DC Structure on Alexandrov Space

(preliminary version)

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1. Generalities

1.1 SC and DC functions on Euclidean spaces. A function $f : \mathbb{R}^n \to \mathbb{R}$ is called semiconcave (SC) if it is locally representable as the difference of a concave function and a smooth function. Clearly SC is closed w.r.t. addition, multiplication by positive numbers and taking minimum. Notice also that if $F : \mathbb{R}^\ell \to \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^n$ have SC components and the components of G are increasing in each argument, then $G \circ F$ has SC components.

We say that $f \in \mathrm{DC}$ if it is locally representable as the difference of two SC functions or, equivalently, as the difference of two concave functions. It is easy to see that DC is an algebra, and $f/g \in DC$ whenever $f, g \in \mathrm{DC}$ and g does not vanish anywhere. Morever, if maps $F : \mathbb{R}^\ell \to \mathbb{R}^m$ and $G : \mathbb{R}^m \to \mathbb{R}^n$ have DC components then so does $G \circ F$. (Indeed, we can (locally) decompose $F = F_2 \circ F_1$, where $F_1 : \mathbb{R}^\ell \to \mathbb{R}^{2m}$ has concave components, and $F_2(x_1, \ldots, x_m, y_1, \ldots, y_m) = (x_1 - y_1, \ldots, x_m - y_m)$. Now $G \circ F_2$ has DC components, and we can write (locally) $G \circ F_2 = G_1 - G_2$, where G_1 and G_2 have concave components, increasing in each argument. It follows that $G_1 \circ F_1$ and $G_2 \circ F_1$ have concave components.) A homeomorphism $F : \mathbb{R}^n \to \mathbb{R}^n$ can be called a DC isomorphism when f is DC if and only if $f \circ F$ is DC. It follows from the previous remark that F is a DC isomorphism iff F and F^{-1} have DC components.

1.2 SC and DC functions on Alexandrov spaces. Let M^n be a (compact when necessary) Alexandrov space with empty boundary. A Lipschitz function $f : M \to \mathbb{R}$ is SC if for each $x \in M$ there is a neighborhood $U_x \ni x$ and $\lambda_x \in \mathbb{R}$ such that for every geodesic γ in U_x the function $f \circ \gamma(t) + \lambda_x t^2$ is concave in t. The basic example of an SC function is dist_x^2 for some $x \in M$. Any continuous function f can be uniformly approximated by SC functions $f_j(x) = \inf_{y \in M}(f(y) + j \cdot |xy|^2)$. If f is SC in some domain $U \subset M$ and $K \subset U$ is compact then there is a function $\bar{f}$ which coincides with f on K and is SC on the whole M. (Indeed, take $\bar{f} = \min(f, a\mathrm{dist}_{\partial U}^2 - b)$ on U and $\bar{f} = a\mathrm{dist}_{\partial U}^2 - b$ on $M \backslash U$ for appropriate large a, b.)

We say that f is DC on M if it is locally representable as the difference of two SC functions. Our previous remark shows that the word "locally" can be dropped. On the other hand, every DC function can be locally represented as the difference of two concave ones — this follows from the existence of very concave functions in small neighborhoods of every point, see [P].

2. Background from [P] and an extension for general SC functions

2.1 Scalar product. Every SC function has a differential at each point; the differential $d_x f$ is a concave homogeneous function on the tangent cone C_x, and its restriction $f'_{(x)}$ to the space of directions $\Sigma_x \subset C_x$ is spherically concave. In particular, if $f = dist_y$, $y \neq x$, then $f'(x) = -\cos \mathrm{dist}_{y'} =: \chi_{y'}$, where y' is the set of directions of all shortest geodesics from x to y.

In general, if Σ is a compact Alexandrov space with curvature ≥ 1, with empty boundary, then a lipschitz function $f : \Sigma \to \mathbb{R}$ is called spherically concave if $f(y)|xz| \geq f(x)|yz| + f(z)|xy|$ whenever y lies on a shortest geodesic between x and z. It is also convenient to consider 0-dimensional Σ, consisting of two points x, y at the distance π, and say that $f : \Sigma \to \mathbb{R}$ is spherically concave if $f(x) + f(y) \leq 0$.

Using induction on dimension, we can define a scalar product of two spherically concave functions f, g by

$$\langle f, g \rangle = \sup_{x \in \Sigma} (f(x)g(x) + \langle f'_{(x)}, g'_{(x)} \rangle) \ ,$$

where the term with derivatives is dropped when dim $\Sigma = 0$. Obviously, $\langle f, g \rangle = \langle g, f \rangle$, $\langle \lambda f, g \rangle = \lambda \langle f, g \rangle$ if $\lambda \geq 0$, $\langle f, f \rangle \geq 0$ for any f. It is easy to check by induction that

$$\langle \min(f, g), h \rangle \leq \max(\langle f, h \rangle, \langle g, h \rangle)$$
$$\langle f + g, h \rangle \leq \langle f, h \rangle + \langle g, h \rangle$$
$$\langle f, g \rangle^2 \leq \langle f, f \rangle \langle g, g \rangle$$
$$\langle \chi_A, h \rangle = -\min_{a \in A} h(a) \quad \text{for any compact } A \subset \Sigma; \quad \text{in particular, } \langle \chi_A, \chi_B \rangle = \cos |A, B|$$
$$\langle -f(a)\chi_a, g \rangle = f(a)g(a) \leq \langle f, g \rangle \quad \text{if } f \text{ attains its minimal value at } a \ .$$

Using quasigeodesics it is also easy to show by induction that $\|f\| := \langle f, f \rangle^{\frac{1}{2}} = -\min_{x \in \Sigma} f(x)$.

Remark. In [P] we used a different scalar product which did not work for general spherical concave functions.

2.2 Consecutive approximations. The scalar product, described in 2.1, can be used to extend all the results of [P] by replacing the admissible functions and functions of class DER with general SC and spherically concave functions respectively. In particular, we have

Lemma. (cf. [P, Lemma 1]) *Let Σ^{n-1} be a compact Alexandrov space with curvature ≥ 1, with empty boundary, and let $f_i : \Sigma \to \mathbb{R}$, $i = 0, 1, \dots, k$ be spherically concave functions. Assume that $\varepsilon = \min_{0 \leq i \neq j \leq k}(-\langle f_i, f_j \rangle) > 0$. Then*

(1) *$k \leq n$, and*

(2) *for each i, $1 \leq i \leq k$, there exists $\xi_i \in \Sigma$ such that $f_j(\xi_i) = 0$ for $j \neq 0, i$, $f_0(\xi_i) \geq \varepsilon$, $f_i(\xi_i) \leq -\varepsilon$.*

Now let $f = (f_1, \dots f_k)$ be a map with SC components. A point $x \in M$ is called regular for F if there exist $\varepsilon_x > 0$ and $U_x \ni x$ such that for each $y \in U_x$ we have $\langle f'_{i(y)}, f'_{j(y)} \rangle < -\varepsilon_p$ for all $1 \leq i \neq j \leq k$, and there exists $\xi^+ \in \Sigma_y$ with $f'_{i(y)}(\xi^+) > \varepsilon_p$ for all $1 \leq i \leq k$. (In fact, the second condition needs to be checked only at $y = x$.) If F is regular at x, then the statement (2) of the lemma allows us to use consecutive approximations to prove that F is open near x. (Indeed, we can increase all the coordinates of $F(y)$ by moving in the direction ξ^+, and we can decrease the i-th coordinate without changing others by moving in the direction ξ_i guaranteed by the lemma.) In case $k = n$, F is in fact a bilipschitz homeomorphism near x. (The proof of local one-to-one property of F is an easy argument based on volume and angle comparison and the statement (1) of the lemma; in [P] it is hidden in the first step of induction (from $k = n + 1$ to $k = n$) in the proof of the Main Theorem.)

3. DC coordinate charts

Proposition. *Let $F = (f_1, \dots, f_n)$ have SC components and be regular in some neighborhood of $x \in M$. Then*

(A) *If f is SC near x and $\langle f'_{(y)}, f'_{i(y)} \rangle < -\varepsilon_x < 0$, $1 \leq i \leq n$, for all y near x, then $f \circ F^{-1}$ is SC near $F(x)$.*

(B) *If $\bar{f}$ is SC and increasing in each argument near $F(x)$ then $\bar{f} \circ F$ is SC near x.*

(C) *f is DC near x iff $f \circ F^{-1}$ is DC near $F(x)$.*

Proof. (A) Let $\bar{\gamma}(t)$ be a straight segment in the image of F, $\gamma(t) = F^{-1} \circ \bar{\gamma}(t)$, $y = \gamma(0)$. From the fact that F is bilipschitz it is easy to see that γ has unique right and left tangent vectors $\gamma^+(0), \gamma^-(0) \in C_y$. Furthermore, $d_y f(\gamma^+(0)) + d_y f(\gamma^-(0)) \leq 0$. Indeed, otherwise, using concavity of $d_y f$ and $d_y f_i$, we could find a vector $v \in C_y$, such that $d_y f(v) > 0$ and $d_y f_i(v) \geq 0$, $1 \leq i \leq n$, which leads to a contradiction with the statement (1) of the lemma in 2.2. Now we'll check that

(1) $f \circ \gamma(t) \leq f(y) + f'_{(y)}(\gamma^+(0))t + Ct^2$ when $t > 0$ is small, for some C independent of t, y, γ; the corresponding statement for $t < 0$ and $\gamma^-(0)$ is checked similarly.

Consider a quasigeodesic σ starting at y in the direction $\gamma^+(0)/|\gamma^+(0)|$. Since f, f_i are SC we have

(2) $f_i \circ \sigma(|\gamma^+(0)|t) \le f_i(y) + f'_{i(y)}(\gamma^+(0))t + Ct^2$, $1 \le i \le n$, and a similar inequality for f.

(See [PPet]; in fact, in this argument we only need the first step of the construction of quasigeodesics, which is not technically complicated.) On the other hand,

(3) $f_i \circ \gamma(t) = f_i(y) + f'_{i(y)}(\gamma^+(0))t,\ \ 1 \le i \le n$ by the definition of γ.

Therefore, using the bilipschitz property of F, we can find a point z in the Ct^2-neighborhood of $\sigma(|\gamma^+(0)|t)$, such that

(4) $f_i \circ \gamma(t) \ge f_i(z)$, $1 \le i \le n$, and

(5) $f(z) \le f(y) + f'_{(y)}(\gamma^+(0))t + Ct^2$.

We claim that

(6) $f(\gamma(t)) \le f(z)$

Indeed, z can be obtained from $\gamma(t)$ by consecutive approximations, as in 2.2, starting from $\gamma(t)$,and (4) guarantees that we only need to use the directions ξ_i in the process, thus increasing the value of f. Now (1) follows from (5) and (6).

(B) This is almost immediate from the definitions.

(C) This follows easily from (A) and (B). For example, if f is SC near x then $\tilde f = f + N\mathrm{dist}_z^2$ satisfies $\langle \tilde f'_{(y)}, f'_{i(y)}\rangle < -\varepsilon_x < 0$ for all $1 \le i \le n$ and all y sufficiently close to x, if z is obtained by moving x a little bit in the direction where all f_i increase, and N is large enough. $\square$

Let S denote the set of singular points of M, and let $M^* \supset M \backslash S$ be the set of all points $x \in M$ such that Σ_x contains $n+1$ directions making obtuse angles with each other. M^* is open, convex, and $\dim_H(M \backslash M^*) \le n-2$. (Convexity follows from Petrunin's work [Pet] on parallel translation, the other properties follow from the results of [BGP].) For each point $x \in M^*$ we can find a map $F : M \to \mathbb{R}^n$ with SC components, which is regular near x. The collection of all such maps form an atlas on M^* and the transition functions are DC according to statement (C) of our Proposition. Following [OS] we can make the transition functions continuously differentiable on $M \backslash S$ by taking the components of the coordinate maps in the form $\int_{y \in B} \mathrm{dist}_y dH_n$.

4. Consequences

4.0 Analytical preliminaries. First we introduce some notation. Let $F : U \subset M^* \to \mathbb{R}^n$ be a DC coordinate chart. We denote by DC_0 the class of DC functions on $F(U)$ which are continuously differentiable on $F(M\backslash S)$, and by BV_0 the class of bounded functions of bounded variation, which are continuous on $F(M\backslash S)$. At the end of the previous section we described a DC_0 atlas on M^*, and we can say that a function f is DC_0 (BV_0) near $x \in M^*$ if $f \circ F^{-1}$ is DC_0 (BV_0) near $F(x)$ for some (and hence for all) DC_0 charts F.

It is well known that the first partial derivatives of the DC (DC_0) function are in BV (BV_0), and the second partial derivatives are signed Radon measures, with $\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}$ as measures. We will also use a classical theorem of Alexandrov, which implies that DC functions have second differential a.e., and their first partial derivatives, considered a.e., are differentiable a.e. It is also known (see [F, 4.5.9(29)]) that since $H_{n-1}(S) = 0$, we have $\|Df\|(A) = 0$ for every $H_{n-1} - \sigma$-finite set A and $f \in BV_0$, in particular we can multiply any first partial derivative of f by a bounded function which is continuous off a $H_{n-1} - \sigma$-finite set, and still get a signed Radon measure.

The following assertions will be used in 4.2 and 4.3. (see [V] for more general results; my thanks to L.C.Evans for this reference)

Lemma.

(1) *If f, g are bounded and BV then fg is BV. Moreover, if g does not change sign and is bounded away from zero, then $f/g \in BV$.*

(2) *If $f, g \in BV_0(U)$ then $(fg)_{x_i} = f_{x_i} \cdot g + f \cdot g_{x_i}$ as measures.*

(3) *Let $g \in BV_0(U)$, $U \subset \mathbb{R}^n_x$, and let $F = (f_1, \ldots, f_n) : U \to F(U) \subset \mathbb{R}^n_y$ be a DC_0 isomorphism. Then*

$$F_\# \left(\frac{\partial g}{\partial x_i} \; dx_1 \wedge \cdots \wedge dx_n \right) = \det(J_F)^{-1} \left(\sum_j \frac{\partial f_j}{\partial x_i} \; \frac{\partial (g \circ F^{-1})}{\partial y_j} \right) dy_1 \wedge \cdots \wedge dy_n \ .$$

Proof. (1) Clear by approximation, using semicontinuity of variation measure.

(2) We need to check that $\int_U (fg) h_{x_i} = \int_U f_{x_i} \cdot gh + \int_U f \cdot g_{x_i} h$ holds for any smooth function h with compact support. Of course this is true if both f, g are smooth, and it is easy to check by approximation if at least one of them is smooth. In general consider a sequence of mollified functions g_j; clearly g_j converges to g at each point where g is continuous, which is $\|Df\|$–a.e. Thus $\int (fg_j) h_{x_i} \to \int (fg) h_{x_i}$ and $\int f_{x_i} g_j h \to \int f_{x_i} g h$ by the dominated convergence theorem. To check the convergence of the remaining term, fix a small $\delta > 0$ and let K be a compact set where f has jumps of size $\geq \delta$. Since $\|Dg\|(K) = 0$, we can find an open neighborhood $V \supset K$ such that $\|Dg\|(\text{clos } V) < \delta$ for large j. (Here we use that $\|Dg_j\|$ weaky converges to $\|Dg\|$.) Since f has no jumps of size $> \delta$ near $U\backslash V$, we can find a continuous function $\bar{f}$ which is uniformly 2δ-close to f on $U\backslash\text{clos } V$. Now

$$| \int_U f \cdot g_{x_i} h - \int_{U\backslash \text{clos } V} \bar{f} \cdot g_{x_i} h | \leq C\delta \ ,$$

the same is true for g_j with large j, and

$$\int_{U\setminus \text{clos } V} \bar f (g_j)_{x_i} h \to \int_{U\setminus \text{clos } V} \bar f g_{x_i} h$$

because $\|Dg_j\|$ converges weakly to $\|Dg\|$.

(3) Arguing similarly to the proof of (2) we can cut off a neighborhood of a compact set where the first derivatives of the components of F have jumps $\geq \delta$, then cover the rest of U by small balls where those derivatives are nearly constant, and check that the left- and right-hand sides of our identity are nearly equal on each ball, using DC functions as test functions. The details are left to the reader. □

4.1 DeRham complex. Differential forms on M can be defined as elements of the vector space generated by monomials of the form $f_0 df_1 \wedge \cdots \wedge df_m$, where all $f_i \in DC$, and two forms can be considered equivalent if they have the same values a.e. The exterior differentiation can be defined by $d(f_0 df_1 \wedge \cdots \wedge df_m) = df_0 \wedge \cdots \wedge df_m$. Correctness of this definition follows from the identity

$$df_0 \wedge df_1 \wedge \cdots \wedge df_m (X_1 \wedge \cdots \wedge X_m) = \sum_{j=0}^{m} (-1)^j \frac{\partial}{\partial x_j} (f_0 df_1 \wedge \cdots \wedge df_m (X_0 \wedge \cdots \wedge \hat X_j \wedge \cdots \wedge X_m))$$

for coordinate vectors $X_0, \ldots, X_m$ of some DC coordinate system F, which holds at each point where all $f_i \circ F^{-1}$ are twice differentiable, which is a.e.

4.2 Metric tensor. Let $F = (f_1, \ldots, f_m)$ be a DC_0 coordinate chart near $x \in M^*$. Then, according to [OS], the metric of M near x can be expressed by a metric tensor, defined and continuous at each nonsingular point. Now let f be a distance function, $f = \mathrm{dist}_y,\ y \neq x$. Then $f \circ F^{-1}$ is DC near $F(x)$, in particular, differentiable a.e., and we have

$$(1) \qquad \sum_{i,j} g^{ij}\ \partial (f \circ F^{-1})/\partial x_i \cdot \partial (f \circ F^{-1})/\partial x_j = 1 \quad \text{a.e.}$$

Now suppose $x \in M^\delta$ for sufficiently small $\delta > 0$, where $M^\delta = \{x \in M : H_{n-1}(\Sigma_x) > (1-\delta) H_{n-1}(S^{n-1})\}$ is an open, convex subset of M, containing $M \setminus S$. Then we can choose a DC_0 coordinate chart F and a collection of distance functions $f^\alpha,\ 1 \leq \alpha \leq n(n+1)/2$, in such a way that the determinant of the system of linear equations (1) with f^α in place of f, with unknowns g^{ij}, is positive and bounded away from zero in a small neighborhood of $F(x)$. (Indeed, this is easy to arrange with some safety margin if $M = \mathbb{R}^n$, and the condition $x \in M^\delta$ guarantees that euclidean inequalities for $\frac{\partial(f^\alpha \circ F^{-1})}{\partial x_i}$ continue to hold in M up to a small error.) Thus the components of the metric tensor can be expressed as rational functions of the first derivatives of $f^\alpha \circ F^{-1}$. In particular, since $f^\alpha \circ F^{-1}$ are DC near $F(x)$, we conclude that the components of the metric tensor are in BV_0 and differentiable a.e.

Remark. This improves the earlier results of Otsu and Shioya [OS].

4.3 Metric connection. Let A, B, C be bounded vector fields on M^δ, and assume in addition that A and C are continuous off an $H_{n-1}-\sigma$-finite set, and $B \in BV_0$ (that is, the coordinates of B in DC_0 charts are BV_0). Then there exists a signed Radon measure $\langle \nabla_A B, C \rangle$ on M^δ which becomes

$$\sum_{i,j,k} A^i C^j \left(\frac{\partial B^k}{\partial x_i}\, g_{kj} + \tfrac{1}{2} B^k \left(\frac{\partial g_{ij}}{\partial x_k} + \frac{\partial g_{kj}}{\partial x_i} - \frac{\partial g_{ik}}{\partial x_j} \right) \right) \cdot \det(g_{ij})^{\frac{1}{2}} dx_1 \wedge \cdots \wedge dx_n$$

in each DC_0 coordinate chart $F : U \subset M^\delta \to \mathbb{R}^n_x$. The correctness of the definition is proved by a standard computation using the observations of 4.0.

4.4 The Hessian of SC functions.

Proposition. *Let F be a DC_0 chart near x and let f be a DC function near x. Assume that $C_x = \mathbb{R}^n$, $f \circ F^{-1}$ has first and second differentials at $F(x)$, and the components of the metric tensor w.r.t. F are differentiable at $F(x)$. Then $d_x f$ is linear on C_x and there exists a quadratic form $H_x f$ on C_x such that*

$$f(y) = f(x) + d_x f(y')|xy| + \tfrac{1}{2} H_x f(y', y')|xy|^2 + o(|xy|^2) \ ,$$

where $y' \in \Sigma_x$ denotes the direction of (any) shortest geodesic xy. Moreover, $H_x f$ can be calculated using standard formulas.

Proof. First of all, we can make a smooth change of coordinates in such a way that in the new coordinate system G the metric tensor at $G(x)$ becomes the identity matrix, and its first derivatives vanish. (Indeed, consider a smooth metric with the same values of the metric tensor and its first derivatives at $F(x)$, find the coordinate transformation that produces normal coordinates, and apply it to F.) Thus we have $|g_{ij}(G(y)) - \delta_{ij}| = o(|xy|)$, and therefore

(1) $|\, |yz| - |G(y)G(z)|\, | = o(r^2)$ for $y, z \in B_x(r)$. Moreover, we have
(2) $|\angle yxz - \angle G(y)G(x)G(z)| = o(|yz|)$ when all angles of the triangle $G(y)G(x)G(z)$ are bounded away from zero.

Indeed, take a point p such that $G(p)$ is in the plane $G(y)G(x)G(z)$, $G(x)$ is contained in the triangle $G(p)G(y)G(z)$ and all angles formed by these four points are bounded away from zero. Then (1) implies that

$$\tilde{\angle}yxz + \tilde{\angle}yxp + \tilde{\angle}zxp \geq \angle G(y)G(x)G(z) + \angle G(y)G(x)G(p) + \angle G(z)G(x)G(p) + o(|yz|)$$
$$= 2\pi + o(|yz|)\ .$$

On the other hand, $\tilde{\angle}yxz \leq \angle yxz$, $\tilde{\angle}yxp \leq \angle yxp$, $\tilde{\angle}zxp \leq \angle zxp$, and $\angle yxz + \angle yxp + \angle zxp \leq 2\pi$, whence $|\tilde{\angle}yxz - \angle yxz| = o(|yz|)$ and (2) follows from (1).

Now let y be close to x. We claim that the angle at $G(x)$ between the directions of the straight segment $G(x)G(y)$ and the image of shortest geodesic $G(xy)$ is $o(|xy|)$ — clearly this proves the proposition. To check the claim, find a point y_1 such that $|xy_1| = |xy|/2$, the direction of $G(xy)$ is between that of $G(x)G(y)$ and of $G(x)G(y_1)$,and the angle between the latter ones is $\pi/2$. Then (2) implies that $\angle(G(xy_1), G(x)G(y_1)) > \angle(G(xy), G(x)G(y)) - o(|xy|)$. Thus if the estimate $\angle(G(xy), G(x)G(y)) = o(|xy|)$ were false, we could construct a sequence $y_i \to x$ with angles $\angle(G(xy_i), G(x)G(y_i))$ bounded away from zero, which is clearly impossible. $\square$

Remark. In a recent work [O], Otsu proves a slightly weaker version of this proposition for distance functions. His technique is completely different, and has the advantage of expressing Hessian of a distance function in terms of derivatives of the norms of Jacobi fields, which is a result of independent interest.

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