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HOMOTOPY THEORY OF DIAGRAMS AND *CW*-COMPLEXES OVER A CATEGORY

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Introduction. The purpose of this paper is to introduce the notion of a CW complex over a topological category. The main theorem of this paper gives an equivalence between the homotopy theory of diagrams of spaces based on a topological category and the homotopy theory of CW complexes over the same base category.

A brief description of the paper goes as follows: in Section 1 we introduce the homotopy category of diagrams of spaces based on a fixed topological category. In Section 2 homotopy groups for diagrams are defined. These are used to define the concept of weak equivalence and J-n equivalence that generalize the classical definition. In Section 3 we adapt the classical theory of CW complexes to develop a cellular theory for diagrams. In Section 4 we use sheaf theory to define a reasonable cohomology theory of diagrams and compare it to previously defined theories. In Section 5 we define a closed model category structure for the homotopy theory of diagrams. We show this Quillen type homotopy theory is equivalent to the homotopy theory of J-CW complexes. In Section 6 we apply our constructions and results to prove a useful result in equivariant homotopy theory originally proved by Elmendorf by a different method.

1. **Homotopy theory of diagrams.** Throughout this paper we let Top be the cartesian closed category of compactly generated spaces in the sense of Vogt [10]. Let J be a small topological category over Top with discrete object space and J-Top the category of continuous contravariant Top valued functors on J. Note that the category J-Top is naturally enriched in Top. See Dubuc [2] for the framework of enriched category theory. We assume the reader is familiar with the standard constructions in Top as in [10] and the standard functor calculus on J-Top as in [5, Section 1].

We let I be the unit interval in Top. If X and Y are diagrams then a homotopy from X to Y is a morphism $H: I \times X \longrightarrow Y$ of J-Top where $I \times X$ is the functor defined on objects $j \in |J|$ by $(I \times X)(j) = I \times X(j)$ and similarly for morphisms of J. In the usual way homotopy defines an equivalence relation on the morphisms of J-Top that gives rise to the quotient homotopy category hJ-Top. We denote the homotopy classes of morphisms

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from X to Y by hJ-Top(X, Y) abbreviated $h(X, Y)$. An isomorphism in hJ-Top is called a J-homotopy equivalence.

A morphism of J-Top is called a *J-cofibration* if it has the J homotopy extension property, abbreviated *J-HEP*. The basic facts about *cofibrations* in Top apply readily to *J-cofibrations.* See [5, Section 2].

The following results from [6] apply formally to the category J-Top.

THEOREM 1.1 (INVARIANCE OF PUSHOUTS). *Suppose that we have a commutative diagram:*

$$\begin{tikzcd}
 & X' \arrow[rr] & & Z' \\
X \arrow[ur, "\alpha"] \arrow[rr] & & Z \arrow[ur, "\theta"] & \\
 & A' \arrow[uu, "i'"'] \arrow[rr, "f'"] & & Y' \arrow[uu] \\
A \arrow[uu, "i"] \arrow[ur, "\beta"] \arrow[rr, "f"'] & & Y \arrow[uu] \arrow[ur, "\gamma"'] &
\end{tikzcd}$$

in which i and i' are J-cofibrations, f and f' are arbitrary morphisms in J-Top. α, β and γ are homotopy equivalences and the front and back faces are pushouts. Then θ is also a homotopy equivalence (θ being the induced map on pushouts).

THEOREM 1.2 (INVARIANCE OF COLIMITS OVER COFIBRATIONS). *Suppose given a homotopy commutative diagram*

$$\begin{tikzcd}
X^0 \arrow[r, "i_0"] \arrow[d, "f^0"] & X^1 \arrow[r, "i_1"] \arrow[d, "f^1"] & \cdots \arrow[r] & X^k \arrow[r, "i_k"] \arrow[d, "f^k"] & \cdots \\
Y^0 \arrow[r, "j_0"] & Y^1 \arrow[r, "j_1"] & \cdots \arrow[r] & Y^k \arrow[r, "j_k"] & \cdots
\end{tikzcd}$$

in J-Top where the i_k and j_k are J-cofibrations and the f^k are homotopy equivalences. Then the map $\operatorname{colim}_k f^k\colon \operatorname{colim}_k X^k \longrightarrow \operatorname{colim}_k Y^k$ *is a homotopy equivalence.*

2. **Homotopy groups.** Let I^n be the topological n-cube and ∂I^n its boundary.

DEFINITION 2.1. By a J-Top pair (X, Y), we mean an object X in J-Top together with a subobject $Y \subseteq X$. Morphisms of pairs are defined in the obvious way. A similar definition will be used for triples, n-ads etc. Let $\varphi\colon j \longrightarrow Y$ be a morphism in J-Top where $j \in |J|$ is viewed as the representable functor J-Top$(\ ,j)$. By Yoneda's theorem φ is completely determined by the point $\varphi(\mathrm{id}_j) = y_0 \in Y(j)$. For each $n \geq 0$, define $\pi_n^j(X, Y, \varphi) = h\big((I^n, \partial I^n, \{0\}) \times j, (X, Y, Y)\big)$ where $y_0 = \varphi(\mathrm{id}_j) \in Y(j)$ serves as a basepoint, and all homotopies are homotopies of triples relative to φ. The reader may formulate a similar definition for the absolute case $\pi_n^j(X, \varphi)$. For $n = 0$ we adopt the convention that $I^0 = \{0, 1\}$ and $\partial I^0 = \{0\}$ and proceed as above. These constructions extend to covariant functors on J-Top. From now on we shall often drop φ from the notation $\pi_n^j(X, Y, \varphi)$.

The proof of the following proposition follows immediately from Yoneda's lemma.

PROPOSITION 2.2. *There are natural equivalences* $\pi_n^j(X) \simeq \pi_n\big(X(j)\big)$ *and* $\pi_n^j(X, Y) \simeq \pi_n\big(X(j), Y(j)\big)$ *which preserve the (evident) group structure when* $n \geq 1$ *(for the absolute case; the relative case requires* $n \geq 2$*).*

As a direct consequence of 2.2 we obtain the long exact sequences:

PROPOSITION 2.3. *For* (X, Y) *and* j *as in 2.1, there exist natural boundary maps* ∂ *and long exact sequences*

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \pi_n^j(X,Y) & \xrightarrow{\partial} & \pi_{n-1}^j(Y) & \longrightarrow & \pi_{n-1}^j(X) & \longrightarrow & \cdots \\ & \longrightarrow & \pi_0^j(Y) & \longrightarrow & \pi_0^j(X) & & & & \end{array}$$

of groups up to $\pi_1^j(Y)$ *and pointed sets thereafter.*

DEFINITION 2.4. A map $e: (X, Y) \longrightarrow (X', Y')$ of pairs in J-Top is called a *J-n-equivalence* if $e(j): \big(X(j), Y(j)\big) \longrightarrow \big(X'(j), Y'(j)\big)$ is an *n-equivalence* in Top for each $j \in |J|$. A map e will be called a *weak equivalence* if e is a J-n-equivalence for each $n \geq 0$. Observe that e is a J-n-equivalence if for every $j \in |J|$ and $\varphi: j \longrightarrow Y$, $e_*: \pi_p^j(X, Y, \varphi) \longrightarrow \pi_p^j(X', Y', e\varphi)$ is an isomorphism for $0 \leq p < n$ and an epimorphism for $p = n$. The reader may easily formulate a similar definition for morphisms $e: X \longrightarrow X'$ of J-Top (the absolute case).

3. **Cellular theory.** In this section we adapt the general treatment of classical homotopy theory and CW-complexes given in [9, Chapter 7] and [6] to develop a good theory of CW-complexes over the topological category J.

Let B^{n+1} be the topological $n+1$-ball and S^n the topological n-sphere. Of course, these spaces are homeomorphic to I^{n+1} and ∂I^{n+1} respectively. We shall construct all complexes over J by the process of attaching cells of the form $B^{n+1} \times j$ by attaching morphisms with domain $S^n \times j$. The formal definition goes as follows:

DEFINITION 3.1. A J-complex is an object X of J-Top with a decomposition $X = \operatorname{colim}_{p \geq 0} X^p$ where $X^0 = \amalg_{\alpha \in A_0} B^{n_\alpha} \times j_\alpha$, $X^p = X^{p-1} \cup_f \Big(\amalg_{\alpha \in A_p} B^{n_\alpha} \times j_\alpha\Big)$ for some attaching morphism $f: \amalg_{\alpha \in A_p} S^{n_\alpha - 1} \times j_\alpha \longrightarrow X^{p-1}$ and for each $p \geq 0$, $\{j_\alpha \mid \alpha \in A_p\}$ is a collection of objects (representable functors) of J. We call X a J-CW-complex if X is a J-complex as above and for all $p \geq 0$ and all $\alpha \in A_p$ we have $n_\alpha = p$.

A J-subcomplex and a relative J complex are now defined in the obvious way. Without further comment we adopt for J-CW-complexes the standard terminology for CW-complexes. See [9, Chapter 7] and [6].

The following technical lemma and its proof are due to May [6,3.5.1].

LEMMA 3.2. *Suppose that* $e: Y \longrightarrow Z$ *is a* J-n-*equivalence. Then we can complete the following diagram in* J-*Top:*

$$\begin{tikzcd}
\partial I^n \times j \arrow[rr, "i_0"] \arrow[dd] & & \partial I^n \times I \times j \arrow[dl, "h"'] \arrow[dd] & & \partial I^n \times j \arrow[ll, "i_1"'] \arrow[dl, "g"'] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"'] & \\
I^n \times j \arrow[rr, "i_0"'] \arrow[ur] & & I^n \times I \times j \arrow[ul, dashed, "\tilde{h}"] & & I^n \times j \arrow[ll, "i_1"] \arrow[ul, dashed, "\tilde{g}"']
\end{tikzcd}$$

THEOREM 3.3 (J-HELP). *If (X,A) is a relative J-CW complex of dimension $\leq n$ and $e\colon Y \longrightarrow Z$ is a J-n-equivalence then we can complete the following diagram in J-Top:*

$$\begin{tikzcd}
A \arrow[rr] \arrow[dd] & & A \times I \arrow[dl] \arrow[dd] & & A \arrow[ll] \arrow[dl] \arrow[dd] \\
& Z & & Y \arrow[ll, "e"'] & \\
X \arrow[rr] \arrow[ur] & & X \times I \arrow[ul, dashed] & & X \arrow[ll] \arrow[ul, dashed]
\end{tikzcd}$$

PROOF. This follows by induction on $\dim(X,A)$, applying 3.2 cell by cell at each stage.

The proofs of the following Whitehead theorem and cellular approximation theorem are formal modifications of the proofs given in [6].

THEOREM 3.4 (WHITEHEAD). *(i) Suppose X is a J-CW complex, and that $e\colon Y \longrightarrow Z$ is a J-n-equivalence. Then $e_*\colon h(X,Y) \longrightarrow h(X,Z)$ is an isomorphism if* $\dim X < n$ *and an epimorphism of* $\dim X = n$. *(ii) If $e\colon Y \longrightarrow Z$ is a weak equivalence, and if X is any J-CW complex, then $e_*\colon h(X,Y) \longrightarrow h(X,Z)$ is an isomorphism.*

THEOREM 3.5 (CELLULAR APPROXIMATION). *Suppose that X is a J-CW complex, and that A is a sub-J-CW complex of X. Then, if $f\colon X \longrightarrow Y$ is a morphism of J-Top which is J-cellular when restricted to A, we can homotope f,* $\mathrm{rel} f|A$ *to a J-cellular morphism $g\colon X \longrightarrow Y$.*

Next we discuss the local properties of J-CW-complexes. First we develop some preliminary concepts. Let X be in J-Top and for each $j \in |J|$ let $t_j\colon X(j) \longrightarrow \operatorname{colim}_J X$ be the natural map of $X(j)$ into the colimit. Observe that for each morphism $s\colon i \longrightarrow j$ of J, $t_j = t_i X(s)$. For each subspace $A \subseteq \operatorname{colim}_J X$ we define $\check{A}(j) = t_j^{-1}(A)$ and for a given $s\colon i \longrightarrow j$ we define $\check{A}(s) = X(s)|\check{A}(j)$, the restriction of the continuous map $X(s)$ to the subspace $\check{A}(j)$. We apply the K-ification functor to assure that all spaces defined above are compactly generated. One quickly checks that $\check{A} \in J$-Top, $\operatorname{colim}_J \check{A} = A$, and there is a natural inclusion morphism $\check{A} \longrightarrow X$. To simplify notation from now on we write X/J for $\operatorname{colim}_J X$.

DEFINITION 3.6. By a special pair in J-Top we mean an ordered pair (X,A) where $X \in J$-Top and $A \subseteq X/J$. We call a special pair (X,A) a *J-neighborhood retract* pair (abbreviated J-NR) if there exist U an open subset of X/J such that $A \subseteq U$ and there exists a retraction morphism $r\colon \check{U} \longrightarrow \check{A}$. (X,A) is called a *J-neighborhood deformation* *retract* pair (abbreviate J-NDR) if (X, A) is a J-NR and the morphism r is a *J-deformation retract.*

Let X be a J-CW complex. The functor colim_J sends cells $B^p \times j$ to cells B^p and preserves the cellular decomposition of X. For this reason X/J has the natural structure of a CW-complex in Top with all its attaching maps being images under colim_J of the corresponding attaching morphisms in J-Top. One may also check that if A is a subcomplex of X/J then $\check{A}$ has the natural structure of a subcomplex of X. In particular if A^p is the p-skeleton of X/J then $\check{A}^p = X^p$ is the p-skeleton of X.

THEOREM 3.7 (LOCAL CONTRACTIBILITY). *Let (X, A) be a special pair in J-Top with X a J-CW complex and $A = \{a\}$, $a \in X/J$. Then there exists a unique object $j \in J$ such that $\check{A} \simeq j$ (j viewed as a representable functor) and (X, A) is a* J-NDR *pair.*

PROOF. Suppose $a \in (X/J)^p \backslash (X/J)^{p-1}$, the p-skeleton minus the $p-1$ skeleton of X/J. Then there is a unique attaching morphism f in J-Top

$$\begin{array}{c} f\colon S^{p-1} \times j \longrightarrow X^{p-1} \\ \downarrow \\ B^p \times j \end{array}$$

with a in the interior of B^p. It follows that $\check{A} \simeq j$ for the unique choice of j given above. To construct the required neighborhood U first take an open ball U_1 contained in the interior of B^p and centered at a. Then U_1 is a neighborhood in $(X/J)^p$ contracting to A. One then extends U_1 inductively cell by cell by a well known procedure to construct the required neighborhood U.

THEOREM 3.8. *Let (X, A) be a special pair in J-Top with X a J-CW complex and A an arbitrary subcomplex of X/J. Then (X, A) is a* J-NDR *pair.*

PROOF. It follows from 3.3 that $\check{A} \subseteq X$ is a J-cofibration. The result then follows from a well known argument of Puppe. See [5, Lemma 4.3, p. 193].

4. **Cohomology.** In this section we use sheaf theory to construct a cohomology theory on J-Top satisfying a suitably formulated set of Eilenberg-Steenrod axioms. We refer the reader to Bredon [1] for the basic definitions and terminology of sheaf theory.

DEFINITION 4.1. By a contravariant coefficient system M on J we mean a continuous contravariant functor $M\colon J \longrightarrow \mathrm{Ab}$ where Ab is the category of discrete abelian groups. Observe that every contravariant coefficient system M is a homotopy invariant functor in the following sense. If $f, g\colon j \longrightarrow j'$ are homotopic (as morphisms of representable functors in J-Top) then $M(f) = M(g)$.

Let $X \in J$-Top and let M be a coefficient system on J. We define a presheaf of abelian groups M^X over X/J as follows: for $A \subseteq X/J$ define $M^X(A) = J\text{-Top}(\check{A}, M)$ equipped with its natural discrete abelian group structure. If $B \subseteq A$ there is a natural restriction homomorphism $M^X(A) \longrightarrow M^X(B)$ and one easily checks that M^X is a sheaf of abelian groups over X/J. Let $f: X \longrightarrow Y$ be a morphism in J-Top with $f/J: X/J \longrightarrow Y/J$ the induced map in Top. There is a natural f/J-cohomomorphism of sheaves $\bar{f}: M^Y \longrightarrow M^X$ given by the obvious composition with f.

DEFINITION 4.2. Let $X \in J$-Top, ψ a family of supports on X/J and M a coefficient system on J. We define $H^n_\psi(X; M) = H^n_\psi(X/J; M^X)$ where the right side is sheaf cohomology with supports ψ as defined in [1, Chapter II]. Given a morphism $f: X \longrightarrow Y$ in J-Top, we let f^* be the homomorphism induced in cohomology by f. Given a special pair (X, A) we define the relative chomology $H^n_\psi(X, A; M) = H^n_\psi(X/J, A; M^X)$ where the right side is relative sheaf cohomology.

EXAMPLE 4.3. Let G be an abelian group and define the constant coefficient system M with value G by setting $M(s) = \mathrm{id}_G$ for any morphism s of J. Then for any $X \in J$-Top one quickly sees that $H^*(X; M) = H^*(X/J; G)$ where the right side is sheaf cohomology with constant coefficients G. Note that absence of a specified support family always means supports in the family of all closed sets.

DEFINITION 4.4. A special pair (X, A) in J-Top is called *acceptable* if for each coefficient system M on J the sheaf M^A over A is the restriction of the sheaf M^X to the subspace A. Note that if (X, A) is a J-NR pair or if X is locally J-NR then (X, A) is acceptable. In particular any special pair (X, A) where X is a J-CW complex is acceptable by 3.7.

All special pairs considered in the rest of this section will be assumed acceptable. We impose this condition to obtain a good theory of relative cohomology.

Note that *a* supports preserving morphism $f: (X, A) \longrightarrow (Y, B)$ naturally induces a homomorphism f^* in relative cohomology. Hence $H^*_\psi(\;\; ; M)$ becomes a candidate for a reasonable cohomology theory on J-Top. The following theorem states and verifies a suitable set of Eilenberg-Steenrod axioms for the theory $H^*(\;\; ; M)$.

THEOREM 4.5.

(1) (Dimension) $H^n(j; M) = \begin{cases} M(j) & n = 0 \\ 0 & n > 0 \end{cases}$ *for each* $j \in J$ *viewed as a representable functor.*

(2) For each special pair (X, A) *in J-Top there is induced a suitable long exact sequence in cohomology with arbitrary supports.*

(3) (Excision) If A and B are subsets of X/J *with* $\bar{B} \subseteq \mathrm{int}A$ *then the inclusion* $i: (X - \check{B}, A - B) \longrightarrow (X, A)$ *induces an isomorphism in cohomology for any support family.*

(4) (Homotopy) If f and g are morphisms of special pairs in J-Top that are homotopic via a support preserving homotopy then $f^* = g^*$.

(5) If $(X, A) = \coprod_\alpha (X_\alpha, A_\alpha)$ *then there is a natural isomorphism induced by the injections into the coproduct,*

$$H^*(X, A; M) \simeq \prod_\alpha H^*(X_\alpha, A_\alpha; M).$$

PROOF. (1) follows from Yoneda's lemma. (2) follows from [1, Chapter 2, Section 12]. (3) follows from [1, Theorem 12.5, p. 61]. (4) follows from [1, Theorem 11.2, p. 55]. (5) is easy to check directly.

If X is a J-CW complex we define cellular cochains $C^n(X; M) = H^n\big(X^n, (X^{n-1}/J); M\big)$. Observe that $C^n(X; M) = \prod_\alpha M(j_\alpha)$ where $B^n \times j_\alpha$, $\alpha \in A_n$ is the family of all n-cells of X. In the usual way one makes $C^*(X; M)$ into a cochain complex using the coboundary operator of a triple. This construction yields the cellular cohomology theory $H^*_{\text{cel}}(\ \ ; M)$ defined for J-CW pairs.

We may adapt the classical proof to show:

PROPOSITION 4.6. *$H^*(\ \ ; M)$ is naturally isomorphic to $H^*_{\text{cel}}(\ \ ; M)$ on the category of J-CW pairs.*

REMARK 4.7. (i) The cellular chomology theory is useful for developing an obstruction theory in J-Top. (ii) Following a well known argument due to Milnor it is possible to prove a uniqueness theorem for cohomology theories defined on the category of J-CW complexes. (iii) In [11] Vogt defines the singular cohomology on J-Top and shows it satisifies a suitable set of axioms. By the above mentioned uniqueness theorem Vogt's singular cohomology agrees with our sheaf cohomology on the category of J-CW complexes.

5. Closed model structure on J-Top.

In [8] Quillen defines a closed model structure for homotopy theory in Top. In this section we emulate this construction to define a closed model category structure on J-Top.

DEFINITION 5.1. A morphism $f: X \longrightarrow Y$ of J-Top is called a *weak fibration,* abbreviated w-fibration, if for each $j \in J$, $f(j): X(j) \longrightarrow Y(j)$ is a Serre fibration in Top. See [9, p. 374] for a discussion of Serre fibrations. Observe that f is a w-fibration if f has the homotopy lifting property for all objects of the form $I^n \times j$. A morphism f is called a weak equivalence if f is a *weak equivalence* as defined in Section 2. A morphism $g: A \longrightarrow B$ is called a *weak cofibration,* abbreviated w-cofibration if g has the left lifting property (LLP) for each trivial w-fibration $f: X \longrightarrow Y$ (a w-fibration that is also a weak equivalence). This means one can always fill in the dotted arrow:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"'] & X \arrow[d, "f"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

REMARK 5.2. (i) The inclusion of a sub-J-complex into a J-complex is always both a J-cofibration and a w-cofibration. (ii) A w-fibration is trivial iff it has the right lifting property (RLP) for each w-cofibration of the form $S^n \times j \longrightarrow B^{n+1} \times j$. [8, 3.2, Lemma 2].

LEMMA 5.3 (QUILLEN'S FACTORIZATION LEMMA). *Any morphism $f: X \longrightarrow Y$ of J-Top may be factored $f = pg$ where g is a w-cofibration and p is a trivial w-fibration.*

PROOF. We construct a diagram

$$\begin{tikzcd} X \arrow[r, "g_0"] \arrow[dr, "f"'] & Z^0 \arrow[r, "g_1"] \arrow[d, "p_0"] & Z^1 \arrow[r] \arrow[dl, "p_1"] & \cdots \\ & Y & & \end{tikzcd}$$

as follows: let $Z^{-1} = X$ and $p_{-1} = f$, and having obtained Z^{n-1} consider the set of all diagrams of the form

$$\begin{array}{ccc} S^{q_\alpha} \times j_\alpha & \xrightarrow{t_\alpha} & Z^{n-1} \\ \downarrow & & \downarrow{\scriptstyle p_{n-1}} \\ B^{q_\alpha+1} \times j_\alpha & \xrightarrow{s_\alpha} & Y \end{array}$$

where we have indexed this set of diagrams by A_n and $\alpha \in A_n$. Define $g_n: Z^{n-1} \longrightarrow Z^n$ by the pushout diagram

$$\begin{array}{ccc} \coprod_{\alpha \in A_n} S^{q_\alpha} \times j_\alpha & \xrightarrow{\coprod t_\alpha} & Z^{n-1} \\ \downarrow & & \downarrow{\scriptstyle g_n} \\ \coprod_{\alpha \in A_n} B^{q_\alpha+1} \times j_\alpha & \longrightarrow & Z^n \end{array}$$

Throughout this construction we have included the use of the trivial sphere i.e., $S^{-1} = \emptyset$, $B^0 = \{\text{pt}\}$. Define $p_n: Z^n \longrightarrow Y$ by $p_n g_n = p_{n-1}$, $p_n in_2 = \coprod s_\alpha$, let $Z = \operatorname{colim} Z^n$, $p = \operatorname{colim} p_n$ and $g = \operatorname{colim} g_n g_{n-1} \cdots g_0$. One may check that g has LLP with respect to each trivial w-fibration and by the small object argument [8, 3.4, Remark] p is a trivial w-fibration.

THEOREM 5.4. *With the structure defined above (Definition 5.1) J-Top is a closed model category.*

PROOF. One quickly checks the axioms for a closed model category [8, 3.1] using 5.3 or its clone to verify the factorization axiom $\underline{M2}$.

We let Ho J-Top be J-Top localized at the weak equivalences. We aim to show that Ho J-Top is equivalent to the homotopy theory of J-CW complexes. First we need the following.

LEMMA 5.5. *Let $X = \operatorname{colim} X_n$ taken over a system of J-cofibrations such that each X_n has the J-homotopy type of a J-CW-complex. Then X has the J-homotopy type of a J-CW complex.*

PROOF. Replace the colimit by the telescope [6, 1.26] and use the homotopy invariance of the homotopy colimit (Theorem 1.2).

The following proposition follows easily.

PROPOSITION 5.6. *Each J-complex is of the J-homotopy type of a J-CW-complex.*

THEOREM 5.7 (APPROXIMATION THEOREM). *There is a functor* $\Gamma: J\text{-}Top \longrightarrow J\text{-}Top$ *and natural transformation* $p: \Gamma \longrightarrow \mathrm{id}$ *such that for each* $X \in J\text{-}Top$, ΓX *is a J-complex, and* p_X *is a trivial w-fibration.*

PROOF. Using 5.3 factor the map $\phi \subseteq X$ into $\phi \subseteq \Gamma X \longrightarrow X$ where ϕ is the empty subfunctor of X. Then by the construction in 5.3 we see that X is a J-complex, p_X is a trivial fibration, Γ is a functor, and p a natural transformation.

The following corollary is immediate from 5.6 and 5.7.

COROLLARY 5.8. *The category* Ho J*-Top is equivalent to the catgory of J-CW complexes modulo homotopy.*

REMARK 5.9. (i) In [9, Theorem 1, p. 412] Spanier makes use of Brown's representability theorem [9, Theorem 11, p. 410] to construct CW approximations in the category Top. In our construction we do not need Brown's theorem and furthermore we construct the useful approximating functor Γ directly on J-Top. We believe this is an improvement over Spanier's construction. (ii) In [5] Heller describes a somewhat different homotopy structure on J-Top. One may check that Heller's localization $\mathrm{Ho}_w\,\mathrm{Top}^J$ of [5, Section 7] is equivalent to our Ho J-Top. It follows that many of the results of [5] (homotopy Kan extensions, etc.) may be applied to Ho J-Top.

6. **Elmendorf's Theorem.** The purpose of this section is to prove a useful result in equivariant homotopy theory originally proved by Elmendorf in [4] by a different method.

Let G be a topological group and let G-Top be the category of right G-spaces in Top. Let O_G be the topological category of canonical right orbits. An object of O_G is a closed subgroup $H \subseteq G$ and $O_G(H, K) = G\text{-Top}(G/H, G/K)$ is given the compact open topology. Observe that there is a natural bijection $G\text{-Top}(G/H, G/K) \simeq [G/K]^H$. Where the right side is the H fixed point set of the right orbit G/K. This bijection is a homeomorphism if we impose (as we always do) the compactly generated topology on all spaces in sight. There is a full and faithful functor $\Phi: G\text{-Top} \longrightarrow O_G\text{-Top}$ which views each $X \in G$-Top as a continuous diagram $\Phi(X)$ of fixed point sets. $\Phi(X)$ is defined by setting $\Phi(X)(H) = G\text{-Top}(G/H, X)$. That is $\Phi(X)$ is the continuous functor $G\text{-Top}(\ \ , X)$ on O_G. Compare [4, Section 1]. We call $f: X \longrightarrow Y$ a G-weak equivalence (G-fibration) if $\Phi(f)$ is a weak equivalence (w-fibration in O_G-Top).

In G-Top there is a well-known theory of G-complexes (G-CW-complexes) that uses cells of the form $B^n \times G/H$. See [12, Section 3] for a discussion of equivariant cellular theory. Observe that under the functor Φ, $B^n \times G/H$ goes to $B^n \times O_G(\ \ , G/H)$, i.e., B^n cross a representable functor.

We need the following lemma for the argument below.

LEMMA 6.1. *If*

$$\begin{array}{ccc} B & \xrightarrow{i} & C \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

is a pushout in G-Top with i a closed inclusion then

$$\begin{array}{ccc} \Phi B & \xrightarrow{\Phi(i)} & \Phi C \\ \downarrow & & \downarrow \\ \Phi Y & \longrightarrow & \Phi X \end{array}$$

is a pushout in O_G-Top.

PROOF. Stripping away the topology we see this holds on the set level since every G-set is a coproduct of orbits. One may then check that the topologies agree.

THEOREM 6.2. *Each O_G-complex (O_G-CW-complex) $Y \in O_G$-Top is isomorphic to ΦX where X is a G-complex (G-CW-complex) in G-Top. It follows that Φ is an isomorphism between the categories of G-complexes (G-CW-complexes) and O_G-complexes (O_G-CW-complexes).*

PROOF. The assertion follows from 6.1 and the fact that Φ is full, faithful and preserves ascending unions.

THEOREM 6.3. *There is a functor A: O_G-Top $\longrightarrow$ G-Top and natural transformation $t\colon \Phi A \longrightarrow \mathrm{id}$ such that ΦAX is an O_G-complex and t_X is a trivial fibration for each $X \in O_G$-Top. It follows that there is an equivalence of categories* Ho O_G*-Top* $\sim$ Ho *G-Top where* Ho *G-Top is G-Top localized at the weak equivalences in G-Top.*

PROOF. We construct A and t using the functor Γ and transformation p given in 5.7. The result follows from 5.8 and 6.2.

COROLLARY 6.4. *Let $Y \in$ G-Top be G homotopically equivalent to a G-CW complex. Then for any $X \in O_G$-Top, hG-Top$(Y, AX) \simeq hO_G$-Top$(\Phi Y, X) \simeq$* Ho O_G*-Top*$(\Phi Y, X)$.

PROOF. This follows from 6.3 and generalities about closed model categories.

REMARK 6.5. (i) In [4] Elmendorf assumes G is a compact Lie group and uses a generalized bar construction to obtain his version of 6.3 and 6.4. Let C: O_G-Top $\longrightarrow$ G-Top be the functor defined by Elmendorf [4, Theorem 1]. For $X \in O_G$-Top there is a natural G weak equivalence $AX \longrightarrow CX$ which is a G homotopy equivalence if X is regular in the sense of Elmendorf. Clearly the functors A and C are closely related.

(ii) The importance of having the approximation functor A given above is demonstrated by several applications given by Elmendorf in [4, Section 2]. For example consider the following. Let $\mathcal{F}$ be an orbit family in G and define $T \in O_G$-Top by:

$$T(H) = \begin{cases} \text{one point} & \text{if } H \in \mathcal{F} \\ \text{empty} & \text{otherwise.} \end{cases}$$

Then $AT = E\mathcal{F}$ is a universal $\mathcal{F}$-space and $B\mathcal{F} = \mathrm{Ho\,colim}\, T = \mathrm{colim}\, \Gamma T = E\mathcal{F}/G$ is a classifying space for the orbit family $\mathcal{F}$. If $\mathcal{F}$ consists of the single trivial subgroup of G then $B\mathcal{F} = BG$ is a classifying space for principal G bundles.

(iii) Let $M\colon O_G \longrightarrow \mathrm{Ab}$ be a coefficient system on O_G. One defines equivariant cohomology with coefficients M denotes $H^*_G(X;M)$ by setting $H^*_G(X;M) = H^*(\Phi X;M)$ for $X \in G$-Top. The results of Section 4 show this definition gives a reasonable cohomology theory on G-Top. Observe that under suitable conditions this theory agrees with Illman's equivariant singular cohomology. See [7, Theorem 3.11].

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