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# Conformal Geodesics in Cartan Calculus

Masterarbeit

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**Abstract:**

The present work deals with conformal geodesics and their description using Cartan calculus.

In the first chapter we recall the definition of Cartan geometry and explain how conformal structures are in 1-1-correspondence with Cartan geometries of type $(G, P)$ with $G = \mathcal{O}(p+1, q+1)$ and $P = \mathrm{Stab}_{\mathbb{R}_+ \cdot l_-} \mathcal{O}(p+1, q+1)$. This is done using an explicit construction of the standard Tractor bundle associated to the conformal Cartan geometry. While the results are not new, existing proofs use different methods and the calculations using the Tractor bundle have not been published before.

We will then review the concepts of canonical curves and conformal geodesics. Following the main source [10], we give different characterizations of canonical curves and show that they are determined by their 2-jet in the $|1|$-graded case, carrying out the calculations for proofs sketched in the literature. We then summarize important properties of conformal geodesics and in particular present the details of a proof sketched in [4], to show that conformal geodesics are precisely those curves, which are locally geodesic and have vanishing Schouten tensor with respect to a metric in the conformal class.

Following this, we prove the main result, that the conformal geodesics of the conformal structure are exactly the canonical curves of the associated Cartan geometry. We give a new proof for this fact using Tractor calculus. This content was announced to appear in a forthcoming paper in [4]. The paper never appeared, so in a way this thesis can be seen as a completion of the survey on conformal geodesics in [4].

In the second chapter we give a proof that conformal embeddings are exactly Cartan embeddings for the associated Cartan geometries, again using Tractor calculus. While this fact certainly served as a motivation to study Cartan geometric embeddings (cf. [25]), we are not aware of an actual proof in the literature so far. We observe some properties of geometric boundaries of geometric embeddings. In patricular we improve a result of [25] to show that not only the accessible points, but even the highly accessible points are dense in the geometric boundary, as expected in [20].

Eventually we use conformal geodesics to show that the $\mathbb{R}^n$ with standard Euclidean metric has a unique conformal compactification, working out the details of a proof previously given in [20].

# Contents

# 1 Conformal Geometry as a Cartan Geometry

## 1.1 The Notion of a Cartan Geometry

In the course of this work we will use conformal geodesics to solve problems of compactification of manifolds. It turns out that Cartan geometries provide convenient means to describe these geodesics. Hence, our first task will be to define Cartan geometries and understand their basic notions. For this we will follow [10]. See also [39] for a concise introduction.

*Definition* 1.1. Let $M$ be a smooth manifold. Let $G$ be a Lie group and $P \subset G$ be a closed subgroup.

A *Cartan geometry of type* $(G, P)$ is a pair $(\mathcal{P}, \omega)$, where $\pi : \mathcal{P} \to M$ is a $P$-principal bundle and $\omega \in \Omega^1(\mathcal{P}, \mathfrak{g})$ is a $\mathfrak{g}$-valued 1-form which

(i) is $P$-equivariant, i.e. $R_p^* \omega = \mathrm{Ad}(p^{-1}) \circ \omega$ for all $p \in P$,

(ii) reproduces the generators of fundamental vector fields, i.e. $\omega(\tilde{X}) = X$ for all $X \in \mathfrak{p}$,

(iii) defines an absolute parallelism, i.e. $\omega_u : T_u\mathcal{P} \to \mathfrak{g}$ is a linear isomorphism for all $u \in \mathcal{P}$.

$\omega$ ist called a *Cartan connection.*

*Definition* 1.2. Let $(\mathcal{P}, \omega)$ be a Cartan geometry of type $(G, P)$ on $M$. For $X \in \mathfrak{g}$ the vector field $\omega^{-1}(X) \in \mathfrak{X}(\mathcal{P})$ given by

$$\omega^{-1}(X)(u) := \omega_u^{-1}(X) \quad \text{for all } u \in \mathcal{P}$$

is called a *constant vector field.*

*Example* 1.3. Given some Lie group $G$ and a closed subgroup $P \subset G$, there is a canonical example of a Cartan geometry of type $(G, P)$–the so called flat model.

Let $M := G/P$, $\mathcal{P} := G$ and $\pi : G \to G/P$ be the canonical projection. Let $\omega := \omega^{MC} \in \Omega^1(G, \mathfrak{g})$ be the *Maurer-Cartan-form*, that is

$$\omega_g^{MC}(X) := dL_{g^{-1}}(X) \in T_eG \text{ for } X \in T_gG. \tag{1.1}$$

Then $(\mathcal{P}, \omega)$ is a Cartan geometry on $M = G/P$.

## 1.2 Conformal Geometry as a Cartan Geometry

Our aim is to study conformal geometry in the language of Cartan geometry. To be able to do this, we will assign a Cartan geometry to a given conformal manifold. Also we will show that in this process no information is lost. That is, we can retrieve the original conformal structure from the newly assigned Cartan geometry.

To achieve this, we introduce the standard Tractor bundle, the idea of which was first described by T. Y. Thomas in [47] and partly named after him. We will follow the construction stated briefly in [10], made explicit in [5] and described in greater detail in [8]. It is worth noting that other canonical constructions exist, that would serve the same purpose, as described in [12] and [31].

Alternatively, one may choose to skip the introduction of a Tractor bundle and directly construct a Cartan geometry from the conformal structure as originally favored by É. Cartan in [17]. For a big class of reductions of the frame bundle N. Tanaka has shown a 1-1-correspondence between such reductions and Cartan connections of a certain type in [45] and [46]. A more condensed publication on these questions is the more recent [13], which uses the same notation as this work for the most part and should be more approachable for the modern reader. Remember that conformal structures or pseudo-Riemannian metrics can be equivalently described through $\mathrm{CO}(p, q)$ and $\mathcal{O}(p, q)$ reductions of the frame bundle respectively. A less abstract construction skipping the Tractor bundle can be found in [8], [22], [24] or [49].

First, let us repeat the basic notions of conformal geometry.

*Definition* 1.4. Let $(M, g)$, $(N, h)$ be two semi-Riemannian manifolds.

(i) A diffeomorphism $f : (M, g) \to (N, h)$ is said to be *conformal*, if some smooth map $\sigma : M \to \mathbb{R}$ exists, such that
$$f^*h = e^{2\sigma} g.$$
In this case $(M, g)$ and $(N, h)$ (or in short $g$ and $h$) are said to be *conformally equivalent* and $e^{2\sigma}$ is called *conformal factor*.

(ii) The relation
$$(M, g) \sim (N, h) :\Leftrightarrow (M, g) \text{ and } (N, h) \text{ are conformally equivalent}$$
is an equivalence relation. An equivalence class with respect to $\sim$ is said to be a *conformal class*. A set $c$ of all metrics that are conformally equivalent to a given metric $g$ on a fixed manifold $M$ is called a *conformal structure* on $M$.

*Example* 1.5. Let $(\mathbb{R}^n, g)$ be the $n$-space with standard metric and $(S^n, h)$ the standard sphere $S^n \subset \mathbb{R}^{n+1}$ with metric induced by $\mathbb{R}^{n+1}$. Consider the *stereographic embedding*
$$\begin{aligned} f : \mathbb{R}^n &\to S^n \\ (y_1, \ldots, y_n) &\mapsto \left( \frac{2y_1}{||y||^2 + 1}, \frac{2y_2}{||y||^2 + 1}, \ldots, \frac{2y_n}{||y||^2 + 1}, \frac{1 - ||y||^2}{||y||^2 + 1} \right). \end{aligned}$$

For this we have

$$
\begin{aligned}
i,j=1,\dots,n,\ i\neq j: &\qquad \frac{\partial f_i}{\partial y_j} = \frac{-2y_i\cdot 2y_j}{(||y||^2+1)^2},\\
i=1,\dots,n: &\qquad \frac{\partial f_i}{\partial y_i} = \frac{2\,||y||^2-4y_i^2+2}{(||y||^2+1)^2},\\
j=1,\dots,n: &\qquad \frac{\partial f_{n+1}}{\partial y_j} = \frac{-4y_j}{(||y||^2+1)^2}
\end{aligned}
$$

and hence for $y\in\mathbb{R}^n$ and $X=(X_1,\dots,X_n)$, $Z=(Z_1,\dots,Z_n)\in T_y\mathbb{R}^n$ we have

$$
\begin{aligned}
(f^*h)_y(X,Z) &= h(df_y(X),df_y(Z))\\
&= \sum_{i,j=1}^n X_iZ_j \underbrace{\left(\frac{\partial f_1}{\partial y_i}(y)\frac{\partial f_1}{\partial y_j}(y)+\dots+\frac{\partial f_{n+1}}{\partial y_i}(y)\frac{\partial f_{n+1}}{\partial y_j}(y)\right)}_{=:\rho_{ij}}
\end{aligned}
$$

and direct computation shows

$$
\rho_{ij} = \begin{cases} \frac{8||y||^4+8||y||^2}{(||y||^2+1)^4}, & \text{for } i=j\\ 0, & \text{for } i\neq j. \end{cases}
$$

So altogether

$$
(f^*h)_y = \frac{8\,||y||^4+8\,||y||^2}{(||y||^2+1)^4}g_y = \frac{8\,||y||^2}{(||y||^2+1)^3}g_y.
$$

That shows, that $f$ is conformal. $\square$

When dealing with conformal geometry it has proved useful to look at the Schouten tensor rather than the standard curvature tensors of pseudo-Riemannian geometry because of its easy transformation behaviour under conformal changes to the metric.

*Definition* 1.6. Let $(M,g)$ be a semi-Riemannian manifold with $\dim M := n\geq 3$ with Ricci-curvature $\operatorname{Ric}^g$ and scalar curvature $\operatorname{scal}^g$.

(i) The *Schouten tensor* is the $(2,0)$-tensor given by

$$
\mathrm{P}^g = \frac{1}{n-2}\left(\operatorname{Ric}^g - \frac{\operatorname{scal}^g}{2(n-1)}g\right).
$$

(ii) We denote the induced $(1,1)$-tensor with the same symbol. That is, for $X\in\mathfrak{X}(M)$ let $\mathrm{P}^g(X)\in\mathfrak{X}(M)$ be the unique vector field such that $g(\mathrm{P}^g(X),Y)=\mathrm{P}^g(X,Y)$ for all $Y\in\mathfrak{X}(M)$.

So for a local pseudo-orthonormal basis $(e_i)_{i=1}^n$ of $TM$ with $g(e_i,e_i)=:\varepsilon_i$ we have

$$
\mathrm{P}^g(X) = \sum_{i=1}^n \varepsilon_i\,\mathrm{P}^g(X,e_i)e_i.
$$

**Lemma 1.7.** *Let* $(M, g)$ *be a semi-Riemannian manifold with* $\dim M \geq 3$ *and* $\widehat{g} = e^{2\sigma} g$ *be a conformally changed metric. Then for* $X$, $Y \in \mathfrak{X}(M)$

$$\nabla^{\widehat{g}}_X Y = \nabla^g_X Y + X(\sigma)Y + Y(\sigma)X - g(X,Y)\operatorname{grad}^g \sigma, \tag{1.2}$$

$$P^{\widehat{g}}(X,Y) = P^g(X,Y) - \frac{1}{2}\,\|\operatorname{grad}^g \sigma\|^2\, g(X,Y) - g(\nabla^g_X \operatorname{grad}^g \sigma, Y) + X(\sigma)Y(\sigma), \tag{1.3}$$

$$P^{\widehat{g}}(X) = P^g(X) - \frac{1}{2}\,\|\operatorname{grad}^g \sigma\|^2\, X - \nabla^g_X \operatorname{grad}^g \sigma + X(\sigma)\operatorname{grad}^g \sigma. \tag{1.4}$$

*Proof.* Equation 1.2 can be derived using the Koszul formula. Equation 1.3 follows after calculating the curvature tensors in the metric $\widehat{g}$ with the help of the previous line. Equation 1.4 follows from dualization of the previous line. □

Explicit calculations for the transformation behavior of these and other standard objects of semi-Riemannian geometry can be found in [33].

As a first step we will assign a vector bundle to a given conformal structure, the *standard Tractor bundle*, see definition 1.11.

*Definition* 1.8. Let $(M, g)$ be a semi-Riemannian manifold with $\dim M \geq 3$ of signature $(p, q)$.

(i) Then the *g-Tractor bundle* is the vector bundle of rank $(n+2)$ given by

$$\mathcal{T}_g := \underline{\mathbb{R}} \oplus TM \oplus \underline{\mathbb{R}},$$

where $\underline{\mathbb{R}}$ denotes the trivial line bundle $M \times \mathbb{R} \to M$.

For some section $\eta : M \to \mathcal{T}_g$ of $\mathcal{T}_g$ we write

$$\eta = \begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix}, \text{ with } \alpha, \beta \in C^\infty(M) \text{ and } Y \in \mathfrak{X}(M).$$

(ii) On $\mathcal{T}_g$ we consider the covariant derivative $\nabla^{\mathcal{T}_g}$ given by

$$\nabla^{\mathcal{T}_g}_X \begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix} = \begin{pmatrix} X(\alpha) - P^g(X,Y) \\ \nabla^g_X Y + \alpha X + \beta P^g(X) \\ X(\beta) - g(X,Y) \end{pmatrix}, \text{ for some section } \begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix} \text{ and } X \in \mathfrak{X}(M).$$

(iii) On $\mathcal{T}_g$ we consider the bundle metric $h^g$ of signature $(p+1, q+1)$ given by

$$h^g\left(\begin{pmatrix} \alpha_1 \\ Y_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ Y_2 \\ \beta_2 \end{pmatrix}\right) := \alpha_1\beta_2 + \beta_1\alpha_2 + g(Y_1, Y_2)$$

for some smooth sections $\begin{pmatrix} \alpha_1 \\ Y_1 \\ \beta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 \\ Y_2 \\ \beta_2 \end{pmatrix} \in \Gamma(\mathcal{T}^g)$.

Now given some conformal manifold $(M,c)$ and some $g \in c$, the $g$-Tractor bundle is by no means canonical to the conformal structure of $M$. However, with the help of the $g$-Tractor bundle we will be able to define the *standard Tractor bundle*, which will turn out to be independent of the choice of $g$.

For this we note two properties of $\nabla^{\mathcal{T}_g}$, which can be checked by direct computation.

**Lemma 1.9.** *Let $(M,g)$ be a semi-Riemannian manifold with $\dim M \geq 3$. For any smooth section $\eta \in \Gamma(\mathcal{T}_g)$ let*

$$\widehat{\eta} = P_{g,\sigma} \cdot \eta \text{ with } P_{g,\sigma} = \begin{pmatrix} e^{-\sigma} & -e^{-\sigma} d\sigma & -\frac{1}{2} e^{-\sigma} \|\operatorname{grad}^g \sigma\|_g^2 \\ 0 & e^{-\sigma} \operatorname{Id} & e^{-\sigma} \operatorname{grad}^g \sigma \\ 0 & 0 & e^{\sigma} \end{pmatrix}.$$

*Then for any $\eta$, $\eta_1$, $\eta_2 \in \Gamma(\mathcal{T}_g)$ we have*

$$\nabla_X^{\mathcal{T}_{\widehat{g}}} \widehat{\eta} = P_{g,\sigma} \nabla_X^{\mathcal{T}_g} \eta \text{ for all } X \in \mathfrak{X}(M) \tag{1.5}$$

*and*

$$h^{\widehat{g}}(\widehat{\eta_1}, \widehat{\eta_2}) = h^g(\eta_1, \eta_2). \tag{1.6}$$

**Lemma 1.10.** *Let $(M,g)$ be a semi-Riemannian manifold with $\dim M \geq 3$. Then $h^g$ is metric with respect to $\nabla^{\mathcal{T}_g}$, that is*

$$X\left(h^g(\eta_1, \eta_2)\right) = h^g\left(\nabla_X^{\mathcal{T}_g} \eta_1, \eta_2\right) + h^g\left(\eta_1, \nabla_X^{\mathcal{T}_g} \eta_2\right)$$

*for any vector field $X \in \mathfrak{X}(M)$ and sections $\eta_1, \eta_2 \in \Gamma(\mathcal{T}_g)$.*

The equivariance properties from Lemma 1.9 suggest the following definition:

*Definition* 1.11. Let $(M,c)$ be a conformal manifold with $\dim M \geq 3$.

(i) For $g \in c$ and $\widehat{g} = e^{2\sigma} g$ we say that $\eta \in \mathcal{T}_g$ and $\widehat{\eta} \in \mathcal{T}_{\widehat{g}}$ are equivalent if

$$\widehat{\eta} = P_{g,\sigma} \eta$$

and we write $\eta \sim \widehat{\eta}$ for that.

(ii) The set

$$\mathcal{T} := \bigcup_{g \in c} \mathcal{T}_g / \sim$$

is called the *standard Tractor bundle* over $M$. For elements $\xi = [\eta] \in \mathcal{T}$ with $\eta \in \mathcal{T}_g$ we use the notation $\xi = [\eta, g]$.

This leads to a structure that has been canonically induced by the conformal structure on $M$. At first we will see that $\mathcal{T}$ is an actual bundle.

**Lemma 1.12.** *Let $(M, c)$ a conformal manifold with $\dim M \geq 3$ and $\mathcal{T}$ be the standard Tractor bundle on $M$.*

*Then $\mathcal{T}$ has a canonical vector bundle structure over $M$.*

*Proof.* For some fixed $g \in c$ the map

$$\begin{aligned}\Phi_g : \mathcal{T} &\to \mathcal{T}_g \\ [\eta, g] &\mapsto \eta\end{aligned}$$

is bijective. $\mathcal{T}_g$ is a vector bundle, hence $\Phi$ can endow $\mathcal{T}$ with the pulled back vector bundle structure. Furthermore, this vector bundle structure is independent of the choice of $g \in c$. To see this, consider two elements $[\eta_1, g]$, $[\eta_2, g]$ in some fixed fiber of $\mathcal{T}$. Writing $[\eta_i, g] = [\widehat{\eta}_i, \widehat{g}]$ for some $\widehat{g} \in c$, we have $P_{g,\sigma}\eta_i = \widehat{\eta}_i$ and therefore

$$\begin{aligned}[\widehat{\eta}_1, \widehat{g}] + [\widehat{\eta}_2, \widehat{g}] &= [\widehat{\eta}_1 + \widehat{\eta}_2, \widehat{g}] \\ &= [P_{g,\sigma}(\eta_1 + \eta_2), \widehat{g}] \\ &= [\eta_1, g] + [\eta_2, g]\,,\end{aligned}$$

so the vector bundle structure on $\mathcal{T}$ is independent of the choice of $g \in c$. □

Further we can see that $\nabla^{\mathcal{T}_g}$ and $h^g$ from definition 1.8 induce analog structures on $\mathcal{T}$:

*Definition* 1.13. Let $(M, c)$ be a conformal manifold with $\dim M \geq 3$ of signature $(p, q)$ with standard Tractor bundle $\mathcal{T}$.

(i) The *Tractor connection* is the covariant derivative on $\mathcal{T}$ given by

$$\nabla^{\mathcal{T}}_X[\eta, g] := [\nabla^{\mathcal{T}_g}_X \eta, g] \text{ for } X \in \mathfrak{X}(M), g \in c, \eta \in \Gamma(\mathcal{T}). \tag{1.7}$$

(ii) The *Tractor metric* is the bundle metric $h$ of signature $(p + 1, q + 1)$ given by

$$h([\eta_1, g], [\eta_2, g]) := h^g(\eta_1, \eta_2). \tag{1.8}$$

**Lemma 1.14.**

*(a) The Tractor connection from equation 1.7 and the Tractor metric from equation 1.8 are well defined.*

*(b) The Tractor connection is metric with respect to the Tractor metric, that is*

$$X(h(\eta_1, \eta_2)) = h(\nabla^{\mathcal{T}}_X \eta_1, \eta_2) + h(\eta_1, \nabla^{\mathcal{T}}_X \eta_2)$$

*for any vector field $X \in \mathfrak{X}(M)$ and smooth sections $\eta_1, \eta_2 \in \Gamma(\mathcal{T})$.*

*Proof.*

(a) We have to show that the definitions in equations 1.7 and 1.8 are independent of the representative $\eta$ of the equivalence class $[\eta, g]$. This follows from Lemma 1.9.

(b) This follows from Lemma 1.10. □

Our goal was to construct a Cartan geometry and so far we have arrived at a vector bundle endowed with a compatible metric and connection. To receive an object like the desired Cartan connection, we consider the principal bundle with principal bundle connection associated to $\mathcal{T}$ and a suitable restriction will then yield a Cartan connection.

Constructing a principal bundle from the vector bundle is a standard process as described in [7]. Since we will make some of the calculations explicit later on, we shall quickly repeat the process used here.

*Definition* 1.15. Let $M$ be a smooth manifold and $E$ be a rank-$k$ vector bundle with a signature $(p, q)$ bundle metric $h$. Then for $x \in M$

$$\begin{aligned}\mathcal{O}(E)_x :=& \{\tau_x = (s_1, \dots, s_k) \mid \tau_x \text{ is a } h_x\text{-orthonormal basis}\},\\ \simeq& \left\{L : \left(\mathbb{R}^k, \langle\cdot,\cdot\rangle_{p,q}\right) \to (E_x, h) \mid L \text{ is linear and orthogonal}\right\}\\ \mathcal{O}(E) :=& \bigcup_{x\in M} \mathcal{O}(E)_x.\end{aligned}$$

Here $\simeq$ simply denotes a 1:1-correspondence. Note that we consider orthonormal bases to be ordered, cf. section 4.1.

Then $\mathcal{O}(E)$ is an $\mathcal{O}(p, q)$-principal bundle over $M$ with right action

$$(e_1, \dots, e_k) \cdot A := \left(\sum_{j=1}^{k} A_{j1} e_j, \sum_{j=1}^{k} A_{j2} e_j, \dots, \sum_{j=1}^{k} A_{jk} e_j\right). \tag{1.9}$$

**Theorem 1.16.** *Let $M$ be a smooth manifold and $E$ be a rank-$k$ vector bundle with a signature $(p, q)$ bundle metric $h$. Then there is a 1:1-correspondence between*

$$Cov := \left\{\text{metric covariant derivates } \nabla \text{ on } E\right\}$$

*and* $$\mathcal{C} := \left\{\text{principal bundle connections } \omega \in \Omega^1(\mathcal{O}(E), \mathfrak{o}(p, q))\right\}.$$

*Proof.*

(i) "$\mathcal{C} \longrightarrow Cov$"

Let $\omega \in \mathcal{C}$. We can then write

$$\omega = \sum_{i,j=1}^{n} \omega_{ij} B_{ij},$$

for suitable $\omega_{ij} : T\mathcal{O}(E) \to \mathbb{R}$, where $B_{ij}$ is the matrix with all entries 0 except for the entry in the $i$-th row and $j$-th column being 1.

Let $s = (s_1, \dots, s_k) : U \to \mathcal{O}(E)$ be a local section on some $U \subset M$. Define the local connection form

$$\omega^s := \omega \circ ds, \tag{1.10}$$
$$\omega^s_{ij} := \omega_{ij} \circ ds. \tag{1.11}$$

Define then

$$\nabla^s_X s_k := \sum_{i=1}^n \omega_{ik}(ds(X))s_i$$

and extend $\nabla^s_X$ by the Leibniz rule. We want to show that this definition is independent of the choice of $s$ and therefore defines a global covariant derivative. Let $t = (t_1, \dots, t_k) : U \to \mathcal{O}(E)$ be another local section. We have

$$s = t \cdot C \tag{1.12}$$

for some $C \in \mathcal{O}(p, q)$. Then for $X \in T_xM \subset TU$

$$ds(X) = dR_{C(x)}(dt(X)) + \widetilde{\mu(X)}(s(x))$$

by the product rule for principal bundles, where $\mu = dL_{C^{-1}(x)}(dC(\cdot))$ is the pulled-back Maurer-Cartan form. Hence

$$\begin{aligned}
\omega^s(X) &= \omega(ds(X)) \\
&= \omega(dR_{C(x)}dt(X)) + \mu(X) \\
&= \mathrm{Ad}(C(x)^{-1})\omega^t(X) + \mu(X).
\end{aligned}$$

For linear groups the adjoint action is given by conjugation, i.e.

$$\omega^s = C^{-1}\omega^t C + C^{-1}dC. \tag{1.13}$$

This shows

$$\begin{aligned}
\nabla^s_X s_k &= \sum_{i=1}^n \omega^s_{ik}(X)s_i \\
&= \sum_{i,l=1}^n C_{li}\omega^s_{ik}(X)t_l && \text{(by equation 1.12)} \\
&= \sum_{i,l=1}^n \omega^t_{li}(X)C_{ik}t_l + \sum_{l=1}^n dC_{lk}(X)t_l && \text{(by equation 1.13)} \\
&= \sum_{i=1}^n C_{ik}\nabla^t_X t_i + dC_{ik}(X)t_i
\end{aligned}$$

$$= \nabla^t_X \left( \sum_{i=1}^n C_{ik} t_i \right)$$
$$= \nabla^t_X s_k. \tag{1.14}$$

i.e. the definition of $\nabla^s$ and $\nabla^t$ coincide and we receive a global covariant derivative $\nabla$.

Note for $X \in T_xM \subset TU$ that

$$\omega^s(X) \in \mathfrak{o}(p,q) = \{Z \in \mathfrak{gl}(n,\mathbb{R}) \mid Z^t J_{p,q} + J_{p,q} Z = 0\}$$

implies $\varepsilon_j \omega^s_{ji} + \varepsilon_i \omega^s_{ij} = 0$ for $\varepsilon_i = h(s_i, s_i)$. Here $J_{p,q} = \begin{pmatrix} -\operatorname{Id}_p & 0 \\ 0 & \operatorname{Id}_q \end{pmatrix}$. Hence

$$X(h(s_i, s_j)) = 0 = \varepsilon_j \omega_{ji}(X)^s + \varepsilon_i \omega_{ij}(X)^s,$$

i.e. $\nabla$ is metric.

(ii) "$Cov \longrightarrow \mathcal{C}$"

Let $\nabla \in Cov$ and $s = (s_1, \ldots, s_k) : U \to \mathcal{O}(E)$ a local section. We then have

$$\nabla s_i = \sum_{j=1}^n \omega^s_{ji} \otimes s_j$$

for some $\omega^s_{ji} : TU \to \mathbb{R}$. Define

$$\omega^s := \sum_{i,j=1}^n \omega^s_{ij} B_{ij}.$$

Let $t = (t_1, \ldots, t_k) : U \to \mathcal{O}(E)$ be a second section with $s = t \cdot C$. We have $\nabla_X s_k = \nabla_X \left( \sum_{i=1}^n C_{ik} t_i \right)$ and using the Leibniz rule shows like in equation 1.14 that

$$\omega^s = \operatorname{Ad}(C^{-1})\omega^t + \mu. \tag{1.15}$$

For $x \in U$, $X \in T_xM$, $g \in \mathcal{O}(p,q)$ and $Y \in \mathfrak{o}(p,q)$ define now

$$\omega_{s(x)} \left( ds(X) + \tilde{Y} \right) := \omega^s(X) + Y, \tag{1.16}$$
$$\omega_{s(x)\cdot g} := \operatorname{Ad}(g^{-1}) \omega_{s(x)} \circ dR_{g^{-1}}. \tag{1.17}$$

Then $\omega$ is a principal bundle connection on $\pi^{-1}(U) \subset \mathcal{O}(E)$.

$\omega$ is independent of the choice of $s$. To see this, let $\widehat{\omega}$ be induced by the section $t$. Then for $X \in T_xM \subset TU$

$$\begin{aligned}
\widehat{\omega}(ds(X)) &= \widehat{\omega} \left( dR_{C(x)}(dt(X)) + \widetilde{\mu(X)}(t(x) \cdot C(x)) \right) \\
&= \operatorname{Ad}(C(x)^{-1}) \widehat{\omega}(dt(X)) + \mu(X) \\
&= \operatorname{Ad}(C(x)^{-1}) \omega^t(X) + \mu(X) \\
&= \omega^s(X) && \text{(by equation 1.15).}
\end{aligned}$$

Hence our construction gives rise to a global connection 1-form $\omega \in \Omega^1(\mathcal{O}(E), \mathfrak{o}(p,q))$.

Because $\nabla$ is metric, we have $\varepsilon_j\omega_{ji} + \varepsilon_i\omega_{ij} = 0$, i.e. for all $X \in T_xM \subset TU$ we have $(\omega^s(X))^t J_{p,q} + J_{p,q}\omega^s(X) = 0$, i.e. $\omega^s(X) \in \mathfrak{o}(p,q)$ and therefore $\omega$ indeed takes values in $\mathfrak{o}(p,q)$ by the definitions made in equations 1.16 and 1.17.

(iii) It is clear that the two procedures are inverses of each other.

□

In particular for the case of the standard Tractor bundle we fix our notation:

*Definition* 1.17. Let $M$ be a smooth manifold with standard Tractor bundle $\mathcal{T}$. Let

$$\begin{aligned} \mathcal{G} &:= \mathcal{O}(\mathcal{T}) \\ &= \left\{ L : \left(\mathbb{R}^k, \langle\cdot,\cdot\rangle_{p+1,q+1}\right) \to (\mathcal{T}_x, h) \mid L \text{ is linear and orthogonal} \right\} \end{aligned}$$

be the associated $G$-principal bundle, where

$$G := \mathcal{O}(p+1, q+1),\ \mathfrak{g} = \mathrm{LA}(G),$$

with right action $L \cdot A := L \circ A$ and induced principal bundle connection $\widehat{\omega} \in \Omega^1(\mathcal{G}, \mathfrak{g})$.

**Lemma 1.18.** *Using the basis*

$$\left( l_- := \frac{1}{\sqrt{2}}(e_{n+1} - e_0), e_1, \ldots, e_n, l_+ := \frac{1}{\sqrt{2}}(e_{n+1} + e_0) \right) \tag{1.18}$$

*of* $\mathbb{R}^{p+1,q+1}$ *and the notation*

$$\begin{aligned} x^\flat &:= x^t J_{p,q} \text{ for a column vector } x \in \mathbb{R}^n, \\ z^\sharp &:= J_{p,q} z^t \text{ for a row vector } z \in (\mathbb{R}^n)^* \end{aligned}$$

*we receive the following representation for the Lie algebra* $\mathfrak{g}$ *of* $G = \mathcal{O}(p+1, q+1)$*:*

$$\mathfrak{g} = \mathfrak{o}(p+1, q+1) = \left\{ \begin{pmatrix} -a & z & 0 \\ x & A & -z^\sharp \\ 0 & -x^\flat & a \end{pmatrix} \middle| \begin{array}{l} a \in \mathbb{R} \\ A \in \mathfrak{o}(p,q) \\ x \in \mathbb{R}^n \\ z \in (\mathbb{R}^n)^* \end{array} \right\}. \tag{1.19}$$

The basis from line 1.18 is called *Witt basis* or *isotropic basis* because its first and last vector are light-like.

*Proof.* The standard scalar product $\langle\cdot,\cdot\rangle_{(p+1,q+1)}$ on $\mathbb{R}^{n+2}$ in the Witt basis is represented by

$$J^{Witt}_{p+1,q+1} := \begin{pmatrix} 0 & & 1 \\ & J_{p,q} & \\ 1 & & 0 \end{pmatrix},$$

and direct computation shows $M^t J^{Witt}_{p+1,q+1} + J^{Witt}_{p+1,q+1} M = 0$ for all matrices $M$ which have the form specified on the right hand side of equation 1.19. Furthermore the dimension of both sides of equation 1.19 is $\dim \mathfrak{o}(p+1,q+1) = \frac{(n+2)(n+1)}{2} = \frac{n(n-1)}{2} + 2n + 1$, i.e. the equality holds. □

We started out with a conformal manifold $(M,c)$ and at this point we have arrived at a principal bundle $\mathcal{G}$. To receive a Cartan connection we need to find a suitable restriction of this principal bundle.

*Definition* 1.19. Let $(M,c)$ be a conformal manifold with standard Tractor bundle $\mathcal{T}$.

(i) Let $g \in c$. The set $\mathcal{L} \subset \mathcal{T}$ given by

$$\mathcal{L} := \mathbb{R}_+ \cdot \left[ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, g \right]$$

is called *canonical Tractor line*. Note that the definition of $\mathcal{L}$ does not depend on the choice of $g \in c$ because of definition 1.11 and the shape of $P_{g,\sigma}$.

(ii) For $x \in M$ let

$$\begin{aligned} \mathcal{P}_x &:= \{L_x \in \mathcal{G}_x \mid L_x(\mathbb{R}_+ \cdot l_-) = \mathcal{L}_x\}, \\ \mathcal{P} &:= \bigcup_{x \in M} \mathcal{P}_x. \end{aligned}$$

Now $\mathcal{P}$ will turn out to be the desired Cartan geometry, as is stated in the next theorem:

**Theorem 1.20.** *With the notation from definition 1.19 we have the following properties of $\mathcal{P}$:*

*(a) $\mathcal{P}$ is a $P$-principal bundle over $M$ with structure group*

$$\begin{aligned} P :&= \mathrm{Stab}_{\mathbb{R}_+ \cdot l_-}\, \mathcal{O}(p+1,q+1) \\ &= \left\{ \begin{pmatrix} a^{-1} & v & -\frac{1}{2} a \langle v, v \rangle_{p,q} \\ 0 & A & -aAv^\sharp \\ 0 & 0 & a \end{pmatrix} \;\middle|\; \begin{array}{l} a \in \mathbb{R}^+ \\ A \in \mathcal{O}(p,q) \\ v \in (\mathbb{R}^n)^* \end{array} \right\} \end{aligned} \tag{1.20}$$

*where elements of $\mathcal{O}(p+1,q+1)$ are represented in the basis $(l_-, e_1, \ldots, e_n, l_+)$ defined in line 1.18.*

*(b) The Lie algebra $\mathfrak{p}$ of $P$ satisfies*

$$\mathfrak{p} = \left\{ \begin{pmatrix} -a & z & 0 \\ 0 & A & -z^\sharp \\ 0 & 0 & a \end{pmatrix} \;\middle|\; \begin{array}{l} a \in \mathbb{R} \\ A \in \mathfrak{o}(p,q) \\ z \in (\mathbb{R}^n)^* \end{array} \right\} \subset \mathfrak{g}.$$

*(c) Let $\omega := \widehat{\omega}\,|_{T\mathcal{P}}$. Then $(\mathcal{P}, \omega)$ is a Cartan geometry of type $(G, P)$ on $M$.*

*Proof.*

(b) We have

$$\mathrm{LA}\left(\mathrm{Stab}_{\mathbb{R}_+ \cdot l_-}\, \mathcal{O}(p+1, q+1)\right) = \mathrm{Stab}_{\mathbb{R}\cdot l_-}\, \mathfrak{o}(p+1, q+1),$$

and using Lemma 1.18 we find that elements in $\mathfrak{g}$ stabilizing $\mathbb{R} \cdot l_-$ are exactly the ones having $x = 0$ (with the notation from equation 1.19).

(a) Direct computation shows that all matrices from the right hand side of equation 1.20 need to be in $P$. Left hand side and right hand side have the same dimension by part (b), hence the equality follows from dimensional reasons.

(c) We first notice that $\omega$ is right-invariant and reproduces fundamental vector fields because it is the restriction of a principal bundle connection.

So it remains to show that $\omega$ also defines an absolute parallelism. To this end let $x \in M$ be arbitrary and consider a neighborhood $U_x \subset M$ of $x$ with local pseudo-orthonormal basis $(s_1, \dots, s_n)$. Now fix some $g \in c$. For ease of notation we write $\begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix}$ instead of $\left[\begin{pmatrix} \alpha \\ Y \\ \beta \end{pmatrix}, g\right]$ and consider the following canonical section:

$$\begin{aligned} \tau = (\tau_0, \dots, \tau_{n+1}) : U_x &\to \mathcal{P} \\ x &\mapsto \left( \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ s_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ s_2 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ s_n \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right) \end{aligned}$$

and the induced local Witt basis

$$\begin{aligned} \widehat{\tau} = (\tau_-, \tau_1, \dots, \tau_n, \tau_+) : U_x &\to \mathcal{P} \\ x &\mapsto \left( \frac{1}{\sqrt{2}}(-\tau_0 - \tau_{n+1}), \tau_1, \dots, \tau_n, \frac{1}{\sqrt{2}}(-\tau_0 + \tau_{n+1}) \right) \\ &= \left( \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ s_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ s_2 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ s_n \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right). \end{aligned}$$

Now in the point $u := \tau(x)$ for any $V \in T_u\mathcal{P}$ we have a representation

$$V = \tilde{Y}(u) + d\tau(X) \text{ with } X \in T_xM, Y \in \mathfrak{p}.$$

Then

$$\omega_u(V) = Y + \sum_{i,j=0}^{n+1} \omega_{ij}^{\tau}(X) B_{ij}$$

where $(\omega_{ij}(X))_{i,j\in\{0,\dots,n+1\}} = \left(\varepsilon_j h\left(\nabla^{\mathcal{T}}_X \tau_i, \tau_j\right)\right)_{i,j\in\{0,\dots,n+1\}} \in \mathfrak{g}$ represented in the canonical basis $(e_0,\dots,e_{n+1})$. Denote

$$\begin{aligned}
\omega_{-,-} &= h\left(\nabla^{\mathcal{T}}_X \tau_-, \tau_-\right),\\
\omega_{-,j} &= \varepsilon_j h\left(\nabla^{\mathcal{T}}_X \tau_-, \tau_j\right) \text{ for } j \in \{1,\dots,n\},\\
\omega_{i,-} &= h\left(\nabla^{\mathcal{T}}_X \tau_i, \tau_-\right) \text{ for } i \in \{1,\dots,n\},
\end{aligned}$$

and $\omega_{+,+}$, $\omega_{+,j}$, $\omega_{i,+}$, $\omega_{-,+}$, $\omega_{+,-}$ accordingly. Then

$$\begin{pmatrix} \omega_{-,-} & \omega_{-,j} & \omega_{-,+} \\ \omega_{i,-} & \omega_{i,j} & \omega_{i,+} \\ \omega_{+,-} & \omega_{+,j} & \omega_{+,+} \end{pmatrix}_{i,j\in\{1,\dots,n\}}$$

is exactly the matrix representation of $(\omega_{ij}(X))_{i,j\in\{0,\dots,n+1\}}$ in the Witt basis.

Setting $X = 0$ we have $\omega_u(V) = Y$ and therefor $\mathfrak{p} \subset \operatorname{Im}\omega_u$.

Now set $Y = 0$ and first calculate $\omega_{i,-}$ for $i \in \{1,\dots,n\}$:

$$\begin{aligned}
\omega_{i,-}(X) &= h\left(\nabla^{\mathcal{T}}_X \tau_i, \tau_-\right)\\
&= h^g\left(\nabla^{\mathcal{T}_g}_X \begin{pmatrix} 0 \\ s_i \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\right)\\
&= h^g\left(\begin{pmatrix} -P^g(X,s_i) \\ \nabla^g_X s_i \\ -g(X,s_i) \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}\right)\\
&= g(X,s_i).
\end{aligned}$$

Now letting $X = s_k$ for $k = 1,\dots,n$ gives us $\omega_{i,-}(X) = \varepsilon_k$. Thus $\dim(\operatorname{Im}\omega_u) = \dim\mathfrak{p} + n = \dim\mathfrak{g}$ and because we have $\operatorname{Im}\omega_u \subset \mathfrak{g}$, the two must be equal. So $\omega_u$ is an isomorphism of vector spaces. By right-invariance we receive that also $\omega_{up}$ is an isomorphism for all $p \in P$. And because $x \in M$ was chosen arbitrarily, we now have that $\omega$ is indeed an absolute parallelism. □

This means we have achieved our first aim, to assign a Cartan geometry to a given conformal structure. It remains to show that no information has been lost. That is: To a Cartan geometry of type $(G,P)$ with $G = \mathcal{O}(p,q)$ and $P = \operatorname{Stab}_{\mathbb{R}_+\cdot l_-} G$ we can assign a conformal structure. And in the case that the Cartan geometry has been induced by a conformal structure in the first place, this process shall reproduce that original conformal structure.

To this end we will first look at the algebraic properties of $G$ and $P$.

*Definition* 1.21. We use the following notation for some special subgroups of $G$ and subalgebras of $\mathfrak{g}$:

$$G_0 := \left\{ \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} \,\middle|\, \begin{array}{l} a \in \mathbb{R}^+ \\ A \in \mathcal{O}(p,q) \end{array} \right\} \subset G,$$

$$\mathfrak{g}_0 := \mathrm{LA}(G_0) = \left\{ \begin{pmatrix} -a & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} \,\middle|\, \begin{array}{l} a \in \mathbb{R} \\ A \in \mathfrak{o}(p,q) \end{array} \right\} \subset \mathfrak{g},$$

$$G_1 := \left\{ \begin{pmatrix} 1 & v & -\frac{1}{2}\langle v,v\rangle_{p,q} \\ 0 & \mathrm{Id} & -v^\sharp \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, v \in (\mathbb{R}^n)^* \right\} \subset G,$$

$$\mathfrak{g}_1 = \mathrm{LA}(G_1) = \left\{ \begin{pmatrix} 0 & z & 0 \\ 0 & 0 & -z^\sharp \\ 0 & 0 & 0 \end{pmatrix} \,\middle|\, z \in (\mathbb{R}^n)^* \right\} \subset \mathfrak{g},$$

$$G_{-1} := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & \mathrm{Id} & 0 \\ -\frac{1}{2}\langle x,x\rangle_{p,q} & -x^\flat & 1 \end{pmatrix} \,\middle|\, x \in \mathbb{R}^n \right\} \subset G,$$

$$\mathfrak{g}_{-1} = \mathrm{LA}(G_{-1}) = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & -x^\flat & 0 \end{pmatrix} \,\middle|\, x \in \mathbb{R}^n \right\} \subset \mathfrak{g}.$$

With these notations we have:

**Lemma 1.22.**

*(a)* $\mathfrak{g}$ *is a* $|1|$*-graded Lie algebra with decomposition* $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$. *That is*

$$[\mathfrak{g}_i, \mathfrak{g}_j]_{\mathfrak{g}} \subset \mathfrak{g}_{i+j} \text{ for } i,j = -1,0,1.$$

*(b) It is* $G_0 \simeq \mathrm{CO}(p,q)$ *and on the level of Lie algebras we have* $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ *and*

$$\mathfrak{g}_1 \simeq (\mathbb{R}^n)^*,$$
$$\mathfrak{g}_{-1} \simeq \mathbb{R}^n.$$

*Proof.*

(a) Can be checked by calculating the ordinary commutator of matrices.

(b) It is $\mathrm{CO}(p,q)$ the conformal group, that is

$$\begin{aligned} \mathrm{CO}(p,q) :&= \{A \in \mathrm{GL}(n) \mid \exists \lambda > 0 \text{ such that } \langle Ax, Ay\rangle_{p,q} = \lambda \langle x,y\rangle_{p,q} \text{ for all } x,y \in \mathbb{R}^n\} \\ &\simeq \mathbb{R}^+ \times \mathcal{O}(p,q), \end{aligned}$$

where the group structure on $\mathbb{R}^+ \times \mathcal{O}(p,q)$ is given by $(A,\lambda) \circ (B,\mu) = (AB, \lambda\mu)$.

Then

$$\begin{aligned} \Phi : G_0 &\to \mathrm{CO}(p,q) \\ \begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & a \end{pmatrix} &\mapsto (A,a) \end{aligned}$$

is an obvious isomorphism.

$\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is clear. The isomorphisms $\mathfrak{g}_1 \simeq (\mathbb{R}^n)^*$ and $\mathfrak{g}_{-1} \simeq \mathbb{R}^n$ are given as

$$\begin{aligned} \Theta_1 : \mathfrak{g}_1 &\to (\mathbb{R}^n)^* \\ \begin{pmatrix} 0 & z & 0 \\ 0 & 0 & -z^\sharp \\ 0 & 0 & 0 \end{pmatrix} &\mapsto z \end{aligned} \tag{1.21}$$

and

$$\begin{aligned} \Theta_{-1} : \mathfrak{g}_{-1} &\to \mathbb{R}^n \\ \begin{pmatrix} 0 & 0 & 0 \\ x & 0 & 0 \\ 0 & -x^\flat & 0 \end{pmatrix} &\mapsto x. \end{aligned} \tag{1.22}$$

□

**Lemma 1.23.** *Let $(\mathcal{P}, \omega)$ be a Cartan geometry of type $(G,P)$ with $G = \mathcal{O}(p+1,q+1)$ and $P = \mathrm{Stab}_{\mathbb{R}_+ \cdot l_-} G$ on some smooth manifold $M$ with $\dim M = n \geq 3$. Then there is a canonically induced conformal structure on $M$.*

*Proof.*

(i) Set
$$\mathcal{P}_0 := \mathcal{P} \times_P (P/G_1).$$
and for elements in $\mathcal{P}_0$ we write $[u, p \cdot \mathbb{R}^n]$. Then
$$[u, p \cdot \mathbb{R}^n] = [u(p')^{-1}, p'p \cdot \mathbb{R}^n] \text{ for } p' \in P.$$
Notice that $P/G_1 \simeq \mathrm{CO}(p,q)$ and $\mathcal{P}_0$ is a $\mathrm{CO}(p,q)$-principal bundle. Define the projection
$$\begin{aligned} \mathrm{pr} : \mathcal{P} &\to \mathcal{P}_0 \\ u &\mapsto [u, e \cdot \mathbb{R}^n], \end{aligned}$$

which makes the following diagram commutative:

$$\begin{tikzcd} \mathcal{P} \arrow[rr, "\mathrm{pr}"] \arrow[dr, "\pi_{\mathcal{P}}"'] & & \mathcal{P}_0 \arrow[dl, "\pi_{\mathcal{P}_0}"] \\ & M & \end{tikzcd} \tag{1.23}$$

Also it satisfies

$$\mathrm{pr} \circ R_g = R_g \circ \mathrm{pr} \tag{1.24}$$

for all $g \in \mathrm{CO}(p,q)$. On the left hand side of that equation $\mathrm{CO}(p,q)$ is considered a subgroup of $P$ by means of the identification explained in Lemma 1.22.

(ii) Define $\theta \in \Omega^1(\mathcal{P}_0, \mathbb{R}^n)$ to be the 1-form which makes the following diagram commutative:

$$\begin{tikzcd} T\mathcal{P} \arrow[r, "\omega"] \arrow[d, "d\,\mathrm{pr}"] & \mathfrak{g} \arrow[d, "\mathrm{proj}_{\mathfrak{g}_{-1}}"] \\ T\mathcal{P}_0 \arrow[r, "\theta"] & \mathbb{R}^n \end{tikzcd} \tag{1.25}$$

That is

$$\theta_{\mathrm{pr}\,u}(d\,\mathrm{pr}(V)) = \mathrm{proj}_{\mathfrak{g}_{-1}}(\omega(V)) \text{ for } u \in \mathcal{P} \text{ and } V \in T_u\mathcal{P}. \tag{1.26}$$

Here $\mathrm{proj}_{\mathfrak{g}_{-1}} : \mathfrak{g} \to \mathfrak{g}_{-1} \simeq \mathbb{R}^n$ denotes the linear projection on the summand $\mathfrak{g}_{-1} \simeq \mathbb{R}^n$.

Then:

- $\theta$ is well-defined:

  First let $u \in \mathcal{P}$ and $V_1, V_2 \in T_u\mathcal{P}$ with $d\,\mathrm{pr}(V_1) = d\,\mathrm{pr}(V_2)$. By definition of pr we have $V_1 - V_2 = \tilde{X}(u)$ for some $X \in \mathfrak{g}_1$, and therefor $\omega(V_1 - V_2) = X$.

  Then

$$\begin{aligned} \theta_{\mathrm{pr}\,u}(d\,\mathrm{pr}_u(V_1)) - \theta_{\mathrm{pr}\,u}(d\,\mathrm{pr}_u(V_2)) &= \mathrm{proj}_{\mathfrak{g}_{-1}}(\omega(V_1)) - \mathrm{proj}_{\mathfrak{g}_{-1}}(\omega(V_2)) \\ &= \mathrm{proj}_{\mathfrak{g}_{-1}}(\omega(V_1 - V_2)) = 0. \end{aligned}$$

  Now let $u_1, u_2 \in \mathcal{P}$ with $\mathrm{pr}(u_1) = \mathrm{pr}(u_2)$. That is $u_2 = u_1 \cdot g$ for some $g \in G_1$. Choose for $V_1 \in T_{u_1}\mathcal{P}$, $V_2 \in T_{u_1 g}\mathcal{P}$ with $d\,\mathrm{pr}(V_1) = d\,\mathrm{pr}(V_2)$ some $\gamma : I \to \mathcal{P}$ with $\gamma'(0) = V_2$. Then

$$\begin{aligned} d\,\mathrm{pr}_{u_1 \cdot g}(V_2) &= \frac{d}{dt}\,\mathrm{pr}(\gamma(t)) \\ &= \frac{d}{dt}\,\mathrm{pr} \circ R_{g^{-1}}(\gamma(t)) \\ &= d\,\mathrm{pr}_{u_1}\left(dR_{g^{-1}} V_2\right) \end{aligned}$$

and the claim follows from the case where $u_1 = u_2 = u$.

- For $g \in G_0 \simeq \mathrm{CO}(p,q)$ we have

$$R_g^* \theta = \mathrm{Ad}(g^{-1}) \circ \theta, \tag{1.27}$$

where the action $\mathrm{Ad} : G \to \mathrm{GL}(\mathbb{R}^n)$ is given by the canonical identification $\mathbb{R}^n \simeq G_1$ from Lemma 1.22.

To see this, let $V \in T\mathcal{P}_0$ and $d\,\mathrm{pr}(\tilde{V}) = V$ for some $\tilde{V} \in T\mathcal{P}$. Then

$$\begin{aligned}
(R_g^*\theta)(V) &= \theta(dR_g\, d\,\mathrm{pr}\,\tilde{V}) \\
&= \theta(d\,\mathrm{pr}(dR_g\tilde{V})) \qquad\qquad \text{(by diagram 1.24)} \\
&= \mathrm{proj}_{\mathfrak{g}_{-1}}\big(\omega(dR_g(\tilde{V}))\big) \\
&= \mathrm{proj}_{\mathfrak{g}_{-1}}\big(\mathrm{Ad}(g^{-1})\omega(\tilde{V})\big)
\end{aligned}$$

and note that $\mathrm{Ad}(g)(\mathfrak{g}_i) \subset \mathfrak{g}_i$ for $i = -1, 0, 1$ by Lemma 1.22. Hence

$$\begin{aligned}
(R_g^*\theta)(V) &= \mathrm{Ad}(g^{-1})(\mathrm{proj}_{\mathfrak{g}_{-1}}\omega(\tilde{V})) \\
&= \mathrm{Ad}(g^{-1})(\theta(V)).
\end{aligned}$$

- For $u \in \mathcal{P}$ we have

$$\mathrm{Ker}\,\theta_u = Tv_u\,\mathcal{P}_0 := \mathrm{Ker}(d\pi_u^{\mathcal{P}_0} : T_u\mathcal{P}_0 \to M).$$

"$\supset$":

Let $V \in Tv\,\mathcal{P}_0$, i.e. $d\pi^{\mathcal{P}_0}(V) = 0$. Let $\tilde{V} \in T\mathcal{P}$ s.t. $V = d\,\mathrm{pr}(\tilde{V})$. Then

$$d\pi^{\mathcal{P}}(\tilde{V}) = d\pi^{\mathcal{P}_0}\big(d\,\mathrm{pr}(\tilde{V})\big) = 0,$$

therefore $\tilde{V} = \tilde{X}(u)$ for some $u \in \mathcal{P}$, $X \in \mathfrak{p}$.

Hence

$$\theta(V) = \mathrm{proj}_{\mathfrak{g}_{-1}}(\omega(\tilde{V})) = \mathrm{proj}_{\mathfrak{g}_{-1}}(X) = 0.$$

"$\subset$":

Conversely, if $V \in \mathrm{Ker}\,\theta_u$ and $\tilde{V} \in T\mathcal{P}$, s.t. $d\,\mathrm{pr}(\tilde{V}) = V$, then

$$\begin{aligned}
\mathrm{proj}_{\mathfrak{g}_{-1}}(\omega(\tilde{V})) &= \theta_u(d\,\mathrm{pr}\,\tilde{V}) \\
&= \theta_u(V) = 0,
\end{aligned}$$

therefore $\omega(\tilde{V}) \in \mathfrak{p}$. Hence again $\tilde{V} = \tilde{X}(u)$ for some $u \in \mathcal{P}$, $X \in \mathfrak{p}$, which implies

$$d\pi^{\mathcal{P}_0}(V) = d\pi^{\mathcal{P}}(\tilde{V}) = d\pi^{\mathcal{P}}(\tilde{X}(u)) = 0.$$

(iii) We now define a $\mathrm{CO}(p,q)$-reduction by

$$\begin{aligned} f:\mathcal{P}_0 &\to \mathrm{GL}(M) \\ u &\mapsto \left(d\pi_u^{\mathcal{P}_0}(\theta_u^{-1}(e_1)),\dots,d\pi_u^{\mathcal{P}_0}(\theta_u^{-1}(e_n))\right), \end{aligned} \tag{1.28}$$

where $(e_1,\dots,e_n)$ is the standard basis in $\mathbb{R}^n$. That is, $f$ makes the following diagram commutative:

$$\begin{tikzcd} \mathcal{P}_0 \times G_0 \arrow[r] \arrow[d, "f\times i"] & \mathcal{P}_0 \arrow[d, "f"] \arrow[dr] & \\ & & M \\ GL(M)\times GL(n) \arrow[r] & GL(M) \arrow[ur] & \end{tikzcd} \tag{1.29}$$

Note that

- $f$ is well-defined:

  The preimage $\theta_u^{-1}(e_i)$ is not unique. But if $\theta_u(V)=\theta_u(W)=e_i$, then $V-W\in \operatorname{Ker}\theta = Tv\mathcal{P}_0$. Hence $d\pi_u^{\mathcal{P}_0}(V)=d\pi_u^{\mathcal{P}_0}(W)$.

  Also the image $f(u)$ indeed defines a basis of $T_{\pi^{\mathcal{P}_0}(u)}M$, because $d\pi_u^{\mathcal{P}_0}\circ\theta_u^{-1}:\mathbb{R}^n\to T_{\pi^{\mathcal{P}_0}(u)}M$ is an isomorphism of vector spaces.

- $f$ is obviously fiber-preserving. That is, $\pi^{\mathrm{GL}(M)}(f(u))=\pi^{\mathcal{P}_0}(u)$.

- $f$ is $G_0$-equivariant, that is $f(u\cdot b)=f(u)\cdot b$ for all $u\in\mathcal{P}_0$ and $b\in G_0\simeq \mathrm{CO}(p,q)$:

  $$f(u\cdot b)=f(u)\cdot b.$$

  Let $b=\begin{pmatrix} a^{-1} & 0 & 0\\ 0 & A & 0\\ 0 & 0 & a\end{pmatrix}\in G_0$. We have

  $$\begin{aligned} f(u)\cdot b &= \left(d\pi_u^{\mathcal{P}_0}\theta_u^{-1}(e_1),\dots,d\pi_u^{\mathcal{P}_0}\theta_u^{-1}(e_n)\right)\cdot(a\cdot A)\\ &= \left(\sum_{j=1}^n aA_{j1}d\pi_u^{\mathcal{P}_0}\theta_u^{-1}(e_j),\dots,\sum_{j=1}^n aA_{jn}d\pi_u^{\mathcal{P}_0}\theta_u^{-1}(e_j)\right) \quad \text{(by equation 1.9)},\\ f(ub) &= \left(d\pi_{u\cdot b}^{\mathcal{P}_0}\theta_{u\cdot b}^{-1}(e_1),\dots,d\pi_{u\cdot b}^{\mathcal{P}_0}\theta_{u\cdot b}^{-1}(e_n)\right). \end{aligned}$$

  Now by equation 1.27 we have $\theta_{u\cdot b}^{-1}(e_i)=dR_b\theta_u^{-1}(\mathrm{Ad}(b)e_i)$ and therefore

  $$\begin{aligned} f(ub) &= \left(d\pi_{u\cdot b}^{\mathcal{P}_0}dR_b\theta_u^{-1}(\mathrm{Ad}(b)e_1),\dots,d\pi_{u\cdot b}^{\mathcal{P}_0}dR_b\theta_u^{-1}(\mathrm{Ad}(b)e_n)\right)\\ &= \left(d\pi_u^{\mathcal{P}_0}\theta_u^{-1}(\mathrm{Ad}(b)e_1),\dots,d\pi_u^{\mathcal{P}_0}\theta_u^{-1}(\mathrm{Ad}(b)e_n)\right). \end{aligned}$$

$G \subset \mathrm{GL}(n+2)$ is a linear group, hence the adjoint action $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ is given by conjugation, so using the implicit identification $\mathfrak{g}_{-1} \simeq \mathbb{R}^n$ we receive

$$\begin{aligned}\mathrm{Ad}(b)e_i &= b\begin{pmatrix} 0 & 0 & 0 \\ e_i & 0 & 0 \\ 0 & -e_i^\flat & 0 \end{pmatrix} b^{-1} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ aAe_i & 0 & 0 \\ 0 & -ae_i^\flat A^{-1} & 0 \end{pmatrix} \\ &= aAe_i \\ &= \sum_{j=1}^n aA_{ji}e_j.\end{aligned}$$

So by linearity of $d\pi^{\mathcal{P}_0} \circ \theta^{-1}$ we find that $f(u) \cdot b = f(u \cdot b)$.

Hence $f$ is a $\mathrm{CO}(p,q)$-reduction of $\mathrm{GL}(n)$ and $\widehat{\mathcal{P}_0} := f(\mathcal{P}_0)$ is a $\mathrm{CO}(p,q)$-bundle.

(iv) In analogy to the case of pseudo-Riemannian metrics we receive a conformal structure $c \in \Gamma(M, T^*M \otimes T^*M)$ on $M$ in the following way: For $x \in M$ let $u \in (\mathcal{P}_0)_x$ be a point over $x$. Write $f(u) = (s_1, \ldots, s_n)$, where $(s_i)$ is a basis of $T_xM$. Then let $g_x^u \in T_x^*M \otimes T_x^*M$ be the metric given by the condition

$$g_x^u(s_i, s_j) = \varepsilon_i \delta_{ij}, \text{ where } \varepsilon_i = \begin{cases} -1, & \text{for } i \leq p, \\ +1, & \text{for } i \geq p+1. \end{cases}$$

Then the conformal class of $g_x$ is independent of the choice of $u$ because for $u \cdot b$ with some $b \in G_0$ we have

$$f(u \cdot b) = f(u) \cdot b = (bs_1, \ldots, bs_n)$$

by $G_0$-equivariance and therefore

$$\begin{aligned} g_x^{ub}(s_i, s_j) &= \alpha \cdot g_x^{ub}(bs_i, bs_j) \qquad \text{(for some } \alpha > 0 \text{, because } b \in \mathrm{CO}(p,q)) \\ &= \alpha \cdot g_x^u(s_i, s_j). \end{aligned}$$

Furthermore $[g_x^u] = [g_x^{ub}] =: c_x$ defines a smooth section by smoothness of $f$. □

Note that for this direction we did not take the detour of constructing a Tractor bundle from the given Cartan geometry. This can be done not only for the particular groups $G$ and $P$ described above, but more generally for any parabolic geometry, as shown in [11]. For the basic notions of parabolic geometry see [10].

The obvious question that comes to mind is now, whether the procedures stated in Theorem 1.20 and Lemma 1.23 are inverse to each other. The positive answer to this question is given by the next theorem:

**Theorem 1.24.** *Let $M$ be a manifold of dimension $\geq 3$. There is a 1:1-correspondence between*

$$\begin{aligned} & A := \{\textit{conformal structures on } M\} \\ \textit{and} \quad & B := \{\textit{isomorphism types of Cartan geometries of type } (G,P) \textit{ on } M\} \end{aligned}$$

*with $G = \mathcal{O}(p+1, q+1)$ and $P = \operatorname{Stab}_{\mathbb{R}_+ \cdot l_-} G$.*

*Proof.* Let $\Phi : A \to B$ be the construction described in Theorem 1.20 and $\Psi : B \to A$ be the construction defined in Lemma 1.23. Let $\Phi(c) = (\mathcal{P}, \omega)$ be the induced Cartan geometry of a conformal class $c \in A$. Recall the notation

$$\begin{aligned} \mathcal{P}_0 &:= \mathcal{P} \times_P (P/G_1), \\ \operatorname{pr} &: \mathcal{P} \to \mathcal{P}_0, \\ \theta &\in \Omega^1(\mathcal{P}_0, \mathbb{R}^n), \\ f = (f_1, \dots, f_n) &: \mathcal{P}_0 \to \operatorname{GL}(M). \end{aligned}$$

Write $\Psi(\Phi(c)) = \tilde{c}$, then $\tilde{c}$ is the conformal class of $\tilde{g}$, which is for some fixed $x \in M$ given by the condition

$$\tilde{g}_x(f_i(u), f_j(u)) = \varepsilon_i \delta_{ij} \text{ for } u \in \mathcal{P}_0 \text{ with } \pi^{\mathcal{P}_0}(u) = x.$$

Fix some $g \in c$. Again we make use of the identification $\mathcal{T} \simeq \mathcal{T}^g$ to simplify notation of sections in $\mathcal{P}$ just as we did in the proof of Theorem 1.20. Using this notation we may assume without loss of generality that

$$u = \left[ \left( \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ s_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ s_2 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ s_n \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right), e \cdot \mathbb{R}^n \right] \in \mathcal{P}_0,$$

where $(s_1, \dots, s_n)$ is an orthonormal basis of $T_xM$ with respect to $g$. That is because the construction of $\tilde{c}_x$ was independent of the choice of $u$.

Extend $(s_1, \dots, s_n)$ to a local pseudo-orthonormal basis $(s_1, \dots, s_n) : U \to \operatorname{GL}(M)$. We receive local sections

$$\begin{aligned} \tilde{\tau} = (\tilde{\tau}_0, \tilde{\tau}_1, \dots, \tilde{\tau}_n, \tilde{\tau}_{n+1}) &= \left( \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ s_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ s_2 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ s_n \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right) : U \to \mathcal{P}, \\ \tau = (\tau_0, \tau_1, \dots, \tau_n, \tau_{n+1}) &= [\tilde{\tau}, e \cdot \mathbb{R}^n] : U \to \mathcal{P}_0. \end{aligned}$$

We receive

$$\begin{aligned}
e_i &= \theta_u \left(d\tau_x(f_i(u))\right) \\
&= \mathrm{proj}_{\mathfrak{g}_{-1}} \left(\omega_{\tilde{\tau}(x)}(d\tilde{\tau}_x(f_i(u)))\right) \\
&= \mathrm{proj}_{\mathfrak{g}_{-1}} \left(\omega_x^{\tilde{\tau}}(f_i(u))\right) \\
&= \mathrm{proj}_{\mathfrak{g}_{-1}} \left(\sum_{k,j=0}^{n+1} \omega_{kj}^{\tilde{\tau}}(f_i(u)) \cdot B_{kj}\right) \\
&= \left(\omega_{1,-}^{\tilde{\tau}}(f_i(u)), \omega_{2,-}^{\tilde{\tau}}(f_i(u)), \ldots, \omega_{n,-}^{\tilde{\tau}}(f_i(u))\right)
\end{aligned}$$

where $\omega^{\tilde{\tau}}$ denotes the local connection 1-form as introduced in equation 1.10. In the last step we recall our particular choice of the identification $\mathfrak{g}_{-1} \simeq \mathbb{R}^n$ from equation 1.22.

Hence for $1 \leq k \leq n$

$$\begin{aligned}
\delta_{ik} &= \omega_{k,-}\left(f_i(u)\right) \\
&= \varepsilon_k \cdot \omega_{-,k}\left(f_i(u)\right) \\
&= \varepsilon_k \cdot h \left( \begin{pmatrix} * \\ f_i(u) \\ * \end{pmatrix}, \tilde{\tau}_k(x) \right) \\
&= \varepsilon_k \cdot g\left(f_i(u), s_k(x)\right)
\end{aligned}$$

which implies $f_i(u) = s_i$ and therefore

$$g(f_j(u), f_i(u)) = g(s_j, s_i) = \varepsilon_i \delta_{ij} = \tilde{g}(f_j(u), f_i(u)).$$

That is $\tilde{c} = c$.

Analog calculation shows $\Phi(\Psi(\mathcal{P}, \omega)) = (\mathcal{P}, \omega)$. □

## 1.3 Canonical Curves in Cartan Geometries

Following [14] and [10] we will introduce canonical curves of Cartan geometries through the development of curves, characterize them as projections of flow lines of constant vector fields and eventually show that in the $|1|$-graded case the 2-jet in one point is enough to pin down a canonical curve.

Throughout the whole section assume $(\mathcal{P}, \omega)$ to be a Cartan geometry of type $(G, P)$ on some manifold $M$. $I$ shall always denote a *connected* interval.

Following section 1.2 of [14] and section 1.5.17 of [10] we introduce the development of curves in a manifold endowed with a Cartan geometry–not necessarily $|1|$-graded.

*Definition* 1.25. Let $(\mathcal{G} = \mathcal{P} \times_P G, \pi, M; G)$ be the extended principal bundle with principal connection $\Phi$ as per Theorem 4.8. Denote by

$$\begin{aligned} j : \mathcal{P} &\to \mathcal{G} \\ u &\mapsto [u, e] \end{aligned} \tag{1.30}$$

the canonical embedding. The Lie group $G$ acts on $G/P$ from the left through

$$\begin{aligned} G \times G/P &\to G/P \\ h, gP &\mapsto hgP. \end{aligned}$$

Let $\mathcal{S} := \mathcal{G} \times_G (G/P)$ be the fibered manifold associated to this action. By Theorem 4.9, $\mathcal{S}$ is equipped with a general connection $\overline{\Phi} \in \Omega^1(\mathcal{S}, Tv\mathcal{S})$ canonically induced by $\Phi$.

$\mathcal{S}$ is called *Cartan's Space.*

We write

$$\begin{aligned} q : \mathcal{G} \times (G/P) &\to \mathcal{S} \\ u, gP &\mapsto [u, gP] \end{aligned} \tag{1.31}$$

for the canonical projection.

*Definition* 1.26. Let $\gamma : I \to M$ be a curve. For $t_0 \in I$ write

$$\begin{aligned} \gamma_{t_0} : \tilde{I} &\to M \\ t &\mapsto \gamma(t_0 + t), \end{aligned} \tag{1.32}$$

where $\tilde{I} = \{t \in \mathbb{R} \mid t + t_0 \in I\}$ denotes the maximum domain for the curve $\gamma_{t_0}$. Denote by $\mathrm{Pt}_{\gamma_{t_0}} : \mathcal{S}_{\gamma(t_0)} \times \tilde{I} \to \mathcal{S}$ the parallel transport induced by the general connection $\overline{\Phi}$ explained in the previous definition 1.25 which exists according to Theorem 4.10 and is defined for all times by the second part of Theorem 4.11.

*Definition* 1.27. Writing $o := eP \in G/P$, the section

$$\begin{aligned} O : M &\to \mathcal{S} \\ x &\mapsto [[u, e], o] \text{ for arbitrary } u \in \mathcal{P}_x \end{aligned} \tag{1.33}$$

is called *canonical section.*

**Lemma 1.28.** *$O$ from definition 1.27 is well-defined.*

*Proof.* It is to show, that $[[u, e], o] \in \mathcal{S}$ is independent of the choice of $u \in \mathcal{P}_x$. To this end let $\tilde{u} \in \mathcal{P}_x$, i.e. $\tilde{u} = u \cdot p$ for some $p \in P$. Then

$$\begin{aligned} [[\tilde{u}, e], o] &= [[u \cdot p, e], o] \\ &= [[u, p], o] \\ &= [[u, e] \cdot p, o] \\ &= [[u, e], pP] \\ &= [[u, e], o]. \end{aligned}$$

□

*Definition* 1.29. For a curve $\gamma : I \to M$ defined on an open interval $I$ with $0 \in I$ with $\gamma(0) = x$ we define the *development of* $\gamma$ $\mathrm{dev}_\gamma$ by

$$\begin{aligned} \mathrm{dev}_\gamma : I &\to \mathcal{S}_x \\ t &\mapsto \mathrm{Pt}_{\gamma_t}(O(\gamma(t)), -t) \end{aligned} \tag{1.34}$$

That is: Follow the curve $\gamma$ to the time $t$, consider the point $O(\gamma(t)) \in \mathcal{S}_\gamma(t)$ and then the parallel transport of that point into the fiber $\mathcal{S}_{\gamma(0)}$ yields $\mathrm{dev}_\gamma(t)$.

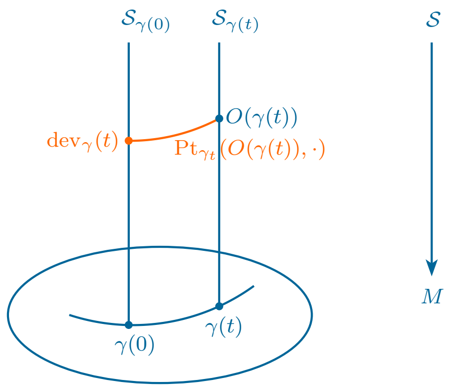


Figure 1: construction of $\mathrm{dev}_\gamma(t)$

While the parallel transport induced by a general connection need not be defined for all times, note again that in this case $\mathrm{Pt} = \mathrm{Pt}^{\overline{\Phi}}$ is the parallel transport induced by a general connection on a fibered manifold associated to a principal bundle, i.e. exists for all times. Thus $\mathrm{dev}_\gamma$ is well defined on all of $I$.

The following result explains the connection between curves in $M$ and their developments:

**Theorem 1.30** (Theorem 1.5.17 of [10])**.**

1. *Let $\gamma : I \to M$ be a smooth curve with $\gamma(0) = x$, let $u \in \mathcal{P}_x$ be a point, and let $g$ be a smooth $G$-valued function defined locally around zero such that $g(0) = e$.*

   *Then, locally around zero, we have $\mathrm{dev}_\gamma(t) = q(j(u), g(t)^{-1} \cdot o)$ if and only if there is a lift $\overline{\gamma} : I \to \mathcal{P}$ of $\gamma$ with $\overline{\gamma}(0) = u$ such that the curve $j(\overline{\gamma}(t)) \cdot g(t)$ in $\mathcal{G} = \mathcal{P} \times_P G$ is horizontal locally around zero.*

2. *Fix $x \in M$. Then*

$$\mathrm{dev} : \left\{ \begin{array}{c} \textit{germs } (I, \gamma) \textit{ of smooth curves} \\ \gamma : I \to M \textit{ with } \gamma(0) = x \end{array} \right\} \to \left\{ \begin{array}{c} \textit{germs } (J, \delta) \textit{ of smooth curves} \\ \delta : J \to \mathcal{S}_x \textit{ with } \delta(0) = O(x) \end{array} \right\}$$
$$\gamma \mapsto \mathrm{dev}_\gamma$$

   *is bijective.*

3. *This map is compatible with having contact to any order. (cf. definition 4.12)*

   *That is: Let $r \in \{0, 1, 2, \dots\}$. Two curves $\gamma$, $\delta : I \to M$ with $\gamma(0) = \delta(0) = x$ are having $r$-th order contact at zero if and only if* $\mathrm{dev}_\gamma$ *and* $\mathrm{dev}_\delta$ *are having $r$-th order contact at zero.*

*Proof.*

1. "$\Leftarrow$":

   Let $\overline{\gamma} : I \to \mathcal{P}$ be the lift, such that $j(\overline{\gamma}(t)) \cdot g(t)$ is locally horizontal in $\mathcal{G}$ and denote by $J$ its domain. For fixed $y \in G/H$ we have that $\alpha_y(t) := q\big(j(\overline{\gamma}(t)) \cdot g(t), y\big) : J \to \mathcal{S}$ is horizontal, because

$$\begin{aligned}\overline{\Phi}(\alpha_y'(t)) &= \overline{\Phi}\left(\frac{d}{dt}q(j(\overline{\gamma}(t))g(t), y)\right) \\ &= dq\Bigg(\underbrace{\Phi\left(\frac{d}{dt}j(\overline{\gamma}(t))g(t)\right)}_{=0 \text{ by assumption}}, \frac{d}{dt}y\Bigg) \qquad \text{(by diagram 4.3)} \\ &= 0.\end{aligned}$$

   Also $\alpha_y$ clearly is a lift of $\gamma : J \to M$. Let $t_0 \in J$ and $y := g(t_0)^{-1} \cdot o$. Then

$$\begin{aligned}\alpha_y(t_0) &= q\left(j(\overline{\gamma}(t_0)) \cdot g(t_0), g(t_0)^{-1} \cdot o\right) \\ &= q\left(j(\overline{\gamma}(t_0)), o\right) \\ &= O(\gamma(t_0)).\end{aligned}$$

   Hence we have for the parallel transport $\mathrm{Pt} = \mathrm{Pt}^{\overline{\Phi}}$ on $\mathcal{S}$:

$$\mathrm{Pt}_{\gamma_{t_0}}(O(\gamma(t_0)), s) = \alpha_{g(t_0)^{-1} \cdot o}(t_0 + s)$$

   for all $s$ with $t_0 + s \in J$. Thus

$$\begin{aligned}\mathrm{dev}_\gamma(t_0) &= \mathrm{Pt}_{\gamma_{t_0}}(O(\gamma(t_0)), -t_0) \\ &= \alpha_{g(t_0)^{-1} \cdot o}(0) \\ &= q\left(j(\overline{\gamma}(0)), g(t_0)^{-1} \cdot o\right) \\ &= q\left(j(u), g(t_0)^{-1} \cdot o\right).\end{aligned}$$

   "$\Rightarrow$":

   Conversely, assume $\mathrm{dev}_\gamma(t) = q(j(u), g(t)^{-1} \cdot o)$ locally around zero. We can always choose a smooth function $\tilde{g} : J \to G$ which represents the same function in $G/P$ as

$g$, i.e. $\tilde{g}(t) \cdot o = g(t) \cdot o$ for all $t \in J$. So without loss of generality assume $g : J \to G$ to be smooth. Define

$$\begin{aligned} \phi : J &\to \mathfrak{g} \\ t &\mapsto -\operatorname{Ad}(g(t))(\omega^{MC}(g'(t))). \end{aligned} \tag{1.35}$$

By the Picard-Lindelöf existence and uniqueness theorem the first order ODE

$$\omega(\overline{\gamma}'(t)) = \phi(t) \tag{1.36}$$

has a solution $\overline{\gamma}$ locally around zero, satisfying $\overline{\gamma}(0) = u$.

Then the curve $j(\overline{\gamma}(t)) \cdot g(t)$ is parallel in $\mathcal{G}$, because

$$\begin{aligned} \Phi\left(\frac{d}{dt} j(\overline{\gamma}(t)) \cdot g(t)\right) &= \Phi\left(dR_{g(t)}\left(\frac{d}{dt} j(\overline{\gamma}(t))\right) + \widetilde{\omega^{MC}(g'}(t))(j(\overline{\gamma}(t)) \cdot g(t))\right) \\ &= \operatorname{Ad}\left(g^{-1}(t)\right)\left(\Phi\left(\frac{d}{dt} j(\overline{\gamma}(t))\right)\right) + \omega^{MC}(g'(t)) \\ &= \operatorname{Ad}\left(g^{-1}(t)\right)\left(\omega(\overline{\gamma}'(t))\right) + \omega^{MC}(g'(t)) \\ &= 0, \end{aligned}$$

where for the second to last equality we used the relationship $j^*\Phi = \omega$, cf. Theorem 4.8. Hence we have constructed the desired lift $\overline{\gamma}$.

2. • dev is surjective:

   Let $\delta : J \to \mathcal{S}_x$ be some smooth curve and write $\delta(t) = [j(u), g^{-1}(t) \cdot o]$ for some fixed $u \in \mathcal{P}_x$ and some $g : J \to G$. In the proof of part 1 we constructed a curve $\overline{\gamma} : I \to \mathcal{P}$. Let $\gamma := \pi \circ \overline{\gamma}$, then by part 1 of the theorem:

   $$\operatorname{dev}_\gamma(t) = q(j(u), g(t)^{-1} \cdot o) = \delta(t).$$

   • dev is injective:

   We have constructed a curve with specified development. After having chosen $u \in \mathcal{P}_x$ and $g : J \to G$, $\overline{\gamma} : I \to \mathcal{P}$ was constructed as the unique solution to an ODE around zero. Obviously the curve $\gamma = \pi \circ \overline{\gamma}$ is the only curve which has $\overline{\gamma}$ as a lift to $\mathcal{P}$.

   Choosing a different representation for $\delta$ (i.e. another $u \in \mathcal{P}$ or another $g : J \to G$), we will receive a different solution $\overline{\gamma}$. If we can show that this solution projects onto the same curve as before, then we know that this projected curve is in fact the *only* curve with the specified development. (That is, because it is the only curve having a suitable lift $\overline{\gamma}$, and therefore, by part 1 of the theorem, it is the only curve to have the specified development)

Every other choice $\tilde{g}$ for $g$ is of the form $\tilde{g} = p \cdot g$ for some $p : J \to P$, because $(p(t) \cdot g(t))^{-1} \cdot o = g(t)^{-1} \cdot o$ if and only if $p(t)^{-1} \in P$. Following the construction from part 1, we get for the function $\tilde{\phi}$ induced by $\tilde{g}$ defined in equation 1.35:

$$\begin{aligned}\tilde{\phi} &= -\operatorname{Ad}(p(t)g(t))\left(\omega^{MC}((p \cdot g)'(t))\right)\\ &= -\operatorname{Ad}(p(t)g(t))\left[\operatorname{Ad}(g(t)^{-1})\left(\omega^{MC}(p'(t))\right) + \omega^{MC}(g'(t))\right]\\ &= -\operatorname{Ad}(p(t))\left(\omega^{MC}(p'(t)) - \phi(t)\right)\end{aligned} \tag{1.37}$$

where in the first step we used that

$$\begin{aligned}\omega^{MC}((p \cdot g)'(t)) &= \omega^{MC}\left(dR_{g(t)}(p'(t)) + \widetilde{\omega^{MC}(g'(t))}(g(t)p(t))\right)\\ &= \operatorname{Ad}(g(t)^{-1})(\omega^{MC}(p'(t))) + \omega^{MC}(g'(t)),\end{aligned}$$

where the last equality holds, because $\omega^{MC}$ is a Cartan connection on $G$.

Denote by $\tilde{\gamma}$ the solution of the differential equation 1.36 for $\tilde{g}$, i.e. $\omega\left(\frac{d}{dt}\tilde{\gamma}(t)\right) = \tilde{\phi}(t)$. The solution is then given explicitly as $\tilde{\gamma}(t) = \overline{\gamma}(t) \cdot p(t)^{-1}$, where $\overline{\gamma}$ is the solution to the original ODE 1.36. To see this, compute

$$\begin{aligned}\omega\left(\frac{d}{dt}(\overline{\gamma}(t) \cdot p(t)^{-1})\right) &= \omega\left(dR_{p(t)^{-1}}\overline{\gamma}'(t) + (\widetilde{\omega^{MC}(p^{-1})'(t))}(\overline{\gamma}(t) \cdot p(t)^{-1})\right)\\ &= \operatorname{Ad}(p(t))\underbrace{\omega(\overline{\gamma}'(t))}_{\phi(t)} + \omega^{MC}((p^{-1})'(t))\\ &= \operatorname{Ad}(p(t))(\phi(t)) - \operatorname{Ad}(p(t))\left(\omega^{MC}(p'(t))\right)\\ &= \tilde{\phi}(t),\end{aligned}$$

where in the second to the last step we used

$$\omega^{MC}((p^{-1})'(t)) = -\operatorname{Ad}(p(t))\left(\omega^{MC}(p'(t))\right), \tag{1.38}$$

which follows from

$$\begin{aligned}0 &= \frac{d}{dt}p(t)p(t)^{-1}\\ &= dR_{p^{-1}(t)}p'(t) + dL_{p(t)}((p^{-1})'(t))\\ &= dL_{p(t)}dR_{p(t)^{-1}}dL_{p(t)^{-1}}p'(t) + \omega^{MC}((p^{-1})'(t))\\ &= \operatorname{Ad}(p(t))dL_{p^{-1}(t)}p'(t) + \omega^{MC}((p^{-1})'(t))\\ &= \operatorname{Ad}(p(t))\left(\omega^{MC}(p'(t))\right) + \omega^{MC}((p^{-1})'(t)).\end{aligned} \tag{1.39}$$

Obviously $\pi \circ \tilde{\gamma} = \pi \circ \overline{\gamma}$, i.e. both choices $g$ and $\tilde{g}$ lead to the same curve $\gamma : I \to M$.

Now consider an alternative choice for $u$, to this end let $\widehat{u} = u \cdot p \in \mathcal{P}_x$ for some $p \in P$. Then $[u, g(t) \cdot o] = [u \cdot p, p^{-1}g(t) \cdot o] = [u \cdot p, p^{-1}g(t)p \cdot o]$. The function from equation 1.35 becomes in this case:

$$\begin{aligned}
\widehat{\phi} &= -\operatorname{Ad}(p^{-1}g(t)p)\left(\omega^{MC}\left(\frac{d}{dt}L_{p^{-1}} \circ R_p(g(t))\right)\right) \\
&= -\operatorname{Ad}(p^{-1}g(t)p)\left(\omega^{MC}\left(dL_{p^{-1}} \circ dR_p(g'(t))\right)\right) \\
&= -\operatorname{Ad}(p^{-1}g(t)p)\left(\omega^{MC}\left(dR_p(g'(t))\right)\right) && (\omega^{MC} \text{ is left-invariant}) \\
&= -\operatorname{Ad}(p^{-1}g(t)p)\left(\operatorname{Ad}(p^{-1})\left(\omega^{MC}\left((g'(t))\right)\right)\right) \\
&= \operatorname{Ad}(p^{-1})(\phi(t)), && (1.40)
\end{aligned}$$

where $\phi$ is the function for the original choice of $u$ and $g$. Denoting again by $\overline{\gamma}$ the solution to the original ODE 1.36, we find the unique solution $\widehat{\gamma}$ to the ODE $\omega(\widehat{\gamma}'(t)) = \widehat{\phi}(t)$ given as

$$\widehat{\gamma} = \overline{\gamma} \cdot p, \tag{1.41}$$

because

$$\begin{aligned}
\omega(\widehat{\gamma}'(t)) &= \omega\left(\frac{d}{dt}R_p(\overline{\gamma}(t))\right) \\
&= \omega\left(dR_p\overline{\gamma}'(t)\right) \\
&= \operatorname{Ad}(p^{-1})\omega\left(\overline{\gamma}'(t)\right) \\
&= \operatorname{Ad}(p^{-1})\phi(t) \\
&= \widehat{\phi}(t).
\end{aligned}$$

And again, this curve $\widehat{\gamma}$ projects onto the same curve $\gamma$ as the original solution $\overline{\gamma}$.

So altogether we have shown, that given any curve $\delta : J \to \mathcal{S}_x$, locally there is exactly one curve $\gamma : I \to M$ with $\operatorname{dev}_\gamma = \delta$. That is, dev is bijective.

3. It remains to show, that dev preserves the contact of curves.

Let $\gamma_1, \gamma_2 : I \to M$ be two curves that have the $r$-th order contact in zero. Consider the horizontal distribution defined by $\Phi \in \Omega^1(\mathcal{G}, \mathfrak{g})$ as described in Lemma 4.7. Fix $u \in \mathcal{P}_x$, then the uniquely determined horizontal lifts $\gamma_1^*, \gamma_2^* : I \to \mathcal{G}$ with $\gamma_1^*(0) = \gamma_2^*(0) = j(u)$ have $r$-th order contact by Lemma 4.16 in zero. Choose arbitrary lifts $\overline{\gamma_1}, \overline{\gamma_2} : J \to \mathcal{P}$ with $\overline{\gamma_1} = \overline{\gamma_2} = u$ which have $r$-th order contact in zero. Now there exist some $g_1, g_2 : J \to G$ such that

$$\gamma_i^* = j(\overline{\gamma_i}(t)) \cdot g_i(t)$$

for $i \in \{1, 2\}$. By the product rule we see that $g_1$ and $g_2$ have $r$-th order contact in zero. Composing with the smooth map $q(u, (\cdot)^{-1} \cdot o)$ preserves this, i.e. the curves $\operatorname{dev}_{\gamma_1} = q(u, g_1^{-1}(t) \cdot o)$ and $\operatorname{dev}_{\gamma_2} = q(u, g_2^{-1}(t) \cdot o)$ have $r$-th order contact.

Conversely, assume $\mathrm{dev}_{\gamma_1} = [u, g_1^{-1}(t)\cdot o]$ and $\mathrm{dev}_{\gamma_2} = [u, g_2^{-1}(t)\cdot o]$ to have $r$-th order contact. Choose lifts $\tilde{g}_i : J \to G$ of $g_i^{-1}(t)\cdot o$ with $\tilde{g}_i(0) = e$ for $i \in \{1,2\}$ which have $r$-th order contact in zero. Then the associated functions $\phi_1$ and $\phi_2$ from equation 1.35 have $(r-1)$-th order contact at zero, which follows from the product rule

$$\begin{aligned}
\frac{d}{dt}\mathrm{Ad}(g(t))(V(t)) &= \frac{d}{ds}\,|_{s=t}\ \mathrm{Ad}(g(s))V(t) + \mathrm{Ad}(g(t))V'(t)\\
&= \frac{d}{ds}\,|_{s=t}\ dL_{g(s)}dR_{g^{-1}(s)}V(t) + \mathrm{Ad}(g(t))V'(t)\\
&= \frac{d}{ds}\,|_{s=t}\ dL_{g(t)}dR_{g^{-1}(t)}dL_{g^{-1}(t)g(s)}dR_{g^{-1}(s)g(t)}V(t) + \mathrm{Ad}(g(t))V'(t)\\
&= \frac{d}{ds}\,|_{s=t}\ \mathrm{Ad}(g(t))\,\mathrm{Ad}\left(L_{g^{-1}(t)}g(s)\right)V(t)\\
&= \mathrm{Ad}(g(t))\,\mathrm{ad}(dL_{g^{-1}(t)}g'(t))V(t) + \mathrm{Ad}(g(t))V'(t)\\
&= \mathrm{Ad}(g(t))\,\mathrm{ad}((\delta g)^{(1)}(t))V(t) + \mathrm{Ad}(g(t))V'(t) \qquad (1.42)
\end{aligned}$$

for any curves $g : I \to G$ and $V : I \to \mathfrak{g}$. Thus also the curves $\overline{\gamma_i}' = \omega^{-1}\circ\phi_i$ have contact of order $(r-1)$ at zero, i.e. $\overline{\gamma_i}$ have contact of order $r$ at zero. By Lemma 4.16 also the projections $\pi\circ\overline{\gamma_i}$ have contact of order $r$ at zero. □

We now aim to define canonical curves as those curves, which have a simple development everywhere. The following definitions make this idea precise:

*Definition* 1.31.

(i) Let $\mathcal{C}$ be a family of smooth curves through $o = eP \in G/P$ in $G/P$ with the following property: For $\gamma : I \to G/P$ an element of $\mathcal{C}$, any $t_0 \in I$ and $g \in G$ such that $\gamma(t_0) = g^{-1}P$ the curve $t \mapsto g\cdot\gamma(t+t_0)$ (wherever defined) is again an element of $\mathcal{C}$.

Then $\mathcal{C}$ is called *admissible.*

(ii) Let $x \in M$ and $\gamma : I \to \mathcal{S}_x$ be a smooth curve defined on an open interval $I$ with $0 \in I$ and $c(0) = O(x)$. Let $\delta : I \to G/P$ be a smooth curve.

Then $\gamma$ *is represented by* $\delta$ *on* $I$ if there exists $u \in \mathcal{P}_x$ such that

$$\gamma(t) = [j(u), \delta(t)] \text{ for all } t \in I.$$

(iii) Let $\mathcal{C}$ be an admissible family of curves. A smooth curve $\gamma : I \to M$ is said to be a *canonical curve of type* $\mathcal{C}$ if for all $t_0 \in I$ the curve $\mathrm{dev}_{\gamma_{t_0}} : \tilde{I} \to \mathcal{S}_{\gamma(t_0)}$ is represented by an element of $\mathcal{C}$ on some neighborhood of zero.

(As in equation 1.32 $\gamma_{t_0}$ denotes the shifted curve and $\tilde{I}$ denotes the shifted domain)

To check whether a curve is canonical it suffices to check whether that curve is canonical *in one point*. This is made explicit in the next lemma:

**Lemma 1.32** (Proposition 1.5.18 of [10]). *Let $\mathcal{C}$ be an admissible family of curves in $G/P$ and $\gamma : I \to M$ a smooth curve.*

*If there exists $t_0 \in I$ such that $\mathrm{dev}_{\gamma_{t_0}}$ is represented by an element of $\mathcal{C}$ on the maximal domain $\tilde{I} = \{t \in \mathbb{R} \mid t + t_0 \in I\}$ then $\gamma$ is a canonical curve of type $\mathcal{C}$.*

*Proof.* Without loss of generality let $t_0 = 0$ and therefore $\tilde{I} = I$.

By assumption we have $\mathrm{dev}_{\gamma_{t_0}} = \mathrm{dev}_\gamma = [j(u), g^{-1}(t) \cdot o]$ for some $u \in \mathcal{P}_{\gamma(0)}$ and $g : I \to G$. On the other hand choose any lift $\overline{\gamma} : I \to \mathcal{P}$ of $\gamma$ with $\overline{\gamma}(0) = u$. By the first part of Theorem 4.11 we have a horizontal lift $\gamma^{hor} : I \to \mathcal{G}$ of $\gamma$. This lift has the form $\gamma^{hor} = j(\overline{\gamma}(t)) \cdot h(t)$ for some $h : I \to G$. By the proof of the first part of Theorem 1.30 we have $\mathrm{dev}_\gamma(t) = [j(u), h(t)^{-1} \cdot o]$ for all $t \in I$. That means that $g^{-1} \cdot o$ and $h^{-1} \cdot o$ are the same curve, so in particular $h^{-1} \cdot o \in \mathcal{C}$.

Now take $t_1 \in I$ and consider the curve $\delta(t) = \gamma^{hor}(t+t_1) \cdot h(t_1)^{-1} \in \mathcal{G}$. Because $\gamma^{hor}$ was horizontal, this curve is again horizontal. It is $\delta(0) = j(\overline{\gamma}(t_1)) \in j(\mathcal{P})$ and $\delta$ is clearly a lift of $\gamma_{t_1}$. By Theorem 1.30 we have $\mathrm{dev}_{\gamma_{t_1}}(t) = [j(\overline{\gamma}(t_1)), h(t_1)h(t+t_1)^{-1} \cdot o]$. Because $\mathcal{C}$ is admissible and $t \mapsto h(t)^{-1} \cdot o$ lies in $\mathcal{C}$, we also have that the curve $t \mapsto h(t_1)h(t+t_1)^{-1} \cdot o$ lies in $\mathcal{C}$. Since $t_1 \in I$ was arbitrary, $\gamma$ is canonical by definition. □

This shows "that the structure of the local canonical curves of type $\mathcal{C}$ through any point $x$ in any Cartan geometry looks exactly as the structure of local curves through $o$ in $G/P$ which are in $\mathcal{C}$." ([10], p. 111) The following corollary makes this precise:

**Corollary 1.33.** *Let $(\mathcal{P}, \omega)$ be a Cartan geometry of type $(G, P)$ on $M$ and let $\mathcal{C}$ be an admissible family of curves in $G/P$. Fix $x \in M$ and $u \in \mathcal{P}_x$.*

*Then the map*

$$\mu : \left\{ \begin{array}{c} \text{germs } (I,\gamma) \text{ of canonical} \\ \text{curves of type } \mathcal{C} \text{ with } \gamma(0) = x \end{array} \right\} \to \left\{ \begin{array}{c} \text{germs } (J,\delta) \text{ of} \\ \text{curves in } \mathcal{C} \end{array} \right\} \\ \gamma \mapsto \delta, \text{ where } \mathrm{dev}_\gamma = [j(u), \delta] \tag{1.43}$$

*is a bijection and is compatible with having contact of any order.*

*Proof.* It is clear that

$$\nu : \left\{ \begin{array}{c} \text{germs } (J,\delta) \text{ of} \\ \text{curves in } \mathcal{C} \end{array} \right\} \to \left\{ \begin{array}{c} \text{germs } (\widehat{J},\widehat{\delta}) \text{ of smooth curves} \\ \widehat{\delta} : J \to \mathcal{S}_x \text{ with } \widehat{\delta}(0) = O(x) \end{array} \right\} \\ \delta \mapsto [j(u), \delta] \tag{1.44}$$

is a bijection that is compatible with having contact to any order.

Theorem 1.30 shows that $\nu^{-1} \circ \mathrm{dev}$ is injective, hence its restriction to a smaller set of curves is still injective.

By the same theorem we know that dev is surjective. I.e. for some curve $\delta : J \to G/P$ we have a curve $\gamma : I \to M$ such that $\mathrm{dev}_\gamma$ is represented by $\delta$. By Lemma 1.32 this means that $\gamma$ is a canonical curve of type $\mathcal{C}$, i.e. $\mu$ is also surjective.

In Theorem 1.30 we showed that dev preserves contact of any order. It is an easy observation that the map $\nu$ from equation 1.44 preserves contact on any order. Hence $\mu = \nu^{-1} \circ \mathrm{dev}$ must also be compatible with contact of any order. □

So in order to understand what jets determine a canonical curve of a Cartan geometry, it will be sufficient to look at the flat model $G/P$.

The general definition of canonical curves is cumbersome and it is difficult to check if a given curve is canonical or not. For a special choice of $\mathcal{C}$, namely if $\mathcal{C}$ is given as a family of one-parameter subgroups, there is an alternative characterization of canonical curves, as the next Lemmas will show.

**Lemma 1.34.** *Let $A \subset \mathfrak{g}$ be an $\mathrm{Ad}(P)$-invariant subset, i.e. $\mathrm{Ad}(p)(A) \subset A$ for all $p \in P$. For $X \in A$ define the curve $c^X$ to be*

$$\begin{aligned} c^X : \mathbb{R} &\to G/P \\ t &\mapsto \exp(tX) \cdot o. \end{aligned}$$

*Define the family of curves induced by $A$ to be*

$$\mathcal{C}_A := \{c^X \mid X \in A\}, \tag{1.45}$$

*then $\mathcal{C}_A$ is admissible.*

*Proof.* Let $c^X \in \mathcal{C}_A$. Clearly $c^X(0) = o$. Now let $t_0 \in \mathbb{R}$ and $c^X(t_0) = g^{-1}o$ for some $g \in G$, i.e. $\exp(t_0X) = g^{-1}p$ for some $p \in P$. Then

$$\begin{aligned} g\exp((t+t_0)X)o &= p\exp(-t_0X)\exp((t+t_0)X)o \\ &= \exp(t\,\mathrm{Ad}(p)X)o \end{aligned}$$

and $\mathrm{Ad}(p)X \in A$ by assumption. Hence $\mathcal{C}_A$ is in fact admissible. □

**Lemma 1.35** (Corollary 1.5.18 of [10])**.** *Given a Cartan geometry $(\mathcal{P}, \omega)$ of type $(G, P)$ on $M$ and $A \subset \mathfrak{g}$, such that $\mathcal{C}_A$ is admissible (cf. equation 1.45).*

*Then a curve $\gamma : I \to M$ is a canonical curve of type $\mathcal{C}_A$ if and only if it locally coincides up to a constant shift of parameter with the projection of a flow line in $\mathcal{P}$ of a constant vector field $\omega^{-1}(X)$ with $X \in A$.*

*Proof.* "$\Leftarrow$":

Let $X \in A$ and $u \in \mathcal{P}$ arbitrary and $\gamma(t) := \pi \circ \mathrm{Fl}_t^{\omega^{-1}(X)}(u)$ the projection of the flow line in $\mathcal{P}$ along the vector field $\omega^{-1}(X)$ starting in $u$ defined on some open interval $I$, assume the parameter to not be shifted without loss of generality. Then

$$\begin{aligned} \gamma^* : I &\to \mathcal{G} \\ t &\mapsto j\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u)\right) \cdot \exp(-tX) \end{aligned}$$

is a horizontal lift of $\gamma$ to $\mathcal{G}$ with respect to the principal connection $\Phi \in \Omega^1(\mathcal{G}, \mathfrak{g})$. That is because

$$\begin{aligned} &\Phi\left(\frac{d}{dt} j\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u)\right) \cdot \exp(-tX)\right) \\ &= \Phi\left(dR_{\exp(-tX)} \frac{d}{dt} j\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u)\right)\right) \\ &\quad + \Phi\left(\omega^{MC}\left(\frac{d}{dt}\exp(-tX)\right)\left(j\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u)\right) \cdot \exp(-tX)\right)\right) \\ &= \mathrm{Ad}(\exp(tX))\Phi\left(\frac{d}{dt} j\left(\mathrm{Fl}_t^{\omega^{-1}(X)}(u)\right)\right) + \omega^{MC}\left(\frac{d}{dt}\exp(-tX)\right) \\ &= \mathrm{Ad}(\exp(tX))\omega\left(\frac{d}{dt}\mathrm{Fl}_t^{\omega^{-1}(X)}(u)\right) + dL_{\exp(tX)} dR_{\exp(-tX)} \underbrace{\frac{d}{ds}\exp(-sX)}_{=-X} \\ &= \mathrm{Ad}(\exp(tX))(X) + \mathrm{Ad}(\exp(tX))(-X) \\ &= 0. \end{aligned}$$

Hence by the first part of Theorem 1.30 we have

$$\mathrm{dev}_\gamma(t) = q(j(u), \exp(tX) \cdot o), \tag{1.46}$$

i.e. $\gamma$ is canonical of type $\mathcal{C}_A$.

"$\Rightarrow$":

Let $\gamma : I \to M$ be a canonical curve of type $\mathcal{C}_A$ with $\mathrm{dev}_\gamma(t) = q(j(u), \exp(tX) \cdot o)$ for some $u \in \mathcal{P}_{\gamma(0)}$ around $t_0 = 0$. By equation 1.46 from the first part of the proof, the curve

$$\begin{aligned} \widehat{\gamma} : J &\to \mathcal{P} \\ t &\mapsto \pi \circ \mathrm{Fl}_t^{\omega^{-1}(X)}(u) \end{aligned}$$

has the development

$$\mathrm{dev}_{\widehat{\gamma}}(t) = q(j(u), \exp(tX) \cdot o) = \mathrm{dev}_\gamma(t).$$

By the second part of Theorem 1.30, the curves $\gamma$ and $\widehat{\gamma}$ coincide locally around zero. Hence $\gamma$ is the projection of a flow line around zero. If choosing another $t_0 \in I$, one finds $\gamma$ to be the projection of a flow line with shifted parameter locally around $t_0$. $\square$

In what follows we will assume $(\mathcal{P}, \omega)$ to be a $|1|$-graded Cartan geometry of type $(G, P)$, i.e. $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a $|1|$-graded Lie algebra and $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. In particular conformal Cartan geometries are of this kind, as shown in Lemma 1.22.

**Lemma 1.36** (Proposition 5.3.1. of [10])**.** *Let $(\mathcal{P}, \omega)$ be a $|1|$-graded Cartan geometry of type $(G, P)$ on a manifold $M$. Let $A_0 \subset \mathfrak{g}_{-1}$ be an $\mathrm{Ad}(G_0)$-invariant subset. Let*

$$A := \{\mathrm{Ad}(\exp Z)(X) \mid X \in A_0, Z \in \mathfrak{g}_1\}. \tag{1.47}$$

*(a) $A$ is $\mathrm{Ad}(P)$-invariant.*

*(b) $\gamma : I \to M$ is a canonical curve of type $\mathcal{C}_A$ if and only if $\gamma$ locally coincides up to a constant shift of parameter with the projection of a flow line in $\mathcal{P}$ of a constant vector field $\omega^{-1}(X)$ with $X \in A_0$.*

The difference of part (b) to the previous Lemma 1.35 is that $X$ may not only be assumed to be in the $\mathrm{Ad}(P)$-invariant set $A$ but even in the smaller set $A_0$.

*Proof.*

(a) Let $p \in P$, $Z \in \mathfrak{g}_1$, $X \in A_0$. We have to show that $\mathrm{Ad}(p)\,\mathrm{Ad}(\exp Z)X \in A$. According to Theorem 4.19 we can write $p = g_0 \exp(Y)$ for some $g_0 \in G_0$, $Y \in \mathfrak{g}_1$. Also note that $[Y, Z] = 0$, hence by Theorem 4.20: $\exp(Y)\exp(Z) = \exp(Y + Z)$. Therefore

$$\begin{aligned}
\mathrm{Ad}(p)\,\mathrm{Ad}(\exp Z)X &= \mathrm{Ad}(g_0 \exp(Y)\exp(Z))X \\
&= \mathrm{Ad}(g_0 \exp(Y + Z))X \\
&= \mathrm{Ad}(\exp(\mathrm{Ad}(g_0)(Y + Z))g_0^{-1})(X) \\
&= \mathrm{Ad}(\exp(\underbrace{\mathrm{Ad}(g_0)(Y + Z)}_{\in \mathfrak{g}_1}))\,\underbrace{\mathrm{Ad}(g_0^{-1})(X)}_{\in A_0}.
\end{aligned}$$

(b) "$\Leftarrow$":

Because $A_0 \subset A$, this direction is a direct consequence of the previous Lemma 1.35.

"$\Rightarrow$":

By Lemma 1.35 we know that $\gamma$ is of the form $\gamma(t) = \pi \circ \mathrm{Fl}_t^{\omega^{-1}(Y)}(u_0)$ for some $u_0 \in \mathcal{P}_{\gamma(0)}$, where $Y = \mathrm{Ad}(\exp Z)(X)$ for $Z \in \mathfrak{g}_1$ and $X \in A_0$. We then have

$$\omega^{-1}(Y)(u) = \omega^{-1}(\mathrm{Ad}(\exp Z)X)(u) = dR_{\exp(-Z)}\omega^{-1}(X)(u \cdot \exp Z),$$

thus $R_{\exp Z} \circ \mathrm{Fl}_t^{\omega^{-1}(Y)}(u_0)$ is a flow line for the constant vector field $\omega^{-1}(X)$. And because

$$\begin{aligned}\gamma(t) &= \pi \circ \mathrm{Fl}_t^{\omega^{-1}(Y)}(u_0)\\ &= \pi \circ R_{\exp Z} \circ \mathrm{Fl}_t^{\omega^{-1}(Y)}(u_0),\end{aligned}$$

we find $\gamma$ to be the projection of a flow line of a constant vector field of $A_0$. □

*Remark* 1.37. An obvious choice for the set $A_0$ from Lemma 1.36 is $A_0 = \mathfrak{g}_{-1}$. Define

$$\widetilde{\mathfrak{g}_{-1}} := \{\mathrm{Ad}(\exp Z)X \mid X \in \mathfrak{g}_{-1}, Z \in \mathfrak{g}_1\}. \tag{1.48}$$

That is, $\widetilde{\mathfrak{g}_{-1}}$ plays the role of $A$ from Lemma 1.36.

*Definition* 1.38. Let $N$ be a manifold. For an arbitrary map $f : N \to G$ we call the function

$$\begin{aligned}\delta f : TN &\to \mathfrak{g}\\ V &\mapsto \omega^{MC}(df(V))\end{aligned} \tag{1.49}$$

the *logarithmic derivative of* $f$.

For a curve $u : \mathbb{R} \to G$ we denote

$$(\delta u)^{(1)}(\cdot) := \omega^{MC}(u'(\cdot)) : \mathbb{R} \to \mathfrak{g} \tag{1.50}$$

and for $i \geq 2$

$$(\delta u)^{(i)}(\cdot) := \frac{d^{i-1}}{dt^{i-1}}(\delta u)^{(1)}. \tag{1.51}$$

Also we set $(\delta u)^{(0)} := u$, which simplifies notation later on.

**Theorem 1.39** (Lemma 5.3.2 of [10])**.** *Let $\gamma_1,\ \gamma_2 : \mathbb{R} \to G$ be smooth curves with $\gamma_1(0) = \gamma_2(0) = e$ and let $u : \mathbb{R} \to G$ be the curve defined by*

$$u(t) := \gamma_2(t)^{-1}\gamma_1(t). \tag{1.52}$$

*(a) Then for each $r \in \mathbb{N}$ with $r \geq 1$, the following conditions are equivalent:*

*(i) $\gamma_1$ and $\gamma_2$ have contact of order $r$ in zero.*

*(ii) $u$ has contact of order $r$ in zero with the constant curve $e$.*

*(iii) The curves $(\delta\gamma_1)^{(1)}$ and $(\delta\gamma_2)^{(1)}$ have contact of order $(r-1)$ in zero.*

*(b) Denote by $\pi : G \to G/P$ the canonical projection. For each $r \in \mathbb{N}$ with $r \geq 1$, the curves $\pi \circ \gamma_1$ and $\pi \circ \gamma_2$ have contact of order $r$ in zero if and only if $(\delta u)^{(i)}(0) \in \mathfrak{p}$ for all $i \in \{1, \dots, r\}$.*

*Proof.* (a) Consider the local parametrization $\exp : \mathfrak{g} \to G$ of $G$. Let $\phi_i : J \to \mathfrak{g}$ such that $\gamma_i(t) = \exp(\phi_i(t))$ for all $t \in J$ and $\phi_i(0) = 0$ for some open interval $J \subset \mathbb{R}$.

"(i) $\Rightarrow$ (iii)":

By Lemma 4.15 we have that all derivatives of the $\phi_i$ in zero agree up to order $r$. By Lemma 4.18 we also have

$$(\delta\gamma_i)^{(1)}(t) = \sum_{p=0}^{\infty} \frac{1}{(p+1)!} \operatorname{ad}(-\phi_i(t))^p(\phi_i'(t)). \tag{1.53}$$

Since $\operatorname{ad} : \mathfrak{g} \times \mathfrak{g} \to \operatorname{gl}(\mathfrak{g})$ is a multilinear map, we have $d\operatorname{ad}_{(X,Y)}(V,W) = \operatorname{ad}(X,W) + \operatorname{ad}(V,Y)$, thus $(\delta\gamma_i)^{(j)}(t)$ is an expression depending on the values $\phi_i(t)$, $\phi_i'(t)$, $\ldots$, $\phi_i^{(j)}(t)$ and therefore $(\delta\gamma_1)^{(j)}(0)$ and $(\delta\gamma_2)^{(j)}(0)$ have to agree by assumption for $j \in \{1, \ldots, r\}$.

"(iii) $\Rightarrow$ (ii)":

If $(\delta\gamma_i)^{(1)}(t)$ has vanishing $(r-1)$-jet in 0, then differentiating equation 1.53 $(s-1)$ times yields

$$\begin{aligned}(\delta\gamma_i)^{(s)}(t) = \phi_i^{(s)}(t) &+ \overbrace{\left\{\begin{array}{c}\text{linear combinations of iterated Lie}\\ \text{brackets with entries } \phi_i^{(k)}(t) \text{ for}\\ 0 \le k \le s \text{ containing one entry } \phi_i(t)\end{array}\right\}}^{=0 \text{ for } t=0} \\ &+ \left\{\begin{array}{c}\text{linear combinations of iterated Lie}\\ \text{brackets with entries } \phi_i^{(k)}(t) \text{ for}\\ 0 \le k \le s-1\end{array}\right\},\end{aligned} \tag{1.54}$$

hence we see via induction over $r$, that $\phi_i^{(s)}(0)$ vanishes for $1 \le s \le r$ if $(\delta\gamma_i)^{(s)}(0)$ vanishes for $1 \le s \le r$. In this case the $r$-jet in zero of the curve $\exp \circ \phi_i$ coincides with the $r$-jet of the constant curve $e$ in zero.

Now assume the general case with the $(r-1)$-jet of $(\delta\gamma_i)^{(1)}$ not necessarily vanishing. Using Lemma 4.17 and equation 1.38 we have

$$(\delta u)^{(1)}(t) = (\delta\gamma_1)^{(1)}(t) - \operatorname{Ad}(u(t)^{-1})((\delta\gamma_2)^{(1)}(t)) \tag{1.55}$$

and another differentiation yields

$$\begin{aligned}(\delta u)^{(2)}(t) = (\delta\gamma_1)^{(2)}(t) &\\ &- \operatorname{Ad}(u(t)^{-1}) \operatorname{ad}(dL_{u(t)}(u^{-1})'(t))(\delta\gamma_2)^{(1)}(t) \\ &- \operatorname{Ad}(u(t)^{-1})((\delta\gamma_2)^{(2)}(t))\end{aligned} \tag{1.56}$$

where we used the product rule from equation 1.42 again. Differentiating equation 1.56 further shows

$$(\delta u)^{(s)}(t) = (\delta\gamma_1)^{(s)}(t) - \left\{ \begin{array}{c} \text{terms containing logarithmic derivatives} \\ \text{of order not higher than } s-1 \text{ of } u \end{array} \right\} - \mathrm{Ad}(u(t)^{-1})((\delta\gamma_2)^{(s)}(t)) \tag{1.57}$$

and, using $u(0) = e$, induction over $s$ shows that $(\delta u)^{(s)}(0)$ vanishes for all $1 \leq s \leq r$. Following equation 1.54 we explained that this implies that the $r$-jet of the curve $u$ in zero vanishes.

"(ii) $\Rightarrow$ (i)":

This follows from writing $\gamma_1 = \gamma_2 \cdot u$ and using the product rule.

(b) As a preparation take some local smooth section $\sigma : G/P \supset U \to G$ defined on an open neighborhood $U$ of $o = eP \in G/P$. Then

$$\begin{aligned} \Phi : U \times P &\to \pi^{-1}(U) \\ (x, b) &\mapsto \sigma(x) \cdot b \end{aligned}$$

is a diffeomorphism, hence locally around zero we have unique curves $b_1,\ b_2 : \mathbb{R} \supset I \to P$ such that

$$\gamma_i(t) = \sigma(\pi(\gamma_i(t))) \cdot b_i(t)$$

for $i \in \{1, 2\}$ and $t \in I$. Thus for $t \in I$:

$$u(t) = b_2(t)^{-1}\sigma(\pi(\gamma_2(t)))^{-1}\sigma(\pi(\gamma_1(t)))b_1(t).$$

Now to proving the claim:

"$\Rightarrow$":

Assume $\pi \circ \gamma_1$ and $\pi \circ \gamma_2$ to have the same $r$-jet in zero, then $\sigma \circ \pi \circ \gamma_1$ and $\sigma \circ \pi \circ \gamma_2$ also have the same $r$-jet in zero. By part (a) of the theorem this implies that $\sigma(\pi(\gamma_2(t)))^{-1}\sigma(\pi(\gamma_1(t)))$ has contact of order $r$ in zero with the constant curve $e$. By the product rule multiplying the two curves with some other curve will preserve contact of any order, i.e. $u(t) = b_2(t)^{-1}\sigma(\pi(\gamma_2(t)))^{-1}\sigma(\pi(\gamma_1(t)))b_1(t)$ and $b_2(t)^{-1}b_1(t)$ have contact of order $r$ in zero.

Again by part (a) of the theorem, this implies that $(\delta u)^{(1)}$ and $\left(\delta b_2(t)^{-1}b_1(t)\right)^{(1)}$ have contact of order $(r-1)$. However by Lemma 4.15 this means exactly

$$\frac{d^i}{dt^i}\,|_{t=0}\ (\delta u)^{(1)}(t) = \frac{d^i}{dt^i}\,|_{t=0}\ \left(b_2(t)^{-1}b_1(t)\right)^{(1)}$$

for $i \in \{0, \dots, r\}$, and the right side is in $\mathfrak{p}$ by definition.

"$\Leftarrow$":

Conversely, assume $(\delta u)^{(i)}(0) \in \mathfrak{p}$ for $i \in \{0, \dots, r\}$. Let $b : I \to P$ be a solution to the ODE

$$(\delta b)^{(1)}(t) = (\delta u)^{(1)}(0) + t \cdot (\delta u)^{(2)}(0) + \frac{1}{2} t^2 \cdot (\delta u)^{(3)}(0) + \dots + \frac{1}{(r-1)!} t^{r-1} \cdot (\delta u)^{(r)}(0) \tag{1.58}$$

with initial value $b(0) = u(0) = e$. Then obviously $(\delta b)^{(i)}(0) = (\delta u)^{(i)}(0)$ for $i \in \{0, \dots, r\}$. By part (a) of the theorem this implies that $b$ and $u$ have contact of order $r$ in zero. Having contact of some order is preserved under composition with another smooth map, hence $u^{-1}$, $b^{-1} : I \to G$ also have contact of order $r$. And again by part (a) we find that the curve $u \cdot b^{-1}$ has the same $r$-jet in 0 as the constant curve $e$. By construction

$$\pi(\gamma_2(t)) = \pi(\gamma_1(t) \cdot u(t)) = \pi(\gamma_1(t) \cdot u(t) \cdot b^{-1}(t))$$

and by the product rule the curves $\gamma_1 \cdot u \cdot b^{-1}$ and $\gamma_1$ have the same $r$-jet in 0, hence the same holds for the composition of the two curves with the projection map $\pi$. This shows that $\pi \circ \gamma_1$ and $\pi \circ \gamma_2$ have the same $r$-jet in 0. □

We will now use part (b) of the above theorem to answer the questions how many derivatives in a point are needed to pin down a canonical curve uniquely.

**Theorem 1.40** (Section 2.7 of [14])**.** *Let $M$ be a manifold and $(\mathcal{P}, \omega)$ be a Cartan geometry of type $(G, P)$ on $M$. If $\mathfrak{g}$ is $|1|$-graded with grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, then a canonical curve of type $C_{\widetilde{\mathfrak{g}_{-1}}}$ is determined by its 2-jet in one point.*

*That is: If two geodesics $\gamma_1$, $\gamma_2 : I \to M$ of type $C_{\widetilde{\mathfrak{g}_{-1}}}$ have the same 2-jet in one point, then they coincide.*

*Proof.*

- Note that it suffices to prove the theorem locally, that is: If two geodesics $\gamma_1$, $\gamma_2 : I \to M$ have the same 2-jet in one point, then there exists an open interval $J \subset I$ such that $\gamma_1 |_J = \gamma_2 |_J$. This implies for

$$A := \{t \in I \mid \gamma_1(t) = \gamma_2(t)\}$$

to be open and closed, hence $\gamma_1(t) = \gamma_2(t)$ everywhere.

- By assumption we have the following developments around $t = 0$:

$$\mathrm{dev}_{(\gamma_1)_0} = [j(u_1), c_1(t)],$$
$$\mathrm{dev}_{(\gamma_2)_0} = [j(u_2), c_2(t)],$$

for some $u_1$, $u_2 \in \mathcal{P}_{\gamma_1(0)}$ and $c_1$, $c_2 : J \to G/P$, $c_1$, $c_2 \in \mathcal{C}_{\widetilde{\mathfrak{g}_{-1}}}$. There is $b \in P$, such that $u_2 = u_1 \cdot b$ and then

$$\mathrm{dev}_{(\gamma_2)_0} = [j(u_1), b \cdot c_2(t)].$$

By admissibility of $\mathcal{C}_{\widetilde{\mathfrak{g}_{-1}}}$ (cf. Lemma 1.34) we have $c_1$, $b \cdot c_2 \in \mathcal{C}_{\widetilde{\mathfrak{g}_{-1}}}$, i.e. there exist $X_1$, $X_2 \in \widetilde{\mathfrak{g}_{-1}}$ such that $c_1(t) = \underbrace{\exp(tX_1)}_{\overline{c}_1(t)} P$ and $bc_2(t) = \underbrace{\exp(tX_2)}_{\overline{c}_2(t)} P$. Multiplying the second equality with $b^{-1}$ yields $c_2(t) = \exp(t \underbrace{\mathrm{Ad}(b^{-1})X_2}_{\in\widetilde{\mathfrak{g}_{-1}}})P$, so without loss of generality assume $b = e$.

Now assume $\gamma_1$, $\gamma_2$ have contact of order 2 in zero. By Corollary 1.33 also the curves $c_1(t)$, $c_2(t)$ have contact of order 2 in zero. Define $u := \overline{c}_2^{-1}\overline{c}_1 : J \to G$. Using part (b) of Theorem 1.39 we find that $(\delta u)^{(s)} \in \mathfrak{p}$ for $s \in \{1, 2\}$.

- We want to show that $(\delta u)^{(s)} \in \mathfrak{p}$ for $s \in \mathbb{N}$.

  First note that

$$\begin{aligned}
(\delta u)^{(1)}(t) &= (\delta \exp(tX_1))^{(1)}(t) + \mathrm{Ad}\left(\exp(-tX_1)\right) \left(\delta \left(\exp(-tX_2)\right)^{(1)}(t)\right) \\
&= X_1 - \mathrm{Ad}\left(\exp(-tX_1)\right)(X_2), && (1.59) \\
(\delta u)^{(2)}(t) &= -\mathrm{Ad}\left(\exp(-tX_1)\right) \mathrm{ad}\left(\left(\delta \exp(-tX_1)\right)^{(1)}(t)\right)(X_2) \\
&= \mathrm{Ad}\left(\exp(-tX_1)\right) \mathrm{ad}\left(X_1\right)(X_2). && (1.60)
\end{aligned}$$

  From that follows the general formula for $s \geq 2$:

$$\begin{aligned}
(\delta u)^{(s)}(t) &= -\mathrm{Ad}\left(\exp(-tX_1)\right) \mathrm{ad}\left(-X_1\right)^{s-1}(X_2) \\
&= \mathrm{Ad}\left(\exp(-tX_1)\right) \mathrm{ad}\left(-X_1\right) \mathrm{Ad}(\exp(tX_1))(\delta u)^{(s-1)}(t). && (1.61)
\end{aligned}$$

- Assume $c_1(t) = \exp(tX)P$ and $c_2 = \exp(Z) \cdot \exp(tX)P = \exp(t\,\mathrm{Ad}(\exp(Z))X)P$ with $X \in \mathfrak{g}_{-1}$ and $Z \in \mathfrak{g}_1$. Plugging this into the equations 1.59 and 1.60:

$$\begin{aligned}
(\delta u)^{(1)}(0) &= X - \mathrm{Ad}(\exp(Z))(X) \\
&= X - \exp(\mathrm{ad}(Z))(X) \\
&= X - X - [Z, X] - \frac{1}{2}[Z, [Z, X]] - \underbrace{\frac{1}{6}[Z, [Z, [Z, X]]] - \ldots}_{=0,\text{ because }|1|\text{-graded}} \\
&= -[Z, X] - \frac{1}{2}[Z, [Z, X]] \in \mathfrak{p}, \\
(\delta u)^{(2)}(0) &= \mathrm{ad}(X)(\mathrm{Ad}(\exp(Z))X) \\
&= \underbrace{[X, [Z, X]]}_{\in\mathfrak{g}_{-1}} + \frac{1}{2}\underbrace{[X, [Z, [Z, X]]]}_{\in\mathfrak{g}_0} \in \mathfrak{p},
\end{aligned}$$

where for the computation we used the fact that $\mathrm{Ad}(\exp Z) = \exp(\mathrm{ad}(Z)) \in \mathfrak{gl}(\mathfrak{g})$. Thus $[X, [Z, X]] = 0$ and therefor by the Jacobi identity also

$$\begin{aligned}[X, [Z, [Z, X]]] &= -[Z, [[Z, X], X]] - [[Z, X], [X, Z]] \\ &= [Z, [X, [Z, X]]] = 0,\end{aligned}$$

i.e. $(\delta u)^{(2)}(0) = 0$. And by equation 1.61 we actually have $(\delta u)^{(s)}(0) = 0$ for $s \geq 2$, so in particular $(\delta u)^{(s)}(0) \in \mathfrak{p}$ for $s \in \mathbb{N}$.

- Assume $c_1(t) = \exp(tX_1)P$, $c_2(t) = \exp(t\tilde{X}_2)P$ for $X_1 \in \mathfrak{g}_{-1}$ and $\tilde{X}_2 \in \widetilde{\mathfrak{g}_{-1}}$. Write $\tilde{X}_2 = \mathrm{Ad}(\exp(Z))X_2$ for some $Z \in \mathfrak{g}_1$ and $X_2 \in \mathfrak{g}_{-1}$, i.e. $c_2(t) = \exp(Z)\exp(tX_2)P$. Then by equation 1.59:

$$\begin{aligned}(\delta u)^{(1)}(0) &= X_1 - \mathrm{Ad}(\exp(Z))X_2 \\ &= X_1 - \exp(\mathrm{ad}(Z))(X_2) \\ &= \underbrace{X_1 - X_2}_{\in\mathfrak{g}_{-1}} - \underbrace{[Z, X_2]}_{\in\mathfrak{g}_0} - \frac{1}{2}\underbrace{[Z, [Z, X_2]]}_{\in\mathfrak{g}_1} - \underbrace{\frac{1}{6}[Z, [Z, [Z, X_2]]] - \ldots}_{=0} \in \mathfrak{p},\end{aligned}$$

  hence the $\mathfrak{g}_{-1}$-coordinate must vanish, i.e. $X_1 - X_2 = 0$. That is, we have reduced this case to the previous case.

- Assume $c_1(t) = \exp(t \cdot \mathrm{Ad}(\exp Z_1)X_1)P = \exp(Z_1) \cdot \exp(tX_1)P$, $c_2(t) = \exp(t \cdot \mathrm{Ad}(\exp Z_2)X_2)P = \exp(Z_2) \cdot \exp(tX_2)P$ with $X_1$, $X_2 \in \mathfrak{g}_{-1}$ and $Z_1$, $Z_2 \in \mathfrak{g}_1$. Note that $\widetilde{c_1}(t) := \exp(tX_1)P$ and $\widetilde{c_2}(t) := \exp(-Z_1)\exp(Z_2) \cdot \exp(tX_2)P$ define the same $u$ as $c_1$ and $c_2$. We have $\exp(-Z_1)\exp(Z_2) = \exp(Z)$ for some $Z \in \mathfrak{g}_1$ by Theorem 4.20. Again we have reduced this case to the previous case.

- By part (b) of Theorem 1.39 we find that $c_1$ and $c_2$ have contact of order infinity in zero. We note that $c_i = \pi \circ \exp(tX_i)$ is the composition of analytic maps for $i \in \{1, 2\}$. Hence there exists some open $J_2 \subset J$ such that $c_1 |_{J_2} = c_2 |_{J_2}$ and therefore by Corollary 1.33 $\gamma_1 |_{J_3} = \gamma_2 |_{J_3}$ for some open $J_3 \subset J_2$, which proves the claim. □

## 1.4 Conformal Geodesics

Given a semi-Riemannian structure on a manifold $M$, one defines the geodesics to be the curves with parallel tangent vector, which makes use of the Levi-Civita connection. The definition cannot be carried out for conformal manifolds because an arbitrary conformal structure $c$ admits no canonical choice of a metric $g \in c$, hence the notion of Levi-Civita connection is not available for conformal manifolds. Furthermore the geodesics of any metric are never invariant under conformal transformation for $\dim M \geq 2$.

However, one may find a certain conformally invariant differential equation in terms of the Levi-Civita connection. The objects satisfying such a differential equation will then be associated to the conformal structure $c$ and not one particular metric and will be called conformal geodesics. A survey on conformal geodesics can be found in [4], extending the earlier [29]. [23] explains in what way the conformal geodesics can be considered as analogues of the ordinary geodesics. [9] provides more detailed results and in particular highlights the connection between conformal geodesics and Weyl structures, which are not mentioned in the present work. Note also [48], which investigates the behaviour of conformal geodesics in special cases quite explicitly. [20] also gives an easily understandable but very brief overview of the basic notions of conformal geodesics. Note that we will later on follow the idea for conformal compactification described there.

Eventually the conformal geodesics will turn out to be exactly the canonical curves of the conformal Cartan geometry. This connection has already been studied not long after Cartan's original publication [17] in [41] and had been further developed by French and Japanese mathematicians. [42] wraps up these classical results; however, it may be somewhat difficult to read at times because of the different notations used therein. Compare also [3] which is concerned with alternative characterizations of the canonical curves of conformal Cartan geometry. In this section we will give an alternative proof for the fact that the conformal geodesics are the canonical curves of the associated Cartan geometry using the Tractor calculus introduced earlier.

Refer to [28] for some context of conformal geodesics in general relativity and an application of the abstract concept of canonical curves to questions with a strong physical motivation at the same time.

It should also be noted that, while we focus on conformal structures and their Cartan geometries only, it is also of interest to study affine geometry, contact geometry and other structures together with their induced Cartan geometries. It turns out that in these cases the canonical curves of the induced Cartan geometry are naturally interesting objects of the underlying geometrical structure as well. Several exemplary spaces are considered in [36] and [19].

*Definition* 1.41. Let $(M, c)$ be a conformal manifold of dimension $n \geq 3$. A spacelike or timelike curve $\gamma : I \to M$ is said to be a *conformal geodesic*, if there is some $g \in c$ such that $\gamma$ satisfies the *conformal geodesic equation* with respect to $g$, that is:

$$\gamma''' = g(\gamma', \gamma')P^g(\gamma') + 3\frac{g(\gamma', \gamma'')}{g(\gamma', \gamma')}\gamma'' + \left(-6\frac{g(\gamma', \gamma'')^2}{g(\gamma', \gamma')^2} + \frac{3}{2}\frac{g(\gamma'', \gamma'')}{g(\gamma', \gamma')} + 2\frac{g(\gamma', \gamma''')}{g(\gamma', \gamma')}\right)\gamma'. \tag{1.62}$$

Note that equation 1.62 is conformally invariant. That is: If $\gamma$ satisfies equation 1.62 for some $g \in c$, it actually satisfies the equation for *all* $g \in c$. Also note that this definition may not easily be generalized to lightlike curves. One may think of Lemma 1.42 as a definition of conformal geodesics which also includes the lightlike case.

We first note some basic properties of conformal geodesics which will be used later on.

**Lemma 1.42** (Proposition 3.3 of [4]). *Let $(M, c)$ be a conformal manifold. Let $\gamma : I \to M$ be a spacelike or timelike curve, then the following are equivalent:*

*(a) $\gamma$ is a conformal geodesic.*

*(b) For every $t_0 \in I$ there exists some interval $(t_0 - \varepsilon, t_0 + \varepsilon) \subset I$ and some $g \in c$ such that $\gamma |_{(t_0 - \varepsilon, t_0 + \varepsilon)}$ is a geodesic with respect to $g$ and the Schouten tensor vanishes along $\gamma$, i.e. $P^g(\gamma'(t)) = 0$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$.*

*Proof.* "(a) $\Rightarrow$ (b)":

Choose some $g \in c$. We start by showing that for *any* regular curve $\gamma : I \to M$ there exists a conformal factor $\sigma : M \to \mathbb{R}$, such that $\gamma$ is a geodesic with respect to the changed metric $e^{2\sigma} g$ locally. To this end let $t_0 \in I$ and without loss of generality assume $t_0 = 0$. Take a chart $(U, \varphi = (x_1, \ldots, x_n))$ around $\gamma(t_0)$ such that $\varphi(\gamma(t)) = (t, 0, \ldots, 0)$ for all $t \in (-\varepsilon, \varepsilon) \subset I$.

We now construct a conformal factor which makes $\gamma$ a unit speed curve. Define

$$\begin{aligned} \varrho : U &\to \mathbb{R} \\ x &\mapsto g(\gamma'(x_1(x)), \gamma'(x_1(x))) \end{aligned}$$

and extend (maybe after shrinking $U$) to a smooth map on $M$. Then for the changed metric $\widehat{g} = \varrho^{-1} g$ we have

$$\widehat{g}(\gamma'(t), \gamma'(t)) = 1 \text{ for all } t \in (-\varepsilon, \varepsilon).$$

So without loss of generality assume that $\gamma$ is a unit speed curve with respect to $g$.

We have $0 = \frac{d}{dt} g(\gamma'(t), \gamma'(t)) = 2g\left(\nabla^g_{\gamma'(t)} \gamma'(t), \gamma'(t)\right)$, i.e. $\nabla^g_{\gamma'(t)} \gamma'(t) \perp \gamma'(t)$. Write

$$\nabla^g_{\gamma'(t)} \gamma'(t) = \sum_{j=1}^{n} \alpha_j(t) \cdot \frac{\partial}{\partial x_j}(\gamma(t)) \text{ for some } \alpha_j : (-\varepsilon, \varepsilon) \to \mathbb{R}. \tag{1.63}$$

Write $g_{ij} := g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$. The orthogonality is then just written as

$$\begin{aligned} 0 &= g\left(\nabla^g_{\gamma'(t)} \gamma'(t), \gamma'(t)\right) \\ &= g\left(\sum_{j=1}^{n} \alpha_j(t) \cdot \frac{\partial}{\partial x_j}(\gamma(t)), \frac{\partial}{\partial x_1}(\gamma(t))\right) \\ &= \sum_{j=1}^{n} g_{1j}(\gamma(t)) \alpha_j(t). \end{aligned} \tag{1.64}$$

Define

$$\begin{aligned} \sigma : U &\to \mathbb{R}^n \\ y &\mapsto 1 + \sum_{i,j=1}^{n} g_{ij}\left(\varphi^{-1}(x_1(y), 0, \dots)\right) x_i(y)\alpha_j(x_1(y)). \end{aligned} \tag{1.65}$$

and $\widehat{g} := e^{2\sigma} g$. Note that

$$\sigma(\gamma(t)) = 1 + \sum_{i=2}^{n}\sum_{j=1}^{n} g_{ij}(\gamma(t))\underbrace{x_i(\gamma(t))}_{=0}\alpha_j(t) + \underbrace{t\sum_{j=1}^{n} g_{1j}(\gamma(t))\alpha_j(t)}_{=0 \text{ by equation 1.64}},$$

i.e. $\sigma$ is constant along $\gamma$. Thus

$$\begin{aligned} \frac{\partial}{\partial x_k}(\gamma(t))\sigma &= \sum_{i=1}^{n}\left(\frac{\partial}{\partial x_k}(\gamma(t))\underbrace{\left[\sum_{j=1}^{n} g_{ij}\left(\varphi^{-1}(x_1(y), 0, \dots, 0)\right)\alpha_j(x_1(y))\right]}_{=0 \text{ for } i=1} \cdot \underbrace{x_i(\gamma(t))}_{=0 \text{ for } i\neq 1}\right) \\ &\quad + \sum_{i,j=1}^{n} g_{ij}(\gamma(t)) \cdot \alpha_j(t) \cdot \underbrace{\frac{\partial}{\partial x_k}(\gamma(t))[x_i(y)]}_{=\delta_{ik}} \\ &= \sum_{j=1}^{n} g_{kj}(\gamma(t)) \cdot \alpha_j(t) \end{aligned}$$

for $k \in \{1, \dots, n\}$, which means

$$\operatorname{grad}^g \sigma(\gamma(t)) = \sum_{j=1}^{n} \frac{\partial}{\partial x_j}(\gamma(t))\alpha_j(t) = \nabla^g_{\gamma'(t)}\gamma'(t). \tag{1.66}$$

Hence by the transformation formula for the Levi-Civita connection (cf. equation 1.2) we have

$$\nabla^{\widehat{g}}_{\gamma'(t)}\gamma'(t) = \nabla^g_{\gamma'(t)}\gamma'(t) + 2\gamma' \cdot \underbrace{\gamma'(t)(\sigma)}_{=0} - \underbrace{g(\gamma'(t), \gamma'(t)) \cdot \operatorname{grad}^g \sigma(\gamma(t))}_{\nabla^g_{\gamma'(t)}\gamma'(t)} = 0.$$

So assume $\gamma : I \to M$ is a solution to equation 1.62. By the above remarks we can assume without loss of generality that $\gamma$ is a geodesic with respect to $g$. Hence $\gamma'' = \gamma''' = 0$ and therefor equation 1.62 reduces to $0 = g(\gamma', \gamma')P^g(\gamma')$. $\gamma$ was assumed to be spacelike or timelike, thus $0 = P^g(\gamma')$.

"(b) ⇒ (a)":

If $\gamma$ is a curve satisfying the assertions of (b), then both left side and right side of 1.62 vanish. □

Different proofs can be found in [9], [40] and (omitting many details) [43]. The following two lemmas are consequences of general ODE theory:

**Lemma 1.43.** *Let $(M, c)$ be a conformal manifold and $\gamma_1$, $\gamma_2 : I \to M$ conformal geodesics defined on a connected interval $I$. If $\gamma_1$ and $\gamma_2$ have contact of order 2 in some point, then $\gamma_1$ and $\gamma_2$ coincide.*

**Lemma 1.44.** *Let $(M, c)$ be a conformal manifold and $x \in M$. For some 2-jet $j \in J^2_x(\mathbb{R}, M)$ denote by $\gamma_j$ the unique conformal geodesic which has the 2-jet $j$ in 0, i.e. $j^2_0\gamma = j$. Let $\alpha \in \mathbb{R}$, then the map*

$$\begin{aligned} \Phi : J^2_x(\mathbb{R}, M) &\to M \\ j &\mapsto \gamma_j(\alpha), \end{aligned} \tag{1.67}$$

*is smooth (wherever defined).*

We are now going to prove the main result of this chapter:

**Theorem 1.45.** *Let $(M, c)$ be a conformal manifold of signature $(p, q)$ and $(\mathcal{P}, \omega)$ the induced Cartan geometry. A spacelike or timelike curve $\gamma : I \to M$ is a conformal geodesic (with respect to c) if and only if $\gamma$ is a canonical curve of type $\mathcal{C}_{\widetilde{\mathfrak{g}_{-1}}}$.*

*Proof.* "$\Rightarrow$":

Let $\gamma : I \to M$ be a conformal geodesic. It suffices to check if $\gamma$ is a canonical curve of type $\mathcal{C}_{\widetilde{\mathfrak{g}_{-1}}}$ locally. To this end, let $t_0 \in I$. According to 1.42 we have some metric $g \in c$, such that $\gamma\mid_{(t_0-\varepsilon,t_0+\varepsilon)}$ is a geodesic with respect to $g$ and $P^g$ vanishes along $\gamma\mid_{(t_0-\varepsilon,t_0+\varepsilon)}$. In a neighborhood $U \subset M$ of $\gamma(t_0)$ we can find a local pseudo-orthonormal basis $(s_1, \ldots, s_n)$ with respect to $g$ which is parallel along $\gamma$ around $t_0$, i.e.

$$\nabla^g_{\gamma'(t)} s_i = 0 \text{ for all } t \in (t_0 - \varepsilon, t_0 + \varepsilon).$$

Here $\nabla^g$ denotes to Levi-Civita connection of $(M, g)$. Now using the canonical identification $\mathcal{T} \simeq \mathcal{T}_g$ we can again consider the canonical local section

$$\begin{aligned} \tau = (\tau_0, \ldots, \tau_{n+1}) &: M \supset U \to \mathcal{P}_c \\ x &\mapsto \left( \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ s_1(x) \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ s_n(x) \\ 0 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right). \end{aligned}$$

Denote $\varepsilon_i := h(\tau_i, \tau_i) \in \{-1, 1\}$. For an arbitrary lift $\tilde{\gamma} : I \to \mathcal{P}$ of $\gamma$ we can write

$$\tilde{\gamma}(t) = \tau(\gamma(t)) \cdot p(t) \text{ for some } p : I \to P.$$

We have

$$
\begin{aligned}
&\omega(d\tau(\gamma'))\\
&=\sum_{i,j=0}^{n+1}\omega^\tau_{ij}(\gamma')B_{ij}\\
&=\sum_{i,j=0}^{n+1}h\left(\nabla^\tau_{\gamma'}\tau_i,\tau_j\right)B_{ij}\\
&=\begin{pmatrix}
\frac{\varepsilon_0}{2}h\left(\begin{pmatrix}0\\ \gamma'-P(\gamma')\\ 0\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix}\right) & \frac{\varepsilon_j}{\sqrt{2}}h\left(\begin{pmatrix}0\\ \gamma'-P(\gamma')\\ 0\end{pmatrix},\begin{pmatrix}0\\s_j\\0\end{pmatrix}\right) & \frac{\varepsilon_{n+1}}{2}h\left(\begin{pmatrix}0\\ \gamma'-P(\gamma')\\ 0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right)\\
\frac{\varepsilon_0}{\sqrt{2}}h\left(\begin{pmatrix}-P(\gamma',s_i)\\ \nabla^g_{\gamma'}s_i\\ -g(\gamma',s_i)\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix}\right) & \varepsilon_j h\left(\begin{pmatrix}-P(\gamma',s_i)\\ \nabla^g_{\gamma'}s_i\\ -g(\gamma',s_i)\end{pmatrix},\begin{pmatrix}0\\s_j\\0\end{pmatrix}\right) & \frac{\varepsilon_{n+1}}{\sqrt{2}}h\left(\begin{pmatrix}-P(\gamma',s_i)\\ \nabla^g_{\gamma'}s_i\\ -g(\gamma',s_i)\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right)\\
\frac{\varepsilon_0}{2}h\left(\begin{pmatrix}0\\ \gamma'+P(\gamma')\\ 0\end{pmatrix},\begin{pmatrix}1\\0\\-1\end{pmatrix}\right) & \frac{\varepsilon_j}{\sqrt{2}}h\left(\begin{pmatrix}0\\ \gamma'+P(\gamma')\\ 0\end{pmatrix},\begin{pmatrix}0\\s_j\\0\end{pmatrix}\right) & \frac{\varepsilon_{n+1}}{2}h\left(\begin{pmatrix}0\\ \gamma'+P(\gamma')\\ 0\end{pmatrix},\begin{pmatrix}1\\0\\1\end{pmatrix}\right)
\end{pmatrix}\\
&=\begin{pmatrix}
0 & \frac{\varepsilon_j}{\sqrt{2}}g(\gamma'-P(\gamma'),s_j) & 0\\
-\frac{1}{\sqrt{2}}(-g(\gamma',s_i)+P(\gamma',s_i)) & \varepsilon_j g(\nabla^g_{\gamma'}s_i,s_j) & \frac{1}{\sqrt{2}}(-g(\gamma',s_i)-P(\gamma',s_i))\\
0 & \frac{\varepsilon_j}{\sqrt{2}}g(\gamma'+P(\gamma'),s_j) & 0
\end{pmatrix}\\
&=\frac{1}{\sqrt{2}}\begin{pmatrix}
0 & \varepsilon_j(g(\gamma',s_j)-P(\gamma',s_j)) & 0\\
g(\gamma',s_i)-P(\gamma',s_i) & 0 & -g(\gamma',s_i)-P(\gamma',s_i)\\
0 & \varepsilon_j(g(\gamma',s_j)+P(\gamma',s_j)) & 0
\end{pmatrix}=:M(t).
\end{aligned}
$$

This is an element in $\mathfrak{so}(p+1,q+1)$. For further computation we calculate the representation in the Witt basis from line 1.18:

$$N(t):=\begin{pmatrix}0 & \varepsilon_j P(\gamma',s_j) & 0\\ -g(\gamma',s_i) & 0 & -P(\gamma',s_i)\\ 0 & \varepsilon_j g(\gamma',s_j) & 0\end{pmatrix}.$$

Hence with all matrices written in the Witt basis we receive

$$
\begin{aligned}
\omega(\tilde{\gamma}'(t)) &= \mathrm{Ad}(p(t)^{-1})N(t)+dL_{p(t)^{-1}}p'(t)\\
&= p(t)^{-1}N(t)p(t)+p(t)^{-1}p'(t)
\end{aligned}
\tag{1.68}
$$

where for the last step we used the fact, that for matrix groups the adjoint action is given by conjugation. In Lemma 1.36 we have identified the geodesics of type $\mathcal{C}_{\widetilde{\mathfrak{g}_{-1}}}$ to be exactly the projections of flow lines of constant vector fields with constant vector in $\mathfrak{g}_{-1}$. Using this we have shown that $\gamma$ is a canonical curve of type $\mathcal{C}_{\widetilde{\mathfrak{g}_{-1}}}$ around $t_0\in I$ if and only if there exist $X\in\mathfrak{g}_{-1}$ and $p:(t_0-\varepsilon,t_0+\varepsilon)\to P$ such that

$$X=p(t)^{-1}N(t)p(t)+p(t)^{-1}p'(t)\text{ for all }t\in(t_0-\varepsilon,t_0+\varepsilon). \tag{1.69}$$

Remember that $\gamma$ is a geodesic of $(M, g)$ and the $s_i$ are parallel along $\gamma$, hence $g(\gamma', s_i)$ are constant for $i = 1, \ldots, n$. Furthermore, by our special choice of $g$ we have $P^g(\gamma', s_i) = 0$ for $i = 1, \ldots, n$. So we have $N(t) \in \mathfrak{g}_{-1}$ and $N(t)$ being a constant matrix. Therefore $X := N(t_0)$, $p(t) = \mathrm{Id}$ is a solution to equation 1.69 and $\gamma$ is a canonical curve of type $\mathcal{C}_{\widetilde{\mathfrak{g}_{-1}}}$.

"$\Leftarrow$":

Consider some canonical curve of type $\mathcal{C}_{\widetilde{\mathfrak{g}_{-1}}}$ $\gamma : I \to M$ and let $\delta : I \to M$ be a conformal geodesic with the same 2-jet in $\gamma(t_0)$. Such a $\delta$ exists by Lemma 1.43. By the first part of this proof $\delta$ is also a canonical curve of type $\mathcal{C}_{\widetilde{\mathfrak{g}_{-1}}}$. By Theorem 1.40 the two curves $\gamma$ and $\delta$ locally coincide. Hence $\gamma$ is locally a conformal geodesic. The property of being a conformal geodesic is a local one, thus $\gamma$ is a conformal geodesic everywhere. $\square$

Note that we established a result about parametrized curves and an analogon for unparametrized curves holds as well.

# 2 Conformal compactifications with Conformal Geodesics

## 2.1 Basic notions of conformal compactification

Compact spaces have a lot of good properties, so it has always been desirable to embed spaces into compact ones to simplify some constructions. Considering a smooth manifold as a topological space, it has several compactifications, e.g. the Stone-Čech compactification which again is a Hausdorff space. However, in general such a compactification does not need to be a manifold again. The problem of compactifying a manifold in a smooth fashion is of interest in itself and discussed e.g. in [15] and [35]. In this work we will not only require a compactification to be smooth, but also to be *conformal* in the following sense:

*Definition* 2.1. Let $(M, c)$, $(N, c')$ be conformal manifolds (without boundary) of the same dimension and $f : M \to N$ a map.

(i) $\sigma$ is called *conformal embedding*, if it is an embedding and $c = \sigma^* c'$.

(ii) $\sigma$ is called *conformal compactification of* $M$, if $N$ is compact. In this case we also call $N$ alone a conformal compactification.

(iii) $\sigma$ is called *trivial conformal compactification*, if $M = N$.

Note that we require the conformal compactification $N$ to be a manifold without boundary. For applications in theoretical physics and global analysis it is at times interesting to broaden the notion of conformal compactification and also allow manifolds with boundaries, as originally suggested by Penrose and described in [6], and further developed over the years. [2] may serve as an up-to-date introduction to these questions. At times one may even drop the requirement for $N$ to be a conformal manifold. For practical questions it may suffice to ask for $\sigma$ to be a conformal map and allowing the conformal factor to be zero on $\partial\sigma(M) \subset N$, as described in [30] and [1]. Because of applications in theoretical physics, the Lorentz case is of particular interest. Take [18] as one example for a compactification procedure and note how in this case lightlike geodesics play the role of the conformal geodesics from chapter 2.3.

The main reference that deals with the same notion of conformal compactification as used in this work is [25], using Cartan methods to expand existence and uniqueness results from [32, 15, 16].

Manifolds may have no conformal compactification, a unique conformal compactification, or even several not conformally equivalent ones. To show that $\mathbb{R}^n$ has a unique conformal compactification will be the subject of the following section. Examples of manifolds that admit no conformal compactification are difficult to construct and discussed in [25]. An example of a manifold that admits several not conformally equivalent conformal compactifications is the following:

*Example* 2.2. Let $\big((0,1)\times(0,1), g\big)$, where $g$ is the standard Euclidean metric on $(0,1)\times$

$(0,1) \subset \mathbb{R}^2$. This manifold has more than one not conformally equivalent conformal compactifications, namely

$$(0,1) \times (0,1) \hookrightarrow [0,1] \times [0,1]/\sim,$$

where $\sim$ identifies points on opposing edges of the unit square. Through this we may receive a torus or a Klein bottle, which are not homeomorphic (hence they are in particular not conformally equivalent).

## 2.2 Cartan embeddings

In what follows, we aim to establish that the conformal boundary of a conformal embedding cannot be too wild. To do this, we will show conformal embeddings to be special cases of Cartan embeddings and then prove a statement about the boundaries of Cartan embeddings. This will be a bit more than we actually need but saves us the cumbersome computations with the conformal structures. For the notions of Cartan embeddings we follow [26].

*Definition* 2.3. Given two Cartan geometries $(\mathcal{P}, \omega)$ and $(\mathcal{P}', \omega')$ of type $(G, P)$ on two manifolds $M$ and $N$ respectively, a map $\sigma : M \to N$ is called a *Cartan embedding with respect to* $(\mathcal{P}, \omega)$ *and* $(\mathcal{P}', \omega')$, if there exists some $\widehat{\sigma} : \mathcal{P} \to \mathcal{P}'$, such that

- $\widehat{\sigma}$ is $P$-equivariant, i.e. $\widehat{\sigma} \circ R_p = R_p \circ \widehat{\sigma}$ for all $p \in P$,
- $\widehat{\sigma}$ is a lift of $\sigma$, i.e. $\sigma \circ \pi_{\mathcal{P}} = \pi_{\mathcal{P}'} \circ \widehat{\sigma}$,

$$\begin{array}{ccc} \mathcal{P} & \xrightarrow{\widehat{\sigma}} & \mathcal{P}' \\ \downarrow{\scriptstyle \pi_{\mathcal{P}}} & & \downarrow{\scriptstyle \pi_{\mathcal{P}'}} \\ M & \xrightarrow{\sigma} & N \end{array} \tag{2.1}$$

- $\widehat{\sigma}$ is a morphism of Cartan geometries, i.e. $\widehat{\sigma}^* \omega' = \omega$.

Note that the $\widehat{\sigma}$ from definition 2.3 is not uniquely determined in general. As an example take some Lie group $G$ with $\dim G \geq 1$ and the closed subgroup $P = G$. By Example 1.3, $(G, \omega^{MC})$ is a Cartan geometry of type $(G, G)$ on $G/G = \{pt\}$. Now obviously $L_g : G \to G$ is a lift of $\mathrm{Id}_{\{pt\}}$ and $L_g^* \omega^{MC} = \omega^{MC}$ for every $g \in G$.

The following theorem justifies to look at Cartan embeddings rather than conformal embeddings:

**Theorem 2.4.** *Let $(M, c)$ and $(N, c')$ be conformal manifolds and $\mathcal{P}_c$ and $\mathcal{P}_{c'}$ be the Cartan geometries induced by $c$ and $c'$ respectively. Let $\sigma : M \to N$ be some map. Then it is equivalent:*

*(i) $\sigma$ is a conformal embedding.*

*(ii) $\sigma$ is a Cartan embedding with respect to $\mathcal{P}_c$ and $\mathcal{P}_{c'}$.*

We defined conformal embeddings to be maps between manifolds of equal dimension (cf. definition 2.1). For Cartan embeddings, this follows automatically from the fact that the Cartan geometries on the two manifolds are of the same type (cf. definition 2.3).

*Proof.* (i) $\Rightarrow$ (ii)

So let $\sigma : M \to N$ be a conformal embedding. By $\mathcal{T}_M$ and $\mathcal{T}_N$ denote the standard Tractor bundles of $M$ and $N$ respectively. Choose $g \in c$ arbitrary and cover $M$ with local sections of the form

$$\begin{aligned}\tau : M \supset U &\to \mathcal{P}_c \\ x &\mapsto \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\s_1(x)\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\s_n(x)\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}\right)\end{aligned}$$

for some local pseudo-orthonormal (with respect to $g$) basis $(s_1, \dots, s_n)$ over $U$. $\sigma$ is conformal, hence there exists some $g' \in c'$, such that $\sigma : (M, g) \to (N, g')$ is an isometry of semi-Riemannian manifolds. For that metric $(d\sigma \circ s_1 \circ \sigma^{-1}, \dots, d\sigma \circ s_n \circ \sigma^{-1})$ is a local pseudo-orthonormal basis of $\sigma(U)$. Also $\sigma$ satisfies

$$d\sigma\left(\nabla^g_X Y\right) = \nabla^{g'}_{d\sigma X} d\sigma(Y) \tag{2.2}$$

which implies

$$P^g(X, Y) = P^{g'}(d\sigma X, d\sigma Y) \tag{2.3}$$

for arbitrary $X, Y \in \mathfrak{X}(M)$.

We can now consider sections of the form

$$\begin{aligned}\tau' : N \supset \sigma(U) &\to \mathcal{P}_{c'} \\ y &\mapsto \left(\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\d\sigma \circ s_1 \circ \sigma^{-1}(y)\\0\end{pmatrix}, \dots, \begin{pmatrix}0\\d\sigma \circ s_n \circ \sigma^{-1}(y)\\0\end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\end{pmatrix}\right),\end{aligned} \tag{2.4}$$

which then cover the whole of $\sigma(M) \subset N$. Now define the lifting $\widehat{\sigma}^\tau : \mathcal{P}_c \to \mathcal{P}_{c'}$ of $\sigma$ by

$$\begin{aligned}\widehat{\sigma}^\tau : \mathcal{P}_c \supset \tau(U) &\to \mathcal{P}_{c'} \\ \tau(x) &\mapsto \tau'(\sigma(x)).\end{aligned} \tag{2.5}$$

Extend $\widehat{\sigma}^\tau$ to the whole of $\pi^{-1}_{\mathcal{P}_c}(U)$ by

$$\widehat{\sigma}^\tau(\tau(x) \cdot p) := \widehat{\sigma}^\tau(\tau(x)) \cdot p \tag{2.6}$$

with $p \in P$. To see that this defines a smooth map on all of $\mathcal{P}_c$ consider another section $\tilde{\tau} : U \to \mathcal{P}_c$ defined using another local basis. Denote by $\tilde{\tau}' : \sigma(U) \to \mathcal{P}_{c'}$ and $\widehat{\sigma}^{\tilde{\tau}}$ the induced maps given by equations 2.4 and 2.5, then we have to check that $\widehat{\sigma}^{\tau} = \widehat{\sigma}^{\tilde{\tau}}$.

We can write $\tilde{\tau}(x) = \tau(x) \cdot p$ for some $p \in P$. Direct computation shows $\tilde{\tau}'(\sigma(x)) = \tau'(\sigma(x)) \cdot p$ and we receive

$$\begin{aligned}\widehat{\sigma}^{\tilde{\tau}}(\tilde{\tau}(x)) &= \tilde{\tau}'(\sigma(x)) \\ &= \tau'(\sigma(x)) \cdot p(x) \\ &= \widehat{\sigma}^{\tau}(\tau(x)) \cdot p(x) \\ &= \widehat{\sigma}^{\tau}(\tilde{\tau}(x)).\end{aligned}$$

So $\widehat{\sigma}^{\tilde{\tau}}$ and $\widehat{\sigma}^{\tau}$ coincide in one point of any fiber and therefore coincide everywhere by equation 2.6. We will denote the resulting map by $\widehat{\sigma} : \mathcal{P}_c \to \mathcal{P}_{c'}$.

Using equations 2.2 and 2.3 we receive through direct calculation

$$h^{\mathcal{T}_M}\left(\nabla^{\mathcal{T}_M}_{s_i(x)}\tau_k, \tau_j(x)\right) = h^{\mathcal{T}_N}\left(\nabla^{\mathcal{T}_N}_{d\sigma(s_i(x))}\tau'_k, \tau'_j(\sigma(x))\right)$$

for $i = 1, \ldots, n$ and $j, k = 0, \ldots, n+1$. Hence

$$\begin{aligned}\omega(d\tau(s_i(x)) &= \omega^{U,\tau}(s_i(x)) \\ &= \sum_{k,j=0}^{n+1} h^{\mathcal{T}_M}\left(\nabla^{\mathcal{T}_M}_{s_i(x)}\tau_k, \tau_j(x)\right) B_{kj} \\ &= \sum_{k,j=0}^{n+1} h^{\mathcal{T}_N}\left(\nabla^{\mathcal{T}_N}_{d\sigma(s_i(x))}\tau'_k, \tau'_j(\sigma(x))\right) B_{kj} \\ &= \omega^{\sigma(U),\tau'}(d\sigma(s_i(x))) \\ &= \omega'(d\widehat{\sigma}(d\tau(s_i(x)))\end{aligned}$$

and furthermore for $X \in \mathfrak{p}$ and the according fundamental vector fields $\tilde{X}_{\mathcal{P}_c} \in \mathfrak{X}(\mathcal{P}_c)$ and $\tilde{X}_{\mathcal{P}_{c'}} \in \mathfrak{X}(\mathcal{P}_{c'})$ in some point $u \in \mathcal{P}$

$$\begin{aligned}\omega'(d\widehat{\sigma}(\tilde{X}_{\mathcal{P}_c}(u))) &= \omega'\left(\frac{d}{dt}\widehat{\sigma}(u \cdot \exp(tX)) \mid_{t=0}\right) \\ &= \omega'\left(\frac{d}{dt}\widehat{\sigma}(u) \cdot \exp(tX) \mid_{t=0}\right) \\ &= \omega'\left(\tilde{X}_{\mathcal{P}_{c'}}(\widehat{\sigma}(u))\right) \\ &= X \\ &= \omega(\tilde{X}_{\mathcal{P}_c}(u)).\end{aligned}$$

Putting the two equations together and using the right-invariance of $\omega$ and $\omega'$ we receive

$$\omega(Y) = \omega'(d\widehat{\sigma}(Y))$$

for arbitrary $Y \in \mathfrak{X}(\mathcal{P}_c)$, i.e. $\widehat{\sigma}^*\omega' = \omega$.

(ii) $\Rightarrow$ (i)

Let $\mathcal{P}_0$ and $\mathcal{P}_0'$ be the $\mathrm{CO}(p,q)$ bundles induced by $\mathcal{P} := \mathcal{P}_c$ and $\mathcal{P}' := \mathcal{P}_{c'}$ and $\theta \in \Omega^1(\mathcal{P}_0, \mathbb{R}^n)$ and $\theta' \in \Omega^1(\mathcal{P}_0', \mathbb{R}^n)$ be the according 1-forms as per equation 1.26. Let $\widehat{\sigma}_0 : \mathcal{P}_0 \to \mathcal{P}_0'$ denote the map canonically induced by $\widehat{\sigma}$, i.e.

$$\begin{aligned} \widehat{\sigma}_0 : \mathcal{P}_0 &\to \mathcal{P}_0' \\ [u, p \cdot \mathbb{R}^n] &\mapsto [\widehat{\sigma}(u), p \cdot \mathbb{R}^n]. \end{aligned}$$

Further let $f : \mathcal{P}_0 \to \mathrm{GL}(M)$ and $f' : \mathcal{P}_0' \to \mathrm{GL}(M)$ be the corresponding $\mathrm{CO}(p,q)$-reductions. Denote by $c_{\mathcal{P}}$ and $c_{\mathcal{P}'}$ the conformal structures induced by $\mathcal{P}$ and $\mathcal{P}'$ respectively.

Let $x \in M$ and $u \in \mathcal{P}_0$ over $x$. Write $f(u) = (s_1, \ldots, s_n)$, $f'(\widehat{\sigma}_0(u)) = (s_1', \ldots, s_n')$. Then $(c_{\mathcal{P}})_x$ is the conformal class of $g_x^u$ given by $g_x^u(s_i, s_j) = \varepsilon_i \delta_{ij}$ and $(c_{\mathcal{P}'})_{\sigma(x)}$ is the conformal class of $g_{\sigma(x)}^{\widehat{\sigma}_0(u)}$ given by $g_{\sigma(x)}^{\widehat{\sigma}_0(u)}(s_i', s_j') = \varepsilon_i \delta_{ij}$. We now aim to show

$$g_{\sigma(x)}^{\widehat{\sigma}(u)}(d\sigma(s_i), d\sigma(s_j)) = g_x^u(s_i, s_j).$$

This will imply $\sigma^* c_{\mathcal{P}'} = c_{\mathcal{P}}$ and therefore by the Correspondence Theorem 1.24 $\sigma^* c' = c$. To see this, it suffices to show that $d\sigma s_i = f_i'(\widehat{\sigma}_0(u))$.

Let $X \in \theta_u^{-1}(e_i)$ be some (not uniquely determined) vector in the pre-image of $e_i$ under the map $\theta_u : \mathcal{P}_0 \to \mathbb{R}^n$. We then have

$$\begin{aligned} \theta'_{\widehat{\sigma}_0(u)}(d\widehat{\sigma}_0(X)) &= \mathrm{proj}_{\mathfrak{g}_{-1}}(\omega'(d\widehat{\sigma}(d\,\mathrm{pr}^{-1} X))) \\ &= \mathrm{proj}_{\mathfrak{g}_{-1}}(\omega(d\,\mathrm{pr}^{-1} X)) \qquad \text{(because } \widehat{\sigma}^*\omega' = \omega\text{)} \\ &= \theta_u X = e_i, \end{aligned} \tag{2.7}$$

where we wrote $d\,\mathrm{pr}^{-1} X$ to denote an arbitrary vector in the pre-image of $X$ under the map $\mathrm{pr} : \mathcal{P} \to \mathcal{P}_0$. Such a vector is not unique but still the notation is justified, because the value of $\theta'$ does not depend on the choice of that lift, as shown in part (ii) of Lemma 1.23. Using $d\widehat{\sigma}_0(X) \in \left(\theta'_{\widehat{\sigma}_0(u)}\right)^{-1}(e_i)$ from equation 2.7, we receive by definition of $f_i'$ (cf. equation 1.28):

$$\begin{aligned} f_i'(\widehat{\sigma}_0(u)) &= d\pi^{\mathcal{P}_0'}(d\widehat{\sigma}_0(X)) \\ &= d\sigma(d\pi^{\mathcal{P}_0}(X)) \\ &= d\sigma f_i(u) \\ &= d\sigma s_i. \end{aligned}$$

As stated before, this implies $\sigma^* c' = c$. $\square$

Our aim is still to show that the boundary of our embedding cannot be too wild. To do this we will introduce some notation following [25]. Even though it is a bit more than what is needed in the present work, we will take a moment to compare the different notions of accessibility.

*Definition* 2.5. Let $\sigma : M \to N$ be a Cartan embedding with respect to $(\mathcal{P}, \omega)$ and $(\mathcal{P}', \omega')$.

(i) The set $\partial_\sigma M := \partial(\sigma(M)) \subset N$ is called *(Cartan) geometric boundary of $M$ with respect to $\sigma$.*

(ii) The Cartan embedding $\sigma$ is said to be *trivial*, if $\partial_\sigma M = \emptyset$.

(iii) $x \in \partial_\sigma M$ is called an *accessible point*, if there exists a $C^1$ curve $\gamma : [0,1] \to N$ such that $\gamma([0,1)) \subset \sigma(M)$ and $\gamma(1) = x$.

(iv) $x \in \partial_\sigma M$ is called a *highly accessible point*, if there exists a hyperplane $H \subset T_x N$, such that for all $C^1$ curves $\gamma : I \to N$ with $\gamma(0) = x$ and $\gamma'(0)$ on one side of the hyperplane there exists an $\varepsilon > 0$ such that the curve $\gamma\,|_{(0,\varepsilon)}$ lies in $\sigma(M)$.

*Remark* 2.6. Obviously a highly accessible point is also an accessible point. The converse does not hold in general. To see this, let

$$M := \mathbb{R}^2 \setminus \left( \left\{ \frac{1}{k} \cdot (\cos\varphi, \sin\varphi) \mid k \in \mathbb{N}, \varphi \in \left[ \frac{\pi}{4}, \frac{7\pi}{4} \right] \right\} \cup \{(0,0)\} \right)$$

the $\mathbb{R}^2$ with some arcs of circles around the origin removed. Let $N = \mathbb{R}^2$ and $\sigma : M \hookrightarrow N$

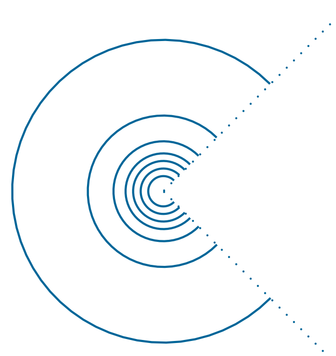

Figure 2: example for an accessible but not highly accessible point

the canonical inclusion. Then $0 \in \partial_\sigma M$ is accessible but not highly accessible.

**Lemma 2.7.** *Let $\sigma : M \to N$ be a non-trivial Cartan embedding with respect to $(\mathcal{P}, \omega)$ and $(\mathcal{P}', \omega')$. Then the highly accessible points are dense in $\partial_\sigma M$.*

Since this is supposed to be true for *any* Cartan embedding, we cannot expect to make use of the property $\sigma^* \omega' = \omega$, because in particular the claim holds for the trivial Cartan geometries $\mathcal{P} \simeq M$ and $\mathcal{P}' \simeq N$. In that case the Cartan embeddings are just the smooth embeddings of one manifold into the other.

*Proof.* We first show that the accessible points are dense in $\partial_\sigma M$.

$\sigma$ is assumed to be non-trivial, i.e. $\partial_\sigma M \neq \emptyset$, so let $p \in \partial_\sigma M$. Choose some Riemannian metric $g$ on $N$.

Consider geodesics emanating from $p$ parametrized by arc length with respect to $g$. We write $\gamma_v$ for the unique geodesic with $\gamma_v(0) = p$ and $\gamma_v'(0) = v$. Assume there exists $\varepsilon > 0$, such that for every $v \in T_pN$ with $||v||_g = 1$ we have $\operatorname{Im}\gamma_v \mid_{[0,\varepsilon)} \subset N \setminus \sigma(M)$. Then $N \setminus \sigma(M)$ contains an open set around $p$, which contradicts $p \in \partial_\sigma M$, hence our assumption must have been wrong.

So we can choose a sequence $\varepsilon_k \to 0$ and unit vectors $v_k \in T_pM$ such that $\gamma_{v_k}(\varepsilon_k) \in \sigma(M)$. Let

$$\varepsilon_k^- := \max\left\{t \in [0, \varepsilon_k) \mid \gamma_{v_k}(t) \in \partial_\sigma(M)\right\}.$$

Then the point $\gamma_{v_k}(\varepsilon_k^-)$ is accessed by the curve $\gamma_{v_k} \mid_{[\varepsilon_k^-, \varepsilon_k)}$. And

$$\begin{aligned} d_g\big(p, \gamma_{v_k}(\varepsilon_k^-)\big) &\leq \varepsilon_k^- && \text{(because } \gamma_{v_k} \text{ is an arc length geodesic)} \\ &\leq \varepsilon_k \to 0, \end{aligned}$$

i.e. the accessible points are dense in $\partial_\sigma M$.

We are now going to prove that also the *highly* accessible points are dense in $\partial_\sigma M$.

We still have $p \in \partial_\sigma M$, and choose $p_k \to p$ some sequence of accessible points converging towards $p$.

For fixed $k \in \mathbb{N}$ let $\delta : [0, 1] \to N$ be a curve accessing $p_k$, i.e. $\delta([0, 1)) \subset \sigma(M)$ and $\gamma(t) = \delta(1 - t)$ the reversed curve. Let $K(p_k, \varepsilon)$ be a convex neighborhood around $p_k$ and choose $t_0 \in (0, 1)$, such that $\gamma \mid_{[0,t_0]} \subset K\left(p_k, \frac{\varepsilon}{2}\right)$.

Now set $t_l := \frac{t_0}{l}$ and $\rho_l := d_g(\gamma(t_l), \partial_\sigma M)$ for $l \in \mathbb{N}$. Note that $\rho_l < \frac{\varepsilon}{2}$ by the choice of $t_0$. Choose $q_l \in \partial_\sigma M$ with $d(\gamma(t_l), q_l) = \rho_l$. Then $q_l$ is highly accessible with the relevant hyperplane being

$$T_{q_l}\partial K(\gamma(t_l), \rho_l) \subset T_{q_l}N.$$

That hyperplane indeed satisfies the requirements from the definition of highly accessible points. To see this, consider some curve emanating from $q_l$ with initial velocity in direction of $\gamma(t_l)$. I.e. let $v \in T_{q_l}N$ be the unique vector such that $\gamma_v(0) = q_l$ and $\gamma_v(1) = \gamma(t_l)$. Such a $v$ exists, because by convexity $K(p_k, \varepsilon)$ is a neighborhood with normal coordinates for $q_l$. That means we are now considering curves $\delta$ with $\delta(0) = q_l$ and $g(\delta'(0), v) > 0$. (see Figure 3)

Such a curve will remain inside $K(\gamma(t_l), \rho_l)$ for a short time after it begins. $K(\gamma(t_l), \rho_l) \cap \partial_\sigma M = \emptyset$, hence it does not intersect $\partial_\sigma M$ for a short time. That is, $q_l$ is highly accessible.

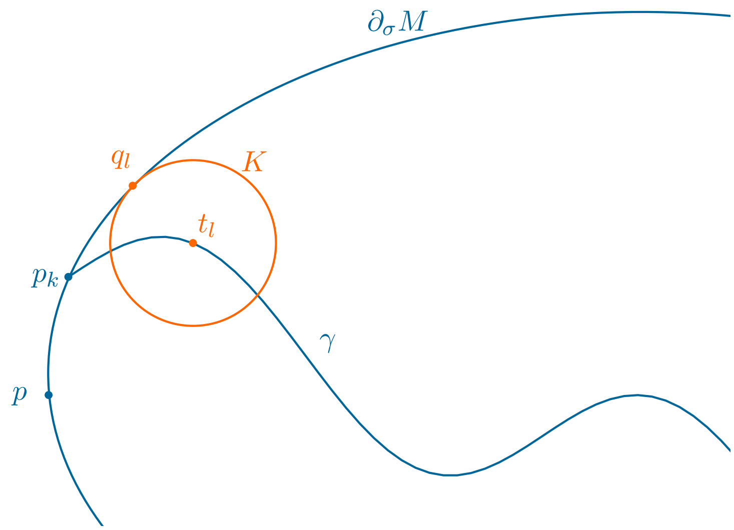


Figure 3: construction of the highly accessible point $q_l$

The construction was made for fixed $k \in \mathbb{N}$, so we write $q_l^k := q_l$, $\varepsilon_k := \varepsilon$ and receive

$$d_g(q_k^k, p) \leq d_g(q_k^k, p_k) + d_g(p_k, p) \leq \varepsilon_k + d_g(p_k, p) \to 0$$

for a suitable choice of $\varepsilon_k$. That shows that also the highly accessible points are dense in $\partial_\sigma M$. □

## 2.3 Conformal Compactification of $\mathbb{R}^n$ with Conformal Geodesics

We have seen in example 1.5 that the stereographic embedding $\mathbb{R}^n \hookrightarrow S^n$ is a conformal compactification of $\mathbb{R}^n$. In this section we aim to show that–in addition to that–it is also the only one. To achieve this, we will use conformal geodesics following the idea discussed in [20].

**Lemma 2.8.** *For any $\alpha, \beta \in \mathbb{R}$ the curve*

$$\begin{aligned}\gamma_{\alpha,\beta} : I &\to M\\ t &\mapsto \frac{2}{(2-\alpha t)^2 + \beta^2 t^2}\big((2-\alpha t)t, \beta t^2\big)\end{aligned}$$

*with*

$$I = \begin{cases} \mathbb{R} \setminus \left\{\frac{2}{\alpha}\right\}, & \text{if } \alpha \neq 0, \beta = 0,\\ \mathbb{R} & \text{otherwise}\end{cases}$$

*is a conformal geodesic of $(\mathbb{R}^2, g)$.*

*Moreover any conformal geodesic $\gamma$ with $\gamma(0) = (0,0)$ and $\gamma'(0) = (1,0)$ is of this form.*

*Proof.* Since $g = g_{st}$ is positive definite, the curve $\gamma$ is nowhere lightlike and direct computation shows $\gamma$ satisfies equation 1.62.

$\gamma_{\alpha,\beta}$ satisfies $\gamma''_{\alpha,\beta}(0) = (\alpha, \beta)$, hence the uniqueness follows from Lemma 1.43. □

*Remark* 2.9. Note that for $\beta \neq 0$ the curve $\gamma_{\alpha,\beta}$ is a circle with center $\left(0, \frac{1}{\beta}\right)$ and a straight line for $\beta = 0$.

**Lemma 2.10.** *For any $\alpha, \beta \in \mathbb{R}$ and any Euclidean motion*

$$\begin{aligned} M : \mathbb{R}^n &\to \mathbb{R}^n \\ x &\mapsto Ax + b \text{ where } A \in \mathcal{O}(n), b \in \mathbb{R}^n \end{aligned}$$

*the curve $M \circ i \circ \gamma_{\alpha,\beta}$, where*

$$\begin{aligned} i : \mathbb{R}^2 &\to \mathbb{R}^n \\ (x_1, x_2) &\mapsto (x_1, x_2, 0, \ldots, 0), \end{aligned} \tag{2.8}$$

*is a conformal geodesic and all conformal geodesics of $\mathbb{R}^n$ are of this form.*

*Proof.* Equation 1.62 is obviously invariant under $i$ and under Euclidean motions, hence $\widehat{\gamma} = M \circ i \circ \gamma_{\alpha,\beta}$ is again a conformal geodesic.

Given $M : x \mapsto Ax + b$ we have

$$\begin{aligned} \widehat{\gamma}(0) &= b, \\ \widehat{\gamma}'(0) &= A\left(di\gamma'_{\alpha,\beta}(0)\right), \\ \widehat{\gamma}''(0) &= A\left(di\gamma''_{\alpha,\beta}(0)\right), \end{aligned}$$

which gives us every possible 2-jet in $\mathbb{R}^n$. So again Lemma 1.43 yields the uniqueness. □

*Remark* 2.11. We have

$$\lim_{t\to\infty} \gamma_{\alpha,\beta}(t) = \frac{2}{\alpha^2 + \beta^2}(-\alpha, \beta) \qquad \text{if } \alpha \neq 0 \text{ or } \beta \neq 0, \tag{2.9}$$

$$\gamma_{\alpha,\beta}(1) = \frac{2}{(2-\alpha)^2 + \beta^2}(2 - \alpha, \beta) \qquad \text{if } \alpha \neq 2 \text{ or } \beta \neq 0. \tag{2.10}$$

We are now ready to prove our main result:

**Theorem 2.12.** *Let $\sigma : \mathbb{R}^n \to M$, $n \geq 3$, be a non-trivial conformal compactification with respect to the conformal structures $c = [g_{st}]$ and $c' = [g']$ on $\mathbb{R}^n$ and $M$ respectively. Assume $M$ to be connected.*

*Then $(M, c')$ is conformally equivalent to the $n$-sphere with standard round metric.*

For this we will make use of the following theorem, which is a consequence of the Liouville theorem for conformal mappings:

**Theorem 2.13.** *Every simply connected, compact, conformally flat Riemannian manifold is conformally equivalent to a sphere with standard round metric.*

The proof of which can be found in [38].

*Proof of Theorem 2.12.* We are going to show that there is exactly one highly accessible point in $\partial_\sigma \mathbb{R}^n$.

Let $q \in \partial_\sigma \mathbb{R}^n$ a highly accessible point and $\gamma : [0,1] \to M$ be any conformal geodesic accessing $q$, i.e. $\gamma(1) = q$ and $\mathrm{Im}\left(\gamma\mid_{[0,1)}\right) \subset \sigma\left(\mathbb{R}^n\right)$. By Lemma 2.10 $\sigma^{-1} \circ \gamma\mid_{[0,1)}$ is either a circle or a straight line in $\mathbb{R}^n$. Since $\gamma(1) \notin \sigma\left(\mathbb{R}^n\right)$, $\sigma^{-1} \circ \gamma$ is unbounded in $\mathbb{R}^n$ and therefore a straight line. We may choose some Euclidean motion $T : \mathbb{R}^n \to \mathbb{R}^n$ and a reparametrization of $\gamma$ such that $(\sigma \circ T)^{-1} \circ \gamma$ is the straight line along the $x$-axis with $(\sigma \circ T)^{-1} \circ \gamma(0) = 0 \in \mathbb{R}^n$, i.e.

$$\begin{aligned} (\sigma \circ T)^{-1} \circ \gamma(t) &= \frac{t}{1-t}(1,0,\dots,0) \qquad &&, \text{ for } t \in [0,1), \\ \gamma(1) &= q. \end{aligned} \tag{2.11}$$

In other words, $(\sigma \circ T)^{-1} \circ \gamma = i \circ \gamma_{2,0}$ with the notation from Lemma 2.10. Note that the map $\sigma \circ T : \mathbb{R}^n \to M$ is still a conformal embedding, i.e. without loss of generality assume $T = \mathrm{Id}$. That is, $\sigma^{-1} \circ \gamma$ shall be of the form specified in equation 2.11. For $\varrho \in \mathbb{R}$ and $N \in \mathrm{Stab}_{(1,0,\dots,0)}\,\mathcal{O}(n)$ we have

$$\begin{aligned} q &= \gamma(1) \\ &= \lim_{t\to 1} \gamma(t) \\ &= \lim_{t\to 1} \sigma \circ N \circ i \circ \gamma_{2,0}(t) \\ &= \lim_{t\to 1}\lim_{\alpha\to 2} \sigma \circ N \circ i \circ \gamma_{\alpha,\varrho(2-\alpha)}(t) && \text{(by Lemma 1.44)} \\ &= \lim_{\alpha\to 2}\lim_{t\to 1} \sigma \circ N \circ i \circ \gamma_{\alpha,\varrho(2-\alpha)}(t) && \text{(by Lemma 1.44)} \\ &= \lim_{\alpha\to 2} \sigma \circ N \circ i \circ \gamma_{\alpha,\varrho(2-\alpha)}(1) \\ &= \lim_{\alpha\to 2} \sigma\left(\frac{2}{2-\alpha}\cdot\frac{1}{1+\varrho^2}\cdot N\left(1,\varrho,0,\dots,0\right)\right). && \text{(by Remark 2.11)} \end{aligned} \tag{2.12}$$

Hence the straight line starting in $0 \in \mathbb{R}^n$ with direction $N \cdot (1,\varrho,0,\dots,0)$ also accesses $q \in \partial_\sigma \mathbb{R}^n$ for any choice of $N$ and $\varrho$, cf. Figure 4. Above we assumed $\gamma$ to be the straight line along the $x$-axis. Repeating the previous argument for the curves of the form from equation 2.12, we find that all straight lines starting in $0 \in \mathbb{R}^n$ and leaving $\mathbb{R}^n$ access $q$.

Now take an arbitrary $p \in \mathbb{R}^n$, $p \neq 0$. There is some straight line connecting $p$ with 0 and the extension will, according to the previous discussion, also access $q$. Repeating the above argument we see that also all straight lines starting in $p$ access $q$.

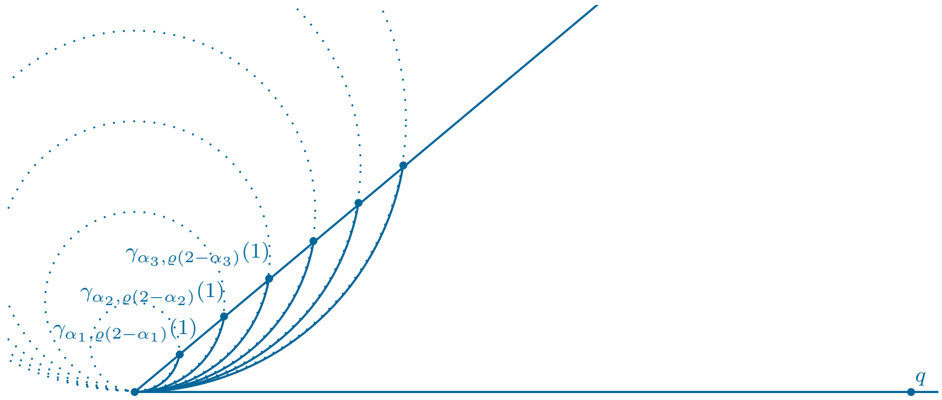


Figure 4: plot of $\gamma_{\alpha,\varrho(2-\alpha)}$ for increasing values $\alpha_1$, $\alpha_2$, …of $\alpha$ and $\varrho = 1$, $\gamma_{\alpha,\varrho(2-\alpha)}(1)$ highlighted, $q$ being the endpoint at infinity

And because any highly accessible point in $\partial_\sigma\mathbb{R}^n$ can be accessed by a conformal geodesic in particular, we find $q$ to be the only highly accessible point in $\partial_\sigma\mathbb{R}^n$. By Lemma 2.7 the highly accessible points are dense in $\partial_\sigma\mathbb{R}^n$, hence $\partial_\sigma\mathbb{R}^n = \{q\}$.

In fact, we even have $M = \sigma(\mathbb{R}^n) \cup \{q\}$. Otherwise $M \setminus \{q\}$ would be covered by the two non-empty disjoint opens $\sigma(\mathbb{R}^n)$ and $\operatorname{int}(M \setminus \sigma(\mathbb{R}^n))$, which contradicts $\dim M > 1$. That is, because we always have the decomposition $M = \sigma(\mathbb{R}^n) \,\dot\cup\, \partial_\sigma\mathbb{R}^n \,\dot\cup\, \operatorname{int}(M \setminus \sigma(\mathbb{R}^n))$.

Now note:

- $M$ is compact.
- $M$ is conformally flat. That is because $\sigma(\mathbb{R}^n)$ is a conformally flat dense subset.
- $M$ is simply connected. To see this, let $\gamma : S^1 \to M$ be any closed curve in $M$. Let $p \in M$ such that $p \notin \operatorname{Im}\gamma$. Let $\varphi : M \to M$ be some diffeomorphism such that $\varphi(p) = q \in M \setminus \sigma(\mathbb{R}^n)$. Then $\varphi \circ \gamma$ is a curve in $\sigma(\mathbb{R}^n)$, i.e. can be contracted, and the same holds for $\gamma$ itself.

Now by Theorem 2.13 we have that $(M, c)$ is conformally equivalent to the sphere with standard round metric. □

While using the Liouville corollary 2.13 is an elegant argument to finish the proof, one may also check that the obvious candidate

$$\begin{aligned} g : M &\to S^n \\ \sigma(x) &\mapsto f(x) \\ q &\mapsto (0, \dots, 0, -1) \end{aligned} \tag{2.13}$$

is a conformal map. Here $f : \mathbb{R}^n \to S^n$ denotes the stereographic embedding as defined in example 1.5.

# 3 Outlook

It would be desirable to give a more direct proof of Theorem 1.45. That is, avoid using that the respective curves are determinded by their 2-jets in one point and instead solve equation 1.69 for a more general matrix $N$. This differential equation is known as the *Sylvester ordinary differential equation*. The problem can be reduced to solving two ODEs of the form $A' = AX$ and $B' = XB$, as described in [21]. The general solution to these equations is difficult to handle, however, one may hope to make use of the $|1|$-gradedness of $\mathfrak{g}$ and equation 4.10 to show that one solution in $P$ exists.

For the reverse direction of the theorem one will have to differentiate equation 1.69 another time and combine the resulting differential equations in each component to the conformal geodesic equation 1.62.

Further, it will be worthwhile to extend Theorem 1.45 to lightlike curves. If one chooses Lemma 1.42 as the defining property for lightlike geodesics, the proof will be very similar to the not lightlike case. However, it would also be of interest to find a differential equation similar to equation 1.62 that also encompasses the lightlike case.

In a more comprehensive program one might want to use conformal geodesics as a method to prove an equivalent of Theorem 2.12 for different manifolds, say the $\mathbb{R}^n$ with indefinite metric or the Poincaré disk.

Considering that many geometric structures (such as pseudo-Riemannian metrics, hermitian metrics, CR structures) are in 1-1-correspondece with Cartan geometries of certain type, one may ask what the geometric meaning of canonical curves in the associated Cartan geometry is, as done in [36] for some cases.

# 4 Appendix

## 4.1 Pseudo-Orthonormal Frames

Let $(M, g)$ be a pseudo-Riemannian manifold of signature $(p, q)$ and dimension $\dim M = n = p+q$. A local pseudo-orthonormal basis $(s_i)_{i=1}^n$ on $M$ is an $n$-tuple of maps defined on some open $U \subset M$ such that $(s_1(x), \dots, s_n(x))$ is a basis of $T_xM$ for all $x \in U$ and such that

$$\varepsilon_i := g(s_i, s_i) = \begin{cases} -1 & , \text{ for } i \in \{1, \dots, p\} \\ 1 & , \text{ for } i \in \{p+1, \dots, n\}. \end{cases}$$

That is, we assume all our psuedo-orthonormal frames to be ordered.

## 4.2 Different Kinds of Connections

Definition 1.1 explains the notion of a Cartan connection. In the following two further notions of connections which are used in the work are explained. For the definitions we follow section 9 of [37].

*Definition* 4.1. Let $(E, \pi, M, F)$ be some fiber bundle. A distribution

$$Th : u \in E \mapsto Th_uE \subset T_uE$$

of dimension $r := \dim M$ is called *general connection* if it

(i) is *geometric*, i.e. for all $u \in E$ exists some neighborhood $U \subset E$ of $u$ and smooth vector fields $X_1, X_2, \dots, X_r$ such that $Th_v = \operatorname{span}(X_1(v), \dots, X_r(v))$ for all $v \in U$,

(ii) is *horizontal*, i.e. for all $u \in E$

$$T_uE = Th_uE \oplus Tv_uE,$$

where $Tv_uE = \operatorname{Ker} d\pi_u$ denotes the canonical vertical distribution.

*Definition* 4.2. Let $(E, \pi, M, F)$ be some fiber bundle. A *general connection form* is a 1-form $\Phi \in \Omega^1(E, VE)$ such that $\Phi \circ \Phi = \Phi$ and $\operatorname{Im} \Phi = TvE$.

**Lemma 4.3.** *Let $(E, \pi, M, F)$ be some fiber bundle. There is a 1:1-correspondence between general connections and general connection forms on $(E, \pi, M, F)$.*

*(a) Given a general connection $Th$, then $\Phi \in \Omega^1(E, TvE)$ given by*

$$\Phi(X \oplus Y) := Y \textit{ for } X \in ThE, Y \in TvE$$

*defines a general connection form on $E$.*

*(b) Given a general connection form* $\Phi \in \Omega^1(E, TvE)$*, then*

$$Th : u \in E \mapsto Th_u E := \operatorname{Ker} \Phi_u$$

*defines a general connection on* $E$*.*

Because of this we will not differentiate between general connections and general connection forms.

A special case of general connections are principal connections. For the definitions we follow section 11 of [37] and section 3.1 of [7].

*Definition* 4.4. Let $G$ a Lie group with Lie algebra $\mathfrak{g}$ and $(\mathcal{G}, \pi, M; G)$ a $G$-principal bundle. A distribution

$$Th : u \in \mathcal{G} \mapsto Th_u\mathcal{G} \subset T_u\mathcal{G}$$

of dimension $r := \dim M$ is called *principal connection* if (considering the principal bundle $\mathcal{G}$ as an ordinary fiber bundle) it is a general connection and additionally

(iii) it is *right-invariant*, i.e. for all $u \in \mathcal{G}$ and $g \in G$ we have

$$dR_g(Th_u\mathcal{G}) = Th_{u \cdot g}\mathcal{G}.$$

**Lemma 4.5.** *By Lemma 4.3 there is a projection* $\Phi \in \Omega^1(\mathcal{G}, Tv\mathcal{G})$ *onto the vertical bundle associated to a principal connection* $Th$*. This projection is then* $G$*-equivariant, i.e.* $R_g^*\Phi = \Phi$*.*

*Conversely, given a* $G$*-equivariant projection* $\Phi \in \Omega^1(\mathcal{G}, Tv\mathcal{G})$*, the general connection* $Th$ *induced by Lemma 4.3 is right-invariant.*

*Definition* 4.6. Let $G$ a Lie group with Lie algebra $\mathfrak{g}$ and $(\mathcal{G}, \pi, M; G)$ a $G$-principal bundle. A 1-form $\Phi \in \Omega^1(\mathcal{G}, \mathfrak{g})$ is called *principal connection form*, if it

(i) is $G$-equivariant, i.e. $R_g^*\Phi = \operatorname{Ad}(g^{-1}) \circ \Phi$ for all $g \in G$,

(ii) reproduces the generators of fundamental vector fields, i.e. $\Phi(\tilde{X}) = X$ for all $X \in \mathfrak{g}$.

**Lemma 4.7.** *Let* $G$ *a Lie group with Lie algebra* $\mathfrak{g}$ *and* $(\mathcal{G}, \pi, M; G)$ *a* $G$*-principal bundle. There is a 1:1-correspondence between principal connections and principal connection forms on* $\mathcal{G}$*.*

*(a) Given a principal connection* $Th$*, then*

$$\Phi_u(\tilde{X}(u) \oplus Y_h) := X$$

*for* $u \in \mathcal{G}$*,* $X \in \mathfrak{g}$*,* $Y_h \in Th_u\mathcal{G}$*, defines a principal connection form on* $\mathcal{G}$*. Here we used an alternative description of the vertical bundle of a principal bundle, namely* $Tv_u\mathcal{G} = \{\tilde{X}(u) \mid X \in \mathfrak{g}\}$*.*

(b) *Given a principal connection form $\Phi \in \Omega^1(\mathcal{G}, \mathfrak{g})$, then*

$$Th : u \in \mathcal{G} \mapsto Th_u\mathcal{G} := \operatorname{Ker} \Phi_u$$

*defines a principal connection.*

Because of Lemma 4.5 and Lemma 4.7 we will not distinguish between principal connections, their induced projections onto the vertical bundle and principal connection forms.

The relations between generalized connections, principal connections and Cartan connections are summarized in the following theorems:

**Theorem 4.8.** *Let $(\mathcal{P}, \pi, M; P)$ be a $P$-principal bundle and let $G$ be a Lie group s.t. $P \subset G$ is a closed subgroup and $\dim M = \dim G - \dim P$. Let $\mathcal{G} := \mathcal{P} \times_P G$ be the $G$-extension of $\mathcal{P}$ with canonical embedding*

$$\begin{aligned} j : \mathcal{P} &\to \mathcal{G} \\ u &\mapsto [u, e]. \end{aligned} \tag{4.1}$$

*Then there exists a bijection*

$$\left\{ \begin{array}{c} \text{Cartan connections } \omega \in \Omega^1(\mathcal{P}, \mathfrak{g}) \\ \text{of type } (G, P) \text{ on } \mathcal{P} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} \text{principal connections } \Phi \in \Omega^1(\mathcal{G}, \mathfrak{g}) \\ \text{on } \mathcal{G} \text{ that act injectively on} \\ T\mathcal{P}, \text{ i.e. } \operatorname{Ker} \Phi \cap di(T\mathcal{P}) = 0 \end{array} \right\}.$$

*This bijection is given by the following constructions:*

(a) *Given $\Phi$ on $\mathcal{G}$ of the specified form, then $\omega := j^*\Phi \in \Omega^1(\mathcal{P}, \mathfrak{g})$ is a Cartan connection of type $(G, P)$ on $\mathcal{P}$.*

(b) *Write $\pi_{\mathcal{P}} : \mathcal{P} \times G \to \mathcal{P}$ and $\pi_G : \mathcal{P} \times G \to G$ for the canonical projections and define $\tilde{\Phi} \in \Omega^1(\mathcal{P} \times G, \mathfrak{g})$ via*

$$\tilde{\Phi}_{(u,g)} := \operatorname{Ad}(g^{-1}) \circ (\pi_{\mathcal{P}}^*\omega)_{(u,g)} + \left(\pi_G^*\omega^{MC}\right)_{(u,g)}$$

*(here $\omega^{MC}$ is the Maurer-Cartan form on $G$). Then this form can be pushed down to a principal connection $\Phi \in \Omega^1(\mathcal{G}, \mathfrak{g})$ and is of the specified form.*

**Theorem 4.9.** *Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$ and $(\mathcal{G}, \pi, M; G)$ a $G$-principal bundle with principal connection $\Phi \in \Omega^1(\mathcal{G}, Tv\mathcal{G})$ as described in 4.5.*

*Let $S$ be some manifold with left action $l : G \times S \to S$ and consider the associated fibered manifold $\mathcal{G} \times_G S$ with canonical projection*

$$\begin{aligned} q : \mathcal{G} \times S &\to \mathcal{G} \times_G S \\ u, s &\mapsto [u, s]. \end{aligned} \tag{4.2}$$

*Then there is a canonically induced general connection* $\overline{\Phi} \in \Omega^1(\mathcal{G} \times_G S, Tv(\mathcal{G} \times_G S))$ *on the fibered manifold* $\mathcal{G} \times_G S$ *making the following diagram commutative:*

$$\begin{array}{ccccc} T\mathcal{G} \times TS & \xrightarrow{\Phi \times \mathrm{Id}} & T\mathcal{G} \times TS & \xrightarrow{\simeq} & T(\mathcal{G} \times S) \\ \downarrow{\scriptstyle dq} & & \downarrow{\scriptstyle dq} & & \downarrow{\scriptstyle dq} \\ T\mathcal{G} \times_{TG} TS & \xrightarrow{\overline{\Phi}} & T\mathcal{G} \times_{TG} TS & \xrightarrow{\simeq} & T(\mathcal{G} \times_G S) \end{array} \tag{4.3}$$

Note that the differential of the group action $G \times G \to G$ provides a Lie group structure on $TG$, thus turning $T\mathcal{G}$ into a $TG$-principal bundle, which allows to define the fibered manifold $T\mathcal{G} \times_{TG} TS$.

The isomorphism $T\mathcal{G} \times TS \simeq T(\mathcal{G} \times S)$ is clear. The isomorphism $T\mathcal{G} \times_{TG} TS \simeq T(\mathcal{G} \times_G S)$ is given by

$$\begin{aligned} T\mathcal{G} \times_{TG} TS &\to T(\mathcal{G} \times_G S) \\ [X, V] &\mapsto dq(X, V). \end{aligned}$$

(See 10.18 of [37] to see that this is a well-defined map) This explains how $T\mathcal{G} \times TS \overset{dq}{\to} T\mathcal{G} \times_{TG} TS$ is to be understood. The proof of Theorem 4.9 can be found in 11.8 of [37].

General connections induce a parallel transports, which in the case of principal connections have particularly good properties:

**Theorem 4.10.** *Let* $(E, \pi, M, F)$ *be some fiber bundle and* $\Phi \in \Omega^1(E, TvE)$ *a general connection. Let* $\gamma : I \to M$ *be a smooth curve with* $t_0 \in I$ *and write* $\gamma(t_0) = x$.

*Then there is a neighborhood* $U \subset E_x \times I$ *of* $E_x \times \{t_0\}$ *and a smooth mapping* $\mathrm{Pt}_\gamma : U \to E$ *satisfying:*

*(i)* $\mathrm{Pt}_\gamma(u_x, \cdot)$ *is a lift of* $\gamma$*, i.e.* $\pi \circ \mathrm{Pt}_\gamma(u_x, t) = \gamma(t)$ *for all* $(u_x, t) \in U$*,*

*(ii)* $\mathrm{Pt}_\gamma(u_x, \cdot)$ *is horizontal, i.e.* $\Phi\left(\frac{d}{dt} \mathrm{Pt}_\gamma(u_x, t)\right) = 0$ *wherever defined.*

*This map is called* parallel transport in $E$ along $\gamma$ with respect to $\Phi$.

The proof can be found in section 9.8 of [37]. Section 3.3. of [7] gives a proof in the case of principal bundles, where an even stronger result holds, as stated in the first part of the following theorem:

**Theorem 4.11.**

1. *Let* $(\mathcal{G}, \pi, M; G)$ *be a* $G$*-principal bundle with principal connection* $\Phi \in \Omega^1(\mathcal{G}, \mathfrak{g})$. *With the notation from Theorem 4.10, one may choose* $U = E_x \times I$*, i.e. the parallel transport along any smooth curve is defined for all times.*

2. *Let $(\mathcal{G}, \pi, M; G)$ be a $G$-principal bundle, $l : G \times S \to S$ a left action of $G$ on some manifold $S$. Denote by $\overline{\Phi} \in \Omega^1(\mathcal{G}\times_G S, Tv(\mathcal{G}\times_G S))$ the canonically induced connection as per Theorem 4.9.*

   *Denote by $\mathrm{Pt}^{\Phi}$ and $\mathrm{Pt}^{\overline{\Phi}}$ the parallel transports induced by $\Phi$ on and $\overline{\Phi}$ respectively, as defined in Theorem 4.10. Let $q : \mathcal{G} \times S \to \mathcal{G} \times_G S$ be the canonical projection (cf. equation 4.2).*

   *Then the two parallel transports are $q$-connected, i.e. for all curves $\gamma : I \to M$, $u \in \mathcal{G}_{\gamma(0)}$, $s \in S$ and $t \in I$ we have:*

$$\mathrm{Pt}^{\overline{\Phi}}_{\gamma}(q(u, s), t) = q\left(\mathrm{Pt}^{\Phi}_{\gamma}(u, t), s\right). \tag{4.4}$$

   *In particular, the parallel transport $\mathrm{Pt}^{\overline{\Phi}}$ is defined for all times.*

The proof for the second statement can be found in section 11.8 of [37].

## 4.3 Jet Bundles

Following [37] we define the jet bundle of a smooth manifold. In the following always let $M$, $N$ be some smooth manifolds and $I$ some open interval containing 0.

*Definition* 4.12 (Definition in 12.1 of [37]). Let $r \in \{0, 1, 2, \ldots\}$. Two curves $\gamma$, $\delta : I \to M$ are said to *have the $r$-th contact at zero*, if for every smooth function $\varphi : M \to \mathbb{R}$ we have

$$\varphi \circ \gamma(t) - \varphi \circ \delta(t) = o(t^r) \text{ for } t \to 0.$$

In this case we write $\gamma \sim_r \delta$ and it is easy to see that $\sim_r$ is an equivalence relation.

**Lemma 4.13** (Lemma in 12.1 of [37])**.** *If $\gamma \sim_r \delta$, then $f \circ \gamma \sim_r f \circ \delta$ for every smooth map $f : M \to N$.*

*Definition* 4.14 (Definition in 12.2 of [37]). Two maps $f$, $g : M \to N$ are said to *determine the same $r$-jet at $x \in M$*, if for every curve $\gamma : I \to M$ with $\gamma(0) = x$ the curves $f \circ \gamma$ and $g \circ \gamma$ have the $r$-th order contact at zero.

It is easy to see that this is an equivalence relation on the set of maps $M \to N$. An equivalence class of this relation is called an *$r$-jet of $M$ into $N$* and we write $j^r_x f$ for such an equivalence class. The set of all $r$-jets of $M$ into $N$ is denoted by $J^r(M, N)$.

**Lemma 4.15** (Proposition in 12.5 of [37])**.** *Given a chart $(U, \varphi = (x_1, \ldots, x_n))$ of $M$ around $x$ and a chart $(V, \psi = (y_1, \ldots, y_m))$ around $f(x)$, two maps $f$, $g : M \to N$ satisfy $j^r_x f = j^r_x g$ if and only if the values of all the partial derivatives up to order $r$ of $\psi \circ f \circ \varphi^{-1}$ and $\psi \circ g \circ \varphi^{-1}$ in the point $\varphi(x)$ coincide.*

**Lemma 4.16.** *Let $(\mathcal{G}, \pi, M; G)$ a $G$-principal bundle with principal connection $\Phi \in \Omega^1(\mathcal{G}, \mathfrak{g})$. Let $\gamma_1$, $\gamma_2 : I \to M$ be two curves with $\gamma_1(0) = \gamma_2(0) = x \in M$. Fix $u \in \mathcal{G}_x$*

*and let* $\gamma_i^* : I \to \mathcal{G}$ *be the uniquely determined horizontal lift of* $\gamma_i$ *with* $\gamma_i^*(0) = u$ *for* $i \in \{1, 2\}$.

*Then* $\gamma_1$, $\gamma_2$ *are having* $r$*-th contact at zero, if and only if* $\gamma_1^*$, $\gamma_2^*$ *are having* $r$*-th contact at zero.*

*Proof.* Clearly, if $\gamma_i^*$ for $i \in \{1, 2\}$ have contact of order $r$ at zero, the compositions with the projection map $\gamma_i = \pi \circ \gamma_i^*$ still have contact of order $r$ at zero. It remains to show the other direction ("$\Rightarrow$").

Consider some local section $\sigma : M \supset U \to \mathcal{G}$. Then a local trivialization of $\mathcal{G}$ is given by

$$\begin{aligned} \theta : U \times G &\to \mathcal{G} \\ (x, g) &\mapsto \sigma(x) \cdot g. \end{aligned} \tag{4.5}$$

For the moment let $\gamma : I \to M$ be any curve with horizontal lift $\gamma^*$ and write $\theta^{-1} \circ \gamma^* = (\gamma, g)$ for some smooth $g : I \to G$. Then

$$\begin{aligned} 0 &= \Phi((\gamma^*)'(t)) \\ &= \Phi((\theta \circ (\gamma, g))'(t)) \\ &= \Phi(d\theta(\gamma'(t), g'(t))) \\ &= \Phi(d\theta(\gamma'(t), 0)) + \Phi(d\theta(0, g'(t))) \end{aligned} \tag{4.6}$$

or $\Phi(d\theta(\gamma'(t), 0)) = -\Phi(d\theta(0, g'(t)))$ in other words. Note that

$$\begin{aligned} \frac{d}{ds} |_{s=0}\, g(t) \cdot \exp\left(s \cdot \omega^{MC}\left(g'(t)\right)\right) &= dL_{g(t)}\left(\omega^{MC}(g'(t))\right) \\ &= g'(t) \end{aligned}$$

and therefore

$$\begin{aligned} d\theta(0, g'(t)) &= d\theta\left(0, \frac{d}{ds} |_{s=0}\, g(t) \cdot \exp\left(s \cdot \omega^{MC}\left(g'(t)\right)\right)\right) \\ &= \frac{d}{ds} |_{s=0}\, \theta\left(\gamma(t), g(t) \cdot \exp\left(s \cdot \omega^{MC}\left(g'(t)\right)\right)\right) \\ &= \frac{d}{ds} |_{s=0}\, \sigma(\gamma(t)) \cdot g(t) \cdot \exp\left(s \cdot \omega^{MC}\left(g'(t)\right)\right) \\ &= \widetilde{\omega^{MC}(g'(t))}(\sigma(\gamma(t)) \cdot g(t)) \end{aligned}$$

which yields for the second term on the right hand side of equation 4.6:

$$\Phi(d\theta(0, g'(t))) = \omega^{MC}(g'(t)). \tag{4.7}$$

The first term of equation 4.6 is given as $\Phi(d\theta(\gamma'(t), 0)) = \Phi(d\sigma(\gamma'(t)))$, so altogether equation 4.6 reads:

$$(\delta g)^{(1)}(t) = -\Phi(d\sigma(\gamma'(t))), \tag{4.8}$$

where $\delta$ denotes the logarithmic derivative as defined in definition 1.38. The two curves $\gamma_1'$ and $\gamma_2'$ have contact of order $(r-1)$ at zero. Composition with the smooth maps $\Phi$ and $d\sigma$ preserves this contact. Writing $\theta^{-1} \circ \gamma_i^* = (\gamma_i, g_i)$ equation 4.8 tells us that also the two curves $(\delta g_1)^{(1)}$ and $(\delta g_2)^{(1)}$ have contact of order $(r-1)$ at zero. Part (a) of Theorem 1.39 implies that $g_1$ and $g_2$ have contact of order $r$ at zero. Thus the curves $\gamma_i^* = \theta(\gamma_i, g_i)$ for $i \in \{1, 2\}$ also have contact of order $r$ at zero. $\square$

## 4.4 Lie groups

**Lemma 4.17** (Satz 1.26 of [7]). *Let $G$ be a Lie group and $M$ a smooth manifold and $\cdot : M \times G \to M$ be a right action. Let $x : I \to M$ be a curve with $x(0) = x \in M$ and $g : I \to G$ be a curve with $g(0) = g$. Then the curve $z(t) := x(t) \cdot g(t) : I \to M$ satisfies*

$$z'(0) = dR_g(x'(0)) + \widetilde{\omega^{MC}(g'(0))}(x \cdot g). \tag{4.9}$$

**Lemma 4.18** (Formula 1.11 of [44]). *Let $G$ be a Lie group with Lie algebra $\mathfrak{g}$. Let $X : I \to \mathfrak{g}$ be a curve. Then*

$$\frac{d}{dt} e^{X(t)} = dL_{e^{X(t)}} \left( \sum_{p=0}^{\infty} \frac{1}{(p+1)!} \operatorname{ad}(-X(t))^p (X'(t)) \right). \tag{4.10}$$

**Theorem 4.19** (Theorem 3.1.3 from [10]). *Let $G$ be a Lie group and $P \subset G$ a closed subgroup. Assume $\mathfrak{g}$ has a $|1|$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ such that $\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. Then*

$$\begin{aligned} \Phi : G_0 \times \mathfrak{g}_1 &\to P \\ (g_0, Z) &\mapsto g_0 \exp(Z) \end{aligned} \tag{4.11}$$

*is a diffeomorphism.*

*In particular for all $p \in P$ there exist $g_0 \in G_0$ and $Z \in \mathfrak{g}_1$ such that $p = g_0 \exp(Z)$.*

Note that the theorem may be generalized to $|k|$-graded Lie algebras.

**Theorem 4.20** (Theorem 3.36 of [34]). *Let $X,\ Y \in \mathfrak{g}$ be such that $[X, Y] = 0$. Then* $\exp(X)\exp(Y) = \exp(X+Y)$.

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