Towards Models for 2-Hilb and 3-Hilb as targets for functorial field theories

Joint with André Henriques and Dave Penneys, based on arXiv:2411.01678 and Work in Progress

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W^*-Categories

Introduction

Algebraic	**Unitary Topological**
Vector space	Hilbert space
Algebra	von Neumann algebra
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W^*- categories: "categorified" Hilbert Spaces.

Bicommutant categories: "categorified" of von Neumann algebras.

Dagger Categories

Definition (Dagger Categories)

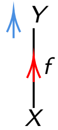
A **dagger category** is a category $\mathcal{T}$ equipped with function $(-)^\dagger : \mathrm{Hom}_{\mathcal{T}}(X, Y) \to \mathrm{Hom}_{\mathcal{T}}(Y, X)$ for all pairs of objects X, Y of $\mathcal{T}$ such that:

- for any object X, $\mathrm{id}_X^\dagger = \mathrm{id}_X$.
- for any morphism $f : X \to Y$, $(f^\dagger)^\dagger = f$
- for any two morphisms f and g such we have $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$.

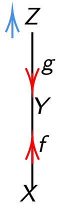
The most important dagger category which will show up throughout this talk is the category of Hilbert spaces Hilb where the $\dagger$-structure is given by taking the adjoint of linear maps.

String Calculus for Dagger Categories

We represent $f: X \to Y$ as



If the direction of f matches the *up* direction (blue arrow) then we read it as f, if it is the opposite we read it as $f^\dagger$.
We represent composition by concatenation and it is read down to up. For example, the diagram



is read as $X \xrightarrow{f} Y \xrightarrow{g^\dagger} Z$.

W*-categories $\sim$ "Von-Neumann Algebra-oids"

A $*$-category is a $\mathbb{C}$-linear $\dagger$-category such that the $\dagger$ is anti-linear on hom-spaces.

Let $\mathcal{T}$ be is a $*$-category, we write $\mathcal{T}^{\oplus}$ for the category whose objects are formal finite sums of objects of $\mathcal{T}$, and whose morphisms are given by $\mathrm{Hom}_{\mathcal{T}}(\oplus X_i, \oplus X_j) := \oplus_{i,j}\mathrm{Hom}_{\mathcal{T}}(X_i, X_j)$.

Definition (W^*-categories)

A W^*-**category** is a $*$-category $\mathcal{T}$ such that $\mathrm{End}_{\mathcal{T}}(X)$ is a von Neumann algebra for every $X \in \mathcal{T}^{\oplus}$.

$\overline{\mathcal{T}}$ is the category with same objects as $\mathcal{T}$, and complex conjugate hom-spaces. We note this as the first involution ($\dagger_0$).

Definition (Idempotent completion)

A W^*-category is called **idempotent complete** if whenever a morphism $p : X \to X$ satisfies $p^2 = p^* = p$, there exists an object Y and a morphism $\iota : Y \to X$ such that $\iota\iota^* = p$ and $\iota^*\iota = \mathrm{id}_Y$.

Definition (Generating set)

A W^*-category $\mathcal{T}$ is said to *admit a set of generators* if there exists a set of objects such that every non-zero object admits a non-zero map from at least one of the generators. It is said to **admit a generator** if the above set may be taken to consist of a single object.

Definition (Orthogonal direct sums)

Given a collection of objects X_i in a W^*-category indexed by some set I, their **orthogonal direct sum** is an object X equipped with morphisms $\iota_i : X_i \to X$ satisfying

$$\iota_i^* \iota_j = \delta_{ij} \mathrm{id}_{X_i} \qquad \sum \iota_i^* \iota_i = \mathrm{id}_X.$$

The orthogonal direct sums, if it exists, is denoted $\oplus_{i \in I} X_i$.
Here, the infinite sum $\sum_{i \in I} \iota_i^* \iota_i$ is defined as the sup over all finite subsets $I_0 \subset I$ of the finite sums $\sum_{i \in I_0} \iota_i^* \iota_i$.

Definition (Cauchy completion)

We call a W^*-category **(Cauchy) complete** if it admits a generator, is idempotent complete, and has all direct sums.

We write $\mathcal{C}^{\hat{\oplus}}$ for the direct sum completion of the idempotent completion of a W^*-category $\mathcal{C}$, and call it the *Cauchy completion* of $\mathcal{C}$.

Theorem

Every complete W^*-category $\mathcal{T}$ is equivalent to $R\text{-}\mathrm{Mod}$ for some von Neumann algebra R.

Let X be a geneartor of $\mathcal{T}$ and $R := \mathrm{End}_{\mathcal{T}}(X)^{\mathrm{op}}$.

$$BR^{\mathrm{op}} \to \mathcal{T} \text{ and } BR^{\mathrm{op}} \to R\text{-}\mathrm{Mod}$$

$$\star_{R^{\mathrm{op}}} \mapsto X \text{ and } \star_{R^{\mathrm{op}}} \mapsto {}_R L^2 R$$

On Cauchy completion, this extends to equivalences,

$$(BR^{\mathsf{op}})^{\hat{\oplus}} \xrightarrow{\simeq} \mathcal{T} \text{ and } (BR^{\mathsf{op}})^{\hat{\oplus}} \xrightarrow{\simeq} R\text{-}\mathrm{Mod}.$$

Therefore, $\mathcal{T} \cong R\text{-}\mathrm{Mod}$.

vN-bimodules and W^*-functors

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be W^*-categories.
A functor between W^*-categories is a $*$-functor $F : \mathcal{T}_1 \to \mathcal{T}_2$ that induces normal homomorphisms $\mathrm{End}_{\mathcal{T}_1}(T) \to \mathrm{End}_{\mathcal{T}_2}(F(T))$ for all $T \in \mathcal{T}_1$.

Theorem

Given von Neumann algebras R_1 and R_2, the functor

$$\begin{array}{rcl} \mathrm{Bim}(R_2, R_1) & \to & \mathrm{Func}\big(R_1\text{-}\mathrm{Mod}, R_2\text{-}\mathrm{Mod}\big) \\ {}_{R_2}X_{R_1} & \mapsto & {}_{R_2}X \boxtimes_{R_1} - \end{array}$$

is an equivalence of categories.
The inverse of sends a functor $F : R_1\text{-}\mathrm{Mod} \to R_2\text{-}\mathrm{Mod}$ to the bimodule ${}_{R_2}\big(F({}_{R_1}L^2 R_1)\big)_{R_1}$.

Intertwiners correspond to natural transformations.

Monoidal Structure

Given Cauchy complete W^*-categories, their *completed tensor product* is given by:

$$\mathcal{T}_1 \hat{\otimes} \mathcal{T}_2 := (\mathcal{T}_1 \otimes \mathcal{T}_2)^{\hat{\oplus}}.$$

where $\otimes$ is the tensor product of $\mathbb{C}$-linear categories.
$\mathrm{Hilb} = (\mathbf{B}\mathbb{C})^{\hat{\oplus}}$ is the unit of the above operation.

Theorem

Given von Neumann algebras R_1 and R_2, the functor

$$(R_1\text{-}\mathrm{Mod}) \,\hat{\otimes}\, (R_2\text{-}\mathrm{Mod}) \to (R_1 \bar{\otimes} R_2)\text{-}\mathrm{Mod}$$
$$({}_{R_1}H) \otimes ({}_{R_2}K) \mapsto {}_{R_1 \bar{\otimes} R_2}(H \otimes K)$$

is an equivalence of categories.

Hilb-valued inner product

Definition (Inner Product)

Every W^*-category $\mathcal{T}$ admits a canonical Hilb-**valued inner product** $\langle -, - \rangle_{\mathrm{Hilb}} : \overline{\mathcal{T}} \times \mathcal{T} \to \mathrm{Hilb}$ given by:

$$\langle X, Y \rangle_{\mathrm{Hilb}} := p_Y L^2(\mathrm{End}(X \oplus Y)) p_X,$$

where $p_X, p_Y \in \text{End}(X \oplus Y)$ are the two projections.

Lemma

Let $\mathcal{T}$ be a Cauchy complete W^*-category. Then

$$\begin{aligned} \mathcal{T} &\to \mathrm{Func}(\overline{\mathcal{T}}, \mathrm{Hilb}) \\ X &\mapsto \langle -, X \rangle \end{aligned} \tag{1}$$

is an equivalence of categories. This corresponds to a statement corresponding to the Riesz representation theorem.

Involution on W^*-functors ($\dagger_1$)

Definition (Adjoint of a W^*-functor)

Given a functor $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ between Cauchy complete W^*-categories, its **adjoint** $F^\dagger$ is defined as the composite

$F^\dagger : \mathcal{T}_2 \xrightarrow{\simeq} \mathrm{Hom}(\overline{\mathcal{T}_2}, \mathrm{Hilb}) \xrightarrow{-\circ \overline{F}} \mathrm{Hom}(\overline{\mathcal{T}_1}, \mathrm{Hilb}) \xrightarrow{\simeq} \mathcal{T}_1.$

Equivalently, the functor $F^\dagger : D \to C$ is specified by the requirement that

$$\langle X, F^\dagger(Y)\rangle_{\mathrm{Hilb}} \;\cong\; \langle F(X), Y\rangle_{\mathrm{Hilb}}, \tag{2}$$

naturally in X and Y.

Definition (For a natural transformation)

Given a natural transformation $\alpha : F \Rightarrow G$ between functors $\mathcal{T}_1 \to \mathcal{T}_2$, the adjoint natural transformation $\alpha^\dagger : F^\dagger \Rightarrow G^\dagger$ is specified,

$$\begin{array}{ccc} \langle X, F^\dagger(Y)\rangle_{\mathrm{Hilb}} & \xrightarrow{\simeq} & \langle F(X), Y\rangle_{\mathrm{Hilb}} \\ {\scriptstyle \langle \mathrm{id}_X, (\alpha^\dagger)_Y\rangle}\downarrow & & \downarrow{\scriptstyle \langle \alpha_X, \mathrm{id}_Y\rangle} \\ \langle X, G^\dagger(Y)\rangle_{\mathrm{Hilb}} & \xrightarrow{\simeq} & \langle G(X), Y\rangle_{\mathrm{Hilb}} \end{array}$$

commutes.

Lemma

The operations $F \mapsto F^\dagger$ and $\alpha \mapsto \alpha^\dagger$ assemble to an antilinear equivalence

$$\dagger : \mathrm{Func}(\mathcal{T}_1, \mathcal{T}_2) \to \mathrm{Func}(\mathcal{T}_2, \mathcal{T}_1), \tag{3}$$

and there are natural unitary isos $\varphi_F : F \to F^{\dagger\dagger}$ and $\nu_{F,G} : F^\dagger \circ G^\dagger \to (G \circ F)^\dagger$.

We also have $\alpha^* : G \Rightarrow F$ defined pointwise as $(\alpha^*)_X := (\alpha_X)^*$, which gives the antilinear involution $* : \mathrm{Func}(\mathcal{T}_1, \mathcal{T}_2) \to \mathrm{Func}(\mathcal{T}_1, \mathcal{T}_2)$. This is $(\dagger_2)$.

2-Hilb

Theorem

The assignment $R \mapsto R\text{-}\mathrm{Mod}$ induces an equivalence of bicategories $\mathrm{Mor}(\mathrm{vN}) \xrightarrow{\simeq} \mathrm{W}^*\mathrm{Cat}$

Remark: It is in fact a equivalence of *tri-involutive monoidal* (fully-dagger monoidal) bi-categories.

	$\mathrm{vN2}$	$\mathrm{W}^*\mathrm{Cat}$
$\dagger_0$	complex conjugation or op of algebra	complex conjugation or op of category
$\dagger_1$	complex conjugation of underlying hilbert space	adjoint functor
$\dagger_2$	adjoint of linear intertwiner	pointwise $*$ (adjoint of linear maps)
	spacial tensor product $\bar{\otimes}$	Complete tensor product of categories $\hat{\otimes}$

2-Hilb-analogies with Hilb

A Hilbert space H	A complete W^*-category $\mathcal{T}$
The one dimensional Hilbert space $\mathbb{C}$	The W^*-category Hilb
Scalar multiplication $\cdot\colon \mathbb{C} \times H \to H$	Canonical tensor $\mathrm{Hilb} \times \mathcal{T} \to \mathcal{T}$
Inner product $\langle\ ,\ \rangle\colon \overline{H} \times H \to \mathbb{C}$	Hilb-valued inner product $\overline{\mathcal{T}} \times \mathcal{T} \to \mathrm{Hilb}$

This is just a glimpse, lots more listed in the beginning of arXiv:2411.01678

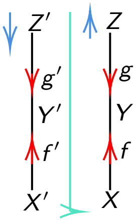
Bi-involutive tensor W^*-categories

String calculus for W*-categories

We stack the string diagrams for $\overline{\mathcal{T}}$ and $\mathcal{T}$, separated by a dividing line. This line comes with a co-orientation, which remembers how the inner product diagram is read. The conjugate category $\overline{\mathcal{T}}$ has the opposite local-up direction.

$$\langle X', X \rangle \xrightarrow{\langle f'^{\dagger}, f \rangle} \langle Y', Y \rangle \xrightarrow{\langle g', g^{\dagger} \rangle} \langle Z', Z \rangle$$

.



There is a unitary isomorphism, $J_{X,Y} \colon \langle X, Y \rangle \xrightarrow{\sim} \overline{\langle Y, X \rangle}$ natural in X, Y. Diagrammatically, $\overline{\langle \ , \ \rangle}$ *swaps* the two strings which makes the isomorphism J manifest.

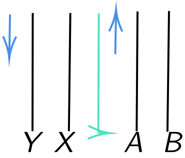
W^*-tensor categories

A **-tensor category* $(\mathcal{T}, \otimes, 1, \alpha, l, r)$ is a *-category with a monoidal structure which is compatible in the sense that

$$\otimes : \mathcal{T} \times \mathcal{T} \to \mathcal{T}$$

is a bilinear functor of *-categories, and the associator α and left and right unitors l, r are unitary.

The string calculus naturally extends to monoidal W^*-categories where the tensor product of objects is represented by placing corresponding strings in parallel left to right in $\mathcal{T}$ and right to left in $\overline{\mathcal{T}}$. The tensor product in $\mathcal{T}$ and $\overline{\mathcal{T}}$ are read outwards starting from the dividing line.



$\langle X \otimes Y,\ A \otimes B \rangle$

Bi-involutive W^*-tensor categories

A **bi-involutive W^*-tensor category** is a W^*-tensor category $\mathcal{T}$ equipped with a covariant anti-linear, anti-tensor functor

$$\overline{} : \mathcal{T} \to \mathcal{T}$$

called the conjugate. The structure data of this anti-tensor functor are denoted

$$\nu_{A,B} : \overline{A} \otimes \overline{B} \xrightarrow{\simeq} \overline{B \otimes A} \qquad \text{and} \qquad j : 1 \to \overline{1}$$

and which satisfy some diagrams.
The functor $\overline{}$ is involutive, meaning that for every $A \in \mathcal{T}$, we are given unitary natural unitary isomorphisms

$$\varphi_A : A \to \overline{\overline{A}}$$

satisfying $\varphi_{\overline{A}} = \overline{\varphi_A}$.

$$
\begin{array}{ccc}
\overline{A} \otimes (\overline{B} \otimes \overline{C}) & \xrightarrow{\mathrm{id}_{\overline{A}} \otimes \nu_{B,C}} & \overline{A} \otimes (\overline{C \otimes B}) \\
{\scriptstyle \alpha_{\overline{A},\overline{B},\overline{C}}} \uparrow & & \downarrow {\scriptstyle \nu_{A,C\otimes B}} \\
(\overline{A} \otimes \overline{B}) \otimes \overline{C} & & \overline{(C \otimes B) \otimes A} \\
{\scriptstyle \nu_{A,B} \otimes \mathrm{id}_{\overline{C}}} \downarrow & & \downarrow {\scriptstyle \overline{\alpha_{C,B,C}}} \\
(\overline{B \otimes A}) \otimes \overline{C} & \xrightarrow{\nu_{B\otimes A,C}} & \overline{C \otimes (B \otimes A)}
\end{array}
$$

$$
\begin{array}{ccc}
1 \otimes \overline{A} & \xrightarrow{j \otimes \mathrm{id}_{\overline{A}}} & \overline{1} \otimes \overline{A} \\
{\scriptstyle l_{\overline{A}}} \downarrow & & \downarrow {\scriptstyle \nu_{1,A}} \\
\overline{A} & \xleftarrow[\overline{r_A}]{} & \overline{A \otimes 1}
\end{array}
\qquad\qquad
\begin{array}{ccc}
\overline{A} \otimes 1 & \xrightarrow{\mathrm{id}_{\overline{A}} \otimes j} & \overline{A} \otimes \overline{1} \\
{\scriptstyle r_{\overline{A}}} \downarrow & & \downarrow {\scriptstyle \nu_{A,1}} \\
\overline{A} & \xleftarrow[\overline{l_A}]{} & \overline{1 \otimes A}
\end{array}
$$

Finally, we require the compatibility conditions $\varphi_1 = \bar{j} \circ j$ and $\varphi_{A\otimes B} = \overline{\nu_{B,A}} \circ \nu_{\overline{A},\overline{B}} \circ (\varphi_A \otimes \varphi_B)$.

Bi-involutive tensor functors

A tensor functor $F\colon \mathcal{T}_1 \to \mathcal{T}_2$ between bi-involutive *-tensor categories is called *a bi-involutive tensor functor* if it comes equipped with a unitary natural transformation

$$\gamma_X : F(\overline{X}) \to \overline{F(X)}$$

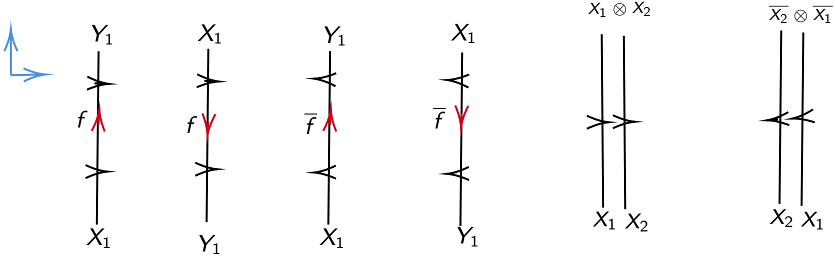
satisfying the following coherences:

$$\begin{array}{ccccc}
F(\overline{X}) \otimes F(\overline{Y}) & \xrightarrow{\mu_{\overline{X},\overline{Y}}} & F(\overline{X} \otimes \overline{Y}) & \xrightarrow{F(\nu^{\mathcal{T}_1}_{x,y})} & F(\overline{Y \otimes X}) \\
\Big\downarrow{\scriptstyle \gamma_X \otimes \gamma_Y} & & & & \Big\downarrow{\scriptstyle \gamma_{Y \otimes X}} \\
\overline{F(X)} \otimes \overline{F(Y)} & \xrightarrow{\nu^{\mathcal{T}_2}_{F(X) \otimes F(Y)}} & \overline{F(Y) \otimes F(X)} & \xrightarrow{\overline{\mu_{Y \otimes X}}} & \overline{F(Y \otimes X)}
\end{array}$$

$$\begin{array}{ccc}
F(x) & \xrightarrow{F(\varphi^{\mathcal{T}_1}_X)} & F(\overline{\overline{x}}) \\
\Big\downarrow{\scriptstyle \varphi^{\mathcal{T}_2}_{F(X)}} & & \Big\downarrow{\scriptstyle \gamma_{\overline{X}}} \\
\overline{\overline{F(X)}} & \xleftarrow[\overline{\gamma_X}]{} & \overline{F(\overline{X})}
\end{array}$$

String calculus for bi-involutive categories

Similar to the chosen up-direction, there is a chosen right direction, together these can be thought as the chosen coordinate axes for a string diagram of a bi-involutive tensor category. We equip our objects with a normal vector or a co-orientation.
We then represent $\overline{X}$ by reversing the co-orientation or reflecting along the up-direction.
This makes ν and φ with the coherences automatically manifest. The unit object is transparent, so j is also in-built.

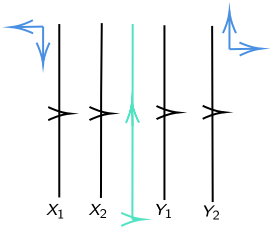


$f : X_1 \to Y_1$ $\quad f^\dagger : Y_1 \to X_1$ $\quad \overline{f} : \overline{X_1} \to \overline{Y_1}$ $\quad \overline{f}^\dagger : \overline{Y_1} \to \overline{X_1}$

For bi-involutive W^*-tensor category

For a bi-involutive tensor category $\overline{\mathcal{T}}$, the local up and right directions are both reversed.

For example, the inner-product $\langle \overline{X_1 \otimes X_2}, Y_1 \otimes Y_2 \rangle \simeq \langle \overline{X_2} \otimes \overline{X_1}, Y_1 \otimes Y_2 \rangle$ is represented as:



We also have the following isomorphism defined by $\overline{\quad\quad}$ being an anti-linear, anti-tensor functor,

$$\begin{aligned}\langle \overline{X}, \overline{Y} \rangle = p_{\overline{Y}} L^2(\mathrm{End}(\overline{X} \oplus \overline{Y})) p_{\overline{X}} &= p_{\overline{Y}} L^2(\overline{\mathrm{End}(X \oplus Y)}) p_{\overline{X}} \\ &= p_{\overline{Y}} \overline{L^2(\mathrm{End}(X \oplus Y))} p_{\overline{X}} = \overline{p_Y L^2(\mathrm{End}(X \oplus Y)) p_X} \\ &= \overline{\langle X, Y \rangle}.\end{aligned}$$

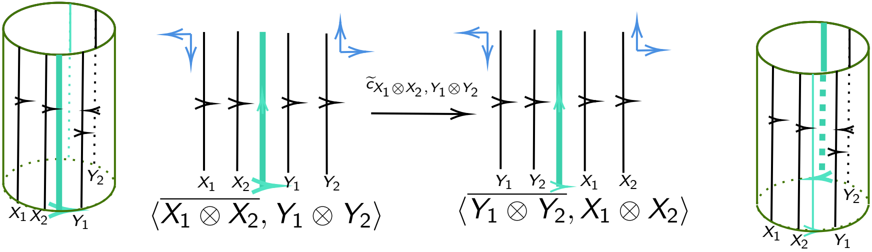
We define an isomorphism $c_{X,Y} \colon \langle X, Y \rangle \to \langle \overline{Y}, \overline{X} \rangle$ as the following composition,

$$\langle X, Y \rangle \xrightarrow{J} \overline{\langle Y, X \rangle} \to \langle \overline{Y}, \overline{X} \rangle$$

and $\tilde{c}_{X,Y} \colon \langle \overline{X}, Y \rangle \to \langle \overline{Y}, X \rangle$ as the following composition,

$$\langle \overline{X}, Y \rangle \to \langle \overline{Y}, \overline{\overline{X}} \rangle \to \langle \overline{Y}, X \rangle$$

using φ_X in the last arrow.


X_1 X_2 Y_1 Y_2
$\langle \overline{X_1 \otimes X_2}, Y_1 \otimes Y_2 \rangle$
$\widetilde{c}_{X_1 \otimes X_2, Y_1 \otimes Y_2}$
Y_1 Y_2 X_1 X_2
$\langle \overline{Y_1 \otimes Y_2}, X_1 \otimes X_2 \rangle$

Commutant of a tensor category

Relative Centre

Let $\iota : \mathcal{T} \to \mathcal{E}$ be a tensor functor between W^*-tensor categories. The *unitary commutant* $\mathcal{Z}(\iota \colon \mathcal{T} \to \mathcal{E})$ of $\mathcal{T}$ inside $\mathcal{E}$ (denoted, $\mathcal{Z}(\iota)$) is the category whose objects are pairs $(X, \{e_X\})$, where X is an object of $\mathcal{E}$ and

$$\{e_X\} = (e_{X,Y} : X \otimes \iota Y \to \iota Y \otimes X)_{Y \in \mathcal{T}}$$

is a collection of unitary isomorphisms, called a half-braiding. The half-braiding is required to be natural in Y, and to satisfy the following in $\mathcal{E}$ for every $Y, Z \in \mathcal{T}$:

$$\begin{array}{ccccc} & & \iota Y \otimes X \otimes \iota Z & & \\ & \overset{e_{X,Y} \otimes \mathrm{id}_{\iota Z}}{\nearrow} & & \overset{\mathrm{id}_{\iota Y} \otimes e_{X,Z}}{\searrow} & \\ X \otimes \iota Y \otimes \iota Z & & & & \iota Y \otimes \iota Z \otimes X \\ \downarrow \wr & & & & \downarrow \wr \\ X \otimes \iota(Y \otimes Z) & & \xrightarrow{\quad e_{X,Y \otimes Z} \quad} & & \iota(Y \otimes Z) \otimes X \end{array}$$

Absorbing objects

An object $\Omega \in \mathcal{T}$ is *left absorbing (right absorbing)* if it is a non-left(right)-zero-divisor and for every non-left(right)-zero-divisor $X \in \mathcal{T}$ we have $\Omega \otimes X \cong \Omega$ ($X \otimes \Omega \cong \Omega$).
The object is *absorbing* if it is both left absorbing and right absorbing.
The absorbing subcategory $\mathcal{T}^{\mathrm{abs}}$ is the completion of the full subcategory on absorbing objects.

Lemma

Given a tensor functor $\iota\colon \mathcal{T} \to \mathcal{E}$, for a complete W^*-tensor category which admits weakly absorbing objects, the map $\mathcal{Z}(\iota) \to \mathcal{Z}(\iota|_{\mathcal{T}^{\mathrm{abs}}})$ is fully-faithful.

Bicommutant Categories

Definition of Bicommutant Category

Let $\mathcal{T}$ be a Cauchy complete bi-involutive W^*-tensor category that admits absorbing objects. This gives us two dagger-monoidal functors,

$$L\colon \mathcal{T} \to \mathsf{End}(\mathcal{T}^{\mathrm{abs}}) \text{ given by } X \mapsto X \otimes -$$
$$R\colon \mathcal{T}^{\mathrm{mop}} \to \mathsf{End}(\mathcal{T}^{\mathrm{abs}}) \text{ given by } X \mapsto - \otimes X$$

A *bicommutant category* $\mathcal{T}$ is such a category equipped with a unitary natural isomorphism $\gamma^L\colon L(\overline{}) \Rightarrow L()^\dagger$ which make L a bi-involutive tensor W^*-functor. Using the associators of $\mathcal{T}$, L and R induce maps

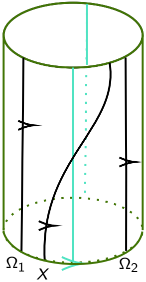
$$\mathcal{T} \to \mathcal{Z}(R\colon \mathcal{T}^{\mathrm{mop}} \to \mathrm{End}(\mathcal{T}^{\mathrm{abs}}))$$
$$\mathcal{T}^{\mathrm{mop}} \to \mathcal{Z}(L\colon \mathcal{T} \to \mathrm{End}(\mathcal{T}^{\mathrm{abs}}))$$

We require these to be equivalences. We require a lot of diagram to commute, which we now list.

For all $X \in \mathcal{T}$, $\Omega_1, \Omega_2 \in \mathcal{T}^{\mathrm{abs}}$, γ^L induces a map

$$\gamma^L_X : \langle \overline{\Omega_1 \otimes X}, \Omega_2 \rangle_{\mathrm{Hilb}} \to \langle \overline{\Omega_1}, X \otimes \Omega_2 \rangle_{\mathrm{Hilb}}$$

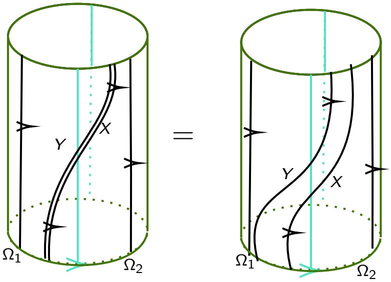
We represent this map diagrammatically by:



The coherences for L being bi-involutive tensor functors are the following:

$$\begin{array}{ccc}
\langle \overline{X} \otimes \overline{Y} \otimes \overline{\Omega}_1, \Omega_2 \rangle & \xrightarrow{\gamma^L_X} & \langle \overline{Y} \otimes \overline{\Omega}_1, X \otimes \Omega_2 \rangle \\
\nu_{X,Y} \big\downarrow & & \big\downarrow \gamma^L_Y \\
\langle \overline{Y \otimes X} \otimes \overline{\Omega}_1, \Omega_2 \rangle & \xrightarrow{\gamma^{\mathrm{L}}_{\mathrm{Y} \otimes \mathrm{X}}} & \langle \overline{\Omega}_1, Y \otimes X \otimes \Omega_2 \rangle
\end{array}$$

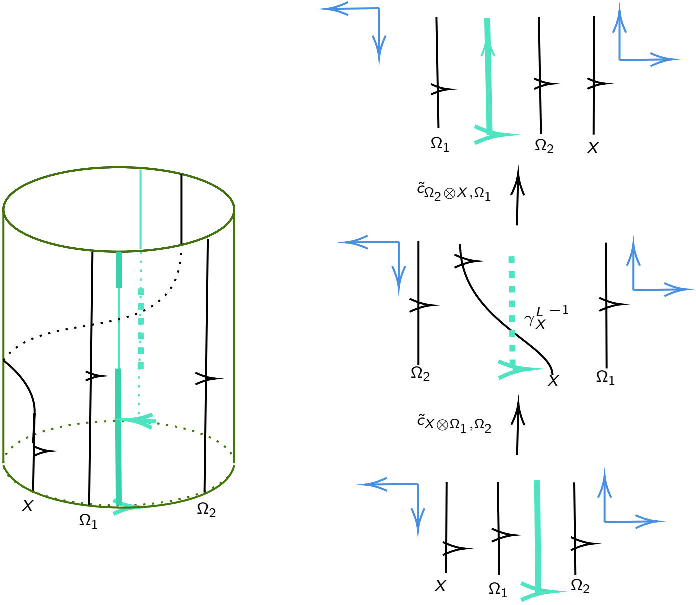
which is manifestly encoded by the diagrammatic calculus.



Using γ^L, we *define* γ^R:

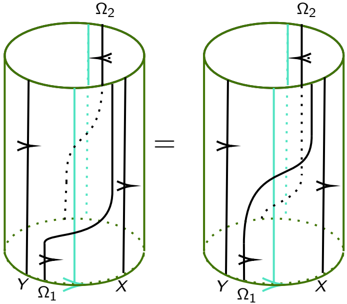
$$\begin{array}{ccc}
\left\langle \overline{X \otimes \Omega_1}, \Omega_2 \right\rangle & \xrightarrow{\gamma^R_X} & \left\langle \overline{\Omega_1}, \Omega_2 \otimes X \right\rangle \\
{\scriptstyle \tilde{c}_{X \otimes \Omega_1, \Omega_2}} \Big\downarrow & & \Big\uparrow {\scriptstyle \tilde{c}_{\Omega_2 \otimes X, \Omega_1}} \\
\left\langle \overline{\Omega_2}, X \otimes \Omega_1 \right\rangle & \xrightarrow[\left(\gamma^L_X\right)^{-1}]{} & \left\langle \overline{\Omega_2 \otimes X}, \Omega_1 \right\rangle
\end{array}$$

Diagrammatically, this involves changing the preferred dividing line, performing ${\gamma^L_X}^{-1}$ and changing the preferred line again.



This makes R a bi-involutive tensor functor.

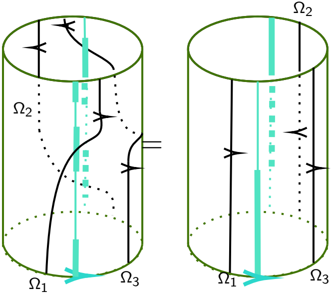
$$
\begin{array}{ccc}
\langle \overline{Y \otimes \Omega_1 \otimes X}, \Omega_2 \rangle & \xrightarrow{\gamma_X^L} & \langle \overline{Y \otimes \Omega_1}, X \otimes \Omega_2 \rangle \\
\Big\downarrow \gamma_Y^R & & \Big\downarrow \gamma_Y^R \\
\langle \overline{\Omega_1 \otimes X}, \Omega_2 \otimes Y \rangle & \xrightarrow[\gamma_{\mathrm{X}}^{\mathrm{L}}]{} & \langle \overline{\Omega}_1, X \otimes \Omega_2 \otimes Y \rangle
\end{array}
$$



Finally, when when all three of $\Omega_1, \Omega_2, \Omega_3$ are absorbing, we require the following diagram to commute.

$$\begin{array}{ccc}
\left\langle \overline{\Omega_1}, \Omega_3 \otimes \Omega_2 \right\rangle & \xrightarrow{\gamma^{R}_{\Omega_2}{}^{-1}} & \left\langle \overline{\Omega_2 \otimes \Omega_1}, \Omega_3 \right\rangle \\
{\scriptstyle \tilde{c}_{\Omega_1, \Omega_2 \otimes \Omega_3}} \downarrow & & \downarrow {\scriptstyle \gamma^{L}_{\Omega_1}} \\
\left\langle \overline{\Omega_3 \otimes \Omega_2}, \Omega_1 \right\rangle & \xleftarrow[\gamma^{R}_{\Omega_3}{}^{-1}]{} & \left\langle \overline{\Omega_2}, \Omega_1 \otimes \Omega_3 \right\rangle
\end{array}$$

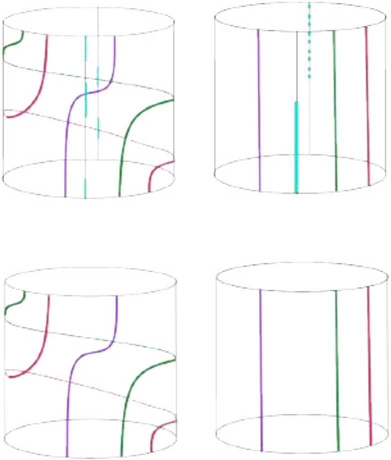
We represent this by two morphisms, the LHS of the diagram below indicates the morphism $\gamma^{R}_{\Omega_3}{}^{-1} \circ \gamma^{L}_{\Omega_1} \circ \gamma^{R}_{\Omega_2}{}^{-1}$, and the RHS is the morphism $\tilde{c}_{\Omega_1, \Omega_2 \otimes \Omega_3}$.



Ω2
Ω1
Ω3
=
Ω2
Ω1
Ω3

Strings on Cylinder

We can put the dividing line anywhere and evaluate, as long as there is at least one string on each side of it.



Concluding Remarks

Examples

$\mathcal{T} = (\mathcal{T}, \otimes, \mathbb{I}, \alpha, l, r, ^{-}, \nu, j, \varphi, \gamma^L)$ be a bicommutant category. Some examples include:

$(\mathrm{Hilb}, \otimes)$

$(\mathrm{Bim}(R), \boxtimes_R)$ for some von Neumann algebra R.

$(\mathrm{Hilb}[G])$ for a discrete group G

$(\mathrm{Rep}_{\mathrm{soliton}}(\mathcal{A}), \boxtimes_{\mathcal{A}})$ for a conformal net $\mathcal{A}$

- When $\mathcal{A}$ is the WZW net for a compact connected group G at level k, this is $\mathrm{Rep}_k(\Omega G)$
- When $\mathcal{A}$ is the Virasoro net, this is $\mathrm{Rep}_c(\mathrm{Diff}(S^1)), \boxtimes)$ at some fixed central charge $c \in \{16/m(m+1) : m2\} \cup [1, \infty)$.

Upcoming Work

Work in Progress includes *understanding modules* over Bicommutant categories and their "categorified" Connes-fusion. A candidate definition is, given M, N right and left $\mathcal{T}$-modules respectively, we can define their fusion as

$$M \boxtimes_{\mathcal{T}} N = p_N \, \mathsf{End}_{\mathcal{T}\text{-Mod}}(\overline{M} \oplus N)^{\mathrm{abs}} p_{\overline{M}}$$

Constructing the 0 and 1 piece of a Segal (Functorial) Chiral CFT, as a functor,

$$\mathrm{Cob}^{\mathrm{conf}}_{0,1,2} \to \mathrm{Mor}(\mathrm{BicommCat})$$

where $\mathrm{Mor}(\mathrm{BicommCat})$ may serve as 3-Hilb in place of $\mathrm{Mor}(\mathrm{TensCat})$ which is usually taken as 3-Vect. We hope to have strictly more fully-dualisable objects and hence it can serve are targets for unitary 3d-TQFTs which were previously known to not fully-extend to a point.

Thank You

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