Higher Categorical Structures in Geometry

General Theory and Applications to Quantum Field Theory

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Warm up: geometry

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$\rightsquigarrow$ consider geometric objects on M.

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   $\longrightarrow$ holonomy groups, curvature
   $\longrightarrow$ general relativity
2. bundles over M (vector- or principal G-bundles, connection)
   $\longrightarrow$ invariants of M $\qquad \mathrm{K}^0(M)$, $\check{\mathrm{H}}^1(M, G)$
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metrics on M form set
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<u>Structural ladder:</u>

metrics on M form	set
bundles over M form	category
bundle gerbes / 2-bundles over M form	2-category

A set consists of

- elements: $\bullet$ (metrics over M)

2-categories

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A category consists of

- objects: $\bullet$ (bundles over M)
- morphisms: $\bullet \to \bullet$ (gauge transformations)

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- morphisms: $\bullet \longrightarrow \bullet$
- 2-morphisms: $\bullet \overset{\longrightarrow}{\underset{\longrightarrow}{\Downarrow}} \bullet$ (higher gauge transformations)

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<u>Examples:</u> homotopies, natural transformations,...

Example: Bundle gerbes

Murray '94:	def. of bundle gerbes $\mathcal{G} = (Y, B, L, \mu)$
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1. Dixmier-Douady class $DD(\mathcal{G}) \in \mathrm{H}^3(M, \mathbb{Z})$
2. locally determined by 2-form B
3. exact sequence $\Omega^2(M) \longrightarrow \mathrm{Iso}\big(\mathcal{G}rb^{\nabla}(M)\big) \xrightarrow{DD} \mathrm{H}^3(M, \mathbb{Z})$

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String theory

Σ: Riemannian surface, M: manifold, g metric, $\mathcal{G}$ bundle gerbe

$$\text{moving string } \phi : \Sigma \to M$$

Feynman amplitude (Witten '84, Gawedzki '88):

$$\mathcal{A}[\phi] = \exp(2\pi i S^{\mathrm{kin}}[\phi]) \cdot \exp(2\pi i S^{WZ}[\phi]) \in \mathbb{C}$$

Wess-Zumino term $\exp(2\pi i S^{WZ}[\phi])$: surface holonomy of $\mathcal{G}$
(defined by local integration)

Example: Bundle gerbes II

Desired surface holonomy $\rightsquigarrow$ 2-category of 2-forms $\mathcal{G}rbtriv^{\nabla}(M)$:

- objects: $B \in \Omega^2(M)$
- morphisms $B \to B'$: $A \in \Omega^1(M)$ s.t. $dA = B' - B$
- 2-morphisms $A \to A'$: $g : M \to U(1)$ s.t $\mathrm{dlog} f = A' - A$

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But: Cannot be glued (descent)!

1. $(U_i)_{i \in I}$ open cover of M
2. B_i objects over U_i
3. $A_{ij} : B_i \to B_j$ morphisms over $U_i \cap U_j$
4. $g_{ijk} : \; B_i \underset{A_{ik}}{\overset{A_{jk} \circ A_{ij}}{\Downarrow}} B_k$ 2-morphisms over $U_i \cap U_j \cap U_k$.

No global 2-form $\rightsquigarrow$ bundle gerbes!

General results

Precise statement:

1. assignment $M \mapsto \mathcal{G}rbtriv^{\nabla}(M)$ is pre-2-stack.
2. pre-2-stack $\mathcal{X}$ is 2-stack if locally defined objects can be glued to global objects (Duskin '89, Breen '94).
3. otherwise 'add' descent objects to $\mathcal{X} \rightsquigarrow$ new pre-2-stack $\mathcal{X}^+$.

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Theorem (2.3.3)

$\mathcal{X}^+$ is a 2-stack (stackification).

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Ingredients for proof:

- extend $\mathcal{X}$ to Lie groupoids Λ (equivariant extension).
- show that $\mathcal{X}(\Lambda)$ is invariant under Morita equivalence in Λ (Theorem 2.2.16)
- prove equivariant descent $\rightarrow$ Theorem 2.3.3. as special case.

Jandl gerbes

Adapt to different situations:

- unoriented surface holonomy (type I String theories)
- Σ non-oriented surface, $\phi : \Sigma \to M$
- want: target space data to define 'holonomy' around ϕ
- replace 2-forms B by 2-densities

  $\to$ local categories $\mathcal{JG}rbtriv^\nabla(M)$.
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Unoriented surface holonomy

1. Each Jandl gerbe $\mathcal{G}$ has an associated $\mathbb{Z}/2$-cover $\mathcal{O}(\mathcal{G}) \to M$ (orientation cover)
2. $\phi : \Sigma \to M$ and $\begin{array}{ccc} \hat{\Sigma} & \xrightarrow{\hat{\phi}} & \mathcal{O}(\mathcal{G}) \\ \downarrow & & \downarrow \\ \Sigma & \xrightarrow{\phi} & M \end{array}$ $\Rightarrow$ holonomy $\mathrm{Hol}_{\mathcal{G}}(\phi, \hat{\phi}) \in U(1)$.
3. Generalizes orientifold backgrounds of Schreiber-Schweigert-Waldorf '07, Distler-Freed-Moore '09 (Prop. 2.4.12)

Supersymmetric sigma models

Remember: Σ world sheet, M target space

- $\phi : \Sigma \to M$ ("worldsheet Boson")
- $\psi \in \Gamma(S\Sigma \otimes \phi^* TM)$ ("worldsheet Fermion")
  where $S\Sigma \to \Sigma$ spinor bundle, in particular Σ spin.

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$\rightsquigarrow$ effective amplitude $\hat{\mathcal{A}}(\phi)$

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Need: trivialisation of Pfaff to get $\hat{A}(\phi) \in \mathbb{C}$.

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Theorem (Bunke '09)

Geometric string structure over M $\to$ trivialization of Pfaff.

String group as Lie-group

Whithead tower of $\mathrm{O}(n)$:

$$\cdots \longrightarrow \mathrm{String}(n) \longrightarrow \mathrm{Spin}(n) \longrightarrow \mathrm{SO}(n) \longrightarrow \mathrm{O}(n)$$

Models:

- Whitehead '52: abstract homotopy theoretic
- Stolz '96, Stolz-Teichner '04: concrete models as top. groups

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Fact: Cannot be finite dimensional.

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Theorem (4.3.6)

There is a model of $\mathrm{String}(n)$ *as a metrizable Fréchet Lie group.*

string structure on (M, g)	$=$	lift of structure group to $\mathrm{String}(n)$
geometric string structure	$=$	lift of Levi-Civita connection

String group as Lie-2-group

String group models as Lie 2-groups:

- BCSS '07, Henriques '08, Schommer-Pries '10.

Lie 2-group Γ = higher categorical analogue of Lie group
Geometric realization $|\Gamma|$: nice topological group (chapter 4.4)

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There is a 2-group $\mathcal{S}\mathtt{TRING}(n)$ such that $|\mathcal{S}\mathtt{TRING}(n)| \simeq \mathrm{String}(n)$.

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principal bundles for Lie groups $\hat{=}$??? for Lie 2-groups
Candidates:

- 2-bundles (Bartels '04, Baas-Bökstedt-Kro '06, Wockel '08)
- Non-abelian bundle gerbes (Breen-Messing '01, Aschieri-Cantini-Jurco '05)

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Theorem (3.7.1)

Γ *a Lie-2-group* $\longrightarrow$ Γ*-2-bundles and non-abelian bundle* Γ*-gerbes form equivalent 2-categories* (*denoted* $2\text{-}\mathcal{B}un_\Gamma(M)$)

2-Bundles

2-group $\mathcal{S}\mathtt{TRING}(n) \rightsquigarrow$ define $\mathcal{S}\mathtt{TRING}(n)$-2-structure

Properties of 2-bundles:

- $M \mapsto 2\text{-}\mathcal{B}un_\Gamma(M)$ is 2-stack Theorem 3.6.8
- $\mathrm{Iso}\big(2\text{-}\mathcal{B}un_\Gamma(M)\big) \overset{1:1}{\longleftrightarrow} \check{\mathrm{H}}^1(M,\Gamma) \overset{1:1}{\longleftrightarrow} \mathrm{Iso}\big(\mathcal{B}un_{|\Gamma|}(M)\big)$ Theorem 3.5.19 & 3.4.6

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Theorem 3.5.19 & 3.4.6

Corollary (4.6.1)

String(n)-structures = $\mathcal{STRING}(n)$-2-structures

More precisely:

natural functor $\mathcal{B}un_{\mathrm{String}(n)}(M) \to 2\text{-}\mathcal{B}un_{\mathcal{STRING}(n)}(M)$

$\rightsquigarrow$ bijection $\check{\mathrm{H}}^1(M, \mathrm{String}(n)) \xrightarrow{\sim} \check{\mathrm{H}}^1(M, \mathcal{STRING}(n))$.

Summary

Framework for higher geometry

- bundle gerbes as 2-stackification
  $\longrightarrow$ equivariance, Jandl gerbes
- structure theory for 2-bundles
  (resp. non-abelian gerbes)
- new models for the string group and (geometric) string structures

$\Rightarrow$ describe and classify sigma models

Outlook

- relation to quantum theory (CFT)
- equivariant DW-theory
- full comparison of geometric string structures

Based on:

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