# Configuration space of intervals with partially summable labels

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# Outline

1. Introduction
2. Configuration spaces
   - Partial abelian monoid
   - Configuration space of intervals
3. Main construction and theorem
   - Configuration space of intervals with partially summable labels
   - Approximation map $\widetilde{I}_M \to \Omega' BM$
   - Quasi-fibration sequence $\widetilde{I}_M \to \widetilde{E}_M \to BM$

# Approximation to a mapping space

**Theorem 1 (Milgram-May-Segal)**

*There exists a weak homotopy equivalence $C(\mathbb{R}^n, X) \to \Omega^n S^n X$, if $X$ is path-connected.*

- An approximation to a mapping space by a configuration space looks like

$$C(M, X) \simeq map(\hat{M}, \overline{(\tau M * X)})$$

(Milgram-May-Segal, McDuff, ...)

- In some cases, a system $\{C(M_n, X)\}$ can approximate a system of mapping spaces, that is, a homology theory. (Segal, Shimakawa, Tamaki,...)

# Approximation to a mapping space

In most cases, an approximation map

$$C(M, X) - \to map(\hat{M}, \overline{(\tau M * X)}))$$

has geometric or physical iterpretation. So this talk is about

a geometric model of a (mapping) space
with geometrically constructed approximation map.

# Group completion

**Theorem 2 (Segal-F.Cohen)**

*$C(\mathbb{R}^n, X) \to \Omega^n S^n X$ is a group completion if $n \geq 2$.*

A group completion of an admissible topological monoid $M$ can be constructed by a homotopy limit of a (possibly huge) diagram $\{\cdots \to M \to M \to \dots\}$ given by multiplication by elements taken from each connected component of $M$.

This talk is also about

A geometric construction of a group completion

# A partial abelian monoid is ...

A partial abelian monoid is

- almost an abelian monoid but with partially defined sum.
- suitable for configuration space construction.
- the additive part of an $\mathbb{F}_1$-algebra.

## Definition 1

*A* **topological partial abelian monoid** *is a space $M$ with base point $0$ equipped with a subspace $M_2$ of $M \times M$ and a map $\mu : M_2 \to M$ which satisfies*

1. $M \vee M \subset M_2$, *and* $\mu(m, 0) = \mu(0, m) = m$, *for all* $m \in M$,
2. $(m, n) \in M_2$ *if and only if* $(n, m) \in M_2$, *and* $\mu(m, n) = \mu(n, m)$,
3. $(\mu(l, m), n) \in M_2$ *if and only if* $(l, \mu(m, n)) \in M_2$, *and* $\mu(\mu(l, m), n) = \mu(l, \mu(m, n))$.

We denote $\mu(m, n) = m + n$.

# Examples

Extreme cases:

1. An abelian monoid is a partial abelian monoid.
2. A based space $X$ can be regarded as a trivial partial abelian monoid by setting $X_2 = X \vee X$ and $\mu : X \vee X \to X$ the folding map. It is called a **trivial partial abelian monoid**.
3. Let $M$ be an abelian monoid and $N$ be a subset which contains 0. Then $N$ is a partial abelian monoid if we set

$$N_2 = \{(n_1, n_2) \mid n_1 + n_2 \in N\}$$

and a sum coming from that in $M$.

# Examples

- $\mathbb{N}_{\leq 1} = \{0, 1\}$ and $\mathbb{N}_{\leq 2} = \{0, 1, 2\}$ have multiplication tables

| | |
|---|---|
| 0 | 1 |
| 1 | × |

and

| | | |
|---|---|---|
| 0 | 1 | 2 |
| 1 | 2 | × |
| 2 | × | × |

# A product of partial abelian monoids

$Y$ : a topological space

$$\mathrm{mul}(Y) = \coprod_{n \geq 0} \mathrm{SP}^n Y$$

— the free abelian monoid generated by $Y_+ = Y \coprod \{0\}$ with an appropriate topology, or equivalently, as $\mathrm{SP}^\infty Y_+$, an infinite symmetric product introduced by Dold and Thom.

—we think of an element of $\mathrm{mul}(Y)$ as a finite multiset — a finite “set” with repeated elements.

# Summability in a pam

For a finite set $S$,

$$\sigma : S \to Y$$

is a multiset. For a subset $T \subset S$,

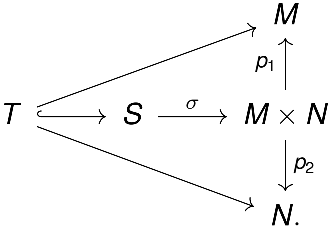
$$\sigma|_T : T \hookrightarrow S \to Y$$

is a submultiset.
When $Y = M$ is a partial abelian monoid, we may speak of a **summable** multiset.
We say that $\sigma$ is **pairwise insummable** if, for any subset $T \subset S$ of cardinality two, $\sigma|T$ is insummable.

# A product of partial abelian monoids

$M, N$ : partial abelian monoids, $S$ : a finite set. Consider the following property for $\sigma : S \to M \times N$ :

for any subset $T$, if one of $p_i \circ (\sigma|T)$ is pairwise insummable then the other is summable.




We denote by $T_{M,N}$ the subspace of mul$(M \times N)$ consisting of $\sigma$ with this property.

# A product of partial abelian monoids

Let $\sim$ be the least equivalence relation on $T_{M,N}$ which satisfies the following three conditions:

(R1) If $m_1$ or $n_1$ is zero then

$$(m_1, n_1) \dot{+} \cdots \dot{+} (m_r, n_r) \sim (m_2, n_2) \dot{+} \cdots \dot{+} (m_r, m_r),$$

(R2) If $m_1 = m_1' + m_1''$ then

$$\begin{aligned}&(m_1, n_1) \dot{+} \cdots \dot{+} (m_r, n_r)\\ &\sim (m_1', n_1) \dot{+} (m_1'', n_1) \dot{+} (m_2, n_2) \dot{+} \cdots \dot{+} (m_r, n_r),\end{aligned}$$

(R3) If $n_1 = n_1' + n_1''$ then

$$\begin{aligned}&(m_1, n_1) \dot{+} \cdots \dot{+} (m_r, n_r)\\ &\sim (m_1, n_1') \dot{+} (m_1, n_1'') \dot{+} (m_2, n_2) \dot{+} \cdots \dot{+} (m_r, n_r).\end{aligned}$$

# A product of partial abelian monoids

Two elements $[\alpha], [\beta]$ in $M \otimes N$ are summable if we can choose their representatives $\alpha, \beta$ in $T_{M,N}$ so that their sum $\alpha \dot{+} \beta$ taken in mul($M \times N$) is contained in $T_{M,N}$. Thus, $M \otimes N$ is a partial abelian monoid in a natural way.
We have a functor

$$\otimes : PAM \times PAM \to PAM \ ; \ (M, N) \mapsto M \otimes N.$$

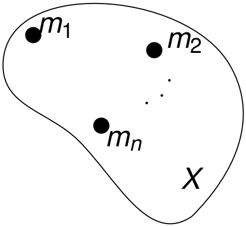
# Examples

1. For abelian groups $A$, $B$, their product $A \otimes B$ defined here is the usual tensor product of modules.
2. For two based spaces $X, X'$, viewed as trivial partial abelian monoids, their product $X \otimes X'$ coincides with their smash product $X \wedge X'$.

# Examples

Intermediate cases:

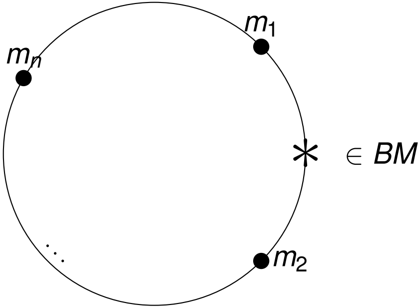
3. $X \otimes \mathbb{N} = \mathrm{SP}^\infty X$, the infinite symmetric product on a based space $X$ of Dold and Thom.
4. Then $X \otimes M$ is the configuration space of finite points in $X$ with labels in $M$ such that only summable labels occur simultaneously.




$(m_1, \ldots, m_n) \in M_n$

# Examples

5. Viewing $S^1$ as a based space, we get $S^1 \otimes M = BM$ the classifying space of a partial abelian monoid. In particular, if $M$ is a monoid this coincides with the McCord model of the classifying space of $M$.




$(m_1, \ldots, m_n) \in M_n$

# Examples

- $\mathbb{N}_{\leq 1} \otimes M \cong M$ for any $M$. (Indeed, $\mathbb{N}_{\leq 1} = S^0$).
- $\mathbb{N}_{\leq 2} \otimes \mathbb{N}_{\leq 2} \cong \mathbb{N}$.
- If $X = \{0, 1, \ldots, n\}$ is a based set, then $BX \cong S^1 \times \cdots \times S^1$($n$ times).

# Examples

6. Let $X$ be a compact based space and $M = Gr := \sqcup Gr_n(\mathbb{R}^\infty)$ be the infinite Grassmannian with a partial sum defined only for two vector spaces which are perpendicular to each other. Then $X \otimes Gr = F(X)$ coincides with the configuration space defined by Segal for connective $K$-homology. Tamaki gave a similar construction, which is enriched by an operad to make twisting on $K$-theory, thus larger than $X \otimes Gr$.

# Examples

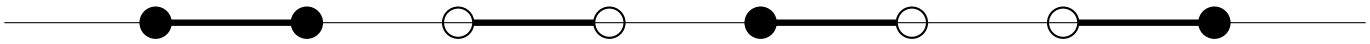
8. Fin($Y$) : ( finite subsets of a space $Y$),
— Fin($Y$) is a partial abelian monoid by disjoint union. If $C_n = \mathrm{Fin}(\mathbb{R}^n)$ then $C_n \otimes X = C_n(X)$ is the configuration space of finite points in $\mathbb{R}^n$ with labels in $X$, introduced by Segal and equivalent to the construction by Milgram and May.
9. $\mathrm{Fin}(\mathbb{R}^\infty) \otimes M = C^M(\mathbb{R}^\infty)$ is the configuration space of finite points in $\mathbb{R}^\infty$ with labels in $M$ defined by Shimakawa.

# Intervals

$H = \{(u, v) \mid u \leq v\} \subset \mathbb{R}^2$, a half-plane in $\mathbb{R}^2$,
$P = \{\pm 1\}$ : the set of "parities",
To any point $(u, v; p, q) \in H \times P^2$ with $u < v$, we assign an interval

$$J = \{x \in \mathbb{R} \mid u <_p x <_q v\} \subset \mathbb{R},$$

where the symbol $<_p$ is interpreted as an inequality $\leq$ or $<$ according as $p = +1$ or $-1$.



$$\mathcal{I} = \{(u, v; p, q) \in H \times P^2 \mid u < v\}.$$

# Intervals

For $J_1, J_2 \in \mathcal{I}$,
we denote $J_1 < J_2$ if $v_1 < u_2$, where $J_k = (u_k, v_k; p_k, q_k)$.
Let $L_r$ be the subspace of $\mathcal{I}^r$ given by

$$L_r = \left\{ (J_1, \ldots, J_r) \in \mathcal{I}^r \;\middle|\; J_1 < \cdots < J_r \right\}.$$

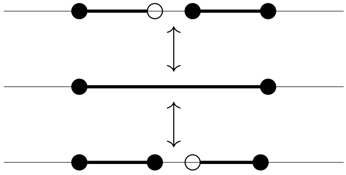
Then $L_r$ is the configuration space of $r$ bounded intervals in $\mathbb{R}$ with mutually disjoint closures.
Now we define

$$I = \coprod L_r$$

and give it a topology such that cutting-pasting and creation-annihilation is allowed.

# Intervals

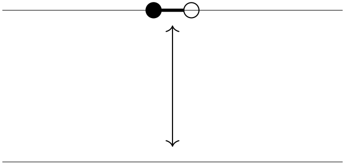
Figure: Cutting-Pasting



–two intervals are pasted when meeting endpoints have opposite parities, that is, one is open and the other is closed,

# Intervals

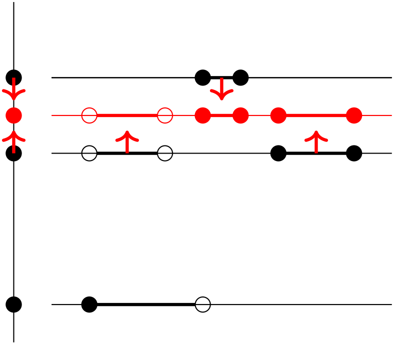
Figure: Creation-Annihilation



— a half-open interval annihilates when its length approaches zero.
Then $I$ has a partial abelian monoid structure by the superimposition of disjoint configurations.

# Example : $I_2 = C_1 \otimes I$

A point of $C_1 \otimes I$ is
— a finite subset of $\mathbb{R}^1$,
— with labels in $I$,
— in which, points can collide,
— in case labels are summable



$C_1 \otimes I$ is a configuration space of horizontal intervals in $\mathbb{R}^2$.
Let's denote this space by $I_2$.

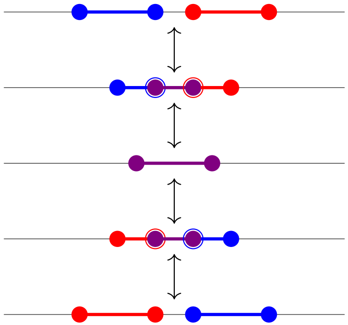
# Partially summable labels

Partially summable labels

- enrich a configuration space in a certain way.
- control a topology of the reproduced configuration space.

# Partially summable labels

Figure: Sum of labels (where red $+$ blue = violet )



# Elementary configuration

$U = (a, b)$ : an open interval in $\mathbb{R}$.
We consider two special types of elements in $\mathrm{mul}(\mathcal{I} \times M)$.

(E1) $e = (J, n)$ with one of the following :

1. $J = (a, b)$
2. $J = (a, w)$ or $J = (a, w]$, $a < w < b$
3. $J = (w, b)$ or $J = [w, b)$, $a < w < b$
4. $J = (w_1, w_2]$ or $J = [w_1, w_2)$, $a < w_1 < w_2 < b$

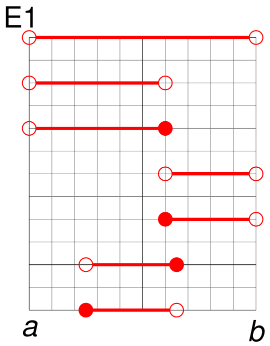
(E2) $e = (J_1, n) \dotplus (J_2, n)$ with one of the following:

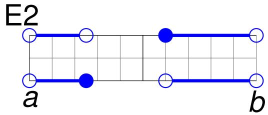
1. $J_1 = (a, w_1], J_2 = (w_2, b)$ and $a \leq w_1 < w_2 < b$, or
2. $J_1 = (a, w_1), J_2 = [w_2, b)$ and $a < w_1 < w_2 \leq b$,

where $n$ is a non-zero element in $M$ for both cases.

We call such $e$ **an elementary configuration in $U$.** In both cases, $n \in M$ is denoted by $n(e)$.

# Elementary configurations



# Admissible multisets

For any $\xi = (J_1, m_1) \dot{+} \cdots \dot{+} (J_r, m_r) \in T_{\mathcal{I},M} \subset \mathrm{mul}(\mathcal{I} \times M)$,
Let $\xi|_U = (J_1 \cap U, m_1) \dot{+} \cdots \dot{+} (J_r \cap U, m_r))$.

$\xi \in T_{\mathcal{I},M}$ is said to be **admissible** if for any $t \in \mathbb{R}$ there exists an open interval $U = (a, b)$ which contains $t$ such that

$$\xi|_U = e_1 \dot{+} \ldots \dot{+} e_r$$

for some elementary configurations $e_1, \ldots, e_r$ in $U$ such that $(n(e_1), \ldots, n(e_r)) \in M_r$.

If, moreover, there exist $\varepsilon > 0$ and an interval $U$ can be taken as $U = (t - \varepsilon, t + \varepsilon)$ for all $t$, then we say that $\xi$ is **$\varepsilon$-admissible.**
It is clear that $\varepsilon$-admissible elements are $\varepsilon'$-admissible if $\varepsilon' < \varepsilon$.

# Admissible multisets

Let $V = (a, b)$ be and open interval with $b - a > \varepsilon$.

We say that an $\varepsilon$-admissible element $\xi$ is supported by $V$ if $\xi|_{(a+\varepsilon/2, b-\varepsilon/2)} = \xi$. If $V \subset V'$ then $\varepsilon$-admissible elements supprted by $V$ are supported by $V'$.

Let $W$, $W(\varepsilon)$, and $W(\varepsilon, V)$ be the subspace of $T_{\mathcal{I},M}$ which consists of admissible elements, $\varepsilon$-admissible elements, and $\varepsilon$-admissble elements supported by $V$, respectively.

# Configuration space of intervals with partially summable labels

Let $I_M$ be the image in $I \otimes M$ of $W$ under the natural map $\pi_{\otimes} \circ \pi_{mul}$. Let also $I_M(\varepsilon)$ and $I_M(\varepsilon, V)$ be the image in $I \otimes M$ of $W(\varepsilon)$ and $W(\varepsilon, V)$, respectively , under $\pi_{\otimes} \circ \pi_{mul}$. Then we alter the topology of $I_M$ by the weak topology of the union

$$I_M = \bigcup_{\varepsilon > 0, V} I_M(\varepsilon, V).$$

Thus, we have defined a **configuration space of intervals with partially summable labels.**

# Thickening — Moore type variant

We define

$$\widetilde{I}_M = \bigcup_{\varepsilon>0, s\geq\varepsilon} I_M(\varepsilon, s) \times \{s\} \times \{\varepsilon\}$$

and give it the topology as a subspace of $I_M \times \mathbb{R}^2$.
If $s = \varepsilon$, $I_M(\varepsilon, \varepsilon)$ consists of one point, the element $\emptyset$ in $I_M$ which represents the empty configuration. As a base point of $\widetilde{I}_M$, we take $(\emptyset, 1, 1)$.

**Proposition 1**

*The projection $\widetilde{I}_M \to I_M$ onto the first component is a weak homotopy equivalence.*

# Approximation map $\widetilde{I}_M \to \Omega' BM$

Approximation map $\widetilde{I}_M \to \Omega' BM$

- is defined in 3 steps : disintegration, scanning, and summing-up.
- is shown to be weak equivalence so to constitute a zig-zag of weak equivalences :

$$I_M \leftarrow \widetilde{I}_M \to \Omega' BM.$$

# Disintegration and scanning

1. By definition, for any element $([\xi], s, \varepsilon) \in \widetilde{I}_M$ and $t \in \mathbb{R}$,

$$\xi|_{U_t} = e_1 \dot{+} \ldots \dot{+} e_r$$

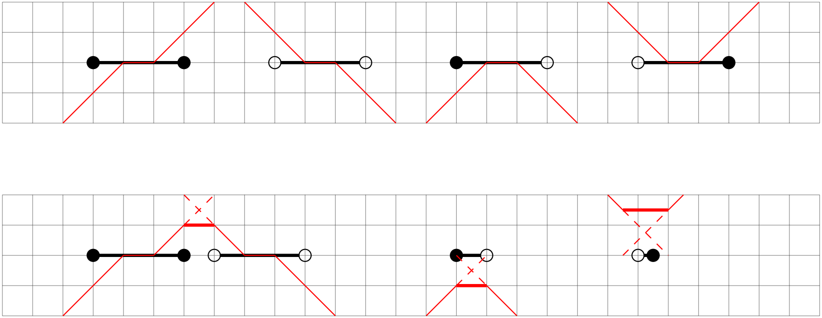
for some elementary configurations $e_1, \ldots, e_r$ in $U_t$ such that $(n(e_1), \ldots, n(e_r)) \in M_r$, where $U_t = (t - \varepsilon, t + \varepsilon)$.

2. For any elementary configuration $e$ in $U_t$, we have a well-defined map $\omega(e) : V_t \to S^1$, where $V_t = \left(t - \frac{\varepsilon}{2}, t + \frac{\varepsilon}{2}\right)$.

So we have an element in $\mathrm{mul}(\mathrm{Map}(V_t, S^1) \times M)$ for each $t \in (0, s)$.
This defines a map

$$\omega_t : W_{s,\varepsilon} \to \mathrm{mul}(\mathrm{Map}(V_t, S^1) \times M).$$

# $\omega'(J)$



# Summing up

3. The composite of $\omega_t$ with the sequence of natural maps

$$\begin{aligned}\mathrm{mul}(\mathrm{Map}(V_t, S^1) \times M) &\to \mathrm{mul}(\mathrm{Map}(V_t, S^1 \times M)) \\ &\to \mathrm{Map}(V_t, \mathrm{mul}(S^1 \times M)),\end{aligned}$$

   maps into $\mathrm{Map}(V_t, T_{S^1,M})$.
   Recalling that $T_{S^1,M}$ is a subset of $\mathrm{mul}(S^1 \times M)$ on which we defined the tensor relations, we have an element in $\mathrm{Map}(V_t, S^1 \otimes M)$.
   This construction is compatible for distinct $t$'s so that we can paste local functions to get a global function in $\mathrm{Map}((0, s), S^1 \otimes M)$.

Thus we get a map $\alpha : \widetilde{I}_M \to \Omega' BM$.

$$\begin{array}{ccccc}
W_{s,\varepsilon} & \xrightarrow{\omega_t} & G & \hookrightarrow & \mathrm{mul}(\mathrm{Map}(V_t, S^1) \times M)\\
 & & \downarrow \mu & & \\
 & & G' & \hookrightarrow & \mathrm{mul}(\mathrm{Map}(V_t, S^1) \times M)\\
 & & \downarrow & & \downarrow\\
 & \alpha_t & & & \mathrm{mul}(\mathrm{Map}(V_t, S^1 \times M))\\
 & & \downarrow & & \downarrow\\
 & & \mathrm{Map}(V_t, T_{S^1,M}) & \hookrightarrow & \mathrm{Map}(V_t, \mathrm{mul}(S^1 \times M))\\
 & \searrow & \downarrow & & \\
 & & \mathrm{Map}(V_t, BM). & &
\end{array}$$

# Total space $\widetilde{E}_M$

A space $E_M$ is almost $I_M$ but

- intervals lie in a half line $[0, \infty)$, and
- The origin works as a "vanishing point".

Then

$$\widetilde{E}_M = \bigcup_{\varepsilon > 0, s \geq \varepsilon} E_M(\varepsilon, s) \times \{s\} \times \{\varepsilon\}$$

is its thickening.

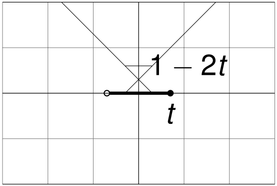
**Proposition 2**

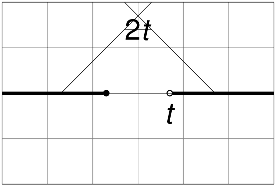
*$\widetilde{E}_M$ is weakly contractible.*

(proof) Push everything into the origin. □

# $\widetilde{I}_M \hookrightarrow \widetilde{E}_M \to BM$

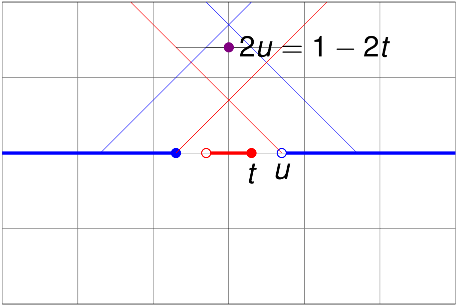
- In $I_M(\varepsilon, s)$, intervals lie in $(0, \infty)$, so we have $I_M(\varepsilon, s) \subset E_M(\varepsilon, s)$, thus we have an inclusion $\widetilde{I}_M \hookrightarrow \widetilde{E}_M$.
- $\widetilde{E}_M \to BM$ is defined using scanning at the origin.



# Scanning at the origin

A configuration consisting of two intervals with “red” and “blue” as a respective label, which maps under $p : \tilde{E}_M \to BM$ to a configuration consisting of a point with “violet” as its label.




“red” $+$ “blue” = “violet” so that (“red”, “blue”) $\in P$(“violet”) .

# Main theorem

**Proposition 3**

*Let $M$ be a partial abelian monoid whose elements are self-insummable. Then the map $p : \widetilde{E}_M \to BM$ is a quasi-fibration with fiber $\widetilde{I}_M$.*

Assuming this, we can state and prove the main theorem :

**Theorem (O.-Shimakawa)**

*Let $M$ be a partial abelian monoid whose elements are self insummable. Then the configuration space $I_M$ of intervals in $\mathbb{R}$ with labels in $M$ is weakly homotopy equivalent to $\Omega BM$.*

# Proof of the main theorem

In the following commutative diagram, lower horizontal line is the Serre's path-loop fibration. The vertical map in the middle is a weak homotopy equivalence, since it is a map between weakly-contractible spaces, hence so is the vertical map on the left.

$$\begin{array}{ccccc}
\widetilde{I}_M & \longrightarrow & \widetilde{E}_M & \longrightarrow & BM \\
\downarrow & & \downarrow & & \Vert \\
\Omega' BM & \longrightarrow & P' BM & \longrightarrow & BM
\end{array}$$

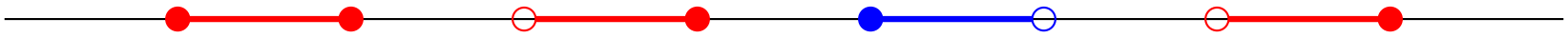
□

# Examples

1. Let $X$ be a based set $\{0, 1, 2\}$ ( trivial partial abelian monoid ).

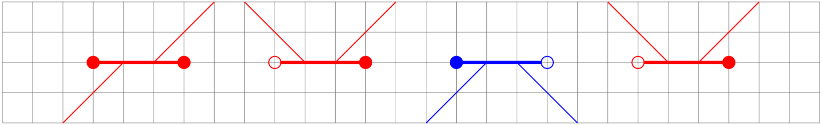
| | | |
|---|---|---|
| 0 | 1 | 2 |
| 1 | $\times$ | $\times$ |
| 2 | $\times$ | $\times$ |

Then $BX \cong S^1 \vee S^1$ and we know from the theorem that $I_X \simeq_w \Omega(S^1 \vee S^1)$.
An element of $I_X$ can be depicted as follows.



# Examples

Scanning map $\widetilde{I}_X \to \Omega(S^1 \vee S^1)$ can be graphed as follows.



$\pi_0(I_X) \cong \pi_1(S^1 \vee S^1) \cong \mathbb{Z} * \mathbb{Z}$

# Examples

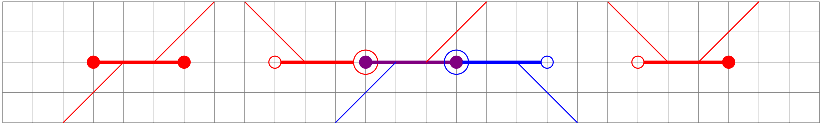
2. Let $M = \mathbb{N}_{\leq 1} \times \mathbb{N}_{\leq 1}$ be the direct product (as a partial abelian monoid !) of $\mathbb{N}_{\leq 1}$ with itself.

| | | | |
|---|---|---|---|
| 00 | 10 | 01 | 11 |
| 10 | $\times$ | 11 | $\times$ |
| 01 | 11 | $\times$ | $\times$ |
| 11 | $\times$ | $\times$ | $\times$ |

Then $BM \cong S^1 \times S^1$ and we know from the theorem that $I_M \simeq_w \Omega(S^1 \times S^1)$.
An element of $I_M$ looks as follows.

# Examples

Scanning map $\widetilde{I}_M \to \Omega(S^1 \times S^1)$ can be graphed as follows.



$\pi_0(I_M) \cong \pi_1(S^1 \times S^1) \cong \mathbb{Z} \oplus \mathbb{Z}$

# Examples

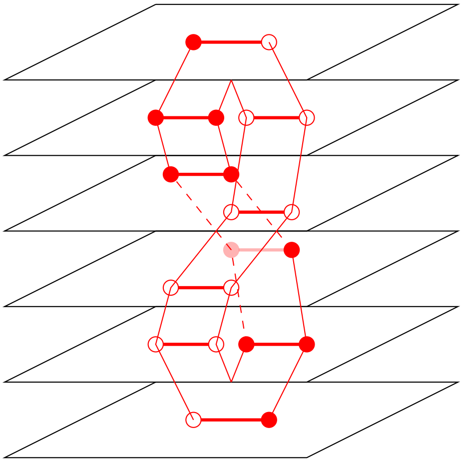
3. Let $M = C_1$ is the configuration space of finite subsets of $\mathbb{R}^1$. Then we know from Milgram-May-Segal's theorem that $BC_1 \cong C_1 \otimes S^1 \simeq_w \Omega SS^1 = \Omega S^2$. Now, our theorem asserts that

$$I_M \simeq_w \Omega^2 S^2.$$

So any element of $\pi_3 S^2$ can be written as a based loop in $I_M$. A generator of $\pi_3 S^2 \cong \mathbb{Z}$ is given by a Hopf map $\eta : S^3 \to S^2$.

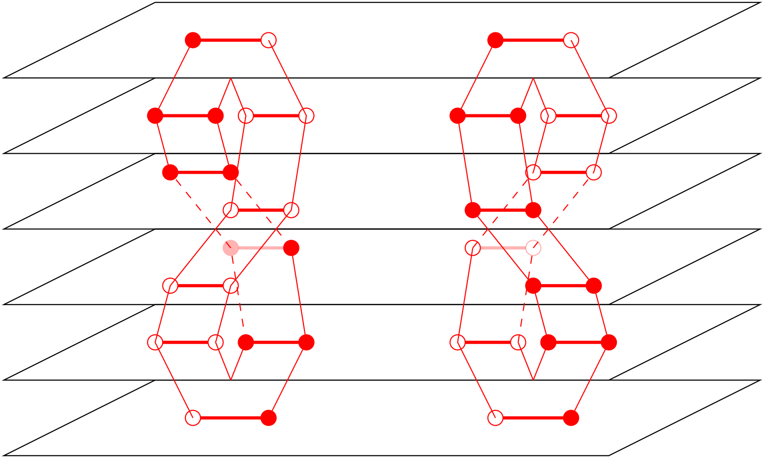
# Examples

Corresponding loop in $I_M$ is given by :



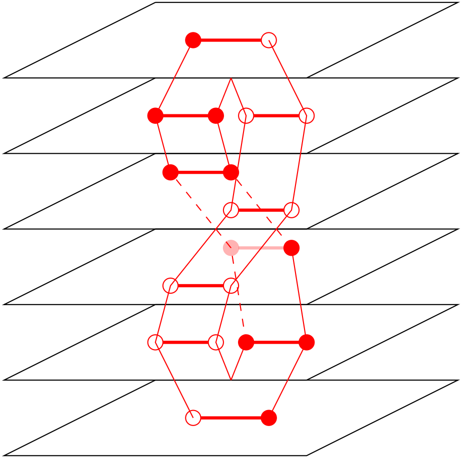
# Examples

Crossing change gives an inverse :



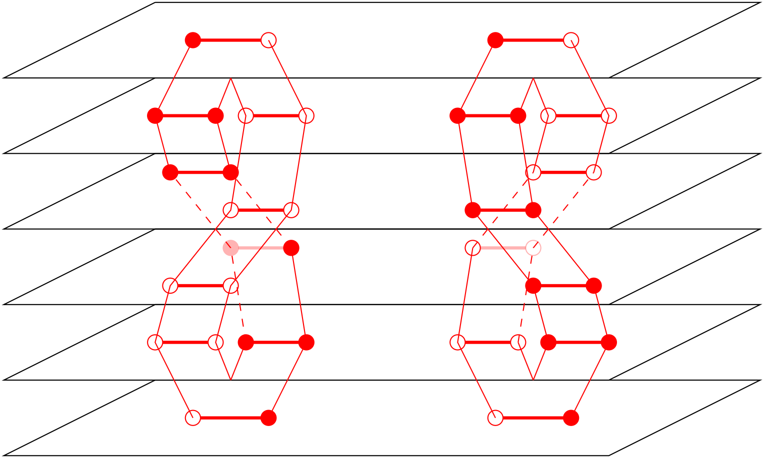
# Examples

Corresponding loop in $I_M$ is given by :



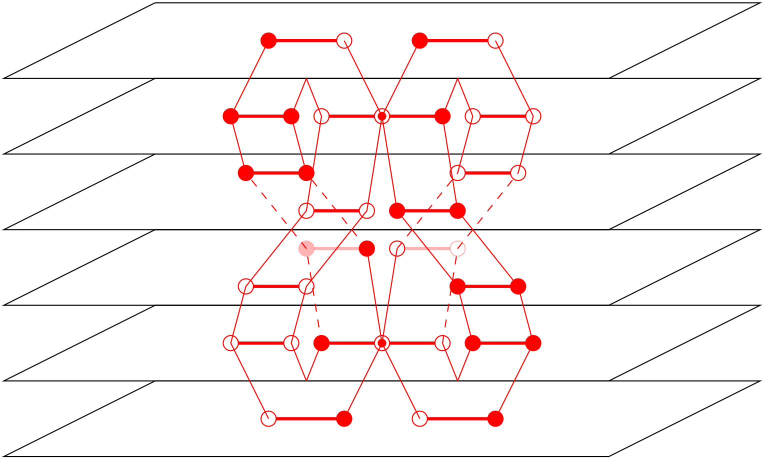
# Examples

Crossing change gives an inverse :



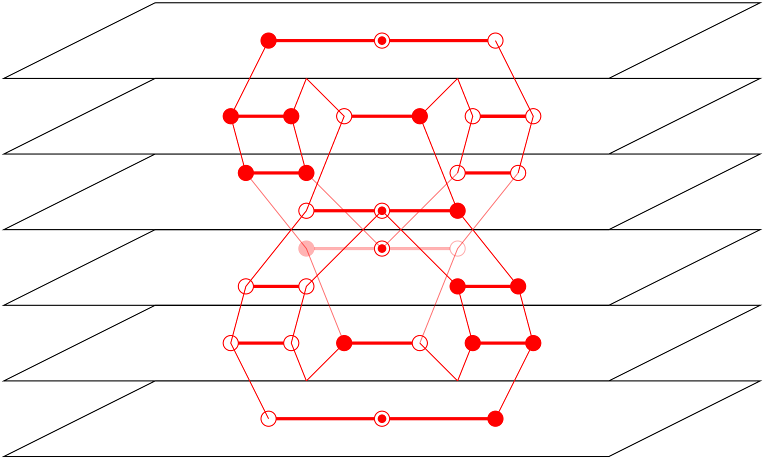
# Examples

Indeed, we can paste two surfaces to remove them:
(Step 1)



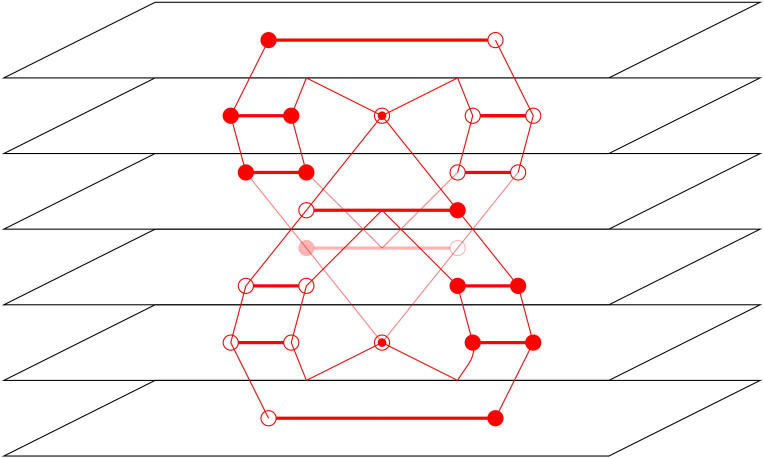
# Examples

Indeed, we can paste two surfaces to remove them:
(Step 2)



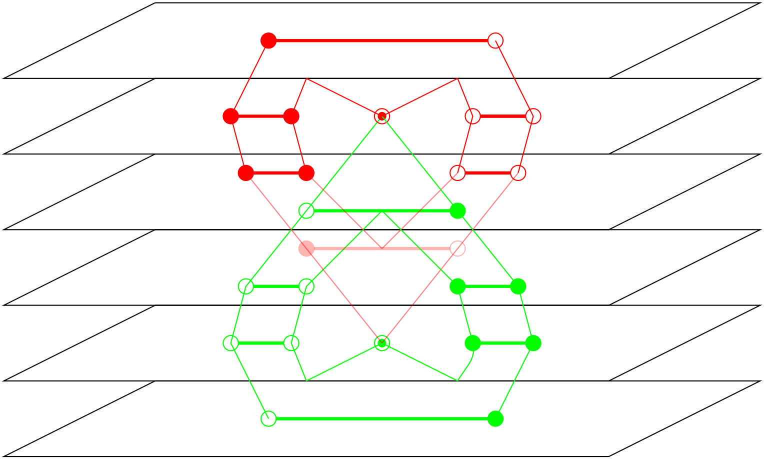
# Examples

Indeed, we can paste two surfaces to remove them:
(Step 3)



# Examples

Indeed, we can paste two surfaces to remove them:
(Step 4)



# Examples

Moreover, if we use the standard embedding $\mathbb{R}^1 \hookrightarrow \mathbb{R}^2$ to get a map $I_{C_1} \to I_{C_2}$, and this amounts to an embedding of configuration of intervals in $\mathbb{R}^2$ into configuration of intervals in $\mathbb{R}^3$ under the standard embedding $\mathbb{R}^2 \hookrightarrow \mathbb{R}^3$. This corresponds to the suspension map $\mathbb{Z} \cong \pi_3(S^2) \to \pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$. So the above pictures also show the vanishing of $2\eta$ in $\pi_4(S^3)$.

# Two lemmas for quasi-fibration

For a proof of proposition 3, we may use the Dold-Thom criterion.

## Lemma 1

*For any open set $V \subset F_j BM - F_{j-1} BM$, there exists a homotopy equivalence $p^{-1} V \simeq V \times \widetilde{I}_M$, so that $V$ is distinguished.*

## Lemma 2

*There exists an open set $O \subset F_j BM$ which contains $F_{j-1} BM$ and homotopies $h_t : O \to O$ and $H_t : p^{-1} O \to p^{-1} O$ such that*

1. $h_0 = id_O$, $h_t(F_{j-1} BM) \subset F_{j-1} BM$ *and* $h_1(O) \subset F_{j-1} BM$,
2. $H_0 = id_{p^{-1} O}$ *and* $p \circ H_t = h_t \circ p$ *for all* $t$, *and*
3. $H_1 : p^{-1} z \to p^{-1} h_1(z)$ *is a weak homotopy equivalence for all* $z \in O$.