

Representations of the Torelli group via the Heisenberg group

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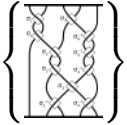
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Joint work with Christian Blanchet and Awais Shaukat

Many definitions of B_n :

- $\pi_1(C_n(D^2))$
- $\{\text{diffeomorphisms of } S = D^2 \setminus \{n \text{ points}\}\} / \text{isotopy}$
- $\left\langle \sigma_1, \dots, \sigma_{n-1} \mid \begin{array}{l} \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \\ \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| \geq 2 \end{array} \right\rangle$
- 

Knot theory:

- $\bigsqcup_{n \geq 1} B_n \twoheadrightarrow \{\text{knots/links in } \mathbb{R}^3\}$ [Alexander, Markov]
- *Burau representation* $\mapsto$ *Alexander polynomial*

Algebraic geometry:

- [Moishezon]: alg. curve in $\mathbb{CP}^2 \mapsto$ *braid monodromy* $F_N \rightarrow B_d$
- [Libgober]: alg. curve in $\mathbb{CP}^2 \mapsto$ invariant
using a representation of B_d

Homotopy theory:

- [Berrick-Cohen-Wong-Wu, 2006]:

$$\pi_*(S^2) \cong \frac{\{\text{Brunnian braids in } S^2 \times [0, 1]\}}{\{\text{Brunnian braids in } D^2 \times [0, 1]\}}$$

Definition

$$\mathrm{Map}(S) = \{\text{diffeomorphisms of } S\} / \text{isotopy}$$

Example: $\mathrm{Map}(D_n) = B_n$ where $D_n = D^2 \setminus \{n \text{ points}\}$

Applications & connections:

- $\mathrm{Map}(S) = \pi_1(\mathcal{M}_S)$
 $\mathcal{M}_S = \text{moduli space of algebraic curves of topological type } S$
- 3-manifold topology — via *Heegaard splittings*
- 4-dimensional symplectic topology [Donaldson]

[Burau] representation (1935):

$$\sigma_i \longmapsto I_{i-1} \oplus \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \oplus I_{n-i-1}$$

- This defines $B_n \longrightarrow GL_n(\mathbb{Z}[t^{\pm 1}]) \subset GL_n(\mathbb{R})$
- Q($\leq$ [Birman'74]): *Is this representation injective?* ($\equiv$ 'faithful')
- A($n \leq 3$): Yes [Magnus-Peluso'69]
- A($n \geq 5$): No [Moody'91, Long-Paton'93, Bigelow'99]
- A($n = 4$): ??
- Q: *Are the braid groups linear?*
 - Does B_n embed into some $GL_N(\mathbb{F})$?

[Lawrence] representation (1990) — geometric definition.

- $\text{Diff}(D_n)$ acts on $C_k(D_n)$ (unordered configuration space)
- $B_n = \text{Map}(D_n) = \text{Diff}(D_n)/\sim$ acts on $H_*(C_k(D_n); \mathbb{Z})$
- Two modifications:
 - Choose $\pi_1(C_k(D_n)) \twoheadrightarrow Q$ invariant under the action.
Then B_n acts on $H_*(C_k(D_n); \mathbb{Z}[Q])$
 - Replace H_* with H_*^{bm} (*Borel-Moore homology*)
Then $H_*^{bm}(C_k(D_n); \mathbb{Z}[Q])$ is a free $\mathbb{Z}[Q]$ -module
concentrated in degree $* = k$

$$\text{Lawrence}_k: B_n \longrightarrow GL_N(\mathbb{Z}[Q]) = \text{Aut}_{\mathbb{Z}[Q]}(H_*^{bm}(C_k(D_n); \mathbb{Z}[Q]))$$

How is the quotient Q defined?

- $\pi_1(D_n) = F_n \longrightarrow \mathbb{Z} = Q$ “total winding number”
- $\pi_1(C_k(D_n)) \longrightarrow \mathbb{Z} \oplus \mathbb{Z}$
 (“total winding number”, “self-winding number”)

Lemma

This quotient is $\text{Map}(D_n)$ -invariant, and hence

$$\text{Lawrence}_k: B_n \longrightarrow GL_N(\mathbb{Z}[Q])$$

is well-defined. Moreover, we have $\text{Lawrence}_1 = \text{Burau}$.

Theorem [Bigelow'00, Krammer'00]

Lawrence_2 is faithful (injective). Hence B_n embeds into $GL_N(\mathbb{R})$.

- Q: Does $\text{Map}(S)$ embed into $GL_N(\mathbb{F})$ for other surfaces S ?
- $\text{Map}(\text{torus}) \cong SL_2(\mathbb{Z}) \subset GL_2(\mathbb{R})$
- $\text{Map}(\Sigma_2) \subset GL_{64}(\mathbb{C})$ [Bigelow-Budney'01]
- In general, wide open!
 - Kontsevich (2006): proposal of a sketch of a construction of a faithful finite-dimensional representation of $\text{Map}(\Sigma_g)$
 - Dunfield (cf. [Margalit'18]): computational evidence suggesting that this will *not* actually be faithful
- From now on, focus on $\Sigma = \Sigma_{g,1}$
(orientable, genus g , one boundary component)

Main result [Blanchet-P.-Shaukat'21]

A new representation of (a central extension of) $\text{Tor}(\Sigma) \subset \text{Map}(\Sigma)$.

Simplest analogue of the Lawrence representations:

$$\text{Map}(\Sigma) \quad \circlearrowright \quad H_k^{bm}(F_k(\Sigma'); \mathbb{Z})$$

- $F_k(\) = \text{ordered configuration space}$
- $\Sigma' = \Sigma \setminus (\text{interval in } \partial\Sigma)$
- *untwisted* $\mathbb{Z}$ coefficients
- $H_k^{bm}(F_k(\Sigma'); \mathbb{Z})$ is a free abelian group of finite rank

Theorem [Moriyama'07]

The kernel of this representation is $\mathfrak{J}(k) \subset \text{Map}(\Sigma)$.

- $\mathfrak{J}(k)$ is the k -th term of the *Johnson filtration* of $\text{Map}(\Sigma)$
- What is this?

- Lower central series: $\pi_1(\Sigma) = \Gamma_0 \supseteq \Gamma_1 \supseteq \Gamma_2 \supseteq \Gamma_3 \supseteq \dots$
- $\Gamma_i = [\pi_1(\Sigma), \Gamma_{i-1}]$ (commutators of length $i + 1$)

Definition [Johnson'81]

$\mathfrak{J}(k)$ = kernel of the action of $\text{Map}(\Sigma)$ on $\pi_1(\Sigma)/\Gamma_k$.

- $\text{Map}(\Sigma) = \mathfrak{J}(0) \supset \mathfrak{J}(1) \supset \mathfrak{J}(2) \supset \mathfrak{J}(3) \supset \dots$
- $\mathfrak{J}(1) = \text{Tor}(\Sigma) = \ker(\text{Map}(\Sigma) \curvearrowright H_1(\Sigma; \mathbb{Z}))$ *Torelli group*

Theorem [Johnson'81]

$$\bigcap_{k=1}^{\infty} \mathfrak{J}(k) = \{1\}$$

Corollary [Moriyama'07]

$\bigoplus_{k=1}^{\infty} H_k^{bm}(F_k(\Sigma'); \mathbb{Z})$ is a faithful (∞ -dim.!) $\text{Map}(\Sigma)$ -representation.

- Idea: Enrich the representation by taking homology with *twisted coefficients* $\mathbb{Z}[Q]$, where $\pi_1(C_k(\Sigma')) = B_k(\Sigma) \twoheadrightarrow Q$.
- $Q = \mathfrak{S}_k$ corresponds to the Moriyama representations:

$$H_k^{bm}(F_k(\Sigma'); \mathbb{Z}) = H_k^{bm}(C_k(\Sigma'); \mathbb{Z}[\mathfrak{S}_k]).$$
- First try abelian quotients Q .

Fact ($k \geq 2$)

$$B_k(S)^{ab} \cong \pi_1(S)^{ab} \oplus \left\{ \begin{array}{ll} \mathbb{Z} & S \text{ planar} \\ \mathbb{Z}/(2k-2) & S = S^2 \\ \mathbb{Z}/2 & \text{otherwise.} \end{array} \right\}$$

- If S is non-planar, we can only “count” the *self-winding number* of braids on S **mod 2**.
(or mod $2k-2$ if $S = S^2$)
- In $\mathbb{Z}[B_k(S)^{ab}]$, the corresponding “variable” t will have order two: $t^2 = 1$.
 $\mapsto$ We get a much “weaker” representation...

Theorem [Bellingeri'04]

$$B_k(\Sigma_{g,1}) \cong \left\langle \sigma_1, \dots, \sigma_{k-1}, \begin{matrix} a_1, \dots, a_g \\ b_1, \dots, b_g \end{matrix} \mid \dots \text{ some relations } \dots \right\rangle$$

Adding the relations saying that σ_1 is *central* (commutes with every element), we obtain:

$$B_k(\Sigma_{g,1}) / \langle\langle [\sigma_1, x] \rangle\rangle \cong \left\langle \sigma, \begin{matrix} a_1, \dots, a_g \\ b_1, \dots, b_g \end{matrix} \mid \begin{array}{l} \text{all pairs commute except} \\ a_i b_i = \sigma^2 b_i a_i \end{array} \right\rangle$$

Definition

$$\mathcal{H}_g = B_k(\Sigma_{g,1}) / \langle\langle [\sigma_1, x] \rangle\rangle$$

This is the *genus-g discrete Heisenberg group*. Note that:

$$\mathcal{H}_1 \cong \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \right\} \subset GL_3(\mathbb{Z})$$

Lemma

The action $\text{Map}(\Sigma) \curvearrowright B_k(\Sigma)$ descends to a well-defined action on the quotient $\mathcal{H}_g$.

Proof

- Aim: $\ker(B_k(\Sigma) \twoheadrightarrow \mathcal{H}_g)$ is preserved by the $\text{Map}(\Sigma)$ -action.
- This is $\langle\langle [\sigma_1, x] \rangle\rangle$, so it is enough to show that σ_1 is fixed by the $\text{Map}(\Sigma)$ -action.
- Let $[\varphi] \in \text{Map}(\Sigma) = \text{Diff}(\Sigma)/\sim$ be represented by a diffeo. φ that fixes *pointwise* a collar neighbourhood of $\partial\Sigma$.
- The loop of configurations $\sigma_1 \in B_k(\Sigma) = \pi_1(C_k(\Sigma))$ can be homotoped to stay inside this collar neighbourhood.

The Heisenberg group fits into a central extension:

$$1 \rightarrow \mathbb{Z} \longrightarrow \mathcal{H}_g \longrightarrow H_1(\Sigma; \mathbb{Z}) \rightarrow 1$$

and the $\text{Map}(\Sigma)$ -action on $\mathcal{H}_g$ lifts the natural action on $H_1(\Sigma; \mathbb{Z})$.

Proposition [Blanchet-P.-Shaukat'21]

$$(a) \ker(\text{Map}(\Sigma) \curvearrowright \mathcal{H}_g) = \text{Chill}(\Sigma)$$

$$(b) \{\varphi \in \text{Map}(\Sigma) \mid \varphi \text{ acts on } \mathcal{H}_g \text{ by conjugations}\} = \text{Tor}(\Sigma)$$

$$\text{Map}(\Sigma) \supset \text{Tor}(\Sigma) \supset \text{Chill}(\Sigma) \supset \mathfrak{J}(2) \supset \mathfrak{J}(3) \supset \dots$$

Idea of proof

$$\Phi: \text{Map}(\Sigma) \longrightarrow \text{Aut}^+(\mathcal{H}_g) \cong H \rtimes Sp(H) \quad H = H_1(\Sigma; \mathbb{Z})$$

(a) Under this identification, $\Phi =$ “Trapp representation”
(= the action of $\text{Map}(\Sigma)$ on $\{\text{unit vector fields on } \Sigma\}$)

$$(b) \begin{array}{ll} \text{alg.:} & \text{Inn}(\mathcal{H}_g) \longleftrightarrow 2H \\ \text{top.:} & \text{image}(\Phi) \subseteq 2H \rtimes Sp(H) \end{array}$$

Theorem [Blanchet-P.-Shaukat'21]

We obtain well-defined representations, defined over $\mathbb{Z}[\mathcal{H}_g]$:

$$(a) \quad \text{Chill}(\Sigma) \quad \curvearrowright \quad H_k^{bm}(C_k(\Sigma'); \mathbb{Z}[\mathcal{H}_g])$$

$$(b) \quad \widetilde{\text{Tor}}(\Sigma) \quad \curvearrowright \quad H_k^{bm}(C_k(\Sigma'); \mathbb{Z}[\mathcal{H}_g])$$

where $\widetilde{\text{Tor}}(\Sigma)$ is a $\mathbb{Z}$ -central extension of $\text{Tor}(\Sigma)$.

Idea of proof

Lemma $\Rightarrow$ we obtain a *twisted* representation, defined over $\mathbb{Z}[\mathcal{H}_g]$:

$$\text{Map}(\Sigma) \quad \curvearrowright \quad H_k^{bm}(C_k(\Sigma'); \mathbb{Z}[\mathcal{H}_g])$$

“Twisted” means:

- the action is $\mathbb{Z}$ -linear
- there is also an action on the ground ring $\mathbb{Z}[\mathcal{H}_g]$
- these are compatible: $\varphi(\lambda.v) = \varphi(\lambda).\varphi(v)$.

Idea of proof (cont.)

Proposition (a) $\Rightarrow$ the action of $\text{Chill}(\Sigma)$ on $\mathbb{Z}[\mathcal{H}_g]$ is trivial
 $\Rightarrow$ the representation is *untwisted* on $\text{Chill}(\Sigma)$.

Untwisting Lemma: A twisted representation of Γ over $\mathbb{Z}[G]$, where the action $\Gamma \curvearrowright G$ is by conjugations, can be *untwisted* by passing to a central extension of Γ by $\mathcal{Z}(G) = \text{Centre}(G)$.

Proposition (b) $\Rightarrow$ we can apply the Untwisting Lemma to $\Gamma = \text{Tor}(\Sigma)$ and $G = \mathcal{H}_g$.

Q: What is the kernel of this representation?

Q': Is it smaller than $\mathfrak{J}(k) = \ker(\text{Moriyama}_k)$?

($k = 2$)

Quotient of $\widetilde{\text{Tor}}(\Sigma)$ -representations / twisted $\text{Map}(\Sigma)$ -representations:

$$H_2^{bm}(C_2(\Sigma'); \mathbb{Z}[\mathcal{H}_g]) \twoheadrightarrow H_2^{bm}(C_2(\Sigma'); \mathbb{Z}[\mathfrak{S}_2])$$

induced by $\mathcal{H}_g \twoheadrightarrow (\mathcal{H}_g)^{ab} = \mathbb{Z}/2 \oplus H_1(\Sigma; \mathbb{Z}) \twoheadrightarrow \mathbb{Z}/2 = \mathfrak{S}_2$.

- The right-hand side is precisely $H_2^{bm}(F_2(\Sigma'); \mathbb{Z}) = \text{Moriyama}_2$, hence

$$\ker(H_2^{bm}(C_2(\Sigma'); \mathbb{Z}[\mathcal{H}_g])) \subseteq \mathfrak{J}(2)$$

- Let $\gamma \subset \Sigma$ be a simple closed curve that separates off a genus-1 subsurface. Then the *Dehn twist* T_γ lies in $\mathfrak{J}(2)$.
- Calculations $\Rightarrow T_\gamma$ acts *non-trivially* in our representation.

Corollary [Blanchet-P.-Shaukat'21]

The kernel of the $\widetilde{\text{Tor}}(\Sigma)$ -representation $H_2^{bm}(C_2(\Sigma'); \mathbb{Z}[\mathcal{H}_g])$ is **strictly smaller** than $\mathfrak{J}(2)$.

Set $k = 2$ and $g = 1$. In this case the representation

$$H_2^{bm}(C_2(\Sigma'); \mathbb{Z}[\mathcal{H}_1])$$

is free of rank 3 over $\mathbb{Z}[\mathcal{H}_1] = \mathbb{Z}[\sigma^{\pm 1}] \langle a^{\pm 1}, b^{\pm 1} \rangle / (ab = \sigma^2 ba)$

Let γ be a curve isotopic to $\partial\Sigma = \partial\Sigma_{1,1}$. Then T_γ acts via:

$$\begin{bmatrix} \sigma^{-8}b^2 + \sigma^{-4}a^{-2} - \sigma a^{-2}b^2 + (\sigma^{-1} - \sigma^{-2})a^{-2}b + (\sigma^{-3} - \sigma^{-4})a^{-1}b^2 + (\sigma^{-4} - \sigma^{-5})a^{-1}b & \begin{pmatrix} \sigma^2 + 1 - 2\sigma^{-1} + \sigma^{-2} + \sigma^{-4} \\ (-\sigma^2 + \sigma + \sigma^{-1} - \sigma^{-2}) \\ (-1 + \sigma^{-1} + \sigma^{-3} - \sigma^{-4}) \end{pmatrix} \begin{pmatrix} a^{-2}b^2 - \sigma a^{-2}b^4 + b^2 - \sigma^{-3}a^{-2} + b \\ (-1 + \sigma^{-1} + \sigma^{-3} - \sigma^{-4})a^{-2}b \end{pmatrix} & \begin{pmatrix} (-1 + 2\sigma^{-1} - \sigma^{-2} - \sigma^{-4} + \sigma^{-5})a^{-2}b + (\sigma^{-1} - \sigma^{-2})a^{-2}b^3 + (\sigma^2 - \sigma - \sigma^{-1} + 2\sigma^{-2} - \sigma^{-3})a^{-2}b^2 + (-\sigma^{-3} + \sigma^{-4})a^{-1}b + (\sigma^{-4} - \sigma^{-5})a^{-1}b^3 + (-\sigma^{-2} + \sigma^{-3} + \sigma^{-5} - \sigma^{-6})a^{-1}b^2 + (-\sigma^{-3} + \sigma^{-4})a^{-2} \end{pmatrix} \\ -\sigma^{-1} - \sigma^{-3} + 2\sigma^{-4} - \sigma^{-5} - \sigma^{-7} + \sigma^{-2}a^2 + (\sigma^{-1} - \sigma^{-2} - \sigma^{-4} + \sigma^{-5})a + \sigma^{-6}a^{-2} + (\sigma^{-3} - \sigma^{-4} - \sigma^{-6} + \sigma^{-7})a^{-1} & \begin{pmatrix} 1 + \sigma^{-2} - \sigma^{-3} + \sigma^{-6} + \sigma^{-6}a^2b^2 - \sigma^{-1}b^2 + (\sigma^{-3} - \sigma^{-4})a^{-1}b^2 + (-1 + \sigma^{-1} + \sigma^{-3} - \sigma^{-4})b + (\sigma^{-2} - 2\sigma^{-3} + \sigma^{-4} - \sigma^{-6} - \sigma^{-7})a^{-1}b - \sigma^{-5}a^{-2} + (-\sigma^{-2} + \sigma^{-3} + \sigma^{-5} - \sigma^{-6})a^{-1} + (\sigma^{-5} - \sigma^{-6})a^{-2}b \end{pmatrix} & \begin{pmatrix} (-\sigma^{-6} + \sigma^{-7})a^{-2}b + (\sigma^{-1} - \sigma^{-2} - \sigma^{-4} + 2\sigma^{-5} - \sigma^{-6})b + (-\sigma^{-3} + 2\sigma^{-4} - \sigma^{-5} - \sigma^{-7} + \sigma^{-8})a^{-1}b + 1 - \sigma^{-1} + \sigma^{-2} - 3\sigma^{-3} + 2\sigma^{-4} + \sigma^{-6} - \sigma^{-7} + (-\sigma^{-2} + 2\sigma^{-3} - \sigma^{-4} + \sigma^{-5} - 2\sigma^{-6} + \sigma^{-7})a^{-1} + (\sigma^{-2} - \sigma^{-3})ab + (-1 + \sigma^{-1} + \sigma^{-3} - \sigma^{-4})a + (-\sigma^{-3} + \sigma^{-6})a^{-2} \end{pmatrix} \\ -\sigma^{-6}ab + (-\sigma^{-3} + \sigma^{-4} - \sigma^{-7})b - \sigma^{-4} + (\sigma^{-1} - \sigma^{-4} + \sigma^{-5})a^{-1}b + \sigma^{-2}a^{-2}b + (-\sigma^{-3} + \sigma^{-6})a^{-1} + \sigma^{-5}a^{-2} & \begin{pmatrix} (-1 - \sigma^{-2} + 2\sigma^{-3} - \sigma^{-6})a^{-1}b + \sigma^{-1}b^3 + \sigma^{-2}a^{-2}b^3 + (1 - \sigma^{-1} - \sigma^{-3} + \sigma^{-4})a^{-1}b^2 + (\sigma^{-1} - \sigma^{-2} + \sigma^{-5})a^{-2}b^2 + (-\sigma^{-1} + \sigma^{-4} - \sigma^{-5})a^{-2}b + (\sigma^{-2} - \sigma^{-3})a^{-1} - \sigma^{-4}a^{-2} \end{pmatrix} & \begin{pmatrix} \sigma^{-3} + (\sigma^{-2} - \sigma^{-3} - \sigma^{-5} + \sigma^{-6})a^{-1} + (-\sigma^{-1} + \sigma^{-2} - \sigma^{-5} + \sigma^{-6})a^{-1}b^2 + (-\sigma^{-2} + \sigma^{-3})a^{-2}b^2 + (-1 + \sigma^{-1} + 2\sigma^{-3} - 3\sigma^{-4} + \sigma^{-7})a^{-1}b + (-\sigma^{-1} + \sigma^{-2} - \sigma^{-5} + \sigma^{-6})a^{-2}b + (-\sigma^{-4} + \sigma^{-5})b^2 + (\sigma^{-2} - \sigma^{-3} - \sigma^{-5} + \sigma^{-6})b + (-\sigma^{-4} + \sigma^{-5})a^{-2} \end{pmatrix} \end{bmatrix}$$

Braid groups

MCGs

Rep of B_n

Rep of MCGs

– Moriyama

– abelian coeff

– non-abelian

– kernel

Thank you!