The geometric cobordism hypothesis

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These slides: https://dmitripavlov.org/nyuad.pdf

Origins of functorial field theory

- 1948 (Feynman): path integral formulation of quantum mechanics
- 1949 (Feynman–Kac): the Feynman–Kac formula
- Later: path integral used in QFT, no longer rigorous
- 1980s (Witten): properties of path integrals for (conformal) field theory
- 1980s (Segal): mathematical formulation of conformal field theory
Further developments

- late 1980s (Atiyah, Kontsevich, ...): topological theories: easier to construct and study, but less relevant for physics
- 1992 (Freed, Lawrence): extended field theories (correspond to locality in physics)
- 1995 (Baez–Dolan): the topological cobordism and tangle hypotheses
- 2002 (Stolz–Teichner): modern formulation of nontopological field theories (including supersymmetry); the Stolz–Teichner program on 2|1-EFTs and TMF
- 2004 (Costello): the \((\infty, 2)\)-category of topological 2-dimensional bordisms
- 2006 (Hopkins–Lurie); 2015 (Calaque–Scheimbauer): the \((\infty, d)\)-category of topological bordisms
Previous results on the topological cobordism hypothesis

- 2008 (Lurie): outline of a proof of the topological cobordism hypothesis
- 2017 (Ayala–Francis): a different approach, conditional on a conjecture
- 2004 (Costello), 2009 (Schommer-Pries): the 2-dimensional topological cobordism hypothesis
- 2006 (Galatius–Madsen–Tillmann–Weiss); 2011 (Bökstedt–Madsen); 2017 (Schommer-Pries): the invertible case
Examples of 2-dimensional nonextended nontopological field theories:

- 2007 (Pickrell): Riemannian 2-dimensional field theory
- 2018 (Runkel–Szegedy): volume-dependent 2-dimensional field theory

Classifications of holonomy maps, transport functors, and 1-dimensional nontopological field theories:

- 1990 (Barrett), 1994 (Caetano–Picken), 2007 (Schreiber–Waldorf): parallel transport for bundles
- 2015 (Berwick-Evans–P.), 2020 (Ludewig–Stoffel): 1-dimensional field theories
Features of the geometric bordism category

- **Locality**: \( k \)-bordisms with corners of all codimensions (up to \( d \)) with compositions in \( d \) directions
  \( \implies \) symmetric monoidal \( d \)-category of bordisms

- **Isotopy**: chain complexes to encode BV-BRST
  \( \implies \) must encode (higher) diffeomorphisms between bordisms
  \( \implies \) symmetric monoidal \((\infty, d)\)-categories

- **Geometric** (nontopological) structures on bordisms:
  Riemannian/Lorentzian metrics,
  complex/conformal/symplectic/contact structures,
  principal \( G \)-bundles with connection and isos,
  higher gauge fields (Kalb–Ramond, Ramond–Ramond)
  \( \implies \) an \((\infty, 1)\)-sheaf of geometric structures

- **Smoothness**: values of field theories depend smoothly on bordisms
  \( \implies \) \((\infty, 1)\)-sheaf of \((\infty, d)\)-categories of bordisms
How to compose bordisms
Definition

Given \( d \geq 0 \), the site \( \text{FEmb}_d \) has

- Objects: submersions \( T \rightarrow U \) with \( d \)-dimensional fibers, where \( U \cong \mathbb{R}^n \) is a cartesian manifold;

- Morphisms: commutative squares with \( T \rightarrow T' \) a fiberwise open embedding over a smooth map \( U \rightarrow U' \);

- Covering families: open covers on total spaces \( T \).
Geometric structures

Definition

Given $d \geq 0$, the site $\text{FEmb}_d$ has
- Objects: submersions with $d$-dimensional fibers;
- Morphisms: fiberwise open embeddings;
- Covering families: open covers on total spaces $T$.

Definition

Given $d \geq 0$, a $d$-dimensional geometric structure is a simplicial presheaf $S: \text{FEmb}_d^{\text{op}} \to \text{sSet}$.

Example:
- $T \to U \mapsto$ the set of fiberwise Riemannian metrics on $T \to U$;
- $(T \to T', U \to U') \mapsto$ the restriction map from $T'$ to $T$. 
Examples of geometric structures

- **fiberwise** Riemannian, Lorentzian, pseudo-Riemannian metrics; positive/negative sectional/Ricci curvature;
- **fiberwise** conformal, complex, symplectic, contact, Kähler structures;
- **fiberwise** foliations, possibly with transversal metrics;
- smooth map to a target manifold $M$ (traditional $\sigma$-model);
- smooth map to an orbifold or $\infty$-sheaf on manifolds;
- **fiberwise** etale map or an open embedding into a target manifold $N$;
- **fiberwise topological** structures: orientation, framing, etc.
- **fiberwise** differential $n$-forms (possibly closed).
Examples of geometric structures: gauge transformations

Definition

Send a $d$-manifold $M$ to (the nerve of) the groupoid $B_{\nabla} G(M)$:

- Objects: principal $G$-bundles on $T$ with a fiberwise connection on $T \rightarrow U$ (gauge fields);
- Morphisms: connection-preserving isomorphisms (gauge transformations).
Examples of geometric structures: (higher) gauge transformations

- Principal $G$-bundles with connection on $M$ (gauge fields, e.g., the electromagnetic field);
- Bundle gerbe with connection on $M$ (B-field, Kalb–Ramond field).
- Bundle 2-gerbe with connection on $M$ (supergravity C-field).
- Bundle $(d-1)$-gerbes with connection on $M$ (Deligne cohomology, Cheeger–Simons characters, ordinary differential cohomology, circle $d$-bundles).
- Geometric tangential structures: geometric Spin$^c$-structure, String (Waldorf), Fivebrane (Sati–Schreiber–Stasheff), Ninebrane (Sati). (Vanishing of anomaly.)
- Differential K-theory (Ramond–Ramond field). Requires $\infty$-groupoids.
The main theorem

Ingredients:

- A dimension \( d \geq 0 \).
- A smooth symmetric monoidal \((\infty, d)\)-category \( \mathcal{V} \) of values.
- A \( d \)-dimensional geometric structure \( \mathcal{S} : \text{FEmb}_d^{\text{op}} \to \text{sSet} \).

Constructions:

- The smooth symmetric monoidal \((\infty, d)\)-category of bordisms \( \mathcal{Bord}_d^{\mathcal{S}} \) with geometric structure \( \mathcal{S} \).
- A \( d \)-dimensional functorial field theory valued in \( \mathcal{V} \) with geometric structure \( \mathcal{S} \) is a smooth symmetric monoidal \((\infty, d)\)-functor \( \mathcal{Bord}_d^{\mathcal{S}} \to \mathcal{V} \).
- The simplicial set of \( d \)-dimensional functorial field theories valued in \( \mathcal{V} \) with geometric structure \( \mathcal{S} \) is the derived mapping simplicial set

\[
\text{FFT}_{d,\mathcal{V}}(\mathcal{S}) = \mathbb{R}\text{Map}(\mathcal{Bord}_d^{\mathcal{S}}, \mathcal{V}).
\]

Can be refined to a derived internal hom.
Conjectures:

- Freed, Lawrence (1992): $\text{FFT}_{d, \mathcal{V}}$ is an $\infty$-sheaf.
- Baez–Dolan (1995), Hopkins–Lurie (2008): if $\mathcal{V}$ is fully dualizable,

$$\text{FFT}_{d, \mathcal{V}}(\mathcal{S}) \simeq \mathbb{R} \text{Map}(\mathcal{S}, \mathcal{V}^\times).$$
The main theorem

Conjectures:
- Freed, Lawrence (1992): \( \text{FFT}_d,\mathcal{V} \) is an \( \infty \)-sheaf.
- Baez–Dolan (1995), Hopkins–Lurie (2008): if \( \mathcal{V} \) is fully dualizable, \( \text{FFT}_d,\mathcal{V}(S) \cong \mathbb{R} \text{Map}(S, \mathcal{V}^\times) \).

Theorem (Grady–P., The geometric cobordism hypothesis)

**Part I:** \( \text{Bord}_d \) is a left adjoint functor:

\[
\mathbb{R} \text{Map}(\text{Bord}_d^S, \mathcal{V}) \cong \mathbb{R} \text{Map}(S, \mathcal{V}_d^\times),
\]

where \( \mathcal{V}_d^\times = \text{FFT}_d,\mathcal{V} \), i.e., \( \mathcal{V}_d^\times (T \to U) = \text{FFT}_d,\mathcal{V}(T \to U) \).

**Part II:** The evaluation-at-points map

\[
\mathcal{V}_d^\times (\mathbb{R}^d \times U \to U) = \text{FFT}_d,\mathcal{V}(\mathbb{R}^d \times U \to U) \to \mathcal{V}^\times(U)
\]

is a weak equivalence of simplicial sets.
Applications (current and future)

- Consequence of the GCH: smooth invertible FFTs are classified by the smooth Madsen–Tillmann spectrum. (Previous work: Galatius–Madsen–Tillmann–Weiss, Bökstedt–Madsen, Schommer-Pries.)
- The Stolz–Teichner conjecture: concordance classes of extended FFTs have a classifying space. (Proof: Locality + the smooth Oka principle (Berwick-Evans–Boavida de Brito–P.).)
- Construction of power operations on the level of FFTs (extending Barthel–Berwick-Evans–Stapleton).
- (Grady) The Freed–Hopkins conjecture (Conjecture 8.37 in Reflection positivity and invertible topological phases)
- Construction of prequantum FFTs from geometric/topological data.
- Quantization of functorial field theories.
Recipe: computing the space of FFTs in practice

Step 1  Compute $\mathcal{V}_d^\times$ (once for every $\mathcal{V}$).

Step 1a  Guess a candidate $W$ for $\mathcal{V}_d^\times$. (Standardized guesses exist.)

Step 1b  Guess a map $W \to \mathcal{V}_d^\times$. (Typically straightforward.)

Step 1c  For every $U \in \text{Cart}$, prove that

$$W(\mathbb{R}^d \times U \to U) \to \mathcal{V}_d^\times(\mathbb{R}^d \times U \to U) \to \mathcal{V}^\times(U)$$

is a weak equivalence. (Easy.)

Step 2  Compute $\mathbb{R}\text{Map}(S, \mathcal{V}_d^\times)$ as $\mathbb{R}\text{Map}(S, W)$. (Like differential cohomology.)
Example: the prequantum Chern–Simons theory

Input data:
- $G$: a Lie group;
- $S = B_{\nabla}G$ (fiberwise principal $G$-bundles with connection);
- $V = B^3U(1)$ (a single $k$-morphism for $k < 3$; 3-morphisms are $U(1)$ as a Lie group).

Output data: a fully extended 3-dimensional $G$-gauged FFT:

\[ \text{Bord}_3^{B_{\nabla}G} \to B^3U(1). \]

- Closed 3-manifold $M \mapsto$ the Chern–Simons action of $M$;
- Closed 2-manifold $B \mapsto$ the prequantum line bundle of $B$;
- Closed 1-manifold $C \mapsto$ the Wess–Zumino–Witten gerbe ($B$-field) of $C$ (Carey–Johnson–Murray–Stevenson–Wang);
- Point $\mapsto$ the Chern–Simons 2-gerbe (Waldorf).
Example: the prequantum Chern–Simons theory

Step 1  Compute $V_3^\times = (B^3U(1))^3$.  
Step 1a  $W$ is the fiberwise Deligne complex of $T \to U$:  

\[ W(T \to U) = \Omega^3 \leftarrow \Omega^2 \leftarrow \Omega^1 \leftarrow C^\infty(T, U(1)). \]

Step 1b  $W \to V_3^\times$: a fiberwise 3-form $\omega$ on $T \to U$  
\[ \mapsto \text{framed FFT: 3-bordism } B \mapsto \exp(\int_B \omega). \]

Step 1c  The composition  

\[ W(T \to U) \to V_3^\times(T \to U) \to V^\times(U) = B^3C^\infty_{\text{fconst}}(T, U(1)) \]

is a weak equivalence by the Poincaré lemma.
Example: the prequantum Chern–Simons theory

Step 1 Result: $\mathcal{V}_3^\times = (B^3 U(1))^\times = B^3 C_{f\text{const}}(-, U(1))$. 

Step 2 Construct a point in

$$R \text{Map}(B_{\nabla} G, W) = R \text{Map}(\Omega^1(-, g) \sslash C^\infty(-, G), B^3 C_{f\text{const}}(-, U(1))).$$


Step 2’ Even better: can compute the whole space $R \text{Map}(B_{\nabla} G, W)$. 

$X$: the prequantum geometric structure
$Y$: the quantum geometric structure (e.g., a point)

\[
\begin{align*}
\text{FFT}_{d,V}(X) & \xrightarrow{\text{GCH}} \mathbb{R} \text{Map}(X, \mathcal{V}_d^\times) \\
\downarrow f & \quad \quad \quad \downarrow Q \\
\text{FFT}_{d,V}(Y) & \xrightarrow{\text{GCH}} \mathbb{R} \text{Map}(Y, \mathcal{V}_d^\times)
\end{align*}
\]

$d = 1$: recover the Spin$^c$ geometric quantization when $X$ is a smooth manifold, $Y = \text{Riem}_{1|1}$, $\mathcal{V} = \text{Fredholm complexes}$. 