Chapter 5

Nilpotent Lie algebras

5.1 Definition

Definition 5.1.1 *A Lie algebra is called* nilpotent *if there exists a decreasing finite sequence* $(\mathfrak{g}_i)_{i\in[0,k]}$ *of ideals such that* $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_k = 0$ *and* $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$ *for all* $i \in [0, k-1]$.

Proposition 5.1.2 *Let* $\mathfrak{g}$ *be a Lie algebra, the following conditions are equivalent:*

(ı) the Lie algebra $\mathfrak{g}$ *is nilpotent;*

(ıı) we have $\mathcal{C}^k\mathfrak{g} = 0$ *for* k *large enough;*

(ııı) we have $\mathcal{C}_k\mathfrak{g} = \mathfrak{g}$ *for* k *large enough;*

(ıv) there exists an integer k *such that* $\operatorname{ad} x_1 \circ \cdots \circ \operatorname{ad} x_k = 0$ *for any sequence* $(x_i)_{i\in[1,k]}$ *of elements in* $\mathfrak{g}$*;*

(v) there exists a decreasing sequence of ideals $(\mathfrak{g}_i)_{i\in[0,n]}$ *with* $\mathfrak{g}_0 = \mathfrak{g}$, $\mathfrak{g}_n = 0$ *and such that* $[\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$ *and* $\dim g_i/g_{i+1} = 1$ *for all* $i \in [0, n-1]$.

Proof. We start with the equivalence of the first three conditions. If *(ıı)* or *(ııı)* holds, then the sequence $(\mathcal{C}^i\mathfrak{g})_{i\in[1,k]}$ or $(\mathcal{C}_{k-i}\mathfrak{g})_{i\in[1,k]}$ satisfy the conditions of the definition and $\mathfrak{g}$ is nilpotent.

Conversely, if the exists a sequence of ideals $(\mathfrak{g}_i)_{i\in[0,k]}$ as in the definition, we prove by induction that $\mathcal{C}^i\mathfrak{g} \subset \mathfrak{g}_i$ and $\mathcal{C}_i\mathfrak{g} \supset \mathfrak{g}_{k-i}$. This is true for $i = 0$. Assume that $\mathcal{C}^i\mathfrak{g} \subset \mathfrak{g}_i$ and $\mathcal{C}_i\mathfrak{g} \supset \mathfrak{g}_{k-i}$, then we have the inclusions $\mathcal{C}^{i+1}\mathfrak{g} = [\mathfrak{g}, \mathcal{C}^i\mathfrak{g}] \subset [\mathfrak{g}, \mathfrak{g}_i] \subset \mathfrak{g}_{i+1}$ and $[\mathfrak{g}/\mathcal{C}_i, (\mathfrak{g}_{k-(i+1)} + \mathcal{C}_i\mathfrak{g})/\mathcal{C}_i\mathfrak{g}] \subset (\mathfrak{g}_{k-i} + \mathcal{C}_i\mathfrak{g})/\mathcal{C}_i\mathfrak{g} = 0$. The last inclusion implies that $(\mathfrak{g}_{k-(i+1)} + \mathcal{C}_i\mathfrak{g})/\mathcal{C}_i\mathfrak{g}$ is in the center of $\mathfrak{g}/\mathcal{C}_i\mathfrak{g}$ and therefore $\mathfrak{g}_{k-(i+1)} \subset \mathcal{C}_{i+1}\mathfrak{g}$. We get $\mathcal{C}^k\mathfrak{g} = 0$ and $\mathcal{C}_k\mathfrak{g} = \mathfrak{g}$.

Now *(ıı)* and *(ıv)* are equivalent. Indeed, the ideal $\mathcal{C}^k\mathfrak{g}$ is composed of the linear combinations of elements of the form $[x_1, [x_2, [\cdots [x_k, y] \cdots]]] = \operatorname{ad} x_1 \circ \cdots \circ \operatorname{ad} x_k(y)$ with $x_i \in \mathfrak{g}$ for all i and $y \in \mathfrak{g}$.

Finally *(ı)* and *(v)* are equivalent. Indeed the last condition imply the first. Conversely, assume that $(\mathfrak{g}_i)_{i\in[0,k]}$ is a sequence of ideals as in the definition of a nilpotent Lie algebra. Then let us complete the sequence $(\mathfrak{g}_i)_{i\in[0,k]}$ to a sequence $(\mathfrak{g}'_i)_{i\in[0,n]}$ with $n = \dim \mathfrak{g}$, $\dim \mathfrak{g}'_i = n-i$, $\mathfrak{g}_{i+1} \subset \mathfrak{g}_i$ and $\mathfrak{g}'_{n-\dim \mathfrak{g}_j} = \mathfrak{g}_j$. We only need to prove that $[\mathfrak{g}, \mathfrak{g}'_i] \subset \mathfrak{g}'_{i+1}$. But, for $i \in [0, n]$, we define $i_s = \max\{j \ / \ \mathfrak{g}'_i \subset \mathfrak{g}_j\}$. We have $\mathfrak{g}_{i_s+1} \subset \mathfrak{g}'_{i+1} \subset \mathfrak{g}'_i \subset \mathfrak{g}_{i_s}$. Therefore $[\mathfrak{g}, \mathfrak{g}'_i] \subset [\mathfrak{g}, \mathfrak{g}_{i_s}] \subset \mathfrak{g}_{i_s+1} \subset \mathfrak{g}'_{i+1}$. □

Corollary 5.1.3 *The center of a non trivial nilpotent Lie algebra is non trivial.*

Proof. Indeed, we must have $\mathcal{C}_1\mathfrak{g} \neq 0$ otherwise there is no k with $\mathcal{C}_k\mathfrak{g} = \mathfrak{g}$. □

Corollary 5.1.4 *The Killing form $\kappa_\mathfrak{g}$ vanishes for $\mathfrak{g}$ nilpotent.*

Proof. For any $(x, y) \in \mathfrak{g}^2$, the element $\mathrm{ad}\, x \circ \mathrm{ad}\, y$ is nilpotent thus $\kappa_\mathfrak{g}(x, y) = \mathrm{Tr}(\mathrm{ad}\, x \circ \mathrm{ad}\, y) = 0$. □

Proposition 5.1.5 *Any subalgebra, any quotient algebra, any central extension a Lie subalgebra is again a Lie subalgebra. A finite product of nilpotent Lie algebras is again a nilpotent Lie algebra.*

Proof. Let $\mathfrak{g}$ be a nilpotent Lie algebra.

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$, then $\mathcal{C}^k\mathfrak{h} \subset \mathcal{C}^k\mathfrak{g}$ and the result follows for $\mathfrak{h}$.

Let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$ and let $\pi : \mathfrak{g} \to \mathfrak{g}/\mathfrak{a}$ be the projection. We proved in Proposition 2.5.6 that $\pi(\mathcal{C}^k\mathfrak{g}) = \mathcal{C}^k(\mathfrak{g}/\mathfrak{a})$ and the result follows for $\mathfrak{g}/\mathfrak{a}$.

Let $0 \to \mathfrak{a} \to \mathfrak{g}' \xrightarrow{p} \mathfrak{g} \to 0$ be a central extension, then $p(\mathcal{C}^k\mathfrak{g}') = \mathcal{C}^k\mathfrak{g}$. Therefore if $\mathcal{C}^k\mathfrak{g} = 0$, then $\mathcal{C}^k\mathfrak{g}' \subset \mathfrak{a}$ and $\mathcal{C}^{k+1}\mathfrak{g}' = 0$ because $\mathfrak{a} \subset \mathfrak{z}(\mathfrak{g}')$.

The last assertion follows from the condition *(iv)* in the previous proposition. □

5.2 Engel's Theorem

Theorem 5.2.1 *Let V be a vector space and $\mathfrak{g}$ be a finite dimensional Lie subalgebra of $\mathfrak{gl}(V)$, such that for all x is nilpotent for all $x \in \mathfrak{g}$, then there is a $v \in V$ with $x(v) = 0$ for all $x \in \mathfrak{g}$.*

Proof. We proceed by induction on $n = \dim \mathfrak{g}$. For $n = 0$, this is clear. We shall need a

Lemma 5.2.2 *Let V be a vector space and $x \in \mathfrak{gl}(V)$ nilpotent, then element f of $\mathfrak{gl}(\mathfrak{gl}(V))$ defined by $y \mapsto [x, y]$ is nilpotent.*

Proof. Indeed, we can compute that $f^m(y)$ is a linear combinaison of terms of the form $x^i y x^{m-i}$ and the result follows. □

Now let $\mathfrak{h}$ be a strict subalgebra of $\mathfrak{g}$. We define a map $\sigma : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ sending $x \in \mathfrak{h}$ to the map $\sigma(x)$ defined by $\overline{y} \mapsto \overline{[x, y]}$ where $\overline{y}$ is the class of $y \in \mathfrak{g}$ in the quotient $\mathfrak{g}/\mathfrak{h}$. By the previous lemma, the map $x \mapsto [x, y]$ is nilpotent so $\sigma(x)$ is nilpotent. Therefore $\sigma(\mathfrak{h})$ satisfies the conditions of the Theorem and $\dim \sigma(\mathfrak{h}) < n$. By induction, there exists $\overline{y}$ a non trivial vector in $\mathfrak{g}/\mathfrak{h}$ with $\sigma(x)(y) = 0$ for all $x \in \mathfrak{h}$. Therefore, there is a y not in $\mathfrak{h}$ with $[x, y] \in \mathfrak{h}$ for all $x \in \mathfrak{h}$. This imples that $\mathfrak{h}$ is an ideal in the subalgebra $\mathfrak{h} \oplus ky$ of $\mathfrak{g}$.

By induction starting with $\mathfrak{h} = 0$, we get a codimension 1 ideal $\mathfrak{h}$ in $\mathfrak{g}$. The result is true for $\mathfrak{h}$ therefore, the subspace W of all $v \in V$ such that $x(v) = 0$ for all $x \in \mathfrak{h}$ is non trivial. Let $y \in \mathfrak{g}$ with $y \notin \mathfrak{h}$, then y stabilises W. Indeed, for $v \in W$, we have $x(y(v)) = y(x(v)) + [x, y](v) = 0$ because $[x, y]$ and x are in $\mathfrak{h}$. Now y is nilpotent on W therefore there exists v non trivial in W with $y(v) = 0$. The vector v does the job. □

Corollary 5.2.3 *A Lie algebra $\mathfrak{g}$ is nilpotent if and only if $\mathrm{ad}\, x$ is nilpotent for all $x \in \mathfrak{g}$.*

Proof. By Proposition 5.1.2, if $\mathfrak{g}$ is nilpotent then $\mathrm{ad}\, x$ is nilpotent for all $x \in \mathfrak{g}$. Conversely, if $\mathrm{ad}\, x$ is nilpotent for all x, then the image of the adjoint representation in $\mathfrak{gl}(\mathfrak{g})$ satisfies the conditions of Engel's Theorem. Therefore, there is a non trivial $x \in \mathfrak{g}$ such that $[x, y] = \mathrm{ad}\, x(y) = 0$ for all $y \in \mathfrak{g}$. Therefore the center of $\mathfrak{g}$ is non trivial. Now the Lie algebra $\mathfrak{g}/\mathfrak{z}(\mathfrak{g})$ satisfies the same hypothesis and we conclude by induction that $\mathcal{C}_k\mathfrak{g} = \mathfrak{g}$ for k large enough. □

Corollary 5.2.4 *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a}$ an ideal of $\mathfrak{g}$. Assume that $\mathfrak{g}/\mathfrak{a}$ is nilpotent and that for all $x \in \mathfrak{g}$, the restriction $\mathrm{ad}\, x|_{\mathfrak{a}}$ is nilpotent, then $\mathfrak{g}$ is nilpotent.*

Proof. Let $x \in \mathfrak{g}$, we prove that $\mathrm{ad}\, x$ is nilpotent. Indeed, it is nilpotent on $\mathfrak{a}$ and on $\mathfrak{g}/\mathfrak{a}$ (there are k and k' such that $\mathrm{ad}^{\,k} x(\mathfrak{g}) \subset \mathfrak{a}$ and $\mathrm{ad}^{\,k'} x(\mathfrak{a}) = 0$ therefore $\mathrm{ad}^{\,k+k'} x(\mathfrak{g}) = 0$). □

Corollary 5.2.5 *Let V be a vector space and $\mathfrak{g}$ a Lie subalgebra of $\mathfrak{gl}(V)$ such that all the elements $x \in \mathfrak{g}$ are nilpotent endomorphisms of V, then $\mathfrak{g}$ is nilpotent.*

Proof. Indeed by Lemma 5.2.2, for any $x \in \mathfrak{g}$, the element $\mathrm{ad}\, x$ is nilpotent. We conclude by applying Corollary 5.2.3 □

Example 5.2.6 For V a vector space and $V_\bullet$ a complete flag, the Lie algebra $\mathfrak{n}(V_\bullet)$ is nilpotent.

5.3 Maximal nilpotent ideal

Definition 5.3.1 *An ideal $\mathfrak{a}$ in $\mathfrak{g}$ is called nilpotent if it is nilpotent as a Lie algebra.*

Lemma 5.3.2 *An ideal $\mathfrak{a}$ of $\mathfrak{g}$ is nilpotent if and only if for all $x \in \mathfrak{a}$, we have that $\mathrm{ad}_{\mathfrak{g}} x$ is nilpotent.*

Proof. The condition is sufficent (we only need that $\mathrm{ad}_{\mathfrak{a}} x$ is nilpotent). Conversely, if $\mathfrak{a}$ is nilpotent, then $\mathrm{ad}_{\mathfrak{a}} x$ is nilpotent and $\mathrm{ad}_{\mathfrak{g}} x(\mathfrak{g}) \subset \mathfrak{a}$ and the result follows. □

We shall need the following general result on representations.

Lemma 5.3.3 *Let V be a finite dimensional representation of the Lie algebra $\mathfrak{g}$, then there exists an increasing sequence $0 = V_0 \subset V_1 \subset \cdots \subset V_n = V$ of subrepresentations of V such that V_i/V_{i-1} is simple for all $i \in [1, n]$.*

Proof. By induction on the dimension of V, we only need to prove that there exists a subrepresentation W of V such that V/W is simple. We also prove this by induction on $\dim V$. Indeed, if V is simple, we are done. Otherwise, there exists a non trivial subrepresentation V' of V and we apply our induction hypothesis on V/V'. We get W/V' a subrepresentation of V/V' (image of the subspace W in V) such that $(V/V')/(W/V')$ is simple. But W is a subrepresentation of V and $V/W \simeq (V/V')/(W/V')$ is simple. □

Lemma 5.3.4 *Let V be a simple representation of $\mathfrak{g}$ and $\mathfrak{a}$ an ideal such that for all $x \in \mathfrak{a}$, the element x_V is nilpotent. Then for all $x \in \mathfrak{a}$, we have $x_V = 0$.*

Proof. By Proposition 4.4.6, the subspace $V^{\mathfrak{a}} = \{v \in V \ / \ x_V \cdot v = 0 \text{ for all } x \in \mathfrak{a}\}$ is a subrepresentation of V. Furthermore, by Engel's Theorem (Theorem 5.2.1), this space is non trivial. Because V is simple we have $V = V^{\mathfrak{a}}$. □

Lemma 5.3.5 *The sum of any two nilpotent ideals is again a nilpotent ideal.*

Proof. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two nilpotent ideals and $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$. We need to prove that if $\mathrm{ad}_{\mathfrak{g}}(x+y)$ is nilpotent. For this consider the sequence of subrepresentations $\mathfrak{g}_0 = 0 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$ of the adjoint representation given by Lemma 5.3.3. Because $\mathrm{ad}_{\mathfrak{g}}x$ and $\mathrm{ad}_{\mathfrak{g}}y$ are nilpotent, for any $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$, we have that $x_{\mathfrak{g}_i/\mathfrak{g}_{i-1}}$ and $y_{\mathfrak{g}_i/\mathfrak{g}_{i-1}}$ are nilpotent for all $i \in [1,n]$. By Lemma 5.3.4 and because $\mathfrak{g}_i/\mathfrak{g}_{i-1}$ is simple, we have the equalities that $x_{\mathfrak{g}_i/\mathfrak{g}_{i-1}} = 0$ and $y_{\mathfrak{g}_i/\mathfrak{g}_{i-1}} = 0$ for all $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$ and for all $i \in [1,n]$. In particular $(x+y)_{\mathfrak{g}_i/\mathfrak{g}_{i-1}} = 0$ for all $x \in \mathfrak{a}$ and $y \in \mathfrak{b}$ and for all $i \in [1,n]$. We have $\mathrm{ad}_{\mathfrak{g}}(x+y)(\mathfrak{g}_i) \subset \mathfrak{g}_{i-1}$ for all $i \in [1,n]$ and $\mathrm{ad}_{\mathfrak{g}}(x+y)$ is nilpotent. □

Corollary 5.3.6 *There exists a maximal nilpotent ideal* $\mathfrak{n}_{\mathfrak{g}}$ *in any finite dimensional Lie algebra* $\mathfrak{g}$.

Remark 5.3.7 The quotient $\mathfrak{g}/\mathfrak{n}_{\mathfrak{g}}$ may have nilpotent ideals.