Introduction. The purpose of this paper is to introduce the notion of a CW complex over a topological category. The main theorem of this paper gives an equivalence between the homotopy theory of diagrams of spaces based on a topological category and the homotopy theory of CW complexes over the same base category.

A brief description of the paper goes as follows: in Section 1 we introduce the homotopy category of diagrams of spaces based on a fixed topological category. In Section 2 homotopy groups for diagrams are defined. These are used to define the concept of weak equivalence and \( J_n \) equivalence that generalize the classical definition. In Section 3 we adapt the classical theory of CW complexes to develop a cellular theory for diagrams. In Section 4 we use sheaf theory to define a reasonable cohomology theory of diagrams and compare it to previously defined theories. In Section 5 we define a closed model category structure for the homotopy theory of diagrams. We show this Quillen type homotopy theory is equivalent to the homotopy theory of \( J \)-CW complexes. In Section 6 we apply our constructions and results to prove a useful result in equivariant homotopy theory originally proved by Elmendorf by a different method.

1. Homotopy theory of diagrams. Throughout this paper we let Top be the cartesian closed category of compactly generated spaces in the sense of Vogt [10]. Let \( J \) be a small topological category over Top with discrete object space and \( J \)-Top the category of continuous contravariant Top valued functors on \( J \). Note that the category \( J \)-Top is naturally enriched in Top. See Dubuc [2] for the framework of enriched category theory. We assume the reader is familiar with the standard constructions in Top as in [10] and the standard functor calculus on \( J \)-Top as in [5, Section 1].

We let \( I \) be the unit interval in Top. If \( X \) and \( Y \) are diagrams then a homotopy from \( X \) to \( Y \) is a morphism \( H: I \times X \rightarrow Y \) of \( J \)-Top where \( I \times X \) is the functor defined on objects \( j \in |J| \) by \( (I \times X)(j) = I \times X(j) \) and similarly for morphisms of \( J \). In the usual way homotopy defines an equivalence relation on the morphisms of \( J \)-Top that gives rise to the quotient homotopy category \( hJ \)-Top. We denote the homotopy classes of morphisms...
from $X$ to $Y$ by $hJ$-Top$(X,Y)$ abbreviated $h(X,Y)$. An isomorphism in $hJ$-Top is called a $J$-homotopy equivalence.

A morphism of $J$-Top is called a $J$-cofibration if it has the $J$ homotopy extension property, abbreviated $J$-HEP. The basic facts about cofibrations in Top apply readily to $J$-cofibrations. See [5, Section 2].

The following results from [6] apply formally to the category $J$-Top.

**Theorem 1.1 (Invariance of Pushouts).** Suppose that we have a commutative diagram:

$$
\begin{array}{c}
X' \\
\downarrow \alpha \\
X \\
\downarrow i \\
A \\
\end{array} 
\xrightarrow{\theta} 
\begin{array}{c}
Z' \\
\downarrow \beta \\
Z \\
\downarrow j \\
Y \\
\end{array} 
\xrightarrow{\gamma} 
\begin{array}{c}
Y' \\
\end{array}

$$

in which $i$ and $i'$ are $J$-cofibrations, $f$ and $f'$ are arbitrary morphisms in $J$-Top. $\alpha$, $\beta$ and $\gamma$ are homotopy equivalences and the front and back faces are pushouts. Then $\theta$ is also a homotopy equivalence ($\theta$ being the induced map on pushouts).

**Theorem 1.2 (Invariance of Colimits Over Cofibrations).** Suppose given a homotopy commutative diagram

$$
\begin{array}{c}
X^0 \\
\downarrow j^0 \\
X^1 \\
\vdots \\
X^k \\
\downarrow j^k \\
Y^0 \\
\downarrow j^0 \\
Y^1 \\
\vdots \\
Y^k \\
\end{array} 
\xrightarrow{f^0} 
\begin{array}{c}
X^1 \\
\downarrow j^1 \\
X^k \\
\downarrow j^k \\
Y^k \\
\end{array}

$$

in $J$-Top where the $i_k$ and $j_k$ are $J$-cofibrations and the $f^k$ are homotopy equivalences. Then the map $\text{colim}_k f^k : \text{colim}_k X^k \to \text{colim}_k Y^k$ is a homotopy equivalence.

2. **Homotopy groups.** Let $P^n$ be the topological $n$-cube and $\partial P^n$ its boundary.

**Definition 2.1.** By a $J$-Top pair $(X,Y)$, we mean an object $X$ in $J$-Top together with a subobject $Y \subseteq X$. Morphisms of pairs are defined in the obvious way. A similar definition will be used for triples, $n$-ads etc. Let $\varphi : j \to Y$ be a morphism in $J$-Top where $j \in |J|$ is viewed as the representable functor $J$-Top$(j)$. By Yoneda’s theorem $\varphi$ is completely determined by the point $\varphi(\text{id}_j) = y_0 \in Y(j)$. For each $n \geq 0$, define $\pi^n_J(X,Y,\varphi) = h((P^n,\partial P^n,\{0\}) \times j,(X,Y,Y))$ where $y_0 = \varphi(\text{id}_j) \in Y(j)$ serves as a basepoint, and all homotopies are homotopies of triples relative to $\varphi$. The reader may formulate a similar definition for the absolute case $\pi^n_J(X,\varphi)$. For $n = 0$ we adopt the convention that $\pi^0_J(X,\varphi) = \{0,1\}$ and $\partial P^0 = \{0\}$ and proceed as above. These constructions extend to covariant functors on $J$-Top. From now on we shall often drop $\varphi$ from the notation $\pi^n_J(X,Y,\varphi)$.

The proof of the following proposition follows immediately from Yoneda’s lemma.
PROPOSITION 2.2. There are natural equivalences \( \pi_n(X) \simeq \pi_n(X(j)) \) and \( \pi_n(X, Y) \simeq \pi_n(X(j), Y(j)) \) which preserve the (evident) group structure when \( n \geq 1 \) (for the absolute case; the relative case requires \( n \geq 2 \)).

As a direct consequence of 2.2 we obtain the long exact sequences:

**PROPOSITION 2.3.** For \((X, Y)\) and \(j\) as in 2.1, there exist natural boundary maps \( \partial \) and long exact sequences

\[
\cdots \rightarrow \pi_n(X, Y) \xrightarrow{\partial} \pi_{n-1}(Y) \rightarrow \pi_{n-1}(X) \rightarrow \cdots
\]

of groups up to \( \pi_1(Y) \) and pointed sets thereafter.

**DEFINITION 2.4.** A map \( e: (X, Y) \rightarrow (X', Y') \) of pairs in \( J\)-Top is called a \( J\)-\( n \)-equivalence if \( e(j): (X(j), Y(j)) \rightarrow (X'(j), Y'(j)) \) is an \( n \)-equivalence in Top for each \( j \in |J| \). A map \( e \) will be called a weak equivalence if \( e \) is a \( J\)-\( n \)-equivalence for each \( n \geq 0 \). Observe that \( e \) is a \( J\)-\( n \)-equivalence if for every \( j \in |J| \) and \( \varphi: j \rightarrow Y, e_*: \pi_p(X, Y, \varphi) \rightarrow \pi_p(X', Y', e\varphi) \) is an isomorphism for \( 0 \leq p < n \) and an epimorphism for \( p = n \). The reader may easily formulate a similar definition for morphisms \( e: X \rightarrow X' \) of \( J\)-Top (the absolute case).

3. **Cellular theory.** In this section we adapt the general treatment of classical homotopy theory and \( CW \)-complexes given in [9, Chapter 7] and [6] to develop a good theory of \( J\)-\( CW \)-complexes over the topological category \( J \).

Let \( B^{n+1} \) be the topological \( n+1 \)-ball and \( S^n \) the topological \( n \)-sphere. Of course, these spaces are homeomorphic to \( f^{n+1} \) and \( \partial f^{n+1} \) respectively. We shall construct all complexes over \( J \) by the process of attaching cells of the form \( B^{n+1} \times j \) by attaching morphisms with domain \( S^n \times j \). The formal definition goes as follows:

**DEFINITION 3.1.** A \( J \)-complex is an object \( X \) of \( J\)-Top with a decomposition \( X = \text{colim}_{p \geq 0} X' \) where \( X' = \coprod_{\alpha \in A_p} B^{n_\alpha} \times j_\alpha \), \( X' = X^{p-1} \cup_f \left( \coprod_{\alpha \in A_p} B^{n_\alpha} \times j_\alpha \right) \) for some attaching morphism \( f: \coprod_{\alpha \in A_p} S^{n_\alpha-1} \times j_\alpha \rightarrow X^{p-1} \) and for each \( p \geq 0 \), \( \{ j_\alpha : \alpha \in A_p \} \) is a collection of objects (representable functors) of \( J \). We call \( X \) a \( J \)-\( CW \)-complex if \( X \) is a \( J \)-complex as above and for all \( p \geq 0 \) and all \( \alpha \in A_p \) we have \( n_\alpha = p \).

A \( J \)-subcomplex and a relative \( J \) complex are now defined in the obvious way. Without further comment we adopt for \( J-CW \)-complexes the standard terminology for \( CW \)-complexes. See [9, Chapter 7] and [6].

The following technical lemma and its proof are due to May [6,3.5.1].

**LEMMA 3.2.** Suppose that \( e: Y \rightarrow Z \) is a \( J \)-\( n \)-equivalence. Then we can complete the following diagram in \( J\)-Top:
\text{THEOREM 3.3 (J-HELP).} If \((X, A)\) is a relative \(J\)-CW complex of dimension \(\leq n\) and \(e: Y \to Z\) is a \(J\)-\(n\)-equivalence then we can complete the following diagram in \(J\)-Top:

\[
\begin{array}{cccc}
\partial \mathbb{I}^n \times j & \overset{i_0}{\longrightarrow} & \partial \mathbb{I}^n \times I \times j & \overset{i_1}{\leftarrow} & \partial \mathbb{I}^n \times j \\
\downarrow & & \downarrow h & & \downarrow \\
Z & \overset{e}{\longleftarrow} & Y & \overset{e_1}{\longrightarrow} & \partial \mathbb{I}^n \times j \\
\downarrow & & \downarrow & & \downarrow \\
I^n \times j & \overset{i_0}{\longrightarrow} & I^n \times I \times j & \overset{i_1}{\leftarrow} & I^n \times j
\end{array}
\]

\text{PROOF.} This follows by induction on \(\dim(X, A)\), applying 3.2 cell by cell at each stage.

The proofs of the following Whitehead theorem and cellular approximation theorem are formal modifications of the proofs given in [6].

\text{THEOREM 3.4 (WHITEHEAD).} (i) Suppose \(X\) is a \(J\)-CW complex, and that \(e: Y \to Z\) is a \(J\)-\(n\)-equivalence. Then \(e_*: h(X, Y) \to h(X, Z)\) is an isomorphism if \(\dim X < n\) and an epimorphism of \(\dim X = n\). (ii) If \(e: Y \to Z\) is a weak equivalence, and if \(X\) is any \(J\)-CW complex, then \(e_*: h(X, Y) \to h(X, Z)\) is an isomorphism.

\text{THEOREM 3.5 (CELLULAR APPROXIMATION).} Suppose that \(X\) is a \(J\)-CW complex, and that \(A\) is a sub-\(J\)-CW complex of \(X\). Then, if \(f: X \to Y\) is a morphism of \(J\)-Top which is \(J\)-cellular when restricted to \(A\), we can homotope \(f\), rel \(\partial A\) to a \(J\)-cellular morphism \(g: X \to Y\).

Next we discuss the local properties of \(J\)-CW-complexes. First we develop some preliminary concepts. Let \(X\) be in \(J\)-Top and for each \(j \in |J|\) let \(t_j: X(j) \to \text{colim}_j X\) be the natural map of \(X(j)\) into the colimit. Observe that for each morphism \(s: i \to j\) of \(J\), \(t_j \circ X(s)\) is the restriction of the continuous map \(X(s)\) to the subspace \(A(j)\). We apply the \(K\)-ification functor to assure that all spaces defined above are compactly generated. One quickly checks that \(\hat{A} \in J\)-Top, \(\text{colim}_j \hat{A} = A\), and there is a natural inclusion morphism \(\hat{A} \to X\). To simplify notation from now on we write \(X/\mathcal{J}\) for \(\text{colim}_j X\).

\text{DEFINITION 3.6.} By a special pair in \(J\)-Top we mean an ordered pair \((X, A)\) where \(X \in J\)-Top and \(A \subseteq X/\mathcal{J}\). We call a special pair \((X, A)\) a \(J\)-neighborhood retract (abbreviated \(J\)-NR) if there exist \(U\) an open subset of \(X/\mathcal{J}\) such that \(A \subseteq U\) and there exists a retraction morphism \(r: \hat{U} \to \hat{A}\). \((X, A)\) is called a \(J\)-neighborhood deformation
retract pair (abbreviate J-NDR) if \((X, A)\) is a J-NR and the morphism \(r\) is a J-deformation retract.

Let \(X\) be a J-CW complex. The functor \(\text{colim}_t\) sends cells \(B^p \times j\) to cells \(B^p\) and preserves the cellular decomposition of \(X\). For this reason \(X/\) has the natural structure of a CW-complex in TOP with all its attaching maps being images under \(\text{colim}_t\) of the corresponding attaching morphisms in J-TOP. One may also check that if \(A\) is a subcomplex of \(X/\) then \(\tilde{A}\) has the natural structure of a subcomplex of \(X\). In particular if \(A^p\) is the \(p\)-skeleton of \(X/\) then \(\tilde{A}^p = X^p\) is the \(p\)-skeleton of \(X\).

**Theorem 3.7 (Local Contractibility).** Let \((X, A)\) be a special pair in J-TOP with \(X\) a J-CW complex and \(A = \{ a \}, a \in X/\). Then there exists a unique object \(j \in J\) such that \(\tilde{A} \simeq j\) (\(j\) viewed as a representable functor) and \((X, A)\) is a J-NDR pair.

**Proof.** Suppose \(a \in (X/)^p \setminus (X/)^p-1\), the \(p\)-skeleton minus the \(p - 1\) skeleton of \(X/\). Then there is a unique attaching morphism \(f\) in J-TOP

\[
f: S^{p-1} \times j \to X^{p-1}
\]

with \(a\) in the interior of \(B^p\). It follows that \(\tilde{A} \simeq j\) for the unique choice of \(j\) given above. To construct the required neighborhood \(U\) first take an open ball \(U_1\) contained in the interior of \(B^p\) and centered at \(a\). Then \(U_1\) is a neighborhood in \((X/)^p\) contracting to \(A\). One then extends \(U_1\) inductively cell by cell by a well known procedure to construct the required neighborhood \(U\).

**Theorem 3.8.** Let \((X, A)\) be a special pair in J-TOP with \(X\) a J-CW complex and \(A\) an arbitrary subcomplex of \(X/\). Then \((X, A)\) is a J-NDR pair.

**Proof.** It follows from 3.3 that \(\tilde{A} \subseteq X\) is a J-cofibration. The result then follows from a well known argument of Puppe. See [5, Lemma 4.3, p. 193].

4. **Cohomology.** In this section we use sheaf theory to construct a cohomology theory on J-TOP satisfying a suitably formulated set of Eilenberg-Steenrod axioms. We refer the reader to Bredon [1] for the basic definitions and terminology of sheaf theory.

**Definition 4.1.** By a contravariant coefficient system \(M\) on \(J\) we mean a continuous contravariant functor \(M: J \to \text{Ab}\) where \(\text{Ab}\) is the category of discrete abelian groups. Observe that every contravariant coefficient system \(M\) is a homotopy invariant functor in the following sense. If \(f, g: j \to j'\) are homotopic (as morphisms of representable functors in J-TOP) then \(M(f) = M(g)\).

Let \(X \in J\)-TOP and let \(M\) be a coefficient system on \(J\). We define a presheaf of abelian groups \(M^X\) over \(X/\) as follows: for \(A \subseteq X/\) define \(M^X(A) = J\)-TOP(\(\tilde{A}, M\)) equipped with its natural discrete abelian group structure. If \(B \subseteq A\) there is a natural restriction homomorphism \(M^X(A) \to M^X(B)\) and one easily checks that \(M^X\) is a sheaf of abelian
groups over $X / J$. Let $f: X \to Y$ be a morphism in $J$-Top with $f / J: X / J \to Y / J$ the induced map in Top. There is a natural $f / J$-cohomomorphism of sheaves $\tilde{f}: M^Y \to M^X$ given by the obvious composition with $f$.

**Definition 4.2.** Let $X \in J$-Top, $\psi$ a family of supports on $X / J$ and $M$ a coefficient system on $J$. We define $H^0_\psi (X; M) = H^0_\psi (X / J; M^X)$ where the right side is sheaf cohomology with supports $\psi$ as defined in [1, Chapter II]. Given a morphism $f: X \to Y$ in $J$-Top, we let $f^*$ be the homomorphism induced in cohomology by $f$. Given a special pair $(X, A)$ we define the relative cohomology $H^\psi_\psi (X, A; M) = H^\psi_\psi (X / J, A; M^X)$ where the right side is relative sheaf cohomology.

**Example 4.3.** Let $G$ be an abelian group and define the constant coefficient system $M$ with value $G$ by setting $M(s) = \text{id}_G$ for any morphism $s$ of $J$. Then for any $X \in J$-Top one quickly sees that $H^\ast (X; M) = H^\ast (X / J; G)$ where the right side is sheaf cohomology with constant coefficients $G$. Note that absence of a specified support family always means supports in the family of all closed sets.

**Definition 4.4.** A special pair $(X, A)$ in $J$-Top is called acceptable if for each coefficient system $M$ on $J$ the sheaf $M^A$ over $A$ is the restriction of the sheaf $M^X$ to the subspace $A$. Note that if $(X, A)$ is a $J$-NR pair or if $X$ is locally $J$-NR then $(X, A)$ is acceptable. In particular any special pair $(X, A)$ where $X$ is a $J$-CW complex is acceptable by 3.7.

All special pairs considered in the rest of this section will be assumed acceptable. We impose this condition to obtain a good theory of relative cohomology.

Note that a supports preserving morphism $f: (X, A) \to (Y, B)$ naturally induces a homomorphism $f^*$ in relative cohomology. Hence $H^\ast_\psi (\cdot; M)$ becomes a candidate for a reasonable cohomology theory on $J$-Top. The following theorem states and verifies a suitable set of Eilenberg-Steenrod axioms for the theory $H^\ast (\cdot; M)$.

**Theorem 4.5.**
1. (Dimension) $H^n(j; M) = \begin{cases} M(j) & n = 0 \\ 0 & n > 0 \end{cases}$ for each $j \in J$ viewed as a representable functor.
2. For each special pair $(X, A)$ in $J$-Top there is induced a suitable long exact sequence in cohomology with arbitrary supports.
3. (Excision) If $A$ and $B$ are subsets of $X / J$ with $B \subseteq \text{int} A$ then the inclusion $i: (X - B, A - B) \to (X, A)$ induces an isomorphism in cohomology for any support family.
4. (Homotopy) If $f$ and $g$ are morphisms of special pairs in $J$-Top that are homotopic via a support preserving homotopy then $f^* = g^*$.
5. If $(X, A) = \bigsqcup_\alpha (X_\alpha, A_\alpha)$ then there is a natural isomorphism induced by the injections into the coproduct,

$$H^\ast (X, A; M) \simeq \bigsqcup_\alpha H^\ast (X_\alpha, A_\alpha; M).$$
PROOF. (1) follows from Yoneda’s lemma. (2) follows from [1, Chapter 2, Section 12]. (3) follows from [1, Theorem 12.5, p. 61]. (4) follows from [1, Theorem 11.2, p. 55]. (5) is easy to check directly.

If X is a J-CW complex we define cellular cochains $C^n(X; M) = H^n\left(X^n, (X^{n-1} / J); M\right)$. Observe that $C^n(X; M) = \prod_\alpha M(j_\alpha)$ where $B^n \times j_\alpha$, $\alpha \in A_n$ is the family of all n-cells of $X$. In the usual way one makes $C^*(X; M)$ into a cochain complex using the coboundary operator of a triple. This construction yields the cellular cohomology theory $H^*_{cel}(\quad; M)$ defined for J-CW pairs.

We may adapt the classical proof to show:

PROPOSITION 4.6. $H^*(\quad; M)$ is naturally isomorphic to $H^*_{cel}(\quad; M)$ on the category of J-CW pairs.

REMARK 4.7. (i) The cellular homology theory is useful for developing an obstruction theory in J-Top. (ii) Following a well known argument due to Milnor it is possible to prove a uniqueness theorem for cohomology theories defined on the category of J-CW complexes. (iii) In [11] Vogt defines the singular cohomology on J-Top and shows it satisfies a suitable set of axioms. By the above mentioned uniqueness theorem Vogt’s singular cohomology agrees with our sheaf cohomology on the category of J-CW complexes.

5. Closed model structure on J-Top. In [8] Quillen defines a closed model structure for homotopy theory in Top. In this section we emulate this construction to define a closed model category structure on J-Top.

DEFINITION 5.1. A morphism $f: X \to Y$ of J-Top is called a weak fibration, abbreviated w-fibration, if for each $j \in J, f(j): X(j) \to Y(j)$ is a Serre fibration in Top. See [9, p. 374] for a discussion of Serre fibrations. Observe that $f$ is a w-fibration if $f$ has the homotopy lifting property for all objects of the form $I^n \times j$. A morphism $f$ is called a weak equivalence if $f$ is a weak equivalence as defined in Section 2. A morphism $g: A \to B$ is called a weak cofibration, abbreviated w-cofibration if $g$ has the left lifting property (LLP) for each trivial w-fibration $f: X \to Y$ (a w-fibration that is also a weak equivalence). This means one can always fill in the dotted arrow:

\[
\begin{array}{ccc}
A & \longrightarrow & X \\
\downarrow g & & \downarrow f \\
B & \longrightarrow & Y
\end{array}
\]

REMARK 5.2. (i) The inclusion of a sub-J-complex into a J-complex is always both a J-cofibration and a w-cofibration. (ii) A w-fibration is trivial iff it has the right lifting property (RLP) for each w-cofibration of the form $S^n \times j \to B^{n+1} \times j$. [8, 3.2, Lemma 2].
LEMMA 5.3 (QUILLEN'S FACTORIZATION LEMMA). Any morphism \( f: X \to Y \) of J-Top may be factored \( f = pg \) where \( g \) is a w-cofibration and \( p \) is a trivial w-fibration.

PROOF. We construct a diagram

\[
\begin{array}{cccccc}
X & \xrightarrow{g_0} & Z^0 & \xrightarrow{g_1} & Z^1 & \to \\
\downarrow f & & \downarrow p_0 & & \downarrow p_1 & \\
Y & & & & &
\end{array}
\]

as follows: let \( Z^{-1} = X \) and \( p_{-1} = f \), and having obtained \( Z^{n-1} \) consider the set of all diagrams of the form

\[
\begin{array}{ccc}
S^{q_\alpha} \times j_\alpha & \xrightarrow{t_\alpha} & Z^{n-1} \\
\downarrow & & \downarrow p_{n-1} \\
B^{q_\alpha+1} \times j_\alpha & \xrightarrow{s_\alpha} & Y
\end{array}
\]

where we have indexed this set of diagrams by \( A_n \) and \( \alpha \in A_n \). Define \( g_n: Z^{n-1} \to Z^n \) by the pushout diagram

\[
\begin{array}{ccc}
\prod_{\alpha \in A_n} S^{q_\alpha} \times j_\alpha & \xrightarrow{\prod_{\alpha} t_\alpha} & Z^{n-1} \\
\downarrow & & \downarrow g_n \\
\prod_{\alpha \in A_n} B^{q_\alpha+1} \times j_\alpha & \to & Z^n
\end{array}
\]

Throughout this construction we have included the use of the trivial sphere i.e., \( S^{-1} = \emptyset, B^0 = \{ \text{pt} \} \). Define \( p_n: Z^n \to Y \) by \( p_n g_n = p_{n-1}, p_n in_2 = \prod s_\alpha \), let \( Z = \text{colim} Z^n, p = \text{colim} p_n \) and \( g = \text{colim} g_n g_{n-1} \cdots g_0 \). One may check that \( g \) has LLP with respect to each trivial w-fibration and by the small object argument [8, 3.4, Remark] \( p \) is a trivial w-fibration.

THEOREM 5.4. With the structure defined above (Definition 5.1) J-Top is a closed model category.

PROOF. One quickly checks the axioms for a closed model category [8, 3.1] using 5.3 or its clone to verify the factorization axiom \( M2 \).

We let \( \text{Ho} J\text{-Top} \) be \( J\text{-Top} \) localized at the weak equivalences. We aim to show that \( \text{Ho} J\text{-Top} \) is equivalent to the homotopy theory of \( J\text{-CW} \) complexes. First we need the following.

LEMMA 5.5. Let \( X = \text{colim} X_n \) taken over a system of \( J\text{-cofibrations} \) such that each \( X_n \) has the \( J\text{-homotopy type} \) of a \( J\text{-CW} \) complex. Then \( X \) has the \( J\text{-homotopy type} \) of a \( J\text{-CW} \) complex.

PROOF. Replace the colimit by the telescope [6, 1.26] and use the homotopy invariance of the homotopy colimit (Theorem 1.2).

The following proposition follows easily.

THEOREM 5.7 (APPROXIMATION THEOREM). There is a functor \( \Gamma : J-\text{Top} \to J-\text{Top} \) and natural transformation \( p : \Gamma \to \text{id} \) such that for each \( X \in J-\text{Top} \), \( \Gamma X \) is a J-complex, and \( p_X \) is a trivial w-fibration.

PROOF. Using 5.3 factor the map \( \phi \subseteq X \) into \( \phi \subseteq \Gamma X \to X \) where \( \phi \) is the empty subfunctor of \( X \). Then by the construction in 5.3 we see that \( X \) is a J-complex, \( p_X \) is a trivial fibration, \( \Gamma \) is a functor, and \( p \) a natural transformation.

The following corollary is immediate from 5.6 and 5.7.

COROLLARY 5.8. The category \( \text{Ho} J-\text{Top} \) is equivalent to the category of J-CW complexes modulo homotopy.

REMARK 5.9. (i) In [9, Theorem 1, p. 412] Spanier makes use of Brown’s representability theorem [9, Theorem 11, p. 410] to construct CW approximations in the category Top. In our construction we do not need Brown’s theorem and furthermore we construct the useful approximating functor \( \Gamma \) directly on \( J-\text{Top} \). We believe this is an improvement over Spanier’s construction. (ii) In [5] Heller describes a somewhat different homotopy structure on \( J-\text{Top} \). One may check that Heller’s localization \( \text{Ho}_w \text{Top} \) of [5, Section 7] is equivalent to our \( \text{Ho} J-\text{Top} \). It follows that many of the results of [5] (homotopy Kan extensions, etc.) may be applied to \( \text{Ho} J-\text{Top} \).

6. Elmendorf’s Theorem. The purpose of this section is to prove a useful result in equivariant homotopy theory originally proved by Elmendorf in [4] by a different method.

Let \( G \) be a topological group and let \( G-\text{Top} \) be the category of right \( G \)-spaces in \( \text{Top} \). Let \( O_G \) be the topological category of canonical right orbits. An object of \( O_G \) is a closed subgroup \( H \subseteq G \) and \( O_G(H, K) = G-\text{Top}(G/H, G/K) \) is given the compact open topology. Observe that there is a natural bijection \( G-\text{Top}(G/H, G/K) \cong [G/K]^H \). Where the right side is the \( H \) fixed point set of the right orbit \( G/K \). This bijection is a homeomorphism if we impose (as we always do) the compactly generated topology on all spaces in sight. There is a full and faithful functor \( \Phi : G-\text{Top} \to O_G-\text{Top} \) which views each \( X \in G-\text{Top} \) as a continuous diagram \( \Phi(X) \) of fixed point sets. \( \Phi(X) \) is defined by setting \( \Phi(X)(H) = G-\text{Top}(G/H, X) \). That is \( \Phi(X) \) is the continuous functor \( G-\text{Top}(\ , X) \) on \( O_G \). Compare [4, Section 1]. We call \( f : X \to Y \) a \( G \)-weak equivalence (\( G \)-fibration) if \( \Phi(f) \) is a weak equivalence (w-fibration in \( O_G-\text{Top} \)).

In \( G-\text{Top} \) there is a well-known theory of \( G \)-complexes (\( G \)-CW-complexes) that uses cells of the form \( B^n \times G/H \). See [12, Section 3] for a discussion of equivariant cellular theory. Observe that under the functor \( \Phi, B^n \times G/H \) goes to \( B^n \times O_G(\ , G/H) \), i.e., \( B^n \) cross a representable functor.

We need the following lemma for the argument below.
LEMMA 6.1. If
\[
\begin{array}{c}
B \\ Y
\end{array} \longrightarrow \begin{array}{c}
C \\ X
\end{array}
\]
is a pushout in $G$-Top with $i$ a closed inclusion then
\[
\begin{array}{c}
\Phi B \\ \Phi Y
\end{array} \longrightarrow \begin{array}{c}
\Phi C \\ \Phi X
\end{array}
\]
is a pushout in $O_G$-Top.

PROOF. Stripping away the topology we see this holds on the set level since every $G$-set is a coproduct of orbits. One may then check that the topologies agree.

THEOREM 6.2. Each $O_G$-complex ($O_G$-CW-complex) $Y \in O_G$-Top is isomorphic to $\Phi X$ where $X$ is a $G$-complex ($G$-CW-complex) in $G$-Top. It follows that $\Phi$ is an isomorphism between the categories of $G$-complexes ($G$-CW-complexes) and $O_G$-complexes ($O_G$-CW-complexes).

PROOF. The assertion follows from 6.1 and the fact that $\Phi$ is full, faithful and preserves ascending unions.

THEOREM 6.3. There is a functor $A : O_G$-Top $\rightarrow G$-Top and natural transformation $\tau : \Phi A \rightarrow \text{id}$ such that $\Phi A X$ is an $O_G$-complex and $\tau_X$ is a trivial fibration for each $X \in O_G$-Top. It follows that there is an equivalence of categories $\text{Ho} O_G$-Top $\sim \text{Ho} G$-Top where $\text{Ho} G$-Top is $G$-Top localized at the weak equivalences in $G$-Top.

PROOF. We construct $A$ and $\tau$ using the functor $\Gamma$ and transformation $\rho$ given in 5.7. The result follows from 5.8 and 6.2.

COROLLARY 6.4. Let $Y \in G$-Top be $G$ homotopically equivalent to a $G$-CW complex. Then for any $X \in O_G$-Top, $hG$-Top$(Y, AX) \simeq hO_G$-Top$(\Phi Y, X) \simeq \text{Ho} O_G$-Top$(\Phi Y, X)$.

PROOF. This follows from 6.3 and generalities about closed model categories.

REMARK 6.5. (i) In [4] Elmendorf assumes $G$ is a compact Lie group and uses a generalized bar construction to obtain his version of 6.3 and 6.4. Let $C : O_G$-Top $\rightarrow G$-Top be the functor defined by Elmendorf [4, Theorem 1]. For $X \in O_G$-Top there is a natural $G$ weak equivalence $AX \rightarrow CX$ which is a $G$ homotopy equivalence if $X$ is regular in the sense of Elmendorf. Clearly the functors $A$ and $C$ are closely related.

(ii) The importance of having the approximation functor $A$ given above is demonstrated by several applications given by Elmendorf in [4, Section 2]. For example consider the following. Let $\mathcal{F}$ be an orbit family in $G$ and define $T \in O_G$-Top by:

\[
T(H) = \begin{cases} 
\text{one point} & \text{if } H \in \mathcal{F} \\
\text{empty} & \text{otherwise.}
\end{cases}
\]
Then \( AT = E\mathcal{F} \) is a universal \( \mathcal{F} \)-space and \( B\mathcal{F} = \text{Ho colim} \, T = \text{colim} \, \Gamma T = E\mathcal{F} / G \) is a classifying space for the orbit family \( \mathcal{F} \). If \( \mathcal{F} \) consists of the single trivial subgroup of \( G \) then \( B\mathcal{F} = BG \) is a classifying space for principal \( G \) bundles.

(iii) Let \( M: O_G \rightarrow \text{Ab} \) be a coefficient system on \( O_G \). One defines equivariant cohomology with coefficients \( M \) denotes \( H^*_G(X; M) \) by setting \( H^*_G(X; M) = H^*(\Phi X; M) \) for \( X \in G\text{-Top} \). The results of Section 4 show this definition gives a reasonable cohomology theory on \( G\text{-Top} \). Observe that under suitable conditions this theory agrees with Illman’s equivariant singular cohomology. See [7, Theorem 3.11].

REFERENCES


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