon_1$ or $\partial_\mu\epsilon_2$ on the right-hand side (super covariantization). Show that this fixes c^2. Note in this regard that, in fact, the law δS itself is supercovariant. Do and should the S^2 terms cancel? For the very brave: evaluate this commutator explicitly to the bitter end.

Exercise 18: Show that there exists a locally supersymmetric cosmological constant term

$$I_{\text{cosm}} = \alpha\int d^3x\, e\,(S + \beta\,\bar{\psi}_\mu\gamma^{\mu\nu}\psi_\nu) \tag{263}$$

and fix β. The SUSY of I_{cosm} also fixes c, which confirms the result for c obtained by requiring closure of the gauge algebra in the previous exercise. Note that by eliminating S from $I_{\text{sugra}} + I_{\text{cosm}}$ one finds a cosmological constant.

3.3 Supergravity in superspace

The treatment of $2+1$ dimensional $N=1$ supergravity in superspace follows in most respects the treatment of Yang-Mills theory, but there is one major difference: the internal symmetry generators (the Lorentz generators) act now on the flat indices of the covariant derivatives. In ordinary general relativity, this amounts to the well-known fact that the inverse vielbeins $e_m{}^\mu$ transform under local Lorentz transformations as $\delta_{\rm L} e_m{}^\mu = \lambda_m{}^n e_n{}^\mu$, but that should now come out of the formalism and not be put in by hand.

We begin by introducing the superalgebra and its generators, then introduce corresponding gauge fields and parameters (all superfields), and define gauge transformations by requiring that covariant derivatives transform covariantly, as in Yang-Mills theory. The gauge fields with a flat bosonic supervector index are again expressed in terms of the gauge fields with a flat fermionic supervector index by the conventional constraint that $\{\nabla_a, \nabla_b\} = 2i\nabla_{ab}$, where a, b are flat fermionic indices; curved fermionic indices will be denoted by α, β. Furthermore, we also impose another conventional constraint which eliminates the spin superconnection as an independent field (second order or Palatini formalism in superspace). This is the analogue of the "no torsion" constraint in ordinary general relativity. The difference is that in general relativity the no torsion constraint is also a field equation, namely the field equation of the spin connection, whereas in superspace it is not a field equation. At least, until now nobody has been able to construct an action in superspace with these constraints as field equations. There are several reasons why one imposes general constraints on the supertorsions and/or on the supercurvatures:

(i) To eliminate as many superfields as possible, so as to simplify the formalism. In three dimensions all constraints are algebraic, whereas in four dimensions some are differential constraints Nieuwenhuizen (1981).

(ii) The particular constraints we adopt below lead to a formalism in which, in a suitable gauge, the $\theta = 0$ part of the fermionic superconnection vanishes while the bosonic superconnection at $\theta = 0$ become the usual spin connection $\omega_\mu{}^{mn}(e, \psi)$ of the x-space theory.

(iii) To remove ghosts and higher-spin fields from the spectrum.

Substituting the constraints into the Bianchi identities shows that all supertorsions and supercurvatures depend only on two superfields R and G_{abc}. Finally we construct a superspace action whose component form reproduces the x-space action (225). It reads

$$\int d^3x\, d^2\theta\, [\,{\rm sdet}\,(E_M{}^A)\,]\, R\ , \tag{264}$$

where $E_M{}^A$ is the supervielbein, which is constructed from the gauge fields $h_A{}^M(x, \theta)$ which gauge the bosonic and fermionic translation generators P_μ

and Q_α. The field equations for this action read

$$G_{abc} = R = 0 \ , \tag{265}$$

and show that there is no gravitational dynamics of the usual kind in three dimensions. Below we see all this in more detail.

3.3.1 Covariant derivatives

As superalgebra we take the super Poincaré algebra, given by

$$\begin{aligned}
&[P_m, P_n] = 0 \\
&[M_{mn}, M_{rs}] = -\eta_{mr} M_{ns} + \eta_{nr} M_{ms} - \eta_{ns} M_{mr} + \eta_{ms} M_{nr} \\
&[P_m, M_{rs}] = \eta_{mr} P_s - \eta_{ms} P_r \\
&\{Q_\alpha, Q_\beta\} = -2i\, (\gamma^m)_{\alpha\beta}\, P_m \\
&[Q_\alpha, P_m] = 0 \\
&[Q_\alpha, M_{rs}] = -\frac{1}{2}\, (\gamma_{[r}\gamma_{s]})_\alpha{}^\beta\, Q_\beta \ .
\end{aligned} \tag{266}$$

The minus sign in $[Q_\alpha, M_{rs}]$ is needed for the Jacobi identities, since

$$(\gamma_{[m}\gamma_{n]})_\alpha{}^\beta\, (\gamma_{[r}\gamma_{s]})_\beta{}^\gamma = -(\gamma_{[m}\gamma_{n]})_{\alpha\beta}\, (\gamma_{[r}\gamma_{s]})^{\beta\gamma} \ . \tag{267}$$

The Lorentz generators are represented on spinors by $\frac{1}{2}\,\gamma^{[r}\gamma^{s]}$, and often we shall use the notation

$$\left(\frac{1}{2}\,\lambda^{rs} M_{rs}\right)\psi_\alpha = \frac{1}{4}\,\lambda^{rs}\,(\gamma_r\gamma_s)_\alpha{}^\beta\,\psi_\beta \equiv \lambda_\alpha{}^\beta\,\psi_\beta \ . \tag{268}$$

Note that for vectors we have

$$\left(\frac{1}{2}\,\lambda^{rs} M_{rs}\right) v_m = -\lambda_m{}^n\, v_n \ , \tag{269}$$

in order that the commutator of two Lorentz transformations

$$[\delta(\lambda_1), \delta(\lambda_2)] = \delta\,[\,(\lambda_{1,m}{}^k\,\lambda_{2,kn} - (1 \leftrightarrow 2)\,] \tag{270}$$

holds both for spinors and for vectors.

We denote the set of generators of the super Poincaré algebra collectively by T_I, so

$$T_I = \{Q_\alpha, P_\mu, M_{rs}\} \ . \tag{271}$$

We take Q_α to be hermitian, and P_μ and M_{rs} antihermitian. The gauge parameters Ω^I and gauge fields $H_A{}^I$, with A a flat superindex, $A = \{a, m\}$, are then given by

$$\Omega^I = \{K^\alpha, K^\mu, L^{rs}\}$$

$$H_A{}^I = \begin{pmatrix} H_a{}^\beta & H_a{}^\mu & \phi_a{}^{rs} \\ H_m{}^\beta & H_m{}^\mu & \phi_m{}^{rs} \end{pmatrix} . \tag{272}$$

All fields and parameters depend on the real coordinates x^μ and θ^α.

We consider now the coset $\{T_I\}/\{M_{rs}\}$, with coset generators Q_α and P_μ. Then we have the usual covariant derivatives and Lie derivatives of rigid superspace. We define covariant derivatives of local superspace as in the Yang Mills case by

$$\nabla_A = D_A + H_A{}^I T_I . \tag{273}$$

The derivatives D_A are the covariant derivatives of rigid superspace, containing in general also a rigid connection term for the subalgebra generators, but in the super Poincaré case this rigid connection vanishes. The gauge fields $H_A{}^I$ are arbitrary local deviations around the rigid vielbeins and rigid connections. We must now distinguish two kind of indices:

(i) Flat supervector indices $A = \{a, m\}$. In the inverse supervielbein that we will construct, they appear as $E_A{}^M$. The supervielbein itself if $E_M{}^A$.

(ii) Curved supervector indices $M = \{\alpha, \mu\}$. Later we shall go from curved to flat and vice-versa by using the supervielbein, as $v_M = E_M{}^A v_A$ and $v_A = E_A{}^M v_M$. This is the standard practice in general relativity, but note the order of contractions: from left-upper to right-lower. For fermionic objects the order will not matter.

The gauge fields consist now of the square supermatrix $H_A{}^M$ and a Lorentz superconnection $\phi_A{}^{rs}$. The complete gauge transformation rules are as usual

$$\nabla_A \;\to\; \nabla'_A = e^{-\Omega} \nabla_A e^{\Omega} , \tag{274}$$

where

$$\Omega = \Omega^I T_I \tag{275}$$

and the generators T_I satisfy

$$[T_I, T_J] = f_{IJ}{}^K T_K \tag{276}$$

Note that Ω is antihermitian and commuting because the term $K^\alpha Q_\alpha$ is antihermitian when both K^α and Q_α are hermitian and anticommuting. Infinitesimally,

$$\delta_{\rm g} \nabla_A = (\delta_{\rm g} H_A{}^I) T_I = (D_A \Omega^I) T_I + [H_A{}^I T_I, \Omega^J T_J] . \tag{277}$$

The covariant derivatives $\nabla_A = D_A + H_A{}^I T_I$ read more explicitly

$$\begin{aligned}\nabla_a &= D_a + H_a{}^\beta Q_\beta + H_a{}^\mu P_\mu + \frac{1}{2}\,\phi_a{}^{rs} M_{rs}\\ \nabla_m &= D_m + H_m{}^\beta Q_\beta + H_m{}^\mu P_\mu + \frac{1}{2}\,\phi_m{}^{rs} M_{rs}\ .\end{aligned} \tag{278}$$

The factor 1/2 ensures that the generators M_{rs} are not counted twice. Since $D_a = \partial_a + i\theta^b \partial_{ba}$ is hermitian, $D_m = \partial_m$, P_μ and M_{rs} are antihermitian, while it is natural to require that the diagonal parts $H_a{}^\beta$ and $H_m{}^\mu$ are also hermitian, we see that Q_β, $\phi_m{}^{rs}$ and $H_m{}^\beta$ are also hermitian. The terms in ∇_a with the Poincaré generators show then that $H_a{}^\mu$ and $\phi_a{}^{rs}$ are imaginary. In the Yang-Mills case we therefore introduced an extra factor of i as $\nabla_\alpha = D_\alpha + iA_\alpha$, but here we will work without any extra factors of i because this actually simplifies matters. Thus, all gauge fields except $H_a{}^\mu$ and $\phi_a{}^{rs}$ are now real.

From eq. (277) it follows that $\delta_{\rm g} H_a{}^\alpha = D_a K^\alpha + \dots$ and $\delta_{\rm g} H_m{}^\alpha = \partial_m K^\alpha + \dots$. Since $D_a K^\alpha = \{D_a, K^\alpha\}$ and $\partial_m K^\alpha = [\partial_m, K^\alpha]$, w= e must take K^α real if $H_a{}^\alpha$ is real, and then $H_m{}^\alpha$ is also real as we already saw. Similarly, from $\delta_{\rm g} H_m{}^\mu = [\partial_m, K^\mu] + \dots$ and the reality of $H_m{}^\mu$ we find that K^μ is real, and then $\delta_g H_a{}^\mu = [D_a, K^\mu]$ confirms that $H_a{}^\mu$ is imaginary. Finally, $\delta_{\rm g}\phi_m{}^{rs} = [\partial_m, L^{rs}] + \dots$ and $\delta_{\rm g}\phi_a{}^{rs} = [D_a, L^{rs}]$ show that L^{rs} is real but $\phi_a{}^{rs}$ is imaginary, as already seen. So all gauge parameters are real.

3.3.2 A new basis for the gauge fields leading to vielbeins

We can now go on as usual for coset manifolds. We replace the generators T_I by minus the covariant Lie derivatives $\mathcal{L}_I = \{\mathcal{L}_M, \mathcal{L}_{rs}\}$, defined by

$$\begin{aligned}\mathcal{L}_\alpha &= \partial_\alpha - i\theta^\beta\,(\gamma^\mu)_{\beta\alpha}\,\partial_\mu\\ \mathcal{L}_\mu &= \partial_\mu\\ \frac{1}{2}\,L^{rs}\mathcal{L}_{rs} &= L^\mu{}_\nu\,x^\nu\partial_\mu + \frac{1}{4}\,L^{rs}\,(\gamma_r\gamma_s)^\alpha{}_\beta\,\theta^\beta\partial_\alpha + \frac{1}{2}\,L^{rs}M_{rs}\end{aligned} \tag{279}$$

and which form a representation of the generators of the superalgebra. Here the M_{rs} is the spin part of the covariant Lie derivative $\mathcal{L}_{rs}$ which acts on the Lorentz indices of matter fields **and by definition also on the indices A of the gauge fields $H_A{}^M$**:

$$\frac{1}{2}\,L^{rs}M_{rs} = [\,\delta_{\rm L}^{(L^{rs})} H_A{}^I\,]\;\frac{\partial}{\partial H_A{}^I}\ . \tag{280}$$

Since it follows from coset theory that $\mathcal{L}_M$ does act on the indices of D_N, and $\delta_A{}^M + H_A{}^M$ is going to be the vielbein field, the definition that M_{rs} acts on $H_A{}^M$ is natural, but it is an extra definition. From the knowledge of how covariant Lie derivatives and the covariant derivatives D_M (anti)commute, we could then deduce the gauge transformation rules for $H_A{}^I$. However, we want to make contact with general relativity, and introduce (super)vielbeins. To this purpose, we rewrite the covariant Lie derivatives $\mathcal{L}_M$ corresponding to the generators P_μ and Q_α as a linear combination of the covariant derivatives of rigid superspace:

$$H_A{}^I T_I \equiv h_A{}^M D_M + \frac{1}{2}\,\phi_A{}^{rs} M_{rs}\ . \tag{281}$$

This amounts to a linear combination of components of H. Note that $H_A{}^M$ and $h_A{}^M$ transform in the same way under local Lorentz transformations. Further, since neither $\mathcal{L}_M$ nor H_N contain terms with M_{rs}, the connections are unchanged. One could also expand $H_A{}^I T_I$ on a basis of ordinary derivatives, as $\tilde{h}_A{}^\alpha \partial_\alpha + \tilde{h}_A{}^\mu \partial_\mu + \frac{1}{2}\,\tilde{h}_A{}^{rs} M_{rs}$. Using the basis with D_M is useful as a starting point for the background field formalism. From (281) we obtain

$$\nabla_A = D_A + h_A{}^M D_M + \frac{1}{2}\,\phi_A{}^{rs} M_{rs}\ . \tag{282}$$

This suggests to define the supervielbein by

$$D_A + h_A{}^M D_M = E_A{}^M D_M \qquad\qquad E_A{}^M = \delta_A{}^M + h_A{}^M\ . \tag{283}$$

The limit of rigid superspace corresponds then to $E_A{}^M = \delta_A{}^M$ and $h_A{}^M = 0$. From now on

$$\nabla_A = E_A{}^M D_M + \frac{1}{2}\,\phi_A{}^{rs} M_{rs} \tag{284}$$

These covariant derivatives ∇_A have no definite reality properties, because under hermitian conjugation the order of $E_A{}^M$ and D_M is reversed and D_M acts then on $E_A{}^M$. Gauge transformations still read

$$\nabla'_A = e^{-\Omega}\,\nabla_A e^{\Omega}\ , \tag{285}$$

but we also expand Ω on the basis with D_M

$$\Omega = k^\alpha D_\alpha + k^\mu \partial_\mu + \frac{1}{2}\,L^{rs} M_{rs}\ . \tag{286}$$

In the covariant approach to Yang-Mills theory, we consistently worked with **curved** superindices $M = \{\alpha, \mu\}$: coordinates x^μ and θ^α, and rigidly covariant derivatives $D_M = \{D_\alpha, D_\mu\}$, where D_α was $D_\alpha = \partial_\alpha + i(\gamma^\mu)_{\alpha\beta}\,\theta^\beta \partial_\mu$ and $D_\mu = \partial_\mu$. In gravitational superspace, on the other hand, we want to interpret the index A of $h_A{}^M$ as a flat index, like the index m of the usual inverse vielbein field $e_m{}^\mu$ of general relativity. In order to be consistent, we should then rewrite the D_A in ∇_A as $\delta_A{}^M D_M$. We shall keep writing D_A, though, but it should be understood that we mean $\delta_A{}^M D_M$. The relation of D_M to the ordinary derivatives $\partial_M = \{\partial_\alpha, \partial_\mu\}$ is via the rigid inverse supervielbein: $D_M = E_{(0)\,M}{}^N \partial_N$. One could introduce yet another type of index for these ∂_N in order to distinguish between the two indices of $E_{(0)\,M}{}^N$. Therefore one sometimes writes $D_M = E^{(0)\,M\Lambda}\partial_\Lambda$ where ∂_Λ are equal to the ordinary derivatives ∂_α and ∂_μ. Then $D_A = \delta_A{}^M E^{(0)\,M\Lambda}\partial_\Lambda$. We shall not introduce the indices Λ, so for us D_M and ∂_M have the same kind of indices: curved indices in local superspace.

3.4 Constraints and Bianchi identities

Having defined covariant superderivatives, we define supertorsions and supercurvatures as usual:

$$[\nabla_A, \nabla_B\} = T_{AB}{}^C\,\nabla_C + \frac{1}{2}\,R_{AB}{}^{rs}M_{rs}\ . \tag{287}$$

As in the case of Yang-Mills theory, we impose the conventional constraint

$$\{\nabla_a, \nabla_b\} = 2i\nabla_{ab}\ . \tag{288}$$

This constraint states that

$$T_{a,b}{}^{cd} = 2i\,\delta_a{}^{(c}\delta_b{}^{d)} \qquad T_{a,b}{}^c = 0 \qquad R_{a,b}{}^{rs} = 0\ . \tag{289}$$

We shall later see that by redefining the bosonic connection $\phi_{ab}{}^{rs}$ one may replace the constraint $R_{a,b}{}^{rs} = 0$ by the more familiar constraint $T_{ab,cd}{}^{ef} = 0$, or equivalently $T_{m,n}{}^r = 0$. In four dimensions one uses $T_{mn}{}^r = 0$ to express the bosonic part of the spin connection in terms of supervielbeins, but in three dimensions we prefer to work with $R_{a,b}{}^{rs} = 0$.

Using (284), we have

$$\begin{aligned}
\{\nabla_a, \nabla_b\} &= \Big[E_a{}^M D_M + \frac{1}{2}\,\phi_a{}^{rs}M_{rs}, E_b{}^N D_N + \frac{1}{2}\,\phi_a{}^{tu}M_{tu}\Big] \\
&= \Big[E_a{}^M (D_M E_b{}^N)\,D_N + \frac{1}{4}\,\phi_a{}^{rs}\,(\gamma_r\gamma_s)_b{}^c\,E_c{}^M D_M \\
&\quad +\frac{1}{2}\,E_a{}^M (D_M\phi_b{}^{rs})\,M_{rs} + (a \leftrightarrow b)\Big] \\
&\quad +\ 2i\,E_a{}^\alpha\,E_b{}^\beta\,D_{\alpha\beta} + \phi_a{}^{rt}\,\phi_b{}^{t's}\,\eta_{tt'}\,M_{rs}\ .
\end{aligned} \tag{290}$$

It is clear from this that the constraint (288) expresses both $E_{ab}{}^M$ and $\phi_{ab}{}^{rs}$ in ∇_{ab} in terms of $E_a{}^M$ and $\phi_a{}^{rs}$. For later purposes we record $T_{a,b}{}^C$ and $R_{a,b}{}^{rs}$:

$$\begin{aligned}
T_{a,b}{}^C &= E_a{}^M (D_M E_b{}^N)\,E_N{}^C + \phi_{ab}{}^C + (a \leftrightarrow b) + T_{(0)\,ab}{}^C \\
R_{a,b}{}^{rs} &= E_a{}^M\,(D_M\phi_b{}^{rs}) + \phi_a{}^r{}_t\,\phi_b{}^{ts} + (a \leftrightarrow b) - T_{a,b}{}^C\,\phi_C{}^{rs}\ ,
\end{aligned} \tag{291}$$

where $T_{(0)\,ab}{}^C$ is the torsion in rigid superspace.

Since the constraint (288) has the same form as in Yang-Mills theory, although here we have also supervielbeins, we obtain from the Jacobi identities the same relation as in (108):

$$[\nabla_a, \nabla_{bc}] = \frac{1}{2}\,\epsilon_{ab}W_c + \frac{1}{2}\,\epsilon_{ac}W_b\ , \tag{292}$$

with anticommuting W_b, but instead of $W_\alpha = W_\alpha{}^a T^a$ (recall that here a was a gauge Lie algebra index) we now have

$$W_a = W_a{}^b\,\nabla_b + \hat{W}_a{}^{bc}\,\nabla_{bc} + \frac{1}{2}\,W_a{}^{rs}M_{rs}\ . \tag{293}$$

Because we are going to use this result in other Bianchi identities, we have expanded W_a in terms of ∇_A and not in terms of D_M. We then again deduce from the Bianchi identity for $\{\nabla_a, [\nabla_b, \nabla_{cd}]\}$ that

$$\nabla^a W_a = 0 \qquad f_{ab} = \frac{1}{2i}\,\nabla_{(a} W_{b)} \tag{294}$$

and that

$$[\nabla_{ab}, \nabla_{cd}] = T_{ab,cd}{}^A \nabla_A + \frac{1}{2}\, R_{ab,cd}{}^{rs} M_{rs} = \epsilon_{bc} f_{ad} + \epsilon_{ad} f_{bc}\ . \tag{295}$$

In four dimensions one finds $\nabla^\alpha W_\alpha + \nabla^{\dot\alpha} W_{\dot\alpha} = 0$ as constraint, but in three dimensions there is no difference between dotted and undotted spinor indices.

To reduce the number of independent superfields, we impose a further constraint on the supertorsions which expresses also the connections $\phi_a{}^{rs}$ in terms of supervielbeins. In Yang-Mills theory this is, of course, not possible. The constraint is

$$T_{a,bc}{}^{de} = 0, \quad \text{or equivalently} \quad T_{am}{}^n = 0\ . \tag{296}$$

One can solve this constraint by expressing the fermionic connection $\phi_a{}^{rs}$ in terms of fermionic inverse supervielbeins $E_a{}^M$. Let us see how. Since $T_{a,bc}{}^{de}$ is the coefficient of ∇_{de} in $[\nabla_a, \nabla_{bc}]$ and $\nabla_{bc} \sim \{\nabla_b, \nabla_c\}$, we begin by dropping all terms with M_{rs} in $[\nabla_a, \nabla_{bc}]$. Then one finds in terms of $E_a \equiv E_a{}^M D_M$ and $\phi_a \equiv \frac{1}{2}\,\phi_a{}^{ms} M_{ms}$

$$\begin{aligned}
\frac{1}{2i}\,[\nabla_a, \nabla_{ab}] &= \Big[E_a + \phi_a, \{E_b + \phi_b, E_c + \phi_c\} \Big] \\
&= \Big[E_a + \phi_a, \{E_b, E_c\} + \phi_{b,c}{}^d E_d + \phi c, b^d E_d + \phi_{bc} \Big] \\
&= \Big[E_a, \{E_b, E_c\} \Big] + \phi_{a,b}{}^d\,\{E_d, E_c\} + \phi_{a,c}{}^d\,\{E_b, E_d\} + (E_a \phi_{b,c}{}^d)\, E_d \\
&+ (E_a \phi_{c,b}{}^d)\, E_d + \phi_{a,b}{}^{b'} \phi_{b',c}{}^d\, E_d + \phi_{a,c}{}^{c'} \phi_{c',b}{}^d\, E_d \\
&- \phi_{b,c}{}^d\,\{E_d, E_a\} - \phi_{c,b}{}^d\,\{E_d, E_a\} - \phi_{bc,a}{}^d\, E_d + M\text{-terms}\ .
\end{aligned} \tag{297}$$

From this expression we must now project out the term with ∇_{de}. Since the leading term in ∇_{de} is $\{E_d, E_e\}$, while the terms with E_d appear in $T_{a,bc}{}^d$ and those with M_{rs} also appear in $R_{a,bc}{}^{rs}$, it is sufficient and easiest to collect only all terms proportional to $\{E_d, E_e\}$. This yields

$$0 = T_{a,bc}{}^{de} = C_{a,bc}{}^{de} + \phi_{a,b}{}^{(d}\,\delta_c{}^{e)} + \phi_{a,c}{}^{(d}\,\delta_b{}^{e)} - \phi_{b,c}{}^{(d}\,\delta_a{}^{e)} - \phi_{c,b}{}^{(d}\,\delta_a{}^{e)}\ , \tag{298}$$

where $C_{a,bc}{}^{de}$ is defined by

$$C_{a,bc}{}^{de}\,\partial_{de} = [E_a, \{E_b, E_c\}] \tag{299}$$

and reads explicitly

$$C_{a,bc}{}^{,de} = E_a{}^M \{D_M\,[E_b{}^N(D_N E_c{}^P)]\}\,E_P{}^{de} \\ - E_b{}^M(D_M E_c{}^N)\,(D_N E_a{}^P)\,E_P{}^{de} + (b \leftrightarrow c)\ . \tag{300}$$

Note that $C_{a,bc}{}^{de}$ depends only on $E_a{}^M$ because $E_P{}^{de}$ depends on $E_a{}^M$ and $E_{ab}{}^M$ and we have already expressed $E_{ab}{}^M$ in terms of $E_a{}^M$. Pairs of terms with ϕ's in eq. (300) combine into terms with an ϵ-symbol

$$C_{a,bc}{}^{de} - \epsilon_{ab}\phi^{(e}{}_c{}^{d)} - \epsilon_{ac}\phi^{(e}{}_b{}^{d)} = 0\ . \tag{301}$$

Contraction with ϵ^{ba} yields

$$C^b{}_{bc}{}^{de} + 3\phi^{(e}{}_c{}^{d)} = 0 \tag{302}$$

We can now express $\phi_{d,ec}$ in terms of $\phi_{(d,e)c}$ by using that $2\phi_{[a,b]\,c} = -\epsilon_{ab}\phi^d{}_{dc}$ and that ϕ is traceless, that is $\phi_{d,e}{}^e = 0$,

$$\phi_{d,ec} = \phi_{(d,e)\,c} + \phi_{[d,e]\,c} = \phi_{(d,e)\,c} + \frac{1}{2}\,\epsilon_{de}\,\phi_{f,c}{}^f = \phi_{(d,e)\,c} + \epsilon_{de}\,\phi_{(f,c)}{}^f\ . \tag{303}$$

We then find

$$\phi_{a,bc} = -\frac{1}{3}\,C^d{}_{dc,ab} - \frac{1}{3}\,\epsilon_{ab}\epsilon^{fe}C^d{}_{de,fc} = \frac{1}{3}\,C^d{}_{da,bc} - \frac{2}{3}\,C^d{}_{d(b,c)\,a}\ , \tag{304}$$

where we have used that $2C^d d[a,b]c = -\epsilon_{ab}\,\epsilon^{fe}\,C^d{}_{de,fc}$.

From eqs. (287), (292) and (293) it follows that the constraint (296) implies that $\hat{W}_{a,}{}^{bc} = 0$. Then $\nabla^a W_a = \{\nabla^a, W_a\} = 0$ reduces to

$$0 = \left\{\,\nabla^a, W_a{}^b\nabla_b + \frac{1}{2}\,W_a{}^{rs}M_{rs}\,\right\} \\ = (\nabla^a W_a{}^b)\,\nabla_b + W_a{}^b\,\{\nabla^a,\nabla_b\} + \frac{1}{2}\,(\nabla^a W_a{}^{rs})\,M_{rs} - W_a{}^a{}_c\nabla^c\ , \tag{305}$$

where

$$W_{a,bc} = \frac{1}{4}\,W_a{}^{rs}\,(\gamma_r\gamma_s)_{bc}\ . \tag{306}$$

Using that $\{\nabla^a,\nabla_b\}$ is symmetric in a,b, and that $\{\nabla_\alpha,\nabla_{ab},M_{rs}\}$ is a basis, we find that

$$\nabla^a W_{ab} + W_a{}^a{}_b = 0 \tag{307}$$

$$W_{(ab)} = 0 \tag{308}$$

$$\nabla^a W_a{}^{rs} = 0\ . \tag{309}$$

Eq. (308) implies that $W_{ab} = \epsilon_{ab}R$, with R a new real commuting superfield. Then, eq. (307) simplifies to

$$\nabla_b R + W_a{}^a{}_b = 0\ . \tag{310}$$

Since the same decomposition as in (104) yields

$$W_{a,bc} = W_{(a,bc)} + \frac{1}{3}\,\epsilon_{ab} W_d{}^d{}_c + \frac{1}{3}\,\epsilon_{ac} W_d{}^d{}_b \ , \tag{311}$$

we find a new totally symmetric real commuting superfield

$$G_{abc} \equiv W_{(a,bc)} \ . \tag{312}$$

The condition (309) becomes in a spinor basis $\nabla^a W_{abc} = 0$ and leads to $\nabla^a G_{abc} = \frac{1}{3}\{\nabla_b, \nabla_c\}R = \frac{2i}{3}\nabla_{bc}R$. Hence we have found that the solution to $\nabla^a W_a = 0$ reads

$$\begin{aligned} W_{ab} &= \epsilon_{ab} R \\ \hat{W}_a{}^{bc} &= 0 \\ W_{a,bc} &= G_{abc} - \frac{1}{3}\,\epsilon_{ab}\nabla_c R - \frac{1}{3}\,\epsilon_{ac}\nabla_b R \\ \nabla^a G_{abc} &= \frac{2i}{3}\,\nabla_{bc} R \ . \end{aligned} \tag{313}$$

and depends on two real commuting superfields R and G_{abc}.

Since we know now the solution for W_a and $[\nabla_a, \nabla_{bc}]$ and $[\nabla_{ab}, \nabla_{cd}]$ are given in terms of W_a, we know now $T_{a,bc}{}^A$ and $R_{a,bc}{}^{rs}$, and also $T_{ab,cd}{}^{ef}$ and $R_{ab,cd}{}^{rs}$. We shall quote them below. First we note that we could have redefined the connection ϕ_{ab} by adding any covariant term $\Delta\phi_{ab}$ to it: $\phi'_{ab} = \phi_{ab} + \Delta\phi_{ab}$. Then

$$\nabla_{bc} = \nabla'_{bc} - \frac{1}{2}\,\Delta\phi_{bc}{}^{rs} M_{rs} \tag{314}$$

and

$$\begin{aligned} \{\nabla_b, \nabla_c\} &= 2i\left(\nabla'_{bc} - \frac{1}{2}\,\Delta\phi_{bc}{}^{rs} M_{rs}\right) \\ [\nabla_a, \nabla'_{bc}] &= \frac{1}{2}\,\epsilon_{ab} W_c + \frac{1}{2}\,\epsilon_{ac} W_b + \frac{1}{2}\,[\nabla_a, \Delta\phi_{bc}{}^{rs} M_{rs}] \\ [\nabla'_{ab}, \nabla'_{cd}] &= T_{ab,cd}{}^A \nabla_A + \frac{1}{2}\,R_{ab,cd}{}^{rs} M_{rs} \\ &\quad + \frac{1}{2}\,[\nabla_{ab}, \Delta\phi_{cd}{}^{rs} M_{rs}] - \frac{1}{2}\,[\nabla_{cd}, \Delta\phi_{ab}{}^{rs} M_{rs}] + \Delta\phi_{ab}{}^{rt}\,\Delta\phi_{cd,t}{}^s\, M_{rs} \end{aligned} \tag{315}$$

We can choose a suitable $\Delta\phi_{ab}$ to make $T_{ab,cd}{}^{ef}$ vanish, *i.e.* $T_{mn}{}^r = 0$. This is a constraint one usually imposes in four dimensions. Note that, since we already fixed ϕ_{ab} by the constraint $\{\nabla_a, \nabla_b\} = 2i\nabla_{ab}$, we cannot further constrain the geometry by imposing $T_{ab,cd}{}^{ef} = 0$ in the same way as we imposed $T_{a,bc}{}^{de} = 0$. What we can do is, by adding a term $\Delta\phi_{ab}$ to ϕ_{ab}, relax the constraint $R_{a,b}{}^{rs} = 0$, which followed from $\{\nabla_a, \nabla_b\} = 2i\nabla_{ab}$ and, instead of $R_{a,b}{}^{rs} = 0$, impose

$T_{ab,cd}{}^{ef} = 0$. To find this $\Delta\phi_{ab}$, we first evaluate $T_{ab,cd}{}^{ef}$ from $[\nabla_{ab}, \nabla_{cd}]$ by using the results for W_a. From (295) we know that

$$[\nabla_{ab}, \nabla_{cd}] = \epsilon_{bc} f_{ad} + \epsilon_{ad} f_{bc} \ , \tag{316}$$

and from (294) we have

$$f_{ab} = \frac{1}{2i} \, \nabla_{(a} W_{b)} \ . \tag{317}$$

Expressing $\nabla_a W_b$ as the anticommutator $\{\nabla_a, W_b\}$, we obtain

$$\nabla_a W_b = (\nabla_a W_b{}^c)\, \nabla_c + W_b{}^c \, \{\nabla_c, \nabla_a\} + \frac{1}{2}\, (\nabla_a W_b{}^{cd})\, M_{cd} + W_{b,a}{}^c \nabla_c \ . \tag{318}$$

According to (313), in $\nabla_a W_b$ there is only one term with ∇_{ab}, namely

$$W_b{}^c \, \{\nabla_a, \nabla_c\} = R \, \{\nabla_a, \nabla_b\} = 2iR \, \nabla_{ab} \ . \tag{319}$$

Hence

$$T_{ab,cd}{}^{ef} \nabla_{ef} = R \, (\epsilon_{bc} \nabla_{ad} + \epsilon_{ad} \nabla_{bc}) \ . \tag{320}$$

Clearly, by redefining

$$\phi_{ab}{}^{rs} = \phi'_{ab}{}^{rs} + \alpha \, (\gamma^{rs})_{ab} \, R \ , \tag{321}$$

with α a constant, we obtain in $[\nabla'_{ab}, \nabla'_{cd}]$ as given in (315) extra terms of the form

$$R \, [M_{ab}, \nabla_{cd}] \sim R \, (\epsilon_{bc} \nabla_{ad} + \epsilon_{bd} \nabla_{ac}) \ , \tag{322}$$

and by choosing α appropriately, we can obtain that $T_{ab,cd}{}^{ef} = 0$. Then, of course,

$$\{\nabla_a, \nabla_b\} = 2i \left[\nabla'_{ab} - \frac{\alpha}{2} \, R \, (\gamma^{rs})_{ab} \, M_{rs} \right] \ . \tag{323}$$

At this point the remaining independent superfield is the inverse fermionic supervielbein $E_a{}^M$. The supertorsions and supercurvatures only depend on R and G_{abc} which themselves depend on $E_a{}^M$. For completeness we record all

$T_{AB}{}^C$ and $R_{AB}{}^{rs}$:

$$\begin{aligned}
T_{ab}{}^{cd} &= 2i\,\delta_a{}^{(c}\delta_b{}^{d)} && (324)\\
T_{ab}{}^{c} &= 0 && (325)\\
T_{am}{}^{n} &= 0 && (326)\\
T_{a,bc}{}^{d} &= \frac{1}{2}\,R\left[\,\epsilon_{ab}\,\delta_c{}^d + (b\leftrightarrow c)\right] && (327)\\
T_{ab,cd}{}^{ef} &= R\left[\,\epsilon_{bc}\,\delta_a{}^{(e}\delta_d{}^{f)} + \epsilon_{ad}\,\delta_b{}^{(e}\delta_c{}^{f)}\right] && (328)\\
T_{ab,cd}{}^{e} &= \frac{1}{2i}\left\{\epsilon_{bc}\left[G_{ad}{}^e + \frac{2}{3}\,\nabla_{(a}R\,\delta_{d)}{}^e\right]\right.\\
&\qquad\left.+\frac{2}{3}\,\epsilon_{ad}\left[G_{bc}{}^e + \nabla_{(b}R\,\delta_{c)}{}^e\right]\right\} && (329)\\
R_{ab}{}^{rs} &= 0 && (330)\\
R_{a,bc}{}^{de} &= \frac{1}{2}\,\epsilon_{ab}\,W_c{}^{de} + (b\leftrightarrow c) && (331)\\
R_{ab,cd}{}^{ef} &= \frac{1}{2i}\left[\,\epsilon_{bc}\,\nabla_{(a}W_{d)}{}^{ef} + \epsilon_{ad}\,\nabla_{(b}W_{c)}{}^{ef}\right] && (332)\\
W_a{}^{ef} &= G_a{}^{ef} - \frac{2}{3}\,\delta_a{}^{(e}\nabla^{f)}R\;. && (333)
\end{aligned}$$

There is a large symmetry group acting on $E_A{}^M$, namely super-Einstein transformations with superparameters k^μ and k^α, and local Lorentz transformations with superparameters L^{rs} [see eqs. (285) and (286)]. As we shall show, one can gauge away all of $E_A{}^M$ except the antisymmetric part of $E_a{}^\alpha$, namely $E_a{}^\alpha = \delta_a{}^\alpha\psi$, and the totally symmetric part of $E_a{}^{\alpha\beta}$, denoted by $E^{(a\alpha\beta)}$. The supergravity action should be gauge invariant, hence it should at most depend on ψ and $E^{(a\alpha\beta)}$. Variation of the action with respect to $E^{(a\alpha\beta)}$ and ψ should then yield the field equations $G_{abc} = 0$ and $R = 0$ (or $R = \Lambda$ if there is a supercosmological term). In what follows we see this.

3.5 Action and field equations

We shall now first derive the field equations in superspace, and then find an action in superspace which reproduces these field equations.

To deduce the field equations in superspace, we use a dimensional argument. We recall that on-shell the field content of $N=1$ x-space supergravity is $e_\mu{}^m$ and $\psi_\mu{}^\alpha$, since the auxiliary field S vanishes on shell. Furthermore, the field equation of the spin connection is that the supercovariantized curl of the vielbein vanishes, and the gravitino field equation is that the supercovariantized curl of the gravitino vanishes. We note that both the supercovariantized curl of the vielbein and the supercovariantized curl of the gravitino have mass dimension 2. From this and the observation that the only covariant objects in x-space are of the form $\partial\partial e$ and $\partial\psi$, or more precisely, supercovariantized Riemann curvatures or gravitino curls, we see that on-shell there exist only covariant objects of mass dimension 2. Supertorsions and supercurvatures[5] with mass dimensions equal to 1/2, 1 or 3/2 must therefore vanish on-shell. From the definition (287) we find the following table of dimensions

$$\begin{array}{ll} \text{dim } 0: & T_{ab}{}^{m} \\ \text{dim } 1/2: & T_{ab}{}^{c},\, T_{am}{}^{n} \\ \text{dim } 1: & T_{am}{}^{b},\, T_{mn}{}^{r},\, R_{ab}{}^{rs} \\ \text{dim } 3/2: & T_{mn}{}^{a},\, R_{am}{}^{rs} \\ \text{dim } 2: & R_{mn}{}^{rs}\ . \end{array} \tag{334}$$

In the approach with $\{\nabla_a, \nabla_b\} = 2i\nabla_{ab}$ we imposed the off-shell constraints (289) and (296), which we repeat for convenience:

$$\text{off shell}: \quad T_{a,b}{}^{cd} = 2i\,\delta_a{}^{(c}\delta_b{}^{d)} \qquad T_{a,b}{}^{c} = T_{am}{}^{n} = R_{a,b}{}^{rs} = 0\ . \tag{335}$$

To this set we now add the constraints that all covariant objects with dimensions below 2 should vanish on-shell

$$\text{on shell}: \quad T_{am}{}^{b} = T_{mn}{}^{r} = T_{mn}{}^{a} = R_{am}{}^{rs} = 0\ . \tag{336}$$

Note that in the approach where we replaced $R_{ab}{}^{rs} = 0$ by $T_{mn}{}^{r} = 0$ off-shell, one finds the same total set of on-shell constraints. From the on-shell condition $T_{am}{}^{b} = 0$ and (327) it follows that $R = 0$ on shell. Similarly, from $R_{am}{}^{rs} = 0$ and (331) we get $W_{abc} = 0$, which togethe= r with (313) and $R = 0$ yields $G_{abc} = 0$. We thus have found that on-shell

$$\text{on shell}: \quad R = G_{abc} = 0\ . \tag{337}$$

Since this implies that **all** supertorsions and supercurvatures vanish on-shell, we see that in 2+1 dimensions there is no (super)gravitational dynamics, a well-known result.

In 3+1 dimensions, not all supertorsions and supercurvatures vanish on-shell. Off-shell they can all be expressed in terms of three superfields R (chiral), G_m (real) and W_{abc} (chiral). Then one can use the Bianchi identities and both the off-shell and on-shell constraints to deduce that certain combinations of mass dimension 3/2 and 2 supertorsions and supercurvatures vanish on-shell. By explicitly evaluating the $\theta = 0$ parts of these combinations, one finds then the vielbein and gravitino field equations. On-shell one has $R = G_m = 0$, but $W_{abc} = 0$ needs not vanish, which leads to nontrivial dynamics.

> **Exercise 19:** Show that in $2+1$ dimensions the curvature $R_{mn}{}^{rs}$ also vanishes on-shell by analyzing the terms with M_{rs} in $[\nabla_{ab}, \nabla_{cd}]$.

[5] The supertorsions and supercurvatures are supercoordinate and local Lorentz covariant tensors. Hence, also their $\theta = 0$ parts should be supercovariant and Einstein and local Lorentz covariant; in particular, they are Einstein scalars. As a result, supertorsions and supercurvatures can only be nonvanishing on-shell if they contain covariant Riemann curvatures or gravitino curls.

Hint: use (316) and (318). Evaluate $R_{\mu\nu}{}^{rs}$ at $\theta = 0$ and relate it to $R_{mn}{}^{rs}$ by using a gauge in which at $\theta = 0$ one has

$$E_M{}^A = \begin{pmatrix} e_\mu{}^m & \psi_\mu{}^a \\ 0 & \delta_\alpha{}^a \end{pmatrix} \qquad \phi_a{}^{rs} = 0 \ . \tag{338}$$

Show that one obtains the supercovariantized Ricci tensor, and show that this is indeed the spin 2 field equation.

If there is a cosmological constant Λ with dimension $[\Lambda] = 1$ in 2+1 dimensions, the supertorsions and supercurvatures with mass dimensions 1 in (334) may not vanish:

$$T_{mn}{}^r = \Lambda\, \epsilon_{mn}{}^r \qquad T_{am}{}^b = \Lambda\, (\gamma_m)_a{}^b \qquad R_{ab}{}^{rs} = \Lambda\, (\gamma^{rs})_{ab} \ . \tag{339}$$

Then $R = \Lambda$ is possible on-shell, instead of $R = 0$. In 3+1, one has $[\Lambda] = 2$ and then $R_{mn}{}^{rs} = \Lambda\, \delta_{[m}{}^r \delta_{n]}{}^s$ is possible. Again this leads to $R = \Lambda$, but still $G_m = 0$ and $W_{abc} =$ not vanishing.

We must now find an action in superspace which is invariant under local symmetries (general supercoordinate transformations and local Lorentz transformations) and which reproduces the field equations $R = G_{abc} = 0$. If we want an action integrand which is a scalar density, the only candidates is

$$I = \frac{1}{\kappa^2} \int d^3x\, d^2\theta \ \ \mathrm{sdet}(E_M{}^A)\ (c_1 R + c_2 \Lambda) \ . \tag{340}$$

We need the gravitational constant $(\kappa^2)^{-1}$ with $[\kappa^2] = -1$ since after integrating over θ we should find the Einstein-Hilbert in three dimensions and the Ricci scalar has mass dimension 2. Since $[d^3x d^2\theta] = -2$, only the superfield R and Λ are possible since $[R] = [\Lambda] = 1$. An integrand $G_{abc}G^{abc}$ has too high mass dimension, but it yields an action if one deletes the factor $1/\kappa^2$; it leads then to conformal supergravity in $2 + 1$.

It is easy to see that the action (340) (re)produces $N = 1$ supergravity. Integration over θ yields a term $\nabla^a \nabla_a R\,|$ and a term with Λ, while using the solution for $R_{ab,cd}{}^{ef}$ in (333) one gets

$$\nabla^a \nabla_a R = -(\, 4i\, R_{mn}{}^{mn} + \nabla_a G_b{}^{ab}\,) \ . \tag{341}$$

Using that $R_{mn}{}^{mn}|$ is the usual scalar Riemann curvature, one recovers the Einstein-Hilbert action. The term $\nabla_a G_b{}^{ab}|$ gives the gravitino action.

Of course, having found a unique candidate for a covariant action does not yet prove it does indeed yield the correct field equations. There are at least three approaches to obtaining the field equations from an action:

(i) One writes the action in terms of the unconstrained superfields $E_a{}^M$, in terms of which $E_M{}^A$, $T_{AB}{}^C$ and $R_{AB}{}^{rs}$ all can be expressed. Then one varies with respect to $E_a{}^M$.

(ii) One chooses a gauge in which $E_a{}^M$ is restricted to $E_a{}^\alpha = \delta_a{}^\alpha \psi$ and $E^{(a\alpha b)}$. If one varies then ψ and $E^{(a\alpha\beta)}$, one should in principle add compensating gauge transformations to stay in this gauge, but since the action is gauge independent, this is not necessary.

(iii) One first deduces which variations of $E_A{}^M$ and $E_M{}^A$ and $\phi_A{}^{mn}$ are allowed by requiring that the variations of constraints on the supertorsion and supercurvatures remain zero. Then one parameterizes the variations of the action in terms of these allowed variations.

In 3+1 dimensions the last method is the easiest because there the action is

$$I \sim \frac{1}{\kappa^2} \int d^4x\, d^4\theta\, \mathrm{sdet}\,(E_M{}^A)\ , \tag{342}$$

leading to the variation

$$\delta I \sim \frac{1}{\kappa^2} \int d^4x\, d^4\theta\, (-)^B\, E_B{}^N\, \delta E_N{}^B\, \mathrm{sdet}\,(E_M{}^A)\ . \tag{343}$$

One can then find the general form of $H_A{}^B = E_A{}^M\, \delta E_M{}^B$ by varying the supertorsions which are constrained to be zero or constant,

$$\begin{aligned} \delta T_{AB}{}^C \quad &= \quad T_{AB}{}^D H_D{}^C - \Big[H_A{}^D T_{DB}{}^C + \nabla_A H_B{}^C \\ &\quad -\frac{1}{2}\, E_A{}^M\, \delta\phi_M{}^{rs} (\gamma_r\gamma_s)_B{}^C + (-)^{AB}\, (A \leftrightarrow B) \Big] \end{aligned} \tag{344}$$

and requiring that these variations all cancel. Crucial in this approach is that total derivatives in $(-)^A H_A{}^A$ vanish upon superintegration. In our case of 2+1 dimensions, these total derivatives are still multiplied by R, so they do not vanish. We shall therefore first choose the gauge in (ii), and then vary as in (iii).

To gauge away as many parts of $E_A{}^M$ as possible while not breaking rigid SUSY, *i.e.* while still ending up with unconstrained superfields, we first deduce how $E_A{}^M$ transforms and then look for field-independent terms in these transformation laws. The transformation law of $E_A{}^M$ can be obtained straightforwardly from (285) and (286):

$$\nabla'_A = \left(E_A{}^M D_M + \frac{1}{2}\, \phi_A{}^{rs} M_{rs} \right)' = \left[\nabla_A\, ,\, k^M D_M + \frac{1}{2}\, L^{rs} M_{rs} \right]\ . \tag{345}$$

One finds

$$\delta E_A{}^M = E_A{}^N (D_N k^M) - k^N (D_N E_A{}^M) - E_A{}^N k^P\, T^{(0)}_{PN}{}^M - L_A{}^B E_B{}^M\ , \tag{346}$$

where[6]

$$L_A{}^B = -\frac{1}{2}\, \{ L^{rs} (\gamma_r\gamma_s)_a{}^b\, ,\, L_m{}^n \} \tag{347}$$

[6]Recall from eq. (268) that on a spinor ψ^a the Lorentz group acts as $-\frac{1}{2}\, \lambda^{mn} (\gamma_m\gamma_n)^a{}_b \psi^b$, so on ψ_a one gets an extra minus sign.

and $T_{MN}^{(0)\ P}$ is the torsion of rigid superspace, due to $[D_M, D_N\}$ and only nonzero for $\{D_\beta, D_\alpha\} = 2i\gamma^\mu_{\beta\alpha}\partial_\mu$. The field-independent terms in (346) are

$$\begin{aligned} \delta E_a{}^\alpha &= E_a{}^\beta D_\beta k^\alpha - L_a{}^b E_b{}^\alpha + \dots \\ \delta E_a{}^\mu &= E_a{}^\alpha D_\alpha k^\mu - 2i\, E_a{}^\alpha k^\beta (\gamma^\mu)_{\beta\alpha} + \dots\ . \end{aligned} \tag{348}$$

We can use a Lorentz transformation with $L_a{}^b = -\frac{1}{2}\, L^{rs}(\gamma_r\gamma_s)_a{}^b$ to make the $E_{a\alpha}$ antisymmetric. Then the most general expression for $E_a{}^\alpha$ id

$$E_a{}^\alpha = \delta_a{}^\alpha\, \psi(x,\theta)\ . \tag{349}$$

This is similar to the practice of making the vielbein in x-space symmetric by a suitable local Lorentz transformation. We can then use k^β to remove some parts of $E_a{}^\mu$. Indeed, for $E^{a,bc} \equiv \epsilon^{aa'} E_{a'}{}^\mu\, (\gamma_\mu)^{bc}$ we find

$$\begin{aligned} \delta E^{a,bc} &= 2i\, \epsilon^{aa'} E_{a'}{}^\alpha\, k^\beta\, (\, \delta_\beta{}^b \delta_\alpha{}^c + \delta_\beta{}^c \delta_\alpha{}^b\,) + \dots \\ &= 2i\, (E^{ac}k^b + E^{ab}k^c\,) + \dots \\ &\simeq 2i\, (\epsilon^{ac}k^b + \epsilon^{ab}k^c) + O\, (E_a{}^c - \delta_a{}^c)\ , \end{aligned} \tag{350}$$

and since one can decompose $E^{a,bc}$ into a totally symmetric part and trace terms, just as $\mathcal{F}_{\alpha,\beta\gamma}$ in (104), we can use k^β to gauge away the trace parts and thus make $E^{a,bc}$ totally symmetric:

$$E^{a,bc} = E_{(a,bc)} \qquad E_a{}^{ac} = 0\ . \tag{351}$$

Note that the parameter $k^\mu(x,\theta)$, or equivalently $k^{\alpha\beta}$ is still left. In spinor notation, its θ expansion reads

$$k^{\alpha\beta}(x,\theta) = \xi^{\alpha\beta}(x) + \theta^{(\alpha}\epsilon^{\beta)}(x) + i\, \theta_\gamma \eta^{(\alpha\beta\gamma)}(x) + i\theta^2 \zeta^{\alpha\beta}(x)\ . \tag{352}$$

We shall identify $\xi^{\alpha\beta}$ as the general coordinate parameter and ϵ^β as the local SUSY parameter.

The symmetry group is restricted by this gauge choice. To stay in this gauge one must satisfy $\delta E_{(a}{}^\alpha \epsilon_{\alpha b)} \equiv \delta E_{(ab)} = 0$ and $\delta E_a{}^{\alpha\beta} \delta_\alpha{}^a \delta_\beta{}^b \equiv \delta E_a{}^{ab} = 0$. From $\delta E_{(ab)} = 0$ one finds

$$\psi D_{(a} k_{b)} - L_{ab}\psi - E_{(a}{}^\mu \partial_\mu k_{b)} = 0\ , \tag{353}$$

and from $\delta E_a{}^{ab} = 0$

$$\psi D_a k^{ab} - 6i\, \psi k^b + E_a{}^\nu \partial_\nu k^{ab} - L_a{}^d E_d{}^{ab} = 0\ . \tag{354}$$

One can solve these equations for L_{ab} and k^a, respectively. Then the remaining transformations of ψ and $E^{(abc)}$ read

$$\begin{aligned} \delta\psi &= \frac{1}{2}\, E_a{}^\mu \partial_\mu k^a + \frac{1}{2}\, \psi D_\alpha k^\alpha - k^\alpha D_\alpha \psi - k^\mu \partial_\mu \psi \\ &= \frac{1}{12}\, \partial_{\alpha\beta} k^{\alpha\beta} + \text{ field-dependent -terms} \end{aligned} \tag{355}$$

$$\begin{aligned} \delta E_a{}^{bc} &= \psi D_a k^{bc} - 4i\, \psi \delta_a{}^{(b} k^{c)} + \dots \\ &= \psi D_{(a} k^{bc)} + \text{ field–dependent terms} \end{aligned} \tag{356}$$

If desired, one can go on and use the parameters $\eta^{\alpha\beta\gamma}$ and $\zeta^{\alpha\beta}$ in $k^{\alpha\beta}$ in (352) to remove the first two terms in $E^{(abc)}$, a so-called Wess-Zumino gauge. One is then left with

$$\begin{aligned} \psi(x,\theta) &= h(x) + i\,\theta^\alpha \psi_\alpha(x) + i\,\theta^2 S(x) \\ E^{(abc)}(x,\theta) &= \theta_d\, h^{(abcd)}(x) + i\,\theta^2 \psi^{(abc)}(x)\ . \end{aligned} \tag{357}$$

We recognize in h and $h^{(abcd)}$ the trace and the traceless part of the symmetric dreibein $e_{\mu m} + e_{m\mu}$, while ψ^α and $\psi^{(abc)}$ constitute the "gamma trace" $\gamma^\mu \psi_\mu$ and the gamma-traceless part of the gravitino ${\psi_\mu}^\alpha$, and S is the auxiliary field. However, we shall not choose this gauge, as it is broken by the supersymmetry generated by $\epsilon^\alpha Q_\alpha$. For example, $\epsilon^\alpha Q_\alpha E^{(abc)}$ produces a term $h^{(abcd)}\epsilon_d$ in the $\theta = 0$ entry of $E^{(abc)}$, and to remove it one would need to add to $\epsilon^\alpha Q_\alpha$ further compensating gauge transformations with $k^{\alpha\beta}$ which maintain the gauge ${E_a}^\alpha = {\delta_a}^\alpha \psi$ and ${E_a}^{ab} = 0$.

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