Modern Physics Letters A, Vol. 3, No. 3 (1988) 325 – 328

FERMI-BOSE TRANSMUTATIONS INDUCED BY GAUGE FIELDS

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Received December 1987

We show that in (2 + 1)-dimensional abelian gauge theory with the Chern-Simons term in the action, charged particles reverse their statistics.

In this letter I will demonstrate that in some theories with gauge fields, statistics of elementary excitations is reversed.

The interest of this result is at least twofold. First, as was shown by Anderson et al.,[1] it is possible that high T_c superconductivity is connected with some unusual properties of Heisenberg antiferromagnets or perhaps, of Hubbard model. The way for quantitative implementation of these ideas was found by Wiegman,[2] who showed that possible confinement r regime in this model may be responsible for the Cooper pairing of electrons. It was noticed in Ref. 3 that, under the same premises, it is highly probable that antiferromagnetic magnons are fermions – the fact envisioned by Pomeranchuck.[4]

Here I shall try to give more technical and quantitative arguments in favor of the physical picture, advocated in Refs. 1–4. Essentially, it will be demonstrated by the explicit computation of the propagators in the gauge fields, that the Chern-Simons term[5] in the gauge action turns bosons to fermions and vice versa. In particular, electrons in this field behave as spin one and zero bosons, which seems to confirm the statements made in Ref. 1.

Let us start with the (2+1)-dimensional antiferromagnet.

We shall accept a conjecture, put forward in Refs. 2 and 3, that antiferromagnetic magnons can be described by the field of unit vector, with ϑ-term, which is Hopf invariant for the map $s^3 \to s^2$. It is convenient to use CP^1 representation for this $\mathbf{n}$-field, and discuss the Lagrangian

$$\mathscr{L} = \frac{1}{\gamma_0} \sum_{k=1}^{2} |\partial_\mu z_k + iA_\mu z_k|^2 + \frac{\vartheta}{16\pi^2} \epsilon_{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \; . \tag{1}$$

Here the field $z = (z_1, z_2)$ defines a point of s^3 such that

$$z^\dagger z \equiv |z_1|^2 + |z_2|^2 = 1 \, .$$

The first term is a standard representation for the $\mathbf{n}$-field, namely if we define $\mathbf{n} = (z^\dagger \sigma z)$, it can be rewritten as $(\partial_\mu \mathbf{n})^2$.

The last term is Chern-Simons invariant, which was first discussed in gauge theories in Ref. 5. Modulo higher derivatives, it is equal to Hopf invariant for the $\mathbf{n}$-field.

The conjecture in Refs. 2 and 3 was that spin-$\frac{1}{2}$ antiferromagnet corresponds to $\vartheta = \pi$ in (1).

We shall now prove that the main long-range effect of the gauge field is that z-quanta become fermions. For the proof, let us look at the expression for the transition amplitude of z-quanta in the form of the path integral. We have

$$G(x,x') = \sum_{(P_{xx'})} e^{-mL(P_{xx'})} \langle e^{i\int_{P_{xx'}} A_\mu dx^\mu} \rangle \tag{2}$$

(Here L is the length of $P_{xx'}$). It will be convenient to start with the partition function

$$z = \sum_{(P)} e^{-mL(P)} \langle \exp\left\{ i \oint_P A_\mu dx^\mu \right\} \rangle$$

where P — are closed paths, and m — the mass. The averaging over A_μ is easily performed

$$\langle e^{i\oint_P A_\mu dx^\mu} \rangle = \exp\left\{ -\frac{1}{2} \oint_P \oint_P dx_\mu dy_\nu \langle A_\mu(x) A_\nu(y) \rangle \right\}$$

$$= \exp\left\{ \frac{2\pi i}{4\pi} \oint_P \oint_P dx_\mu dy_\nu e_{\mu\nu\lambda} \frac{(x_\lambda - y_\lambda)}{(x-y)^3} \right\} . \tag{3}$$

We performed averaging here using only the last term in (1) because vacuum polarization by z-quanta produce terms $\propto F_{\mu\nu}^2$ with higher number of derivatives. The integral in (3) would be Gauss integral for linking number, if we had integrated over two different loops P and $\tilde{P}$. In our case, we have to regularize this integral carefully. We do it by

$$I = \frac{1}{4\pi} \oint_P \oint_P dx_\mu dy_\nu\, e_{\mu\nu\lambda} \partial_\lambda \frac{1}{|x-y|}$$
$$= \oint_P dx_\mu \oint_{\Sigma_P} d^2 y_\mu\, \delta(x-y) . \tag{4}$$

(Here we used Stokes' theorem. The Σ_P is an arbitrary surface bounded by P. Let us replace

$$\delta(x-y) \Rightarrow \delta_\epsilon(x-y) = \frac{1}{(2\pi\epsilon)^{3/2}} e^{-(x-y)^2/\epsilon} . \tag{5}$$

The integral (5) is now well defined and is dominated at $\epsilon \to 0$ by the region close to the boundary. Parametrizing this region, we get

$$I = \frac{1}{2\pi} \oint_P d\mathbf{x} \cdot [\mathbf{n} \times \dot{\mathbf{n}}]$$
$$\mathbf{x} = \mathbf{x}(s); \quad \dot{\mathbf{n}} = \frac{d\mathbf{n}}{ds} \tag{6}$$

Here $d\mathbf{x}$ is a tangent vector to the path, while $\mathbf{n}$ is a normal along the Σ_P. It is easy to rewrite this in terms of normal connection defined by the relations

$$\dot{\mathbf{e}} = \sum_{i=1}^{2} B_i \mathbf{n}_i\,; \quad \dot{\mathbf{x}} \propto \mathbf{e}$$
$$\dot{\mathbf{n}}_i = C\epsilon_{ik}\mathbf{n}_k - B_i \mathbf{e}\,. \tag{7}$$

The integral (6) is expressed in terms of normal connection C which is also called a torsion of the curve,

$$I = \frac{1}{2\pi}\int_0^L C(s)ds\,. \tag{8}$$

It is easily seen from (7) that C is essentially a Dirac potential on a sphere $\mathbf{e}^2 = 1$ with a magnetic monopole in the center. Namely, one can represent C in the form

$$C(s) = \int_0^1 du\,\mathbf{e}\cdot\left[\frac{\partial\mathbf{e}}{\partial s}\times\frac{\partial\mathbf{e}}{\partial u}\right] \tag{9}$$

where we introduced interpolating field

$$\mathbf{e}(s,u) = \begin{cases}\mathbf{e}(s) & u = 1\\ \text{const} & u = 0\end{cases}\,.$$

The integral (9) is defined modulo some integer, which is irrelevant, due to Dirac quantization.

So, our result is

$$\phi(P) = \langle e^{i\oint A_\mu dx_\mu}\rangle = \exp\left(i\int_0^L ds\int_0^1 du\,\mathbf{e}\cdot\left[\frac{\partial\mathbf{e}}{\partial s}\times\frac{\partial\mathbf{e}}{\partial u}\right]\right) \tag{10}$$

We shall now show that the right-hand side of (10) is precisely equal to the spin factor for the spin-$\frac{1}{2}$ particle. Namely, the sum over closed paths for the Dirac electron in three dimensional space time is given by

$$z = \sum_{(P)} e^{-mL(P)}\phi(P)\,. \tag{11}$$

The main reason why (11) is true lies in the fact that if we consider $\mathbf{e}(s)$ as a dynamical variable, and the Dirac potential in (10) as an action, defining Poisson brackets, then it is obvious that

$$[e^a, e^b] = \epsilon^{abc}e^c \tag{12}$$

(Structure constants in Poisson brackets are in general defined by the closed 2-form in the action.)

More technically, it means that

$$\int \mathscr{D}\mathbf{e}\, e^{i\int_0^L ds\, C(s)}\,\delta(\mathbf{e}^2 - 1)\; e_{\alpha_1}(s_1)\ldots e_{\alpha_N}(s_N)$$
$$= \mathrm{Tr}\,(\sigma_{\alpha_1}\ldots\sigma_{\alpha_N}) \tag{13}$$

where σ_α are Pauli matrices (for the monopole of the charge $2s$, we would have $(2s + 1)$-dimensional representation in (13)).

Therefore, we can replace

$$\mathbf{e}(s) \Rightarrow \sigma \ . \tag{14}$$

The propagator in the momentum space for our particle is given by

$$G(\mathbf{p}) = \int_0^\infty dL e^{-mL} \int D\mathbf{e}\, e^{i\int_0^L C ds} \delta(\mathbf{e}^2 - 1) \cdot e^{i\mathbf{p}\int_0^L \mathbf{e} ds} \ . \tag{15}$$

Using (14), we get

$$G(\mathbf{p}) = \int_0^\infty dL e^{-mL} e^{i(\sigma \cdot \mathbf{p})L} = \frac{1}{m - i(\sigma \cdot \mathbf{p})}$$

which is correct fermion propagator. Strictly speaking, we used somewhat more than (13) which refers to traces. But careful treatment of boundary terms in (13) permits us to use it for matrix elements as well.

So, we have proved that dressing of z-quanta by the gauge fields turns them into Dirac fermions χ. In this proof, we neglected higher derivative terms in the action. They will lead to short range interaction among these fermions. Let us notice also, that the mass term for z-quanta, $z^\dagger z$ turns into $\bar{\chi}\gamma_\mu \partial_\mu \chi$ because the $\bar{\chi}\chi$ term violates p-parity.

So, we arrived at the rather peculiar picture: z-quanta being bosons at large momenta behave as fermions at small momenta. Vice-versa, if we had a charged fermion in our theory it would transmute into two bosons with spins one and zero.

It is very tempting to find similar phenomena in different dimensionalities and also for strings rather than for particles. At the moment, this problem is not solved.

Acknowledgments

I am grateful to P. B. Wiegman for many stimulating discussions.

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