

# Noncommutative Geometry and Generalized GMP Algebra for Three-Dimensional Quantum Hall Fluids of Extended Objects

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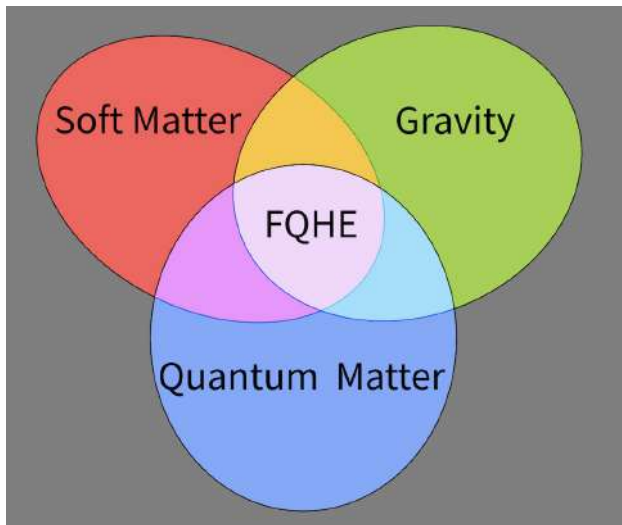
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# Outline

- ▶ Introduction: Fractional Quantum Hall Effect in 2D
- ▶ Guiding-center variables and noncommutative geometry
- ▶ Extended objects and bundle gerbes in 3D
- ▶ Generalized GMP algebra for 3D quantum Hall fluids
- ▶ Connection to topological BF+BB theory
- ▶ Conclusions and Outlook

# Fractional Quantum Hall Effect (FQHE)



# Quantum Hall Effect in 2D: A Brief Review

The quantum Hall effect (QHE) is one of the most remarkable quantum phenomena in condensed matter physics.

- ▶ Strong perpendicular magnetic field  $\mathcal{B}$  applied to a 2D electron gas
- ▶ Quantized Hall conductance:  $\sigma_{xy} = \nu \frac{e^2}{h}$  where  $\nu$  is the filling factor
- ▶ Landau levels: discrete energy levels  $E_n = \hbar\omega_c(n + 1/2)$
- ▶ In the strong field regime, fast cyclotron motion is frozen
- ▶ Low-energy physics described by guiding-center coordinates

# Chern-Simons theory in the Abelian Laughlin states

Coulomb interactions give rise to the fractionalization of the electric charge,  $\nu = \frac{1}{q}$ , with  $q$  a positive odd number.

$$S_{CS}[a, A] = \int d^3x \left( \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda - \frac{q}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + \dots \right).$$

By integrating out the hydro-dynamical field  $a_\mu$  we obtain the correct fractional Hall conductivity

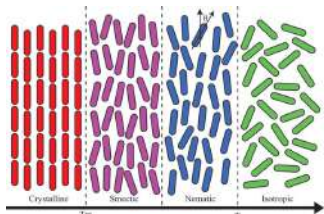
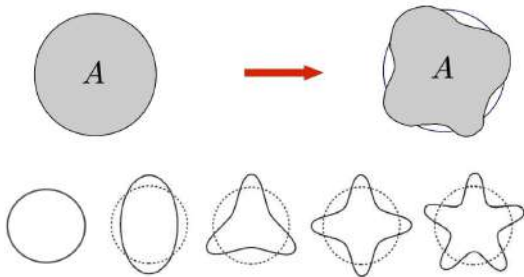
$$S_{CS}[A] = \int d^3x \left( \frac{1}{4\pi q} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \right).$$

On a disk geometry, the edge theory is given by a chiral boson action in polar coordinates  $(t, \phi)$

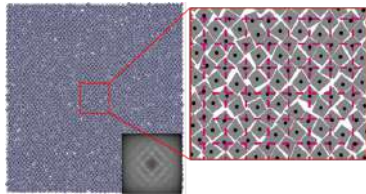
$$S_\varphi = \frac{1}{2} \int d\phi dt \left[ \partial_t \varphi \partial_\phi \varphi + v_b (\partial_\phi \varphi)^2 \right],$$

namely a chiral CFT with  $c = 1$ .

# Incompressibility and area-preserving diffeomorphisms



(a) Nematic Order



(b) Tetratic Order

Emergent geometry in the FQHE induced by interactions  
(Haldane, PRL (2011); B. Yang et al., PRL (2012)).

- ▶ Chern-Simons theories cannot encode dynamical degrees of freedom because they are topological.
- ▶ Rotational symmetry is not relevant to form FQH states.
- ▶ Non-relativistic bimetric theories:  $(g_{ab}, \bar{g}_{ab})$   
(Maciejko, Hsu, Kivelson, Park, Sondhi, PRB (2013),  
Gromov and Son, PRX (2017)).

## Guiding-Center Formalism in 2D

In the strong magnetic field regime, electrons undergo circular cyclotron motion.

The position of the center of the cyclotron orbit defines the **guiding-center coordinates**:

$$X^i = x^i + \epsilon^{ij} \frac{q_j}{eB},$$

where  $x^i$  are the physical coordinates and  $q_j$  are the canonical momenta.

These guiding-center coordinates satisfy a noncommutative algebra:

$$[X^i, X^j] = i\ell_B^2 \epsilon^{ij},$$

where  $\ell_B = \sqrt{\hbar/(eB)}$  is the magnetic length.

This noncommutativity is fundamental to understanding incompressible fractional quantum Hall states.

# Girvin-MacDonald-Platzman (GMP) Algebra

Girvin, MacDonald, and Platzman (1985) studied the algebra of projected density operators in the lowest Landau level (LLL). The projected density operator in momentum space:

$$\rho_{\mathbf{q}} = \sum_i e^{i\mathbf{q} \cdot \mathbf{X}_i},$$

where the sum is over all electrons and  $\mathbf{X}_i$  are guiding-center coordinates.

The GMP algebra:

$$[\rho_{\mathbf{q}}, \rho_{\mathbf{q}'}] = 2i \sin\left(\frac{\ell_B^2}{2} \mathbf{q} \times \mathbf{q}'\right) \rho_{\mathbf{q}+\mathbf{q}'},$$

This closed algebra is central to the theory of fractional quantum Hall states and has been proven to be the unique closed algebra for projected densities.

# Why Go to 3D?

Extending quantum Hall physics to three dimensions is both challenging and fascinating.

- ▶ In 2D: magnetic field is a 2-form  $\mathcal{B} = \mathcal{B} dx \wedge dy$
- ▶ In 3D: we need a 3-form background field  $H = H dx \wedge dy \wedge dz$
- ▶ Such 3-form fields appear naturally in string theory and higher gauge theory
- ▶ They are associated with **bundle gerbes**, a higher categorical generalization of line bundles
- ▶ Instead of point particles, we consider **extended objects** (quasi-strings) coupled to the 3-form field

# Extended Objects and Quasi-Strings

In 3D fractional quantum Hall systems, the fundamental excitations are extended objects rather than point particles.

## Quasi-strings:

- ▶ One-dimensional extended objects in 3D space
- ▶ Couple to a 3-form background field  $H$  via their worldsheet
- ▶ Action includes a coupling term:  $\int_{\Sigma} B$ , where  $B$  is a 2-form potential with  $dB = H$
- ▶ Analogous to how point particles couple to electromagnetic fields via  $\int A \cdot dx$

This is the natural generalization of electrons in a magnetic field to higher dimensions!

A **bundle gerbe** is a geometric object that generalizes principal bundles to higher gauge theory.

## Key features:

- ▶ Classified by a 3-form curvature  $H \in H^3(M, \mathbb{Z})$
- ▶ The topological invariant is the **Dixmier-Douady class**  $[H] \in H^3(M, \mathbb{Z})$
- ▶ Generalizes the first Chern class for line bundles
- ▶ Locally:  $H = dB$  where  $B$  is a 2-form connection

2D QHE	3D QHE
Point particles	Quasi-strings
2-form $B$	3-form $H$
Line bundle	Bundle gerbe
Chern class	Dixmier-Douady

# Closed quasi-strings

Let us consider the dynamics of a single closed string parametrized by  $\sigma \in [0, 2\pi)$  and embedded in three-dimensional space via the worldsheet coordinates  $X^i(\tau, \sigma)$ ,

$$S = \frac{m}{2} \int d\tau d\sigma \partial_\tau X^i \partial_\tau X_i + \frac{e}{2} \int d\tau d\sigma B_{ij}(X) \partial_\alpha X^i \partial_\beta X^j \epsilon^{\alpha\beta}, \quad (1)$$

$$\Pi_i(\sigma) = \frac{\delta L}{\delta \partial_\tau X^i} = m \partial_\tau X_i + e B_{ij}(X) \partial_\sigma X^j, \quad (2)$$

and take the strong-field limit  $m \rightarrow 0$ .

In this limit, the momentum is no longer an independent variable but becomes a constraint, i.e.

$$\phi_i(\sigma) \equiv \Pi_i(\sigma) - e B_{ij}(X) \partial_\sigma X^j \approx 0. \quad (3)$$

These are *second-class constraints* in the sense of Dirac.

The fundamental equal-time Poisson brackets are

$$\{X^i(\sigma), \Pi_j(\sigma')\} = \delta_j^i \delta(\sigma - \sigma'), \quad \{X^i, X^j\} = \{\Pi_i, \Pi_j\} = 0. \quad (4)$$

# Generalized Guiding-Center Variables

We now compute the Poisson brackets between constraints

$$\{\phi_i(\sigma), \phi_j(\sigma')\} = -e H_{ijk} \partial_\sigma X^k \delta(\sigma - \sigma') - 2e B_{ij}(X(\sigma)) \partial_\sigma \delta(\sigma - \sigma'), \quad (5)$$

and remind that the Dirac brackets between two functionals  $A, B$  are defined as

$$\{A, B\}_D = \{A, B\} - \iint d\sigma d\sigma' \{A, \phi_i(\sigma)\} C_{ij}^{-1}(\sigma, \sigma') \{\phi_j(\sigma'), B\}, \quad (6)$$

where  $C_{ij}(\sigma, \sigma') \equiv \{\phi_i(\sigma), \phi_j(\sigma')\}$ . From Eq. (5), one finds

$$C_{ij}^{-1}(\sigma, \sigma') = \frac{1}{eH} \epsilon_{ijk} \frac{\partial_\sigma X^k}{|\partial_\sigma X|^2} \delta(\sigma - \sigma'), \quad (7)$$

# Noncommutative Geometry in 3D

We can now define the collective (guiding-center) variables, given by

$$x^m \equiv \frac{1}{2\pi} \oint d\sigma X^m(\sigma), \quad (8)$$

$$X^{ij} \equiv \frac{1}{2} \oint d\sigma (X^i \partial_\sigma X^j - X^j \partial_\sigma X^i). \quad (9)$$

The only non-vanishing Dirac bracket is

$$\{x^m, X^{ij}\}_D = - \iint d\sigma d\sigma' \{x^m, \phi_k(\sigma)\} C_{kl}^{-1}(\sigma, \sigma') \{\phi_l(\sigma'), X^{ij}\}, \quad (10)$$

such that

$$\{x^m, X^{ij}\}_D = -\frac{1}{2\pi} \oint d\sigma \frac{1}{eH} \epsilon^{mlp} \frac{\partial_\sigma X^p(\sigma)}{|\partial_\sigma X(\sigma)|^2} \left( \delta^i_\ell \partial_\sigma X^j(\sigma) - \delta^j_\ell \partial_\sigma X^i(\sigma) \right).$$

## 3D Generalization of GMP Algebra

$$\langle 0_{\text{int}} | \{x^m, X^{ij}\}_D | 0_{\text{int}} \rangle = \ell_H^3 \epsilon^{mij}, \quad (11)$$

where we have introduced the characteristic volume scale

$$\ell_H^3 \equiv \frac{1}{eH}. \quad (12)$$

Upon quantization, the projected Dirac brackets are promoted to commutators,  $\{A, B\}_D \rightarrow -i[A, B]$ , yielding the universal generalized guiding-center algebra

$$[x^m, X^{ij}] = i\ell_H^3 \epsilon^{mij}, \quad [x^i, x^j] = [X^{ij}, X^{kl}] = 0, \quad (13)$$

$$\rho(\mathbf{q}) = e^{iq_m x^m}, \quad \Upsilon(\mathbf{Q}) = e^{iQ_{ij} X^{ij}}, \quad (14)$$

$$[\rho(\mathbf{q}), \Upsilon(\mathbf{Q})] = -2i \sin\left(\frac{\ell_H^3}{2} q_m Q_{ij} \epsilon^{mij}\right) e^{iq_m x^m + iQ_{ij} X^{ij}}. \quad (15)$$

This is a **closed algebra**, generalizing the 2D GMP algebra to three dimensions!

# Strong-Field Regime in 3D

In the presence of a strong and constant 3-form background field  $H$ :

- ▶ The internal dynamics of quasi-strings becomes very fast
- ▶ Fast degrees of freedom are effectively frozen (similar to cyclotron motion in 2D)
- ▶ Low-energy kinematics is governed by **generalized guiding-center variables**
- ▶ These include both:
  - ▶ **Vectorial coordinates**  $X^i$  (position of string element)
  - ▶ **Tensorial coordinates**  $X^{ij}$  (orientation/shape of string)
- ▶ This is a key difference from 2D: we need tensor-valued coordinates!

# Connection to Topological Field Theory

The generalized GMP algebra can be related to the canonical quantization of a topological field theory.

We consider a **BF + BB theory** in 3+1 dimensions:

$$S_{BF} = \frac{\kappa_{DD}}{2\pi} \int d^4x \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} \partial_\rho A_\sigma + \frac{\kappa_{DD}}{4\pi} \int d^4x \epsilon^{\mu\nu\rho\sigma} B_{\mu\nu} B_{\rho\sigma},$$

where:

- ▶  $B$  is a 2-form gauge field
- ▶  $A$  is a 1-form gauge field
- ▶  $\kappa_{DD}$  is coupling constants related to the DD invariant

The **BF term** couples different forms, while the **BB term** is a self-interaction of the 2-form field.

This generalizes Chern-Simons theory (relevant for 2D QHE) to higher dimensions.

# Canonical Quantization

$$\Pi_A^m = \frac{\delta S_{BF}}{\delta(\partial_t A_m)} = \frac{\kappa_{DD}}{2\pi} \epsilon^{mij} B_{ij}, \quad (16)$$

such that the canonical equal-time commutation relation is given by

$$[A_m(r), B_{ij}(r')] = \frac{2\pi i}{\kappa_{DD}} \epsilon_{mij} \delta^{(3)}(r - r'), \quad (17)$$

$A_m(r)$  and  $B_{ij}(r)$  are operator-valued distributions, physical observables are defined via smearing with smooth test functions. Localized excitations correspond to wavepackets, and the associated collective operators are obtained by projecting onto these modes. Let  $f(r)$  and  $g(r)$  be normalized wavepackets and define

$$\hat{A}^m = \int d^3r f(r) A^m(r), \quad \hat{B}^{ij} = \int d^3r g(r) B^{ij}(r). \quad (18)$$

By employing Eq. (17), we obtain

$$[\hat{A}^m, \hat{B}^{ij}] = \frac{2\pi i}{\kappa_{DD}} V_{fg} \epsilon^{mij}. \quad (19)$$

By choosing  $f(r)$  and  $g(r)$  such that

$$\int d^3r f(r) g(r) = 1, \quad (20)$$

or, equivalently, by absorbing the overlap factor  $V_{fg}$  into the normalization of the collective operators, we can express the smeared commutator in a simplified canonical form.

A direct comparison with the microscopic three-dimensional guiding-center algebra  $[x^m, X^{ij}] = i\ell_H^3 \epsilon^{mij}$ , allow us to obtain the following identifications

$$x^m \leftrightarrow \ell_H \sqrt{\frac{\kappa_{DD}}{2\pi}} \hat{A}^m, \quad X^{ij} \leftrightarrow \ell_H^2 \sqrt{\frac{\kappa_{DD}}{2\pi}} \hat{B}^{ij}. \quad (21)$$

# Physical Interpretation

What does this all mean physically?

- ▶ The 3D fractional quantum Hall fluid has an **incompressible ground state**
- ▶ Excitations above the ground state are gapped
- ▶ The Dixmier-Douady invariant quantizes the “magnetic flux” through 3D space
- ▶ This topological invariant is robust against smooth deformations
- ▶ The GMP algebra structure ensures the stability of the incompressible phase
- ▶ This provides a framework for understanding fractional statistics of quasi-strings

# Topological Features

The 3D quantum Hall fluids we describe have rich topological features:

## Global properties:

- ▶ Topological invariant: Dixmier-Douady class
- ▶ Quantized response functions
- ▶ Protected edge states (2D surfaces)
- ▶ Bulk-boundary correspondence

## Local properties:

- ▶ Noncommutative geometry
- ▶ Generalized GMP algebra
- ▶ Projected density operators
- ▶ Guiding-center dynamics

The interplay between local and global structures characterizes the quantum Hall phase!

# Geometric Framework Summary

Our geometric framework unifies several key concepts:

1. **Bundle gerbes** provide the geometric arena (replacing principal bundles)
2. **Quasi-strings** are the fundamental excitations (replacing point particles)
3. **Generalized guiding-center variables** describe the low-energy kinematics
4. **Noncommutative geometry** emerges from the strong 3-form field
5. **Generalized GMP algebra** governs the density operator commutators
6. **BF+BB topological field theory** provides the effective field-theoretic description

# Comparison: 2D vs 3D Quantum Hall

Property	2D QHE	3D QHE
Objects	Point particles	Quasi-strings
Background field	2-form $B$	3-form $H$
Geometry	Line bundle	Bundle gerbe
Topological class	Chern class	Dixmier-Douady
Coordinates	Vectors $X^i$	Vectors + Tensors
Algebra	GMP algebra	Generalized GMP
Field theory	Chern-Simons	BF + BB

The 3D case involves a systematic categorification of 2D concepts!

# Key Technical Results

- 1:** In the strong 3-form field limit, the guiding-center variables  $X^i$  and  $X^{ij}$  form a closed noncommutative algebra.
- 2:** The projected density operators satisfy the generalized GMP algebra with structure functions determined by the 3-form curvature  $H$ .
- 3:** The generalized GMP algebra is equivalent to the canonical quantization of BF+BB topological field theory with level  $k$  equal to the Dixmier-Douady invariant.

These results provide a complete characterization of the algebraic structure of 3D quantum Hall fluids.

# Implications for Quantum Hall Physics

Our results have several important implications:

- ▶ Clarifies the mathematical structure of hypothetical 3D quantum Hall states
- ▶ Extends the successful framework of 2D QHE to higher dimensions
- ▶ Provides tools for studying incompressible states of extended objects
- ▶ Connects microscopic dynamics to topological field theory
- ▶ Opens pathway to understanding fractional statistics in 3D
- ▶ May have applications to string theory and higher gauge theory

# Experimental Realizations?

Can we observe 3D quantum Hall physics in real systems?

## Challenges:

- ▶ No natural 3-form electromagnetic fields in nature
- ▶ Need to engineer effective 3-form backgrounds
- ▶ Quasi-string excitations are exotic

## Potential platforms:

- ▶ Synthetic gauge fields in ultracold atoms
- ▶ Engineered metamaterials with effective higher-form gauge theories
- ▶ Topological phases in certain strongly correlated systems
- ▶ String theory and holographic systems

While challenging, the theory provides a clear target for experimental efforts!

# Open Questions and Future Directions

Several interesting questions remain:

- ▶ Can we construct explicit wavefunctions for 3D quantum Hall ground states?
- ▶ What are the properties of edge states on 2D boundaries?
- ▶ How does the algebra generalize to fractional filling factors?
- ▶ What is the structure of quasi-particle (quasi-string) excitations?
- ▶ Can we identify material candidates or synthetic implementations?
- ▶ How does this relate to other higher-dimensional topological phases?

# Connections to Other Fields

This work connects to several active research areas:

- ▶ **Higher category theory:** Bundle gerbes and categorification
- ▶ **Topological field theory:** BF+BB theories and their quantization
- ▶ **String theory:** Worldsheet coupling to higher-form gauge fields
- ▶ **Condensed matter:** Generalization of fractional quantum Hall states
- ▶ **Mathematical physics:** Noncommutative geometry and operator algebras

# Conclusions

- ▶ We developed a geometric framework for 3D quantum Hall fluids of extended objects (quasi-strings)
- ▶ Bundle gerbes and 3-form fields provide the natural geometric setting
- ▶ Generalized guiding-center variables (vectors and tensors) describe the low-energy dynamics
- ▶ These variables obey a noncommutative geometry leading to a generalized GMP algebra
- ▶ The algebra is equivalent to canonical quantization of BF+BB topological field theory
- ▶ The Dixmier-Douady invariant plays the role of topological charge

This work establishes the algebraic and geometric structure of 3D quantum Hall phases, extending the geometric framework of 2D quantum Hall physics to higher dimensions.

**Key message:** The generalized GMP algebra is the natural structure governing incompressible quantum Hall states in three dimensions.

**Looking forward:** Many exciting questions remain about wavefunctions, collective modes and potential physical realizations.

G. Palumbo, [arXiv:2602.15664](https://arxiv.org/abs/2602.15664) (2026)

Thank you for your attention!

Questions?

# Nematic order and massive spin-2 modes

The nematic order in liquid crystals is characterized by a symmetric traceless matrix order parameter  $Q_{ab}(x, t) \approx \eta_{ab} - g_{ab}(x, t)$ .

$$Q_{ab} = Q(2d_a d_b - \delta_{ab}), \quad \mathcal{Q} = Q_{11} + iQ_{12} = Qe^{i\varphi},$$

where  $d = (\cos \varphi/2, \sin \varphi/2)$  is the planar nematic director.  
Landau-de Gennes Lagrangian

$$L_N = \frac{k_1}{8} \epsilon^{bd} Q_{ab} \partial_t Q_{ad} - \frac{k_2}{8} (\partial_c Q_{ab})^2 - \frac{k_3}{2} (Q_{ab})^2 - \dots,$$

$$H_N = \frac{k_2}{8} |\partial_i \mathcal{Q}|^2 + \frac{k_3}{2} |\mathcal{Q}|^2,$$

can be easily diagonalized in momentum space

$$H_N = \int \frac{dp_x dp_y}{(2\pi)^2} E_p \beta_p^\dagger \beta_p, \quad [\beta_p, \beta_{p'}^\dagger] = (2\pi)^2 \delta(p - p'),$$

where the mode spectrum is given by  $E_p = k_3 + (k_2/4)p^2$ . This spectrum identifies a gapped spin-2 mode.

# FQHE on curved space and Hall viscosity

The Hall viscosity represents the response of the FQH system to shear or strain. Given  $S[a, A, \omega]$ , with  $\omega$  a  $SO(2)$  spin connection, we integrate out  $a$  and obtain

$$S[\omega, A] = \int \left[ \left( \frac{\nu \bar{s}^2}{4\pi} - \frac{c}{48\pi} \right) \omega d\omega + \frac{\nu}{4\pi} A dA + \frac{\nu \bar{s}}{2\pi} A d\omega \right],$$

$$\rho = \frac{\nu}{2\pi} B + \frac{\nu \bar{s}}{4\pi} R, \quad J^i = \frac{\nu}{2\pi} \epsilon^{ij} E_j + \frac{\nu \bar{s}}{2\pi} \epsilon^{ij} \mathcal{E}_j,$$

where  $\bar{s} = q/2$  is the average orbital spin (shift).

$$s^\mu = \frac{\nu \bar{s}}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda + \left( \frac{\nu \bar{s}^2}{2\pi} - \frac{c}{24\pi} \right) \epsilon^{\mu\nu\lambda} \partial_\nu \omega_\lambda,$$

where

$$\frac{\nu \bar{s}}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda = \eta_H u^\mu,$$

with  $u^\mu$  the covariant drift velocity and  $\eta_H = \nu \bar{s} B / 4\pi$  the Hall viscosity.