DIFFERENTIAL COHOMOLOGY AND GERBES: AN INTRODUCTION TO HIGHER DIFFERENTIAL GEOMETRY

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Abstract. This is a lecture note for a minicourse given at the IUT Mathematics and Statistics Research Seminar. This version contains the notes for the lecture given on December 11th, 18th, and 25th, 2023.

1. Čech cohomology and characteristic classes

Definition 1.1. Let G be a Lie group. A **principal G-bundle** over a smooth manifold M is a smooth map $\pi\colon P \to M$ and a right G-action on P satisfying

(1) π is G-invariant; i.e., $\pi(p \cdot g) = \pi(p)$ for all $p \in P$ and $g \in G$.
(2) On each fiber G acts freely and transitively from the right.
(3) P is locally trivial via G-equivariant trivialization; i.e., at every $m \in M$ there exists an open subset $U \subset M$ and a diffeomorphism $\varphi\colon \pi^{-1}(m) \to U \times G$ such that $p \mapsto (\pi(p), \phi(p))$ satisfying $p \cdot g \mapsto (\pi(p), \phi(p) \cdot g)$.

The conditions (1) and (2) mean that the G-orbits are fibers of π. This is equivalent to saying $P \times G \to P \times_M P$, $(p, g) \mapsto (p, p \cdot g)$ is a diffeomorphism; i.e., P is a *G-torsor*.

Definition 1.2. A **bundle map** of principal G-bundles from $\pi_1\colon P_1 \to M$ to $\pi_2\colon P_2 \to M$ is a diffeomorphism $f\colon P_1 \to P_2$ that preserves the fiber and G-equivariant; i.e., $f(p \cdot g) = f(p) \cdot g$ and $\pi_2 \circ f = \pi_1$.

Principal G-bundles over M with maps form a groupoid (a category whose morphisms are invertible) and it is denoted by $\mathrm{Prin}_G(M)$. We will also use the notation $\mathrm{Bun}_{\mathbb{C}^n}(M)$ to denote the groupoid of rank n complex vector bundles over M.

Example 1.3. Let $G = GL_n(\mathbb{C})$. Consider $\pi\colon P \to M$ and take an associated fiber bundle $E(P) \to M$ with a fiber $\mathbb{C}^n$ defined by $E(P) := (P \times \mathbb{C}^n)/G$ with a diagonal G-action: $(p, v) \mapsto (pg, g^{-1}v)$. The bundle $E(P)$ is a complex vector bundle over M of rank n. On the other hand, let $E \in \mathrm{Bun}_{\mathbb{C}^n}(M)$. At each $x \in M$, consider the set $\mathrm{Fr}(E)_x$ of all bases of the vector space E_x; equivalently the set of all $\mathbb{C}$-linear maps $p\colon \mathbb{C}^n \to E_x$. Then the smooth map $\pi\colon \mathrm{Fr}(E) \to M$ with

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$\pi^{-1}(x) = P(E)_x$ and a right G-action on $\mathrm{Fr}(E)$ defined by $p \mapsto p \circ g$ is a principal G-bundle over M. It leads to the following equivalence of categories.

$$\mathrm{Prin}_{GL_n(\mathbb{C})}(M) \underset{\mathrm{Fr}}{\overset{E}{\rightleftarrows}} \mathrm{Bun}_{\mathbb{C}^n}(M)$$

For this reason, in what follows, we don't distinguish a $\mathbb{C}^\times$-, S^1-, or a U_1-bundle and a complex line bundle.

Notation 1.4. We shall use the notation $U_{i_1\cdots i_n}$ to denote the n-fold intersection $U_{i_1} \cap \cdots \cap U_{i_n}$.

Definition 1.5. Let G be an abelian group, M a topological space, and $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ an open cover of M. The set $\check{C}^p(\mathcal{U}; G) = \{f_{i_0\cdots i_p} \colon U_{i_0\cdots i_p} \to G\}_{i_0,\cdots,i_p\in\Lambda}$ inherited the operation of the group G is degree p **Čech cochain group**. Together with the map $\delta_p : \check{C}^p(\mathcal{U}; G) \to \check{C}^{p+1}(\mathcal{U}; G)$, $(f)_{i_0\cdots i_p} \mapsto (\delta f)_{i_0\cdots i_{p+1}} := f_{\widehat{i_0} i_1\cdots i_{p+1}} - f_{i_0\widehat{i_1}\cdots i_{p+1}} + \cdots + (-1)^{p+1}_{i_0 i_1\cdots i_p\widehat{i_{p+1}}}$, the sequence of groups $(\check{C}^\bullet(\mathcal{U}; G), \delta_\bullet)$ is the **Čech cochain complex**. (It is easy to verify that $\delta^2 = 0$. Here the hat means an omission). The cohomology of this complex $\check{H}^\bullet(\mathcal{U}; G) := \ker(\delta_\bullet)/\mathrm{Im}(\delta_{\bullet-1})$ is the Čech cohomology of M defined on an open cover $\mathcal{U}$.

Now if the group G in the definition above is not abelian, in general, the coboundary maps δ are not group homomorphisms, neither $\ker\delta$ nor $\mathrm{Im}\delta$ form a group, and if we apply δ to a cocycle, we do not get $\delta^2 = 1$. We shall see below what goes on starting from the lowest degree.

- $p = 0$: There is no problem. $\check{H}^0(\mathcal{U}; G) = \{f \in \check{C}^0(\mathcal{U}; G) : \delta(f)_{ij} = 0\} = \mathrm{Map}(M, G)$. This is a group under a pointwise group multiplication.
- $p = 1$: Neither $\ker\delta_1$ nor $\mathrm{Im}\delta_0$ form a group. On the set $\ker\delta_1$, we may impose an equivalence relation defined by the action of 0-cochains

  $$g_{ij} \sim g'_{ij} \quad \text{if and only if} \quad g'_{ij} = f_i^{-1} g_{ij} f_j.$$

  So we may define $\check{H}^1(\mathcal{U}; G)$ as the pointed set $\ker\delta_1/\sim$ with a distinguished element the constant map $g_{ij} \equiv 1$. Notice where set $\check{H}^1(\mathcal{U}; G)$ is precisely the set of isomorphism classes of principal G-bundles over M defined on the open cover U (see Remark below). For this reason, principal G-bundles are geometric models of a degree 1 nonabelian cohomology of M with coefficients in a group G.
- $p \geq 2$: There is no reasonable way to make sense of $\check{H}^p(\mathcal{U}; G)$.

Remark 1.6. We shall closely look into how the set $\check{H}^1(M; G)$ classifies principal G-bundle over M up to isomorphism. Recall that every principal G-bundle is locally trivial and diffeomorphic to $U \times G$ for some open $U \subset M$. That means if we are given a family of *transition functions* on every double overlap $U_{ij} \in \mathcal{U} = \{U_{ij}\}_{i,j\in\Lambda}$, i.e., $\{g_{ij} \colon U_{ij} \to G : i, j \in \Lambda\}$, we can rebuild the principal G-bundle. Since the transition functions satisfy

$$g_{ij}(x) \cdot g_{jk}(x) \cdot g_{ki}(x) = 1, \text{ for all } x \in U_{ijk} \tag{1.1}$$

The equation (1.1) is called the *cocycle condition* of a principal G-bundle. So if we have a principal bundle P over M, we have a family of transition functions $\{g_{ij}\}_{i,j\in\Lambda}$ satisfying the condition (1.1) and vice versa (under a mild condition). Likewise, if we have a bundle map $f\colon P\to P'$ covering M, we have a family of functions on open sets in the cover $\{f_i\}_{i\in\Lambda}$ satisfying that $g'_{ij}(x)=f_j^{-1}(x)g_{ij}(x)$ for all $x\in U_{ij}$ and vice versa (under the same mild condition). Here the mild condition is that the open cover $\mathcal{U}$ has to be a good cover. A **good cover** (a.k.a. Leray's covering) is an open cover of M if all open sets and their intersections are contractible. Such a covering always exists (see [9, Prop. A.1] and references therein). An open cover $(\mathcal{V},\imath)$ is a **refinement** of $\mathcal{U}$ if $\imath\colon\mathcal{V}\to\mathcal{U}$ such that $V\subseteq\imath(V)$ for all $V\in\mathcal{V}$. A refinement induces a map $\mathrm{res}_{\mathcal{V},\mathcal{U}}\colon\check{H}^1(\mathcal{U};G)\to\check{H}^1(\mathcal{V};G)$, and it satisfies $\mathrm{res}_{\mathcal{W},\mathcal{U}}=\mathrm{res}_{\mathcal{W},\mathcal{V}}\circ\mathrm{res}_{\mathcal{V},\mathcal{U}}$. So we can define the set $\check{H}^1(M;G)$ a direct limit over refinements of open cover; i.e.,

$$\check{H}^1(M;G)=\varinjlim_{\mathcal{U}}\check{H}^1(\mathcal{U};G).$$

If the cover $\mathcal{U}$ is good, the restriction map $\check{H}^1(\mathcal{U};G)\xrightarrow{\cong}\check{H}^1(M;G)$ is an isomorphism. Therefore, we conclude that

$$\begin{aligned}\pi_0\mathrm{Prin}_G(\mathcal{U})&\to\check{H}^1(\mathcal{U};G)\\ [P]&\mapsto(g_{ij}).\end{aligned}\tag{1.2}$$

If we remove the abelian assumption of groups, the long exact sequence induced by a short exact sequence of groups cannot go any further than the degree $p=1$.

Proposition 1.7. Let

$$1\longrightarrow K\xrightarrow{i}\widetilde{G}\xrightarrow{j}G\longrightarrow 1\tag{1.3}$$

be a short exact sequence of groups. We have the following long exact sequence of groups and pointed sets

$$\begin{tikzcd} 1 \arrow[r] & \check{H}^0(\mathcal{U};K) \arrow[r, "i_*"] & \check{H}^0(\mathcal{U};\widetilde{G}) \arrow[r, "j_*"] & \check{H}^0(\mathcal{U};G) \arrow[dll, out=-10, in=170] \\ & \check{H}^1(\mathcal{U};K) \arrow[r, "i_*"] & \check{H}^1(\mathcal{U};\widetilde{G}) \arrow[r, "j_*"] & \check{H}^1(\mathcal{U};G) \end{tikzcd}$$

However, in a special case that the second term in the sequence is an abelian group whose image is in the center of the third, we can extend the long exact sequence just one term further. We have the following propositions.

Proposition 1.8. If the group K in the short exact sequence (1.3) is abelian and $i(K)$ belongs to the center of $\widetilde{G}$, then the long exact sequence in the Proposition 1.7 extends to $\check{H}^2(\mathcal{U};K)$:

$$\begin{tikzcd} 1 \arrow[r] & \check{H}^0(\mathcal{U};K) \arrow[r, "i_*"] & \check{H}^0(\mathcal{U};\widetilde{G}) \arrow[r, "j_*"] & \check{H}^0(\mathcal{U};G) \arrow[dll, out=-10, in=170] \\ & \check{H}^1(\mathcal{U};K) \arrow[r, "i_*"] & \check{H}^1(\mathcal{U};\widetilde{G}) \arrow[r, "j_*"] & \check{H}^1(\mathcal{U};G) \arrow[dll, dashed, out=-10, in=170, "\alpha"] \\ & \check{H}^2(\mathcal{U};K) \end{tikzcd}$$

Proposition 1.9 (Dixmier–Douady)**.** If the sheaf $\underline{\widetilde{G}}_M$ is soft, then

$$\alpha : \check{H}^1(\mathcal{U}; G) \to \check{H}^2(\mathcal{U}; K)$$

is a bijection.

Proof. See Dixmier–Douady [7, Lemme 22, p.278] or Brylinski [3, Prop. 4.1.8, p.162] $\square$

In the above, $\underline{G}_M$ is a sheaf such that $\underline{G}_M(U)$ is a group of smooth functions $f\colon U \to G$ for each open $U \subseteq M$. A sheaf $\underline{G}_M$ is **soft** if $\underline{G}_M(M) \to \underline{G}_M(C)$ is onto for every closed $C \subset M$. Here, we can think of $\underline{G}_M(C) = \lim_U \underline{G}_M(U)$ (since M is paracompact) where the direct limit is taken over all open neighborhoods of C.

Example 1.10. (1) Consider a short exact sequence

$$1 \longrightarrow SO_n \xrightarrow{\;i\;} O_n \xrightarrow{\det} \mathbb{Z}_2 \longrightarrow 1.$$

The induced map $w_1\colon \check{H}^1(M; O_n) \to \check{H}^1(M; \mathbb{Z}_2)$ is a correspondence $[P] \in \pi_0 \mathrm{Prin}_{O_n}(M) \mapsto w_1([P])$ which is the *first Stifel–Whitney class.* So $w_1([P]) = 0$ if and only if P comes from an SO_n-bundle; i.e., P is orientable. Equivalently the obstruction for transition maps of a Euclidean vector bundle lift to SO_n is the first Stifel–Whitney class.

(2) Consider a short exact sequence

$$1 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathrm{Spin}_n \longrightarrow SO_n \longrightarrow 1.$$

The induced map $w_2\colon \check{H}^1(M; SO_n) \to \check{H}^2(M; \mathbb{Z}_2)$ is a correspondence $[P] \in \pi_0 \mathrm{Prin}_{SO_n}(M) \mapsto w_2([P])$ which is the *second Stifel–Whitney class.* So $w_2([P]) = 0$ if and only if P comes from a Spin_n-bundle. Equivalently the obstruction for transition maps of an oriented Euclidean vector bundle lift to Spin_n is the second Stifel–Whitney class. Here one can think of Spin_n as a double cover of SO_n, which is also a universal cover. For a construction of Spin_n in terms of Clifford algebras, see [14, Section 1.2].

Remark 1.11. The Whitehead tower of O_n is of particular interest. The **Whitehead tower** of a space X is a factorization of the point inclusion $\mathrm{pt} \to X$

$$\mathrm{pt} \simeq \lim_{n\to\infty} X_n \longrightarrow \cdots \longrightarrow X_2 \longrightarrow X_1 \longrightarrow X_0 \simeq X$$

such that each X_n is $(n-1)$-connected (i.e., all homotopy groups π_k vanish for $k \leq n-1$) and each map $X_n \to X_{n-1}$ is a fibration which is an isomorphism on all π_k for $k \geq n$. For the space O_n, we have a Whitehead tower as follows:

$$\mathrm{pt} \longrightarrow \cdots \longrightarrow \mathrm{FiveBrane}_n \longrightarrow \mathrm{String}_n \longrightarrow \mathrm{Spin}_n \longrightarrow SO_n \longrightarrow O_n$$

Here String_n is a 6-connected cover of Spin_n

$$1 \longrightarrow K(\mathbb{Z}, 2) \longrightarrow \mathrm{String}_n \longrightarrow \mathrm{Spin}_n \longrightarrow 1.$$

and $\mathrm{FiveBrane}_n$ is a 7-connected cover of String_n

$$1 \longrightarrow K(\mathbb{Z},6) \longrightarrow \mathrm{FiveBrane}_n \longrightarrow \mathrm{String}_n \longrightarrow 1.$$

It is known that the obstruction to lift a Spin_n-bundle to a String_n-bundle is the *first fractional Pontryagin class* $\frac{1}{2}p_1$ and a String_n-bundle to a $\mathrm{FiveBrane}_n$-bundle the *second fractional Pontryagin class* $\frac{1}{6}p_2$ and so on. See [8] for more details.

Example 1.12. (3) Consider a short exact sequence

$$1 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{R} \longrightarrow S^1 \longrightarrow 1.$$

Note that $\underline{\mathbb{R}}_M$ is a soft sheaf (recall Tietze extension theorem). The induced map $c_1 \colon \check{H}^1(M;S^1) \xrightarrow{\cong} \check{H}^2(M;\mathbb{Z})$ is a correspondence $[L] \in \pi_0\mathrm{Prin}_{S^1}(M) \mapsto c_1([L])$ which is the *first Chern class.* Note that if the group G is abelian and $\mathcal{G}$ is a sheaf of locally constant functions in G, $\check{H}^p(M;\mathcal{G})$ and $H^p(M;G)$ the degree p singular cohomology with coefficients in G is the same. Since the group $\mathbb{Z}$ is discrete, we can identify $\check{H}^p(M;\mathbb{Z})$ and $H^p(M;\mathbb{Z})$ for any degree p.

Proposition 1.13. (Dixmier–Douady) Let $\mathcal{H}$ be a complex separable Hilbert space. The sheaf $\underline{U(\mathcal{H})}_M$ is soft.

Proof. See Dixmier–Douady [7, Lemme 4, p.252] or Brylinski [3, Cor. 4.1.6, p.162] □

Example 1.14. (4) Consider a short exact sequence

$$1 \longrightarrow U_1 \longrightarrow U(\mathcal{H}) \longrightarrow PU(\mathcal{H}) \longrightarrow 1.$$

Since $U(\mathcal{H})$ is a soft sheaf, the induced map $DD \colon \check{H}^1(M;PU(\mathcal{H})) \xrightarrow{\cong} \check{H}^2(M;S^1) \xrightarrow{\cong} H^3(M;\mathbb{Z})$ is a correspondence $[P] \in \pi_0\mathrm{Prin}_{PU(\mathcal{H})}(M) \mapsto DD([P])$ which is the *Dixmier–Douady class* of a gerbe.

Definition 1.15. A **characteristic class** of a principal G-bundle P over M is an assignment

$$\begin{aligned} c \colon \pi_0\mathrm{Prin}_G(M) &\to H^\bullet(M;A) \\ [P] &\mapsto c(P) \end{aligned}$$

that is natural; i.e. $f^*c(P) = c(\overline{f}^*P)$ for

$$\begin{tikzcd} P' \arrow[r, "\overline{f}"] \arrow[d, "\pi'"'] \arrow[dr, phantom, "\circlearrowright"] & P \arrow[d, "\pi"] \\ M' \arrow[r, "f"] & M \end{tikzcd}$$

Here A is an abelian group.

Since $\mathrm{Prin}_G(-) : \mathbf{Man}^{\mathrm{op}} \to \mathbf{Sets}$ is representable by BG, by Yoneda Lemma (See MacLane [15]) we have the following proposition

Proposition 1.16. An assignment

$$\{\text{Characteristic class of principal } G\text{-bundles}\} \longrightarrow H^\bullet(BG;A)$$

is one-to-one and onto.

Remark 1.17. There is an alternative way to define characteristic classes using a "geometric datum ," i.e. a connection ∇ on $P \in \mathrm{Prin}_G(M)$. This is the *Chern–Weil theory.* For example, given a line bundle with connection (L,∇), the first Chern class of ∇ is defined by a Chern–Weil form $\frac{i}{2\pi}\mathrm{curv}(\nabla)$. Here $\mathrm{curv}(\nabla)$ is the curvature 2-form of the connection ∇. Chern–Weil theorem shows that the cohomology class of a Chern–Weil form does not depend on the choice of connection. So $\left[\frac{i}{2\pi}\mathrm{curv}(\nabla)\right] \in H^2(M;\mathbb{R})$ is a topological invariant of a line bundle. A priori the class $\left[\frac{i}{2\pi}\mathrm{curv}(\nabla)\right]$ is a class in $H^2(M;\mathbb{C})$, but it can be shown that it is actually a class in $H^2(M;\mathbb{R})$. The realification of the first Chern class Example 1.12 above is equal to the first Chern class $\left[\frac{i}{2\pi}\mathrm{curv}(\nabla)\right]$ from the Chern–Weil theory. See Morita [17, Chapter 5] to learn more about Chern–Weil theory of characteristic classes.

We have seen that, up to isomorphism, complex line bundles are classified by $H^2(M;\mathbb{Z})$ via the first Chern class (Example 1.12) and principal $PU(\mathcal{H})$-bundles are by $H^3(M;\mathbb{Z})$ via the Dixmier–Douady class (Example 1.14). We can ask the following question: What classifies (higher) line bundles with connection? For example, if we consider a groupoid $\mathrm{Bun}_{\mathbb{C}}^{\nabla}(M)$ whose objects are line bundles with connection (L,∇) and morphisms are bundle isomorphism preserving the connection, what classifies the isomorphism classes of $\mathrm{Bun}_{\mathbb{C}}^{\nabla}(M)$? This question leads us to "differential cohomology." Up to isomorphism, line bundles with connection are classified by the degree 2 differential cohomology $\widehat{H}^2(M)$, gerbes with connection by $\widehat{H}^3(M)$, 2-gerbes with connection by $\widehat{H}^4(M)$, and so on.

2. Cheeger–Simons differential characters

In this section, we introduce a differential extension of singular cohomology theory $H^*(-;\mathbb{Z})$ on the site of smooth manifolds. Among various known models, we shall introduce the model by Cheeger and Simons [6] which is one of the historical landmarks. Interested readers are referred to the homotopy theoretic model by Hopkins and Singer [13], a spark complex model by Harvey, Lawson, and Zweck [11], and a novel construction using ∞-sheaves of spectra by Bunke, Nikolaus, and Völkl [4].

Notation 2.1. We shall define some notations which will be used throughout this section. Let M be a smooth manifold and R a commutative ring with unity.

- $C^k(M;R)$: smooth singular k-cochains in M with coefficients in R.
- $Z^k(M;R)$: smooth singular k-cocycles in M with coefficients in R.
- $\Omega^k(M)$: differential k-forms on M.
- $\int : \Omega^k(M) \to C^k(M;\mathbb{R})$ is a $\mathbb{R}$-linear map $\omega \mapsto \int \omega$, where $\int\omega \colon C_k(M;\mathbb{R}) \to \mathbb{R}$ is a pairing of a singular k-chain and a differential k-form.

- $\Omega^k_{\text{cl}}(M)_{\mathbb{Z}}$: closed differential k-forms with integral periods; i.e., $\omega \in \Omega^k_{\text{cl}}(M)_{\mathbb{Z}}$ if and only if $d\omega = 0$ and $\int \omega\big|_{Z_k(M)} \in \mathbb{Z}$.
- $\sim$ is the natural map $\mathbb{R} \to \mathbb{R}/\mathbb{Z}$.

A nonvanishing differential form does not take its values in a proper subring $\Lambda \subset \mathbb{R}$. Hence we have the following:

Proposition 2.2. The map

$$\begin{aligned} \int : \Omega^k(M) &\to C^k(M;\mathbb{R}/\mathbb{Z}) \\ \omega &\mapsto \widetilde{\int \omega} \end{aligned}$$

is one-to-one.

Definition 2.3 (Cheeger and Simons [6])**.** Let M be a smooth manifold. The group $\widehat{H}^k(M)$ of **differential characters** of degree k consists of pairs (χ, ω) where $\chi \in \mathrm{Hom}_{\mathbb{Z}}(Z_{k-1}(M), \mathbb{R}/\mathbb{Z})$ and $\omega \in \Omega^k(M)$ satisfying that

$$\chi \circ \partial D = \int_D \omega \mod \mathbb{Z}, \text{ for all } D \in C_k(M;\mathbb{Z}),$$

where the group structure is the componentwise addition.

Remark 2.4. The degree of the $\widehat{H}^k(M)$ in the above definition is different from the one that appears in Cheeger and Simons [6] which defines the same group as degree $k+1$. A consequence of adopting their convention would be a mismatch of degree in the group of differential characters and real cohomology, so the forgetful map (see below for a definition) would be $I\colon \widehat{H}^k(M) \to H^{k+1}(M;\mathbb{R})$. We stick to our convention for the sake of consistency with literature in recent years.

The main goal of this section is to understand the following diagram known as the *differential cohomology hexagon diagram.*

Proposition 2.5. The group of differential characters $\widehat{H}^k(M)$ satisfies the following diagram; i.e. all squares and triangles are commutative and the diagonal, upper, and lower sequences of arrows are exact sequences.

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \\
& H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \arrow[rr, "-B"] \arrow[dr, hook, "e"] & \arrow[d, phantom, "\circlearrowleft"] & H^k(M;\mathbb{Z}) \arrow[ur] \arrow[dr, "r"] & \\
H^{k-1}(M;\mathbb{R}) \arrow[ur, "\sim"] \arrow[dr, "\text{rep}"'] \arrow[rr, phantom, "\circlearrowleft"] & & \widehat{H}^k(M) \arrow[ur, "I"'] \arrow[dr, "R"] \arrow[d, phantom, "\circlearrowleft"] \arrow[rr, phantom, "\circlearrowleft"] & & H^k(M;\mathbb{R}) \\
& \frac{\Omega^{k-1}(M)}{\Omega^{k-1}_{\text{cl}}(M)_{\mathbb{Z}}} \arrow[rr, "d"'] \arrow[ur, "a"] & \, & \Omega^k_{\text{cl}}(M)_{\mathbb{Z}} \arrow[ur, "\int \circ \text{dR}"'] \arrow[dr] & \\
0 \arrow[ur] & & & & 0
\end{tikzcd}$$

Proof. We shall divide the proof into several parts and enumerate them.

(1) I and R maps: We begin with some algebra facts.

A1. A subgroup of a free abelian group is free.

A2. An abelian group G is **divisible** if, for any $x \in G$ and any $n \in \mathbb{Z}^+$, there exists $y \in G$ such that $x = ny$.

A3. An abelian group G is divisible if and only if the group G is an injective object in the category of abelian groups; If $f\colon A \to G$ and $A \subset B$, there exists a map $\widetilde{f}\colon B \to G$ that satisfies $\widetilde{f}|_A = f$.

Take $(\chi, \omega) \in \widehat{H}^k(M)$ and consider $\chi\colon Z_{k-1}(M) \to \mathbb{R}/\mathbb{Z}$. Since $Z_{k-1}(M)$ is a subgroup of a free abelian group $C_{k-1}(M;\mathbb{Z})$, it is a free (**A1**), and hence projective. We have the following commutative diagram:

$$\begin{tikzcd}
& \mathbb{R} \arrow[d] \\
Z_{k-1}(M) \arrow[ur, dotted, "\overline{\chi}"] \arrow[r, "\chi"] \arrow[ur, phantom, "\circlearrowleft"'] & \mathbb{R}/\mathbb{Z} \arrow[d] \\
& 0
\end{tikzcd}$$

Now since $\mathbb{R}$ is divisible (**A2**), it is injective (**A3**). Hence $\overline{\chi} : Z_{k-1}(M) \to \mathbb{R}$ lifts to the map T satisfying the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[d] & \\
Z_{k-1}(M) \arrow[r, "\overline{\chi}"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{R} \\
C_{k-1}(M) \arrow[ur, dotted, "T"'] &
\end{tikzcd}$$

So $\widetilde{T|_{Z_{k-1}(M)}} = \chi$. It follows that $\widetilde{\delta T} = \delta\widetilde{T} = \widetilde{T} \circ \partial = \int \omega \mod \mathbb{Z}$ Here the first equality is simply $\sim \circ(T \circ \partial_k) = (\sim \circ T) \circ \partial_k$. Thus, there exists $c \in C^k(M;\mathbb{Z})$ such that

$$\delta T = \int \omega - c. \tag{2.1}$$

Note that $0 = \delta^2 = \int d\omega - \delta c$, so $\int d\omega = \delta c$. Since a real differential form cannot take its value in a proper subring of $\mathbb{R}$, this means $d\omega \equiv 0 = \delta c$. It is readily seen that ω has an integral period. We define the maps I and R as follows:

$$I : \widehat{H}^k(M) \to H^k(M;\mathbb{R}) \qquad R : \widehat{H}^k(M) \to \Omega^k_{\mathrm{cl}}(M)_{\mathbb{Z}}$$
$$(\chi, \omega) \mapsto [c] \qquad (\chi, \omega) \mapsto \omega$$

Let's verify that these maps are well-defined. Since the choice of lifts is not unique, we have to verify that the above definition does not depend on the choices we made. Suppose T' is another lift satisfying $\delta T' = \int \omega' - c'$. Then $\widetilde{T' - T|_{Z_{k-1}(M)}} = 0$, so $T' = T + \delta s + d$ for some $d \in C^{k-1}(M;\mathbb{Z})$ and $s \in C^{k-2}(M;\mathbb{R})$. So $\delta T' = \delta T + 0 + \delta d$ if and only if $\int \omega' - c' = \int \omega - c + \delta d$ if and only if $\int(\omega' - \omega) = c' - c + \delta d$. Again, since the real differential form cannot take its value in a proper subring of $\mathbb{R}$, this means $\omega \equiv \omega'$ and $[c'] = [c]$.

We show that R is surjective. Let $r\colon H^k(M;\mathbb{Z}) \to H^k(M;\mathbb{R})$ be the realification map (which is from the universal coefficient theorem for cohomology; see [12, Section 3.1]). Notice that, given $\omega \in \Omega^k_{\mathrm{cl}}(M)_{\mathbb{Z}}$, there exists $u \in H^k(M;\mathbb{Z})$ such that $r(u) = [\int \omega]$. Since ω has integral periods, $\delta \int \omega = \int \omega \circ \partial \in \mathbb{Z}$ is an integral cochain and since ω is closed, $\delta \int \omega = \int d\omega = 0$ (Stokes' theorem). Now let $u = [c]$ for some $c \in C^k(M;\mathbb{Z})$. Then $\int \omega - c = \delta\lambda$ for some $\lambda \in C^{k+1}(M;\mathbb{R})$. Define $\chi := \widetilde{\lambda|_{Z_{k-1}(M)}}$. So R is surjective.

The map I is also surjective. Given any $[c] \in H^k(M;\mathbb{Z})$, $\delta c = 0$ as real cochains. By the de Rham theorem, there exists a $\omega \in \Omega^k_{\mathrm{cl}}(M)$ such that $\int \omega - c = \delta\mu$ for some $\mu \in C^{k-1}(M;\mathbb{R})$. Define $\chi := \widetilde{\mu|_{Z_{k-1}(M)}}$. So the map I is surjective.

(2) The e map: We define the e map as follows:

$$e : H^{k-1}(M;\mathbb{R}/\mathbb{Z}) \to \widehat{H}^k(M)$$
$$[x] \mapsto (x|_{Z_{k-1}(M)}, 0)$$

The map e is well-defined. If we take a different representative $x + \delta y$, the restriction of δy to $Z_{k-1}(M)$ vanishes. The map e is one-to-one: Let $\Lambda \subset \mathbb{R}$ a proper subring. From the universal coefficient theorem we have $H^k(X;\mathbb{R}/\Lambda) \cong \mathrm{Hom}_{\mathbb{Z}}(H_k(X), \mathbb{R}/\Lambda)$, since $\mathrm{Ext}(H_{n-1}(X), \mathbb{R}/\Lambda) = 0$, from $n(\mathbb{R}/\Lambda) = (n\mathbb{R})/\Lambda = \mathbb{R}/\Lambda$, for any $n \in \mathbb{Z}$. Since $B_k \to Z_k \to H_k \to 0$ is exact if and only if $B_k^* \leftarrow Z_k^* \leftarrow H_k^* \leftarrow 0$ is exact, $\mathrm{Hom}_{\mathbb{Z}}(H_k(X), \mathbb{R}/\Lambda) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(Z_k(X), \mathbb{R}/\Lambda)$ is an injection.

(3) The a map: We define the a map as follows:

$$a : \frac{\Omega^{k-1}(M)}{\Omega^{k-1}_{\mathrm{cl}}(M)_{\mathbb{Z}}} \to \widehat{H}^k(M)$$
$$[\alpha] \mapsto (\widetilde{\int \alpha|_{Z_{k-1}(M)}}, d\alpha)$$

It is obvious that the map a is well-defined and the subgroup $\Omega^{k-1}_{\mathrm{cl}}(M)_{\mathbb{Z}}$ is the kernel of the map $\Omega^{k-1}(M) \to \widehat{H}^k(M)$, $\alpha \mapsto (\int \alpha|_{Z_{k-1}(M)}, d\alpha)$.

(4) Diagonals are exact: First Im$e = \ker R$. The inclusion $\subseteq$ is clear. To see $\supseteq$, take (χ, ω) such that $\omega = 0$. Then $\chi = T|_{Z_{k-1}(M)}$ satisfying that $\delta T = c$, so T is a $\mathbb{R}/\mathbb{Z}$-valued cocycle, representing a class in $H^{k-1}(M; \mathbb{R}/\mathbb{Z})$, and $T|_{Z_{k-1}(M;\mathbb{R}/\mathbb{Z})} = \chi$.

Now Im$a = \ker I$. Again the inclusion $\subseteq$ is clear. To see $\supseteq$, take (χ, ω) such that $\chi = T|_{Z_{k-1}(M)}$ satisfying $\delta T = \int \omega - c$. By assumption, $c = \delta d$ for some $d \in C^{k-1}(M; \mathbb{Z})$. From $\int \omega = \delta(T + d)$, we have $\omega = d\alpha$ for some $\alpha \in \Omega^{k-1}(M)$, and $\int \alpha = T + d + \delta f$ for some $f \in C^{k-2}(M; \mathbb{R})$. Then δf vanishes when we restrict it to $Z_{k-1}(M)$ and d also vanishes modulo $\mathbb{Z}$. Thus, the preimage of I is $(\int \alpha|_{Z_{k-1}(M)}, d\alpha)$.

(5) Squares commute: The map rep is defined as follows.

$$\begin{aligned} \mathrm{rep} : H^{k-1}(M; \mathbb{R}) &\to \frac{\Omega^{k-1}(M)}{\Omega^{k-1}_{\mathrm{cl}}(M)_{\mathbb{Z}}} \\ [\beta] &\mapsto \beta + \Omega^{k-1}_{\mathrm{cl}}(M)_{\mathbb{Z}} \end{aligned}$$

which does not depend on the choice of representatives since all exact forms are closed forms with integral periods. From this, it is clear that the square on the left is commutative. Notice that the Equation (2.1) shows the commutativity of the square on the right.

(6) Triangles commute: Two triangle diagrams below commute.

$$\begin{tikzcd}
H^{k-1}(M; \mathbb{R}/\mathbb{Z}) \arrow[rr, "-B"] \arrow[dr, "e"'] & {} \arrow[d, phantom, "\circlearrowright"] & H^k(M; \mathbb{Z}) \\
& \widehat{H}^k(M) \arrow[ur, "I"'] \arrow[dr, "R"] \arrow[d, phantom, "\circlearrowright"] & \\
\frac{\Omega^{k-1}(M)}{\Omega^{k-1}_{\mathrm{cl}}(M)_{\mathbb{Z}}} \arrow[ur, "a"] \arrow[rr, "d"] & {} & \Omega^k_{\mathrm{cl}}(M)_{\mathbb{Z}}
\end{tikzcd}$$

The commutativity of the lower triangle is obvious. Take a $\mathbb{R}/\mathbb{Z}$-valued cocycle x and consider $(x|_{Z_{k-1}(M)}, 0) \in \widehat{H}^k(M)$. There exists $T \in C^{k-1}(M; \mathbb{R})$ such that $x|_{Z_{k-1}(M)} = T|_{Z_{k-1}(M)}$ satisfying $\delta T = -c$ for some $c \in C^k(M; \mathbb{Z})$, so $I(x|_{Z_{k-1}(M)}, 0) = c = -\delta T = -B([x])$. $\square$

(7) Upper and lower sequences are exact: It is readily seen that the following are exact sequences.

$$H^{k-1}(M; \mathbb{R}) \xrightarrow{\sim} H^{k-1}(M; \mathbb{R}/\mathbb{Z}) \xrightarrow{-B} H^k(M; \mathbb{Z}) \xrightarrow{r} H^k(M; \mathbb{R})$$

$$H^{k-1}(M; \mathbb{R}) \xrightarrow{\mathrm{rep}} \frac{\Omega^{k-1}(M)}{\Omega^{k-1}_{\mathrm{cl}}(M)_{\mathbb{Z}}} \xrightarrow{d} \Omega^k_{\mathrm{cl}}(M)_{\mathbb{Z}} \xrightarrow{\int \circ \mathrm{dR}} H^k(M; \mathbb{R})$$

Immediately from the definition, $\widehat{H}^0(M) = 0$ and $\widehat{H}^1(M) = C^\infty(M, \mathbb{R}/\mathbb{Z})$. Also note that $\widehat{H}^k(M) = 0$ if $k > \dim(M)$. When $k = 2$, we have the following proposition:

Proposition 2.6. The following assignment is a one-to-one correspondence:

$$\pi_0 \mathrm{Prin}_{S^1,\nabla}(M) \longrightarrow \widehat{H}^2(M)$$
$$[(P,\theta)] \mapsto (\chi, \frac{1}{2\pi} d\theta)$$

where, for any loop γ in M, χ is defined by the holonomy of the loop γ; i.e.,

$$\chi(\gamma) := \mathrm{Hol}(\gamma)$$

and for any $D \in C_2(M;\mathbb{Z})$ bounding γ,

$$\chi(\partial D) = \frac{1}{2\pi}\int_D d\theta \mod \mathbb{Z}$$

which is extended to all $Z_1(M)$ by setting $\chi(x) = \chi(\gamma) + \frac{1}{2\pi}\int d\theta(y)$ for any $x = \gamma + \partial y$.

Given $d\theta \in \Omega^2_{\mathrm{cl}}(M)_{\mathbb{Z}}$, as we have seen in the surjectivity of R, there exists $[c] \in H^2(M;\mathbb{Z})$ such that $[\int d\theta] = r([c])$. The class $[c]$ is the characteristic class that classifies P; i.e., the first Chern class.

The above proposition addresses the question at the end of Section 1 at least for degree 2. What is a higher analogue of Proposition 2.6? How can one define a map? In the following section, we shall see that the isomorphism classes of gerbes with connection are in one-to-one correspondence with $\widehat{H}^3(M)$, and To establish the correspondence one has to construct χ; i.e., a holonomy of gerbe.

Remark 2.7. Although we do not go into details, the differential cohomology group $\widehat{H}^\bullet(M)$ has a ring structure (See Cheeger and Simons [6, p.56, Theorem 1.11]).

In differential cohomology, the hexagon diagram plays an important role. One uses the hexagon diagram in Proposition 2.5 to compute differential cohomology groups. Furthermore, it is known that the hexagon diagram uniquely characterizes the differential cohomology. Phrasing slightly differently, If there are two $\widehat{H}^k(M)$ fitting into the middle of the hexagon diagram, then they are naturally isomorphic. This is a theorem of Simons and Sullivan [20] which is generalized by Bunke and Schick [5] and Stimpson [22] to the uniqueness of the differential extension of all exotic cohomology theories under some mild assumptions.

3. S^1-banded gerbes with connection

Throughout this section, M is a smooth manifold. In Section 1 we have seen that elements of $H^2(M;\mathbb{Z})$ are represented by complex line bundles, and in Section 2 differential cohomology classes in $\widehat{H}^2(M;\mathbb{Z})$ are represented by complex line bundles with connection. What are corresponding geometric objects representing $H^n(M;\mathbb{Z})$ and $\widehat{H}^n(M;\mathbb{Z})$? The answer is $(n-2)$-gerbes with connection.

Remark 3.1. For a generalized cohomology theory $E^\bullet$ and its differential extension $\widehat{E}^\bullet$, investigating geometric cocycles representing (differential) cohomology classes is a very interesting research topic that is not fully understood yet to this date. For example, elements of even complex K-group

$K^0(M)$ are represented by vector bundles over M and odd complex K-group $K^{-1}(M)$ by Ω-vector bundles, but other interesting generalized cohomology theories such as elliptic cohomology and topological modular forms it is largely unknown which geometric objects in the space M represent cohomology classes. Also, note that this question is closely related to the Stolz–Teichner program [23] wherein they have conjectured a hypothetical equivalence between the totality of supersymmetric field theories of degree n over M modulo concordance and the group $E^n(M)$. There are differential and twisted refinements of this conjecture as well (See for example Stoffel [24, 25] and references therein).

Let us observe how a gerbe arises. Consider the short exact sequence of groups

$$1 \longrightarrow U_1 \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1.$$

In Example 1.14 above, we considered the map $DD\colon \check{H}^1(\mathcal{U};G) \to \check{H}^2(\mathcal{U};U_1)$ when $G = PU(\mathcal{H})$. In the proposition below we shall closely look at how this map is defined.

Proposition 3.2. A principal G-bundle P over M lifts to a principal $\widetilde{G}$-bundle if and only if the cocycle representing $DD(P)$ is trivializable.

Proof. We look at how the map $DD\colon \check{H}^1(\mathcal{U};G) \to \check{H}^2(\mathcal{U};U_1)$ is defined. Choose a good cover $\mathcal{U}$ on M. Over each U_{ij}, consider the transition map $g_{ij}\colon U_{ij} \to G$ of P. Since $\mathcal{U}$ is a good cover, U_{ij} is contractible. Hence there is a homotopy between the map g_{ij} and a constant map, which lifts by the homotopy lifting property, since the map $\widetilde{G} \xrightarrow{r} G$ is a fibration. Let $\widetilde{g} : U_{ij} \to \widetilde{G}$ be a lift of g_{ij}. The cocycle condition $g_{ij}g_{jk}g_{ki} = \lambda_{ijk} \cdot \mathbf{1}_{\widetilde{G}}$, for some $\lambda_{ijk} \in \check{C}^2(\mathcal{U};U_1)$. It is an easy exercise to verify that $\lambda = \{\lambda_{ijk}\}_\Lambda$ is a degree 2 Čech cocycle on $\mathcal{U}$ and the class $[\lambda]$ does not depend on the choice of lifting $\widetilde{g}_{ij}$. So the map DD is a correspondence $[P] \mapsto [\lambda]$ and using the isomorphism (due to softness of $\underline{\mathbb{R}}$), it is valued in $H^3(M;\mathbb{Z})$. $\square$

There are several models representing gerbes. The degree 2 U_1-valued Čech cocycle λ considered above as an obstruction to lift a principal G-bundle to $\widetilde{G}$-bundle is one model, and there are other ways to represent it as a stack. We refer the reader to Giraud [10], Brylinski [3], Behrend and Xu [1], and Moerdijk [16]. In this section, we will specialize on a model called *bundle gerbe* of Murray [18] which is presumably the most widely used model in literature.

Let $\pi\colon Y \to M$ be a surjective submersion. The p-fold *fiber product* of $\pi\colon Y \to M$ is

$$Y^{[p]} := \{(y_1, \cdots, y_n) \in Y^p : \pi(y_1) = \cdots = \pi(y_p) \text{ for } y_i \in Y\}.$$

The projection of $Y^{[p]}$ onto the $(i_1, \cdots, i_k)^{\text{th}}$ copy of $Y^{[k]}$ is $\pi_{i_1\cdots i_k}\colon Y^{[p]} \to Y^{[k]}$. For example, let $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ be an open cover of M. Then consider

$$Y_{\mathcal{U}} := \{(x, i) \in M \times \Lambda : x \in U_i\} \subset M \times \Lambda.$$

The map $\pi\colon Y_{\mathcal{U}} \to M$ is a surjective submersion which is an open cover.

Remark 3.3. Recall that a fiber product $X \times_M Y$ of $X \xrightarrow{\phi} M \xleftarrow{\pi} Y$ is in general not a smooth manifold. If ϕ, π are submersions, then the fiber product is a smooth manifold. So a surjective submersion is not only a generalization of an open cover, it lets us stay within the category of smooth manifolds.

Definition 3.4 (Murray [18])**.** A **bundle gerbe** is a triple $\mathcal{L} = (L, \pi, \mu)$ where

(1) $\pi \colon Y \to M$ is a surjective submersion.
(2) $L \in \mathrm{Prin}_{S^1}(Y^{[2]})$.
(3) $\mu \colon \pi_{12}^* L \otimes \pi_{23}^* L \to \pi_{13}^* L$ is an S^1-bundle isomorphism.
(4) μ is associative over $Y^{[4]}$: i.e.,

$$\begin{tikzcd}
\pi_{12}^*L \otimes \pi_{23}^*L \otimes \pi_{34}^*L \arrow[r, "\pi_{123}^*\mu \otimes \mathbf{1}"] \arrow[d, "\mathbf{1}\otimes\pi_{234}^*\mu"'] \arrow[dr, phantom, "\circlearrowleft"] & \pi_{13}^*L \otimes \pi_{34}^*L \arrow[d, "\pi_{134}^*\mu"] \\
\pi_{12}^*L \otimes \pi_{24}^*L \arrow[r, "\pi_{124}^*\mu"] & \pi_{14}^*L
\end{tikzcd}$$

Let us construct the *Dixmier–Douady class*, the characteristic class of a bundle gerbe. Let $\mathcal{L} = (L, \pi, \mu)$ be a bundle gerbe over M. Take a good open cover (cf. Remark 1.6) $\mathcal{U} = \{U_i\}_{i \in \Lambda}$ of M. Then local sections on each open set $\sigma_i \colon U_i \to Y$ and on each double intersection $(\sigma_i, \sigma_j) \colon U_{ij} \to Y^{[2]}$ can be defined. We consider the pullback of $L \to Y^{[2]}$ along $(\sigma_i, \sigma_j) \colon U_{ij} \to Y^{[2]}$.

$$\begin{tikzcd}
(\sigma_i, \sigma_j)^* L \arrow[r] \arrow[d] & L \arrow[d] \\
U_{ij} \arrow[u, dotted, bend left, "s_{ij}"] \arrow[ur, dotted, "s_{ij}"] \arrow[r, "(\sigma_i, \sigma_j)"] & Y^{[2]}
\end{tikzcd}$$

Take a section $s_{ij} \colon U_{ij} \to (\sigma_i, \sigma_j)^* L$ equivalently a map $\mu \colon s_{ij} \colon U_{ij} \to L$. Over triple intersections we have

$$s_{ij}(x) \otimes s_{jk}(x) \mapsto \lambda_{ijk}(x) s_{ik}(x), \quad x \in U_{ijk}.$$

Here the associativity of μ implies that λ_{ijk} is a degree 2 Čech cocycle in M defined on $\mathcal{U}$.

Definition 3.5. Let $\mathcal{L} = (L, \pi, \mu)$ be a bundle gerbe over M. The **Dixmier–Douady class** $DD(\mathcal{L})$ is the cohomology class $[\lambda] \in \check{H}^2(\mathcal{U}; U_1)$.

It is not difficult to verify that $DD(\mathcal{L})$ does not depend on the choices we made.

Let's recall connections and curvatures on a principal G-bundle. A **connection** θ on a principal G-bundle $\pi \colon P \to M$ is a differential 1-form on P valued in $\mathfrak{g}$ satisfying that

(1) $\theta(X^*) = X$ where $X \in \mathfrak{g}$ and $X_x^* := \frac{d}{dt}\big|_{t=0} x \cdot e^{tX}$ for each $x \in P$.
(2) $R_g^* \theta = \mathrm{Ad}_{g^{-1}} \circ \theta$

The **curvature** of $(P.\theta)$ is a $\mathfrak{g}$-valued 2-form $\mathrm{Curv}(\theta) := d\theta + \frac{1}{2}[\theta, \theta]$ on P.

Now we define connection and curving of a bundle gerbe.

Definition 3.6. A **connection** on $\mathcal{L} = (L, \pi, \mu)$ is a connection ∇ on L compatible with μ, i.e., $\pi_{12}^*(L, \nabla) \otimes \pi_{23}^*(L, \nabla) \xrightarrow{\mu} \pi_{13}^*(L, \nabla)$ is a connection preserving isomorphism.

So a connection on $\mathcal{L}$ has to be a $\mathbb{R}$-valued differential 1-form on $Y^{[2]}$.

Definition 3.7. A **curving** B of a bundle gerbe with connection (L, π, μ, ∇) is a differential 2-form on Y satisfying $\mathrm{Curv}(\nabla) = \pi_2 B - \pi_1 B$.

A connection and a curving on a bundle gerbe are called the *connective structure*. By a *bundle gerbe with connection* we mean a bundle gerbe with connective structure.

To work with curvatures and curvings we need the following proposition.

Proposition 3.8. Let $\pi\colon Y \to M$ be a surjective submersion. The following sequence is a long exact sequence

$$0 \longrightarrow \Omega^k(M) \xrightarrow{\pi^*} \Omega^k(Y) \xrightarrow{\delta} \Omega^k(Y^{[2]}) \xrightarrow{\delta} \cdots$$

where $\delta = \sum_{k=1}^p (-1)^{k-1} \pi^*_{i_1 \cdots \hat{i_k} \cdots i_p}$

Proof. See Murray [18, Section 8]. Compare Bott and Tu [2, Prop. 8.5]. □

Note that $0 = d\mathrm{Curv}(\nabla) = d\delta B = \delta dB$ so there exists a unique $H \in \Omega^3(M; \mathbb{R})$ such that $\pi^* H = dB$. The differential form H is closed, so it represents a degree 3 real cohomology class in M. The Proposition 3.8 shows that the cohomology class of M does not depend on the choices involved.

Definition 3.9. Let $\widehat{\mathcal{L}} = (L, \pi, \mu, \nabla, B)$ be a bundle gerbe with connection. The **3-curvature** (also known as the 3-form flux or the Dixmier–Douady form) of $\widehat{\mathcal{L}}$ is a real differential 3-form on M satisfying that $\pi^* H = dB$.

Remark 3.10. In literature, H is defined as a real-valued differential form in some places and $i\mathbb{R}$-valued differential form in some other places. Recall that, in Definitions 3.6 and 3.7, connection forms and curving forms are $\mathbb{R}$-valued, as the Lie algebra of the Lie group S^1 is $\mathbb{R}$. If we consider the Lie group U_1, its Lie algebra is $i\mathbb{R}$ (here $i = \sqrt{-1}$) and we consider differential forms valued in $i\mathbb{R}$.

It turns out the 3-curvature of a gerbe represents the corresponding de Rham cohomology class of the Dixmier–Douady class above.

Proposition 3.11. Let $\widehat{\mathcal{L}} = (L, \pi, \mu, \nabla, B)$ be a bundle gerbe with connection. The de Rham cohomology class of its 3-curvature form H is equal to the realification of its Dixmier–Douady class $DD(\mathcal{L})$; i.e., $r(DD(\mathcal{L})) = [H]_{\mathrm{dR}}$, where r is the realification map $r\colon H^3(M; \mathbb{Z}) \to H^3(M; \mathbb{R})$ considered in the proof of Proposition 2.5.

Proof. See Murray [18, Section 11]. □

Example 3.12. Consider the short exact sequence of groups

$$1 \longrightarrow U_1 \longrightarrow \widetilde{G} \longrightarrow G \longrightarrow 1.$$

Let $\pi\colon Y \to M$ be a principal G-bundle. There is a natural map $g\colon Y^{[2]} \to \widetilde{G}$ coming from the transitivity of the right G-action. Pull back the fibration $\widetilde{G} \to G$ to obtain a U_1-bundle L over $Y^{[2]}$. Note that the fiber of $(y_1, y_2) \in Y^{[2]}$ is the coset $U_1 g(y_1, y_2)$ in $\widetilde{G}$. So the multiplication map $\mu\colon \pi_{12}^* L \otimes \pi_{23}^* L \to \pi_{13}^* L$ is defined by the coset multiplication $U_1 g(y_1, y_2) \cdot U_1 g(y_2, y_3) = U_1 g(y_1, y_3)$ and is readily seen to be associative. So $\mathcal{L} = (L, \pi, \mu)$ is a bundle gerbe over M called the *lifting bundle gerbe* of the principal G-bundle $\pi\colon Y \to M$. The Dixmier–Douady class $DD(\mathcal{L})$ is precisely the obstruction for the lifting the G-valued cocycle to $\widetilde{G}$ considered in Proposition 3.2.

Definition 3.13. Let $\mathcal{U} = \{U_i\}_{i\in\Lambda}$ be a good cover of M. The **Deligne complex** is the double complex $\check{C}^\bullet(\mathcal{U}; \Omega^\bullet)$ endowed with total differential $D = d + (-1)^q \delta$ on $\check{C}^p(\mathcal{U}; \Omega^q)$ where the Čech differential δ and the exterior derivative d; i.e.,

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle -\delta} & & \uparrow{\scriptstyle \delta} & & \\
\check{C}^2(\mathcal{U}; \underline{U_1}) & \xrightarrow{d\log} & \check{C}^2(\mathcal{U}; \Omega^1) & \xrightarrow{d} & \check{C}^2(\mathcal{U}; \Omega^2) & \xrightarrow{d} & \cdots \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle -\delta} & & \uparrow{\scriptstyle \delta} & & \\
\check{C}^1(\mathcal{U}; \underline{U_1}) & \xrightarrow{d\log} & \check{C}^1(\mathcal{U}; \Omega^1) & \xrightarrow{d} & \check{C}^1(\mathcal{U}; \Omega^2) & \xrightarrow{d} & \cdots \\
\uparrow{\scriptstyle \delta} & & \uparrow{\scriptstyle -\delta} & & \uparrow{\scriptstyle \delta} & & \\
\check{C}^0(\mathcal{U}; \underline{U_1}) & \xrightarrow{d\log} & \check{C}^0(\mathcal{U}; \Omega^1) & \xrightarrow{d} & \check{C}^0(\mathcal{U}; \Omega^2) & \xrightarrow{d} & \cdots
\end{array}$$

The cohomology of the total complex with the total degree n is the degree n **Deligne cohomology group** $\check{H}^n_D(\mathcal{U})$ of M defined on $\mathcal{U}$.

Proposition 3.14. A bundle gerbe with connection $\widehat{\mathcal{L}} = (L, \pi, \mu, \nabla, B)$ determines a total degree 2 cocycle in the Deligne complex.

Proof. Recall notations in the paragraph between Definitions 3.4 and 3.5. In it, we have obtained a Čech 2-cocycle $\{\lambda_{ijk}\}$. Let us take $A_{ij} = \sigma_{ij}^* \nabla$ and $B_i = \sigma_i^* B$. It is readily seen that the triple $\widehat{\lambda} := (\lambda_{ijk}, A_{ij}, B_i)$ satisfies $D\widehat{\lambda} = 0$ and its cohomology class $[\widehat{\lambda}]_D \in H^2_D(M)$ is independent of the choice of local sections σ_i. □

It is natural to ask if the isomorphic bundle gerbes with connection have Deligne-cohomologous cocycles in the Deligne complex. The answer is yes, but there is a subtlety in isomorphisms of bundle gerbes. One might guess that it is a U_1-bundle isomorphism compatible with the bundle gerbe structure μ, but this is not a notion we want. We will then get non-isomorphic bundle gerbes having the same Dixmier–Douady class. Stevenson [21] and Murray and Stevenson [19] have found that the correct notion of bundle gerbe isomorphism is the “stable isomorphism.” We will introduce a version that Waldorf [26] came up with.

Definition 3.15 (Waldorf [26])**.** For $\widehat{\mathcal{L}}_i = (L_i, \pi_i, \mu_i, \nabla_i, B_i)$, an **isomorphism** $\widehat{\mathcal{L}}_1 \xrightarrow{\widehat{\mathcal{K}}} \widehat{\mathcal{L}}_2$ is a quadruple $(\zeta, K, \nabla_K, \alpha)$ consists of the following.

1. A surjective submersion $\zeta \colon Z \to Y_1 \times_M Y_2$
2. $(K, \nabla_K) \in \mathrm{Prin}^{\nabla}_{S^1}(Z)$ such that $\mathrm{Curv}(\nabla_K) = \zeta^*(B_2 - B_1) \in \Omega^2(Z)$.
3. An isomorphism $\alpha \colon (L_1, \nabla_1) \otimes \zeta_2^*(K, \nabla_K) \to \zeta_1^*(K, \nabla_K) \otimes (L_2, \nabla_2)$ of S^1-bundles with connection over $Z \times_M Z$ compatible with μ_1 and μ_2.

Remark 3.16. When $\zeta = \mathbf{1}$, we recover the *stable isomorphism* of Murray and Stevenson [19].

Proposition 3.17 (Waldorf [26])**.** There is an equivalence of groupoids between the 1-groupoid of 1-morphisms of $\mathrm{Grb}(M)$ and the 1-groupoid of stable isomorphisms of $\mathrm{Grb}_{\mathrm{st}}(M)$.

Definition 3.18 (Waldorf [26])**.** A transformation $\widehat{\mathcal{J}} \colon \widehat{\mathcal{K}}_1 \Rightarrow \widehat{\mathcal{K}}_2$, which is an isomorphism between isomorphisms from $\widehat{\mathcal{L}}_1$ to $\widehat{\mathcal{L}}_2$ (i.e., a **2-morphism**) is an equivalence class of triples (W, ω, β_W) consists of the following:

1. A surjective submersion $\omega : W \to Z_1 \times_{Y_1 \times_M Y_2} Z_2$
2. An isomorphism $\beta_W : (K_1, \nabla_1) \to (K_2, \nabla_2)$ over W compatible with α_1 and α_2.

$$\begin{tikzcd}
L_1 \otimes \omega_2^* K_1 \arrow[r, "\alpha_1"] \arrow[d, "\mathbf{1} \otimes \omega_2^* \beta_W"] & \omega_1^* K_1 \otimes L_2 \arrow[d, "\omega_1^* \beta_W \otimes \mathbf{1}"] \\
L_1 \otimes \omega_2^* K_2 \arrow[r, "\alpha_2"] & \omega_1^* K_2 \otimes L_2
\end{tikzcd}$$

$(W, \omega, \beta_W) \sim (W', \omega', \beta_{W'})$ if there is a smooth manifold X with surjective submersions to W and W' such that the following diagram commutes

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & W \arrow[d, "\omega"] \\
W' \arrow[r, "\omega'"] & Z_1 \times_{Y_1 \times_M Y_2} Z_2
\end{tikzcd}$$

and β_W and $\beta_{W'}$ coincides if pulled back to X.

Proposition 3.19. The category $\mathrm{Grb}_{\nabla}(M)$ consisting of bundle gerbes with connection $\widehat{\mathcal{L}}$ as objects, morphisms as defined in Definition 3.15, and 2-morphisms as defined in Definition 3.18 is a 2-groupoid (i.e. a category whose morphisms are invertible and whose morphism between morphisms are invertible).

Now we go back to our discussion on Deligne cohomology. Since the cover $\mathcal{U}$ of M is good, we can define the Deligne cohomology group $H^k_D(M)$ as a direct limit over refinements which is isomorphic to the one defined on $\mathcal{U}$. We have the following result.

Proposition 3.20. Let $\widehat{\mathcal{L}}_i \in \mathrm{Grb}_\nabla(M)$. If $\widehat{\mathcal{L}}_1$ and $\widehat{\mathcal{L}}_2$ are stably isomorphic if and only if they define the same Deligne cohomology class in $\check{H}^2_D(M)$.

Proof. See Murray and Stevenson [19, Theorem 4.1]. □

Proposition 3.21. Let M be a smooth manifold. The following correspondence is an isomorphism:

$$H^k_D(M) \to \widehat{H}^{k+1}(M)$$

Proof. See Brylinski [3, Prop. 1.5.7] and references therein. □

Corollary 3.22. Let M be a smooth manifold. The following are isomorphic as groups

$$\pi_0 \mathrm{Grb}_\nabla(M) \cong \widehat{H}^3(M)$$

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