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A SIMPLE SOLUTION FOR A GROUP COMPLETION PROBLEM

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1. Introduction

Let X be a space with base point. We denote by ΣX the reduced suspension on X and by ΩX the space of based loops on X. Σ and Ω are self functors on the category of compactly generated spaces and left and right adjoint to each other. Thus the composite of their iterations $\Omega^n\Sigma^n$ constitute a monad and plays an important role in algebraic topology. For example, consider the colimit

$$\Omega^\infty\Sigma^\infty X = \lim_{\rightarrow}\{\cdots \to \Omega^{n-1}\Sigma^{n-1}X \to \Omega^n\Sigma^n X \to \cdots\}$$

of the sequence of the inclusion maps $\Omega^{n-1}\Sigma^{n-1}X \to \Omega^n\Sigma^n X$ given by $\lambda \mapsto (t\wedge u \mapsto t\wedge\lambda(u))$, where $\lambda \in \Omega^{n-1}\Sigma^{n-1}X, t \in S^1$ and $u \in S^{n-1}$. Then the homotopy group $\pi_*(\Omega^\infty\Sigma^\infty X)$ is canonically isomorphic to the stable homotopy group $\pi_*^S(X)$ of X.

$\Omega^n\Sigma^n X$ can be approximated by the configuration space of finite points in $\mathbb{R}^n$ with labels in X. More precisely, this fact can be stated as follows : Consider a set

$$C_n(X) = \left\{(S,x) \;\middle|\; \begin{array}{l} S : \text{a finite subset} \subset \mathbb{R}^n \\ x : S \to X : \text{a map} \end{array} \right\} \Big/ \sim$$

where $\sim$ is the equivalence relation defined by

$$(S,x) \sim (T,y) \iff \begin{cases} x(c) = y(c) & \text{if } c \in S\cap T \\ x(c) = *, y(c) = * & \text{otherwise.} \end{cases}$$

We topologize this set in a natural way. Then a classical theorem of Segal is,

Theorem 1 (G.Segal [5]). *If X is path-connected, $C_n(X) \simeq_w \Omega^n\Sigma^n X$.*

The assumption on the path-connectivity of X is crucial in the above theorem. $C_n(X)$ has a H-space structure under the multiplication given by the juxtaposition of configurations, which corresponds to the usual H-space structure on $\Omega^n\Sigma^n X$. If X is not path-connected, $\pi_0(C_n(X))$ is not necessarily a group, while $\pi_0(\Omega^n\Sigma^n X)$ is always a group by virtue of the existence of the homotopy inverse in $\Omega^n\Sigma^n X$.

Without the assumption on the connectivity, the following fact is known:

Theorem 2 (J.P.May, G.Segal). *There exist a space $\widetilde{C}_n(X)$ which is weakly equivelent to $C_n(X)$ and a map $g : \widetilde{C}_n(X) \to \Omega^n\Sigma^n X$ which induces an isomorphism*

$$g_* : H_*(\widetilde{C}_n(X))[\pi^{-1}] \xrightarrow{\cong} H_*(\Omega^n\Sigma^n X),$$

where $[\pi^{-1}]$ denotes the localization of the Pontrjagin ring $\widetilde{C}_n(X)$ by the multiplicative subset $\pi = \pi_0(\widetilde{C}_n(X))$.

Above theorems suggest that the non-existence of a homotopy inverse in $C_n(X)$ is the obstruction for $C_n(X)$ to be weakly equvalent to $\Omega^n\Sigma^n X$. So we condider, a

Problem . *Install a homotopy inverse into the H-space $C_n(X)$ to get a group completion Z_n ; $Z_n \simeq_w \Omega^n\Sigma^n X$.*

We recall two precedent models related to the above problem. One is the space of positive and negative particles [3] :

$$C^{\pm}(\mathbb{R}^n) = \left\{ (S,p) \,\middle|\, \begin{array}{l} S : \text{a finite subset} \subset \mathbb{R}^n \\ p : S \to \{\pm 1\} \end{array} \right\} \Big/ \sim$$

where the topology is given so that two points with the opposite parity in $\{\pm 1\}$ can collide and annihilate. By the annihilation of oppositely charged particles, this space can be considered as a space constructed from $C_n(S^0)$ by putting a homotopy inverse to it. But it does not approximate $\Omega^n\Sigma^n S^0 = \Omega^n S^n$, indeed, it is showed by McDuff that $C^{\pm}(\mathbb{R}^n) \simeq_w \Omega^n(S^n \times S^n/\Delta)$, where Δ denotes the diagonal subspace of $\Sigma^n X \times \Sigma^n X$.

The other is the space of signed subcubes merged along the first coordinate defined by Caruso and Waner [1]. They showed that this space approximates $\Omega^n\Sigma^n X$ for any X, but it is based on the space of little cubes and quite intricate than our construction.

In this paper, we outline the construction of the space of intervals in $\mathbb{R}^n$ and show that it is a simple solotion to the above theorem. Details of these lines are given in [4]. We also consider the space of intervals in S^1 and indicate a proof that it is weakly equivalent to $L\Sigma X$, where LY denotes the space of free loops on Y.

2. A simple solution

We consider the space of bounded intervals in $\mathbb{R}^n$ parallel to the first axis with labels in X. It is denoted by $I_n(X)$; As a set

$$I_n(X) = \left\{ \{(J_1,x_1),\cdots,(J_k,x_k)\} \,\middle|\, \begin{array}{l} J_i \text{ are disjoint intervals in } \mathbb{R}^n \\ x_i \in X \end{array} \right\} \Big/ \sim$$

This set is topologized so that

- Any open and closed ends of two intervals can be attached, if their labels in X coincide, and
- Any half-open interval can vanish when its length comes to be zero.

Then we can show that

Theorem 3. *(O-.) $I_n(X) \simeq_w \Omega^n\Sigma^n X$.*

More precisely, $I_n(X)$ is formulated as follows. Consider k-tuples $((J_1,x_1),\cdots,(J_k,x_k))$ where J_i is a bounded interval and $x_i \in X$. Let $I_{(k)}(X)$ be the set of such k-tuples satisfying

(1) J_i are pairwise disjoint

(2) $s \in J_{i-1}$ and $t \in J_i$ implies $s \leq t$.
(3) if $\bar{J}_{i-1} \cap \bar{J}_i \neq \emptyset$ implies $x_{i-1} = x_i$.

The topology of $I_{(k)}(X)$ is the one as a subspace of $\mathbb{R}^{2k} \times \{\pm 1\}^{2k} \times X^k$, by the correspondence

$$((J_1, x_1), \cdots, (J_k, x_k)) \mapsto (u_1, \cdots, u_{2k}; p_1, \cdots, p_{2k}; x_1, \cdots, x_k).$$

We call p_{2i-1} (p_{2i}) the parity of the left(right) hand side of J_i.

We define

$$I_1(X) = \coprod_{k \geq 0} I_k(X)/\sim,$$

where $\sim$ denotes the equivalence relation generated by the relation shown below. Suppose

$$\iota = ((J_1, x_1), \cdots, (J_k, x_k)) \in I_k(X)$$

and

$$\iota' = ((K_1, y_1), \cdots, (K_{k-1}, y_{k-1})) \in I_{k-1}(X).$$

Then $\iota' \sim \iota$ if one of the following holds:

(1) $K_i = \begin{cases} J_i & \text{if } i < j \\ J_j \cup J_{j+1} & \text{if } i = j \\ J_{i+1} & \text{if } i > j \end{cases}, \quad y_i = \begin{cases} x_i & \text{if } i < j \\ x_j = x_{j+1} & \text{if } i = j \\ x_{i+1} & \text{if } i > j. \end{cases}$

(2) $K_i = \begin{cases} J_i & \text{if } i < j \\ J_{i+1} & \text{if } i \geq j \end{cases}, \quad y_i = \begin{cases} x_i & \text{if } i < j \\ x_{i+1} & \text{if } i \geq j, \end{cases}$ and $J_j = \emptyset$ or $x_j = *$.

We define

$$I_n(X) = C_{n-1}(I_1(X)).$$

In the above, $I_1(X)$ is considered as a partial monoid by superimposition and $C_{n-1}(I_1(X))$ is the configuration space with partially summable labels.[6]

As for $C_n(X)$, $I_n(X)$ is too small to define a map to $\Omega^n \Sigma^n X$, and we need the thickening of $I_n(X)$. We fix a real number $\delta > 0$. For any $\varepsilon < \delta$, we denote by $I_1(X)_\varepsilon$ the subspace of $I_1(X)$ consisting of ε-separated elements. Here ε-separated means that any pair of end points of (the same or another) interval(s) which have the same parity are separated by ε. Then we define

$$\widetilde{I}_n(X) = \cup_{0 < \varepsilon \leq \delta} C_{n-1}(I_1(X)_\varepsilon) \times \{\varepsilon\} \subset I_n(X) \times (0, \delta).$$

We can show that the projection map $\widetilde{I}_n(X) \to I_n(X)$ is a weak equivalence.

3. Constructions

We need a few more constructions to outline the proof of the Theorem 3.

Firstly we construct a map

$$\alpha : \widetilde{I}_n(X) \to \Omega C_{n-1} \Sigma X.$$

To do this, we regard an interval as a physical object with extraordinary electricity. Then we can think of a point in $\widetilde{I}_n(X)$ as a configuration of such electric objects. α is a map which corresponds to each configuration an electric field it generates. More precisely, α is defined as follows. Let ι be an element of $I_1^\varepsilon(X)_s$. Suppose ι is represented by a k-tuple $((J_1, x_1), \cdots, (J_k, x_k))$ where J_i is an interval with end points u_{2i-1} and u_{2i}. We also assume that $u_{i-1} \leq u_i$ for all i. If u_j ($j = 2i-1$ or $2i$) is a closed(open) end of J_i, we put $p_j = 1(-1)$.

We define subintervals $N_i \subset [0, s]$ $(i = 1, \cdots, 2k)$ as

$$N_1 = [u_1 - \varepsilon/2, \mathrm{Min}(u_1 + \varepsilon/2, u_2 - \varepsilon/2)],$$

$$N_i = [\mathrm{Max}(u_i - \varepsilon/2, u_{i-1} + \varepsilon/2), \mathrm{Min}(u_i + \varepsilon/2, u_{i+1} - \varepsilon/2)], \text{ for } 1 < i < 2k,$$

and

$$N_{2k} = [\mathrm{Max}(u_{2k} - \varepsilon/2, u_{2k-1} + \varepsilon/2), u_{2k} + \varepsilon/2].$$

We define a function $f : \bigcup_{i=1}^{2k} N_i \to S^1 \wedge X$ by

$$f(t) = [p_i((t - u_i)/\varepsilon + (-1)^i/2)] \wedge x_{G((i+1)/2)}, \text{ if } t \in N_i$$

where S^1 is regarded as $[-1, 1]/\{\pm 1\}$ and $G(q)$ denotes the largest integer which does not exceed q. We can extend f continuously to $[0, s]$ in such a way that it is piecewise constant outside $\bigcup_{i=1}^{2k} N_i$.

This definition does not depend on the choice of a representative, so we obtain a map

$$\alpha_s^\varepsilon : I_1^\varepsilon(X)_s \to \Omega_s(\Sigma X),$$

which is clearly an abelian partial monoid homomorphism. Then we define a map $\alpha : \widetilde{I}_1(X) \to \Omega\Sigma X$ by $(\xi, \varepsilon, s) \mapsto \alpha_s^\varepsilon(\xi)$, which is also an abelian partial monoid homomorphism, if we regard $\Omega\Sigma X$ as an abelian partial monoid appropriately.

Then we define a map $\alpha : \widetilde{I}_n(X) \to \Omega C_{n-1}(\Sigma X)$ by the composite

$$\widetilde{I}_n(X) \to C_{n-1}(\widetilde{I}_1(X)) \overset{C_{n-1}(\alpha)}{\longrightarrow} C_{n-1}(\Omega\Sigma X) \to \Omega C_{n-1}(\Sigma X),$$

where the first map is given by an inclusion $I_1^\varepsilon(X) \to \widetilde{I}_1(X)$, while the last map is given by

$$[v_1, l_1; \cdots; v_k, l_k] \mapsto (t \mapsto [v_1, l_1(t); \cdots; v_k, l_k(t)]), \ l_i \in \Omega\Sigma X.$$

Next, let $E_n(X)$ be a space of bounded intervals in $\mathbb{R}_{\geq 0} \times \mathbb{R}^{n-1}$ parallel to the first axis with labels in X. We can construct the thickening of $E_n(X)$ similarly to the definition of $\widetilde{I}_n(X)$ and denoted by $\widetilde{E}_n(X)$.

Then we can regard $\widetilde{I}_n(X)$ as a subspace of $\widetilde{E}_n(X)$ by a homeomorphism $(0, \infty) \times \mathbb{R}^{n-1} \approx \mathbb{R}^n$. Furthermore, the composite map

$$\widetilde{I}_n(X) \overset{\alpha}{\to} \Omega C_{n-1}\Sigma X \overset{i}{\to} PC_{n-1}\Sigma X$$

can be extended to $\beta : \widetilde{E}_n(X) \to PC_{n-1}\Sigma X$, where $PC_{n-1}\Sigma X$ denotes the path space on $C_{n-1}\Sigma X$ and $i : \Omega C_{n-1}\Sigma X \to PC_{n-1}\Sigma X$ is the standard inclusion.

4. Outline of the proof of Theorem 3

It is easily shown that

Lemma 4. *$E_n(X)$ is weakly contractible.*

This ensures that β is a weak homotopy equivalence. By using the Dold-Thom criterion[2], we can also show that

Lemma 5. *The sequence*

$$\widetilde{I}_n(X) \xrightarrow{\alpha} \widetilde{E}_n(X) \xrightarrow{i} C_{n-1}\Sigma X$$

is a quasi-fibration.

Then we have a commutative diagram

$$\begin{array}{ccccc} \widetilde{I}_n(X) & \xrightarrow{i} & \widetilde{E}_n(X) & \xrightarrow{p} & C_{n-1}\Sigma X \\ {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\beta}\big\downarrow & & \big\| \\ \Omega C_{n-1}\Sigma X & \longrightarrow & PC_{n-1}\Sigma X & \longrightarrow & C_{n-1}\Sigma X, \end{array}$$

with horizontal rows quasi-fibration and fibration, β weak equivalence. It follows that α is a weak equivalence. By the Segal's theorem mentioned above, $C_{n-1}\Sigma X \simeq_w \Omega^{n-1}\Sigma^{n-1}\Sigma X$. Thus we have shown that $I_n(X) \simeq_w \Omega^n\Sigma^n X$.

5. The space of intervals in S^1

In this section we will consider the space of intervals in S^1.

The following seems the easiest way to formulate this space : Let $J_1(X)$ be the space of intervals in $\mathbb{R}$ as defined above, but allowing inifinitely many intervals. Then the space of intervals in S^1 is defined as

$$I_{S^1}(X) = J_1(X)^{\mathbb{Z}},$$

the $\mathbb{Z}$-fixed point set of $J_1(X)$, where $\mathbb{Z}$ acts on $J_1(X)$ by the shift.

Theorem 6. *We have a weak homotpy equivalence*

$$I_{S^1}(X) \simeq_w L\Sigma X.$$

Proof. We may regard $I_1(X)$ as the space of intervals in $(0,1)$ via a homeomorphism $\mathbb{R} \approx (0,1)$. Then we have an obvious injection $I_1(X) \to I_{S^1}(X)$.

We can define the thickening $\widetilde{I}_{S^1}(X)$ of $I_{S^1}(X)$ similarly as $\widetilde{I}_1(X)$. We still have an injection $i : \widetilde{I}_1(X) \to \widetilde{I}_{S^1}(X)$. Furthermore, we can define a scanning map $\gamma : \widetilde{I}_{S^1}(X) \to L\Sigma X$ and a projection $p : \widetilde{I}_{S^1}(X) \to \Sigma X$. Indeed, we have a commutative diagram

$$\begin{array}{ccccc} \widetilde{I}_1(X) & \xrightarrow{i} & \widetilde{I}_{S^1}(X) & \xrightarrow{p} & \Sigma X \\ {\scriptstyle\alpha}\big\downarrow & & {\scriptstyle\gamma}\big\downarrow & & \big\| \\ \Omega\Sigma X & \longrightarrow & L\Sigma X & \longrightarrow & \Sigma X, \end{array}$$

where the lower row is the usual fibration. We can show that the upper row is a quasi-fibration, and thus γ is a weak equivalence. As $\widetilde{I}_{S^1}(X)$ is weakly equivalent to $I_{S^1}(X)$, we have proved the theorem. □

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