数理解析研究所講究録 1343 巻 2003 年 61-64

THE SPACE OF INTERVALS AND AN APPROXIMATION TO $\Omega^n\Sigma^n X$

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1. Introduction

Let $C_n(X)$ be the configuration space of finite points in $\mathbb{R}^n$ with labels in X. As a set, $C_n(X)$ is given by

$$C_n(X)=\left\{(S,x)\;\middle|\;\begin{array}{l}S:\text{a finite subset}\subset\mathbb{R}^n\\ x:S\to X\end{array}\right\}\Big/\sim$$

where the relation is the base point relation, that is

$$(S,x)\sim(T,y)\iff\begin{cases}x(c)=y(c) & \text{if } c\in S\cap T\\ x(c)=*,y(c)=* & \text{otherwise.}\end{cases}$$

We say X group-completes to Y if there exist an admissible H-space $\widetilde{X}$ which is weakly equivalent to X and a group completion $\widetilde{X}\to Y$ [?]. By definition, if X group-completes to Y then $H_*(X)[\pi_0(X)^{-1}]\simeq H_*(Y)$.

Then Segal showed the following

Theorem 1 (Segal [?]). $C_n(X)$ group-completes to $\Omega^n\Sigma^nX$. When X is connected, $C_n(X)\simeq_w\Omega^n\Sigma^nX$.

By the above theorem, it is natural to expect that if we could put a Hopf inverse to $C_n(X)$ in some nice way, then we get a model which approximates $\Omega^n\Sigma^nX$ even when X is not connected. We recall the following special case of D.McDuff's construction[?]. We put

$$C^{\pm}(\mathbb{R}^n)=\left\{(S,p)\;\middle|\;\begin{array}{l}S:\text{a finite subset}\subset\mathbb{R}^n\\ p:S\to\{\pm1\}\end{array}\right\}\Big/\sim$$

where the topology is given so that two points with the opposite parity in $\{\pm1\}$ can collide and annihilate. By the annihilation of oppositely charged particles, this space can be considered as a space constructed from $C_n(S^0)$ by putting a homotopy inverse to it. But it does not approximate $\Omega^n\Sigma^nS^0=\Omega^nS^n$, indeed, it is showed by McDuff that $C^{\pm}(\mathbb{R}^n)\simeq_w\Omega^n(S^n\times S^n/\Delta)$, where Δ denotes the diagonal subspace of $\Sigma^nX\times\Sigma^nX$

By an interval in $\mathbb{R}^n$ we mean a subspace $J\times v\subset\mathbb{R}\times\mathbb{R}^{n-1}$ where $J\subset\mathbb{R}^1$ is a bounded interval and $v\in\mathbb{R}^{n-1}$. We put

$$I_n(X)=\left\{\{(J_1,x_1),\cdots,(J_k,x_k)\}\;\middle|\;\begin{array}{l}\{J_i\}:\text{disjoint intervals in }\mathbb{R}^n\\ x_i\in X\end{array}\right\}\Big/\sim$$

This set is topologized so that

- Any two intervals can be connected into one interval if they are of different type in meeting ends (*i.e.* one is closed and the other is open) and their labels in X coincide, and
- Any half-open interval can vanish when its length comes to be zero.

Theorem 2 (Main theorem). $I_n(X)\simeq_w\Omega^n\Sigma^nX$ (even for non-connected X)

The construction of $I_n(X)$ and the above theorem is inspired by the idea of Prof. K.Shimakawa concerning the same problem generalized to the G-equivariant setting. Let G be a group acting on X, and V be an orthogonal G-module. which contains all the irreducible G-representations inifinitely many times as direct summands. We denote the space $Map_*(V^c, V^c \wedge X)$ of based maps by $\Omega^V \Sigma^V X$, where V^c denotes the one point compactification of V. When G is finite, it is known that the configuration space $C_V(X)$ of finite points in V with labels in X is weakly equivalent to $\Omega^V \Sigma^V X$, but when G is infinite, $C_V(X)$ is too small to give such an approximation. His idea is to substitute $C_V(X)$ by a space of some class of manifolds embedded in V and get a weak equivalence to $\Omega^V \Sigma^V X$. Especially, the manifolds can be cut and pasted in his space of manifolds, which specializes to the connection of intervals in our space.

We give an outline of the proof of the main theorem in §2. In §3, we explain how $I_n(X)$ is related to $\Omega^n \Sigma^n X$ by observing the idea of physical analogue behind the definition of $\alpha : \widetilde{I}_n(X) \to \Omega C_{n-1} \Sigma X$. We also give the explicit definition of α in §3.

2. Outline of the proof of the main theorem

Let U be a subspace of $\mathbb{R}^1$ and $I_1(X)_U$ denote the space of intervals in U. We denote $I_1(X)_s = I_1(X)_{(0,s)}$. $I_1(X)$ is homeomorphic to $I_1(X)_s$ for any $s > 0$. We say that $\iota \in I_1(X)_U$ is ε-separated if it consists of intervals which satisfy the following conditions.

(1) they are subinterval in $U - \partial\bar{U}_{\varepsilon/2}$, where $\bar{U}$ denotes the closure of U and $\partial\bar{U}_{\varepsilon/2}$ denotes the $\varepsilon/2$-neighborhood of its boundary,
(2) any two ends (of the same or distinct intervals) with the same parity are separated more than or equal to ε, and
(3) any two intervals with the distinct labels in X are separated more than or equal to ε.

Let $I_1^\varepsilon(X)_U$ be the subspace of $I_1(X)_U$ consisting of all the ε-separated elements. We define

$$I_n^\varepsilon(X) = C_{n-1}(I_1^\varepsilon(X)).$$

(We agree here, that C_{n-1} is a continuous self-functor on the category of topological abelian partial monoids [?],[?]. Abelian partial monoid structure of $I_1^\varepsilon(X)$ is given by superimposition.)

Then we define

$$\widetilde{I}_n(X) = \{(\xi, \varepsilon, s) \mid 0 < \varepsilon \leq \delta,\ s \geq 0,\ \xi \in I_n^\varepsilon(X)_s\},$$

with a topology considered as a subspace of $I_n(X)_\infty \times (0, \delta] \times [0, \infty)$. Then the following lemma holds.

Lemma 3. $I_n(X) \simeq_w \widetilde{I}_n(X)$.

We can define a space $\widetilde{E}_n(X)$ and maps i and p appropriately so that the following proposition holds.

Proposition 4. $\widetilde{I}_n(X) \xrightarrow{i} \widetilde{E}_n(X) \xrightarrow{p} C_{n-1}\Sigma X$ is a quasifibration.

Proof. We follow the Dold-Thom criterion for a quasifibration.[?] So it suffices to show two lemmas below. □

Lemma 5. (Dold-Thom criterion 1)
For any open set $V \subset F_j C_{n-1}\Sigma X - F_{j-1} C_{n-1}\Sigma X$, V is distinguished.

Lemma 6. (Dold-Thom criterion 2)
There exist an open neighborhood U of $F_{j-1}C_{n-1}\Sigma X$ in $F_jC_{n-1}\Sigma X$ and a homotopies $h_t : U \to U$ and $H_t : p^{-1}U \to p^{-1}U$ such that

(1) $h_0 = id_U$ and $h_1(U) \subset F_{j-1}C_{n-1}\Sigma X$,
(2) $H_0 = id_{p^{-1}U}$ and $pH_t = h_t p$ for all t,
(3) $H_1 : p^{-1}z \to p^{-1}h_1 z$ is a homotopy equivalence for all $z \in U$.

Filtration $F_jC_{n-1}\Sigma X$ is given in [?]. The proof of two lemmas above are quite lengthy and we refer the reader to [?].

We have maps α and β which make the following diagram commutative.

$$\begin{array}{ccccc} \widetilde{I}_n(X) & \xrightarrow{i} & \widetilde{E}_n(X) & \xrightarrow{p} & C_{n-1}\Sigma X \\ \alpha\downarrow & & \beta\downarrow & & \| \\ \Omega C_{n-1}\Sigma X & \longrightarrow & PC_{n-1}\Sigma X & \longrightarrow & C_{n-1}\Sigma X. \end{array} \tag{1}$$

The above quasifibration approximates the path loop fibration

$$\Omega C_{n-1}\Sigma X \to PC_{n-1}\Sigma X \to C_{n-1}\Sigma X,$$

once we have proved the following

Lemma 7. $\widetilde{E}_n(X)$ is weakly contractible.

As $\widetilde{E}_n(X)$ is weakly contractible, β is a weak equivalence, and so is α. Then the main theorem follows from Lemma ??, since, by the Segal's theorem, $C_{n-1}\Sigma X \simeq \Omega^{n-1}\Sigma^n X$.

3. The map $\alpha : \widetilde{I}_n(X) \to \Omega C_{n-1}\Sigma X$

Before giving the explicit definition of α, we observe that there exists an idea of physical analogue behind it.

To relate $\widetilde{I}_n(X)$ with $\Omega^n\Sigma^n X$, we regard an interval to be a string which produces an electric field. Essentially, the electric property of a string differs from an ordinary electricity only in the direction of its tangent vector at two ends, so we first concentrate on the case $n = 1$.

We introduce some extraordinary electricity of particles, the notion of a particle 'charged half'. A particle charged $\pm\frac{1}{2}$ on the left(right) make the same effect as a particle charged ± 1 on the left(right) side and does nothing on its right(left) side.

A string works as a pair of particles each of which is charged $+\frac{1}{2}$ or $-\frac{1}{2}$. If the string has a closed(open) end on its left side, then we take it as if there is a particle charged $+\frac{1}{2}(-\frac{1}{2})$ on the left located at $\frac{\epsilon}{2}$ inside the end. We understand the exsistence of another particle charged $\pm\frac{1}{2}$ on the right similarly for the right end of the string.

Then a point in $\widetilde{I}_1^\epsilon(X)$ is considered to be a configuration of finite number of such strings in $\mathbb{R}$. $\widetilde{I}_1^\epsilon(X) \to Map^{cpt}(\mathbb{R}, \mathbb{R} \times X)$ is defined by assigning to the configuration of strings a field it produces, making the labels in X in consideration. Taking one point compactification of $\mathbb{R}$ and a quotient appropriately, we get a map $\alpha^\epsilon : \widetilde{I}_1^\epsilon(X) \to Map_*(\mathbb{R}\cup\{\infty\}, (\mathbb{R}\cup\{\infty\})\wedge X) \approx \Omega\Sigma X$. (Regard that, in $Map_*(\mathbb{R}\cup\{\infty\}, (\mathbb{R}\cup\{\infty\})\wedge X)$, ∞ is the base point of $\mathbb{R}\cup\{\infty\}$ on the source, while 0 is on the target.) α^ϵ for all $0 < \epsilon \leq \delta$ constitutes a map $\alpha : \widetilde{I}_n(X) \to \Omega^n\Sigma^n X$.

If we prefer, we may define $\widetilde{I}_n(X) \to \Omega^n\Sigma^n X$ by assignment of the field to the configuration of strings in $\mathbb{R}^n$, by regarding the effect of each string in the direction orthogonal to the first axis as the effect given by a 'string charged $+1$'.

Now we give the explicit definition of $\alpha : \widetilde{I}_n(X) \to \Omega C_{n-1}\Sigma X$. Let ι be an element of $I_1^\varepsilon(X)_s$. Suppose ι is represented by a k-tuple $((J_1, x_1), \cdots, (J_k, x_k))$ where J_i is an interval with end points u_{2i-1} and u_{2i}. We also assume that $u_{i-1} \le u_i$ for all i. If u_j ($j = 2i-1$ or $2i$) is a closed(open) end of J_i, we put $p_j = 1(-1)$.

We define subintervals $N_i \subset [0, s]$ $(i = 1, \cdots, 2k)$ as

$$N_1 = [u_1 - \varepsilon/2, \mathrm{Min}(u_1 + \varepsilon/2, u_2 - \varepsilon/2)],$$

$$N_i = [\mathrm{Max}(u_i - \varepsilon/2, u_{i-1} + \varepsilon/2), \mathrm{Min}(u_i + \varepsilon/2, u_{i+1} - \varepsilon/2)], \text{ for } 1 < i < 2k,$$

and

$$N_{2k} = [\mathrm{Max}(u_{2k} - \varepsilon/2, u_{2k-1} + \varepsilon/2), u_{2k} + \varepsilon/2].$$

We define a function $f : \bigcup_{i=1}^{2k} N_i \to S^1 \wedge X$ by

$$f(t) = [p_i((t - u_i)/\varepsilon + (-1)^i/2)] \wedge x_{G((i+1)/2)}, \text{ if } t \in N_i$$

where S^1 is regarded as $[-1, 1]/\{\pm 1\}$ and $G(q)$ denotes the largest integer which does not exceed q. We can extend f continuously to $[0, s]$ in such a way that it is piecewise constant outside $\bigcup_{i=1}^{2k} N_i$.

This definition does not depend on the choice of a representative, so we obtain a map

$$\alpha_s^\varepsilon : I_1^\varepsilon(X)_s \to \Omega_s(\Sigma X),$$

which is clearly an abelian partial monoid homomorphism. Then we define a map $\alpha : \widetilde{I}_1(X) \to \Omega\Sigma X$ by $(\xi, \varepsilon, s) \mapsto \alpha_s^\varepsilon(\xi)$, which is also an abelian partial monoid homomorphism, if we regard $\Omega\Sigma X$ as an abelian partial monoid appropriately.

Then we define a map $\alpha : \widetilde{I}_n(X) \to \Omega C_{n-1}(\Sigma X)$ by the composite

$$\widetilde{I}_n(X) \to C_{n-1}(\widetilde{I}_1(X)) \xrightarrow{C_{n-1}(\alpha)} C_{n-1}(\Omega\Sigma X) \to \Omega C_{n-1}(\Sigma X),$$

where the first map is given by an inclusion $I_1^\varepsilon(X) \to \widetilde{I}_1(X)$, while the last map is given by

$$[v_1, l_1; \cdots; v_k, l_k] \mapsto (t \mapsto [v_1, l_1(t); \cdots; v_k, l_k(t)]), \ l_i \in \Omega\Sigma X.$$

References

[1] Dold and Thom, Quasifaserungen und unendliche symmetrische Produkte, Ann. of Math. (2) **67** (1958), 239–281.

[2] J. May, *The geometry of iterated loop spaces*, Springer Lecture Notes in Math. **271** (1972).

[3] J. May, E_∞ spaces, group completions, and permutative categories, London Math. Soc. Lecture Note **11** (1974).

[4] D. McDuff, Configuration spaces of positive and negative particles, Topology **14** (1975), 91–107.

[5] S. Okuyama, The space of intervals in a Euclidean space, preprint.

[6] G. Segal, Configuration spaces and iterated loop-spaces, Inventiones math. **21** (1973), 213–221.

[7] K.Shimakawa, Configuration spaces with partially summable labels and homology theories, Math.J.Okayama Univ. **43** (2001), 43–72.