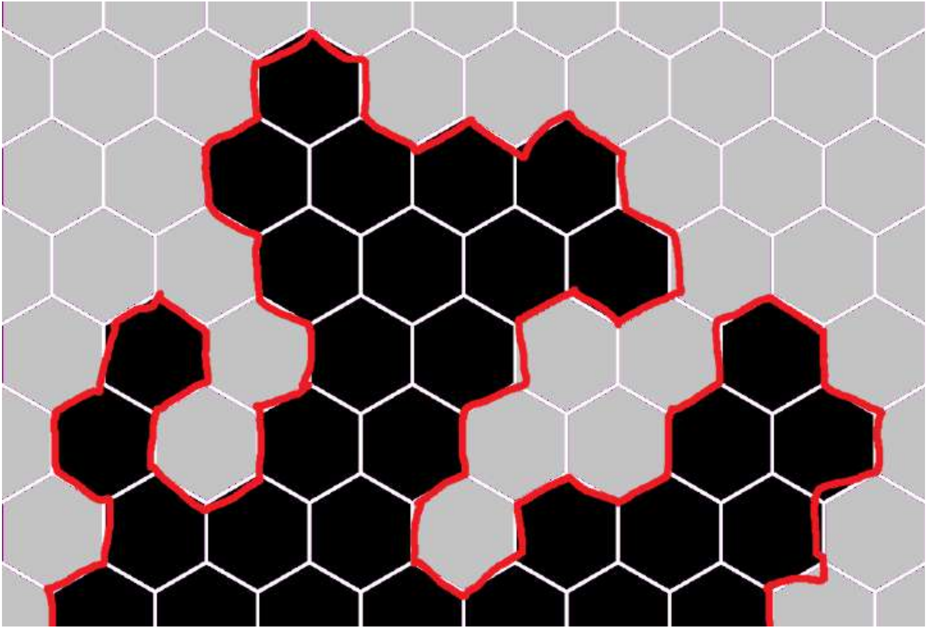
Three-point connectivities of interfaces in 2D critical statistical models

Rongvoram Nivesvivat

New York University Abu Dhabi

Based on a series of (ongoing) work with

- Jesper Jacobsen (ENS Paris, IPhT Saclay, Sorbonne Université)
- Sylvain Ribault (IPhT Saclay)
- Hubert Saleur (IPhT Saclay, University of Southern California)
- Paul Roux (ENS Paris, IPhT Saclay)



Interfaces in 2D critical statistical models

- The Ising interfaces
- Hull percolation
- Self-avoiding loops in SAW
- Several more statistical systems: The Q-state Potts model, Spanning tree, Loop-erased random walk etc.

Probability theory

- The scaling limit of interface in 2D critical statistical models is expected to converge to **Schramm-Loewner Evolution** SLE_κ Schramm 99
- **Conformal Loop Ensemble** $\mathrm{CLE}_\kappa \sim$ Collection of SLE_κ
  Camia, Newman, Sheffield, Werner, Ang, Holden, Sun, ...

QFT and Integrability

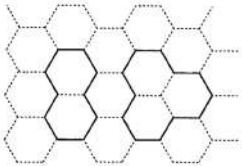
- Interfaces in statistical models are described by **Loop models**
  Temperley, Lieb 71; Baxter, Kelland, Wu 76
- Critical loop models can be described by conformally invariant QFT, known as **2D CFT**
  Nienhuis, Cardy, Saleur, Zuber, Di Francesco, ...

1. The model
2. Three-point functions on the lattice
3. Three-point functions in the critical limit
4. Main results and outlook

Loop model on the hexagonal lattice

a.ka. "the $O(n)$ loop model" by Nienhuis 82

$\sim n^2 K^{22}$

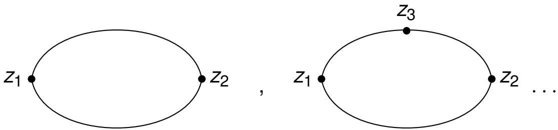


$$Z_{\text{loop}}(K, n) = \sum_{\text{non-intersecting loops}} n^{\#(\text{loops})} K^{\#(\text{bonds})}$$

$$= \int_{S^{n-1}} \prod_i dS_i \prod_{\langle i,j \rangle} (1 + K S_i S_j)$$

- $n \to 1$: Interfaces of the Ising spins.
- $n \to 0$: self-avoiding walk

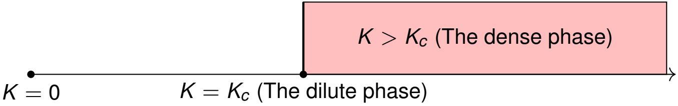
Connection probabilities/ Correlation functions:



(1)

We are interested in finding their closed expressions.

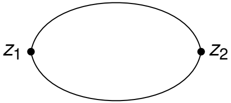
The scaling limit and phase transition



- Second-order phase transition at

$$K_c = (2 + \sqrt{2 - n})^{-\frac{1}{2}} \tag{2}$$

Proof for K_c at $n = 0$ by Duminil-Copin and Smirnov 10



$$\sim \frac{1}{(z_1 - z_2)^{\Delta(n)}} \tag{3}$$

- Changes in critical exponents for $K > K_c$
- **Expect full conformal symmetry in both dilute and dense phases**

- Critical loop models can be described by 2D CFT (Conformal field theory)

$$c = 13 - 6\beta^2 - 6\beta^{-2} \quad \text{and} \quad \beta^2 = \begin{cases} \frac{1}{\pi}\arccos(-n/2) \in [1,2] & \text{dilute} \\ 2 - \frac{1}{\pi}\arccos(-n/2) \in (0,1) & \text{dense} \end{cases} \tag{4}$$

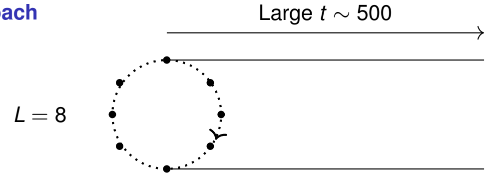
- The corresponding 2D CFT is **non-unitary**.
- Expected to converge to CLE_κ.

n	c	$\kappa = 4/\beta^2$	Models
0	0	$\frac{8}{3}$	Self-avoiding walk (Dilute)
1	0	6	Percolation (Dense)
1	$\frac{1}{2}$	$\frac{16}{3}$	Ising domain walls (Dilute)
-2	-2	2	Loop-erased random walk (Dilute)

(5)

Another universality class at $K = \infty$ where we expect $\mathcal{W}_3$ symmetry Reshetikhin 91 and Dupic, Estienne, Ikhlef 2016

Transfer matrix approach



(6)

- Rewrite the partition function as $Z_{\text{loop}} = \text{Tr}(e^{-H})$,

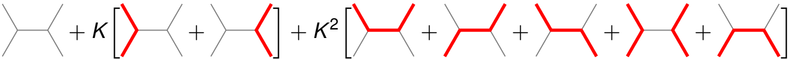
$$H = -K \sum_{i=1}^{L} e_i \quad \text{where } e_i \text{ are generators of the dilute Temperley-Lieb algebra} \tag{7}$$

For instance, see Grimm 95 and Belletête, Saint-Aubin 11.

- Rewrite the partition function Z as the product of the transfer matrix T

$$Z = \langle \text{final}| T^t |\text{initial}\rangle \quad \text{with} \quad T = \left(\prod_{i=1}^{L} R_{2i,2i+1} \right) \left(\prod_{i=1}^{L} R_{2i-1,2i} \right) . \tag{8}$$

where

$R_{k,k+1} =$ 

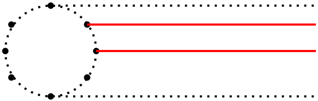
(9)

Three-point functions

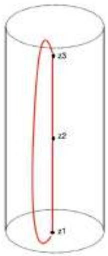
$\hat{Z}(l_1, l_2, l_3; K) = Z$ with

- Insert l_1 consecutive lines at the bottom
- Insert l_2 consecutive lines at the middle
- Insert l_3 consecutive lines at the top

For example, $l_1 = 2$



$\lim_{L\to\infty} \frac{\hat{Z}(2,2,2;K_c)}{Z}$



- We are interested in the universal ratios

$$U(l_1, l_2, l_3; K) = \hat{Z}(l_1, l_2, l_3; K)\sqrt{\frac{Z}{\hat{Z}(l_1, l_1, 0; K)\hat{Z}(l_2, l_2, 0; K)\hat{Z}(l_3, l_3, 0; K)}} \quad (10)$$

- $\lim_{L\to\infty} U(0, 0, 0; K_c) \propto$ Liouville 3pt Jacobsen, Saleur, Ikhlef 15

Computing $\lim_{L\to\infty} U(l_1, l_2, l_3; K_c)$

- Use CFT to predict the closed expression.
- Use transfer matrix approach to compute $U(l_1, l_2, l_3; K_c)$ for large L.
- Compare results from the two different approaches.

- 2D Quantum field theories with conformal symmetry described by

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \quad (11)$$

  where c is the central charge.
- **What are CFTs exactly?** Let $O_j(z,\bar{z})$ be local operators that transform in Virasoro reps,

$$\mathrm{CFT} = \{O_i(z,\bar{z}) | O_i(z,\bar{z}) \times O_j(z,\bar{z}) \propto \sum_{k \subset \mathrm{CFT}} C_{ijk} O_k(z,\bar{z})\} \quad (12)$$

- Correlation functions are strongly constrained,

$$\langle O_{\Delta_1}(z_1) V_{\Delta_1}(z_2) \rangle = \delta_{\Delta_1,\Delta_2} |z_{12}|^{-2\Delta}$$

$$\langle O_{\Delta_1}(z_1) V_{\Delta_2}(z_2) O_{\Delta_3}(z_3) \rangle = \frac{C_{123}}{|z_{12}|^{\Delta_1+\Delta_2-\Delta_3} |z_{13}|^{\Delta_1+\Delta_3-\Delta_2} |z_{23}|^{\Delta_2+\Delta_3-\Delta_1}}$$

  All higher-point functions can be written in terms of the three-point functions

CFT describing critical loop models

Spectrum from the Coloumb Gas formalism

Di Francesco, Saleur, Zuber 87; Ikhlef, Jacobsen, Saleur 15

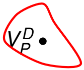
$$V_P^D : (\Delta(P), \Delta(P)) \quad \text{for} \quad P \in \mathbb{C}\,, \tag{13a}$$

$$V_{(r,s)} : (\Delta_{(r,s)}, \Delta_{(r,-s)}) \quad \text{for} \quad r \in \frac{\mathbb{N}^*}{2} \quad \text{and} \quad s \in \frac{\mathbb{Z}}{r}\,, \tag{13b}$$

where

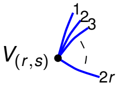
$$\Delta(P) = \frac{c-1}{24} + P^2 \quad \text{with} \quad \Delta_{(r,s)} = \Delta(P_{(r,s)}) \quad \text{and} \quad P_{(r,s)} = \frac{1}{2}(r\beta - s\beta^{-1})\,. \tag{14}$$

- Diagonal field V_Δ^D = loop-insertion operator



$$w(P) = 2\cos(2\pi P) \tag{15}$$

- Non-diagonal field $V_{(r,s)}$ = line-insertion operator



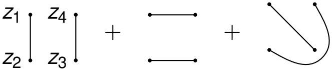
(16)

Building correlation functions

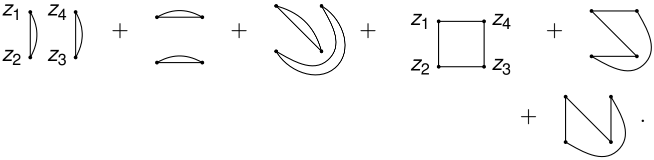
Conjecture: Correlation functions are parametrized by **combinatorial maps**
Grans-Samuelsson, RN, Jacobsen, Ribault, Saleur 23

Combinatorial maps = graphs with no crossing + cyclic symmetry of incident edges

$$\left\langle \prod_{i=1}^{4} V_{(\frac{1}{2},0)}(z_i,\bar{z}_i) \right\rangle = \tag{17}$$



$$\left\langle \prod_{i=1}^{4} V_{(1,0)}(z_i,\bar{z}_i) \right\rangle = \tag{18}$$

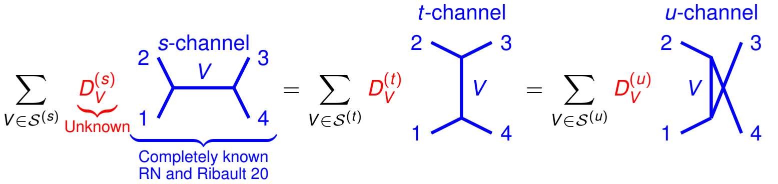


The critical limit of $U(l_1, l_2, l_3; K_c)$ is expected to be

$$\lim_{L\to\infty} U(l_1, l_2, l_3; K_c) \propto \left\langle V_{(\frac{l_1}{2},0)}(0) V_{(\frac{l_2}{2},0)}(\infty) V_{(\frac{l_3}{2},0)}(1) \right\rangle \tag{19}$$

Computing correlation functions

- Solve the crossing-symmetry equation of $\langle V_1 V_2 V_3 V_4 \rangle$



(20)

$\lim_{L\to\infty} U(l_1, l_2, l_3; K_c)$ **can be extracted from** D_V

- The crossing equation + Other constraints $\Longrightarrow$
  $D_{(r,s)} =$ rational functions in $n \times$ Barnes' double Gamma functions
  Jacobsen, RN, Ribault 23
- Other constraints from the assumptions:
  - Single-valuedness of $\langle V_1 V_2 V_3 V_4 \rangle$
  - Analyticity of model's parameters
  - Conformal symmetry

Three points on the same loop $\langle V_{(1,0)} V_{(1,0)} V_{(1,0)} \rangle$

$$\lim_{L\to\infty} U(2,2,2;K_c) = \frac{\Gamma_\beta(\beta+\beta^{-1})^6 \Gamma_\beta(2\beta)^3}{\Gamma_\beta\left(\frac{1}{2\beta}+\beta\right)^6 \Gamma_\beta\left(\frac{1}{2\beta}+2\beta\right)^2 \Gamma_\beta(2\beta+2\beta^{-1})} \frac{1}{\pi(\beta^{-2}-\beta^2)} \times \sqrt{\frac{\sin(\pi\beta^2)\sin(\pi\beta^{-2})}{2\cos(\pi\beta^2)}} \quad (21)$$

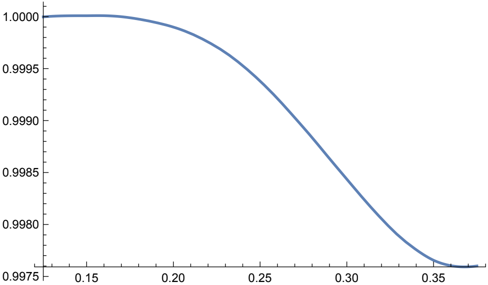
where $\Gamma_\beta(x)$ has the integral representation

$$\Gamma_\beta(x) = \exp\left\{ \int_0^\infty \frac{dt}{t} \left(\frac{e^{-xt} - e^{-\frac{1}{2}(\beta+\beta^{-1})}}{(1-e^{-\beta t})(1-e^{-\beta^{-1}t})} - \frac{(\frac{1}{2}(\beta+\beta^{-1})-x)^2}{2} e^{-t} - \frac{\frac{1}{2}(\beta+\beta^{-1})-x}{t} \right) \right\}. \quad (22)$$

- $\lim_{L\to\infty} U(2,2,2;K_c)$ perfectly agrees with the $CLE_{\frac{4}{\beta^2}}$ result by Xin Sun et al.

Comparison with other methods

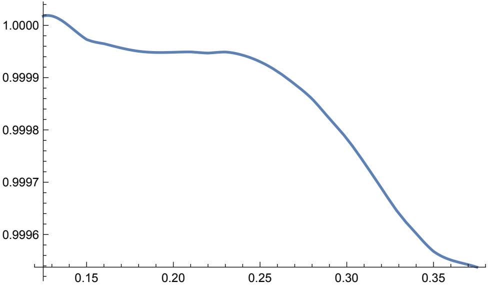
- $\lim_{L\to\infty} U(2,2,2;K_c)$ perfectly agrees with the transfer matrix result on lattice of size $L \geq 7$.



$$\text{Y-axis} = \frac{\lim_{L\to\infty} U(2,2,2;K_c)}{U(2,2,2;K_c) \text{ on lattice of size } L=7} \quad , \quad \text{X-axis} = \frac{\beta^2}{4}$$

The case $\beta^2 = \frac{2}{3}$ describes the probability of having 3 points on the boundary of percolation cluster.

Walking through 3 points $\langle V_{(\frac{1}{2},0)} V_{(1,0)} V_{(\frac{1}{2},0)} \rangle$

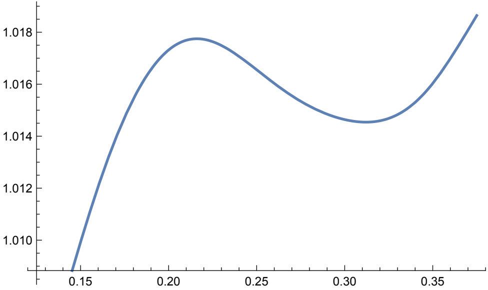


$$\text{Y-axis} = \frac{\lim_{L\to\infty} U(1,2,1;K_c)}{U(1,2,1;K_c) \text{ on lattice of size } L=7},$$

$$\text{X-axis} = \frac{\beta^2}{4}$$

The case $\beta^2 = \frac{3}{2}$ describes the probability of self-avoiding walk through 3 points.

3 Pivotal points $\langle V_{(2,0)} V_{(2,0)} V_{(2,0)} \rangle$



$$\text{Y-axis} = \frac{\lim_{L\to\infty} U(4,4,4;K_c)}{U(4,4,4;K_c) \text{ on lattice of size } L=7},$$

$$\text{X-axis} = \frac{\beta^2}{4}$$

For $\beta^2 = \frac{2}{3}$, this case describes the probability of 3 points sit at the pivot points of percolation clusters.

Conjecture :
Jacobsen, RN, Ribault, Roux

$$\lim_{L\to\infty} U(l_1, l_2, l_3; K_c) = C_{(\frac{l_1}{2},0)(\frac{l_2}{2},0)(\frac{l_3}{2},0)} \sqrt{\frac{C_{(0,1-\beta^2)(0,1-\beta^2)(0,1-\beta^2)}}{\prod_{i=1}^{3} C_{(\frac{l_i}{2},0)(\frac{l_i}{2},0)(0,1-\beta^2)}}} \qquad (23)$$

with

$$C_{(r_1,s_1)(r_2,s_2)(r_3,s_3)} = \prod_{\epsilon_1,\epsilon_2,\epsilon_3=\pm} \Gamma_\beta^{-1}\left(\frac{\beta+\beta^{-1}}{2} + \frac{\beta}{2}\left|\sum_i \epsilon_i r_i\right| + \frac{\beta^{-1}}{2}\sum_i \epsilon_i s_i\right) \qquad (24)$$

- $C_{(r_1,s_1)(r_2,s_2)(r_3,s_3)}$ reduce to Liouville three-point functions for $r_1 = r_2 = r_3 = 0$
- $C_{(r_1,s_1)(r_2,s_2)(r_3,s_3)}$ also appears in the E-series minimal models.
- **Rewrite higher-point functions as these 3-point functions**
  - 1-point functions on the torus.
  - 2-point functions on the disk.