Fractured Structure on Condensed Anima

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Abstract

We compare notions generalizing features of topology, namely condensed mathematics and cohesive resp. fractured ∞-topoi. After showing that cohesion is not sensible on the ∞-topos of condensed anima $\mathbf{Cond}(\mathbf{An})$ we provide a fractured structure on $\mathbf{Cond}(\mathbf{An})$. We apply this to compare sheaf cohomology with condensed cohomology and show that for the corporeal objects of the fractured structure which simultaneously are also topological spaces the cohomologies agree.

Contents

0 Introduction

Topology is useful, basic, and ubiquitous. Nonetheless, it was only in 1940 that today's axiomatic definition of a topological space as a collection of open subsets of a set was first formulated by the Bourbaki group [Bru18, p. 134] while an axiomatization of topological spaces through neighborhoods was defined by Felix Hausdorff in his *Grundzüge der Mengenlehre* in 1914 [Hau49, Chapter 7.1]. On the other hand, topological ideas were floating around long before: first instances of the idea of a topology go back to Euler and his polyhedron formula in 1750 [Bru18, p. 124, 125] leading to the modern notion of the Euler characteristic χ. Needless to say, the idea of 'being close', as topology is supposed to axiomatize, has existed for a long time, even before Euler.

Today, the notion of topology is used in almost every part of mathematics – topology is useful and ubiquitous. Just to name some examples, algebraic geometers adhere to their famous Zariski topologies, number theorists consider p-adic topologies, functional analysts use different topologies on function spaces and probability theorists refer to their Polish spaces.

Topology is also basic: already in calculus, we draw back to topological notions. Weierstraß's Extreme Value Theorem depends on the notion of compactness and the Intermediate Value Theorem depends on the notion of connectedness. So nowadays every mathematician encounters topological ideas early on in their mathematical lives.

However, even a notion as basic and ubiquitous as topology can be flawed. Sometimes, the wishful thinking of marrying topological ideas with other ideas in mathematics can lead to unwanted complications. We present problems occurring in mathematical areas that conflict with the notion of topology.

1. Algebra: The problem of classifying certain algebraic structures is important but hopeless. It is impossible to classify all abelian groups. So often mathematicians instead endow this algebraic category with more structure and, for instance, consider the category of topological abelian groups **TopAb** or possibly of Lie groups or group schemes. The additional structure on **TopAb** now opens up possibilities for more tools involving topology. For example, if the topological group is moreover also locally compact and Hausdorff, then we have Pontryagin Duality in our arsenal.

   This is a helpful observation but unfortunately, the category **TopAb** admits a fatal flaw: it is not an abelian category. The map $\mathrm{id}_{\mathbb{R}} : \mathbb{R}_{\mathrm{disc}} \to \mathbb{R}_{\mathrm{eucl}}$ from $\mathbb{R}$ with the discrete topology to $\mathbb{R}$ with the Euclidean topology has trivial kernel and cokernel but is not an isomorphism! So a priori, it is not possible to use homological algebra in **TopAb**.

2. Homotopy Theory: Especially with the emergence of higher category theory and homotopy type theory, working with homotopy types became more relevant than ever. A homotopy type can be modeled with actual spaces, i.e. CW complexes or Kan complexes, so for example the unit disc D^2 models a particular homotopy type, namely the trivial homotopy type $*$.

   However, this process of passing from an actual space to its homotopy type forgets the topological information that can potentially be useful. For example, Brouwer's Fixed Point Theorem famously states that every continuous map $D^2 \to D^2$ admits a fixed point but this becomes an obvious statement in the homotopy type context: a map $* \to *$ surely has a fixed point.

At long last, a topology is by definition a collection of open subsets of a set as defined by Bourbaki. Hence, we necessarily need an underlying set to talk about a topology. However, we sometimes even wish for properties of topology on objects without a naturally underlying set.

All of these problems prompt the quest for a more well-suited structure or an axiomatization of topological properties. This goal occupies much of contemporary mathematics.

Topos theory is the first step in this procedure. A Grothendieck topos is a category of sheaves that abstracts the notion of a covering from topology. Topos theory already appeared in the 1960s [AGV71] and is hence by now a well-studied topic. It is the stepping stone to two modern notions that we are interested in.

Simultaneously and independently of each other, Dustin Clausen and Peter Scholze introduced condensed mathematics [Sch19] while Clark Barwick and Peter Haine introduced pyknotic mathematics [BH19] in 2019. These are – up to a set-theoretic technicality – the same notion and use sheaves on compact Hausdorff spaces to give a framework in which algebra and topology can actually be combined.

Another, purely topos-theoretic attempt, lies in cohesion. In 2007, William Lawvere [Law07] first gave a formal axiomatization of cohesive topoi which generalizes the adjunction quadruple

$$\begin{tikzcd}
\mathbf{Top}^{\text{locally path-connected}} \arrow[r, shift left=4.5ex, "\pi_0"] \arrow[r, phantom, shift left=3ex, "\perp"] \arrow[r, leftarrow, shift left=1.5ex, "\mathrm{Disc}" description] \arrow[r, phantom, "\perp"] \arrow[r, shift right=1.5ex, "U" description] \arrow[r, phantom, shift right=3ex, "\perp"] \arrow[r, leftarrow, shift right=4.5ex, "\mathrm{CoDisc}"'] & \mathbf{Set}
\end{tikzcd}$$

of connected components, discrete topology, forgetful functor, and codiscrete topology. This forms the basis to encode how points are held together. Urs Schreiber then generalized this to the world of ∞-categories in 2013 [Sch13] allowing us to remember much more homotopical information. Even more recently, Jacob Lurie [Lur18, Chapter 20] and David Carchedi [Car20] developed a local version of cohesion that Lurie coined fractured ∞-topoi in 2018 in order to axiomatize a suitable notion of gluings of sheaves on schemes [Lur18, Section 20.1].

Condensed mathematics and cohesive resp. fractured topoi both being notions axiomatizing topological features leads to the natural question of relationships between these notions. The main goal of this thesis is to compare the two concepts.

Here's a linear overview of the thesis.

- Section 1: We give a crash course in higher topos theory where we begin by recalling basic notions in ∞-category theory. In particular, the thesis is readable without prior knowledge of ∞-category theory when taking on certain parts on faith. Many results don't change if one omits the ∞. Afterward, we will define ∞-topoi and will introduce the relevant concepts about them: Grothendieck topologies, descent, object classifier, and hypercompleteness.

- Section 2: We begin by enhancing ∞-topoi with the property of cohesion and give numerous examples of cohesion. Then, we discuss fractured structures and introduce machinery called admissibility structures to construct non-trivial fractured ∞-topoi.

- Section 3: After an interlude on point-set topology regarding the subcategories **Stone** and **Stonean** of the category of compact Hausdorff spaces **CHaus**, we define condensed objects $\mathbf{Cond}(\mathscr{C})$ of an ∞-category $\mathscr{C}$. In particular, we obtain the ∞-topos of condensed anima $\mathbf{Cond}(\mathbf{An})$ and will formulate a number of results ∞-categorically which are often only phrased 1-categorically in the literature.

- Section 4: After proving that $\mathbf{Cond}(\mathbf{An})$ is not a cohesive ∞-topos, we obtain the main result of the thesis, namely a fractured structure on condensed anima.

  Theorem 0.1. There exists a suitable ∞-topos $\mathbf{Cond}^{\text{open}}(\mathbf{An})$ and an adjunction triple

$$\begin{tikzcd}
\mathbf{Cond}(\mathbf{An}) \arrow[r, "i^*" description] & \mathbf{Cond}^{\mathrm{open}}(\mathbf{An}) \arrow[l, shift right=5, "i_!"'] \arrow[l, phantom, shift right=2.5, "\perp"] \arrow[l, phantom, shift left=2.5, "\perp"] \arrow[l, shift left=5, "i_*"]
\end{tikzcd}$$

which yields the structure of a fractured ∞-topos.

With a computational argument, we prove that this is a non-trivial fractured structure.

- Section 5: A concrete application of this formal machinery lies in cohomology. We discuss the setting in which sheaf cohomology $H^\bullet_{\text{sheaf}}$ agrees with condensed cohomology $H^\bullet_{\text{cond}}$ and give an approach via our fractured structure on condensed anima. In particular, we show that they agree on what we call corporeal spaces.
- Section 6: Certain unanswered questions concerning our investigations remain. It is unclear for which spaces the ansatz in section 5 yields positive results. Possibly adjusting the fractured structure on condensed anima could yield further viewpoints. At the very least, this provides a readable summary for the author to take up the problem again in the future.

There are three subsections that seemed too important to omit but which will not be explicitly required later for our comparison of fractured structures with condensed mathematics. For completeness we will include them and will decorate the respective subsections with a star *. The reader can feel free to skip them and will still be able to read the following sections.

The goals of this thesis are two-fold.

1. We give an expository account of higher topos theory, cohesion and fractured structure, and condensed mathematics. Much of this is modern mathematics with only few write-ups, so any additional text about them is useful. To our knowledge, some parts are more detailed here than presented anywhere else in the literature. A comparison of cohesion with fractured ∞-topoi also seems to be new.
2. We present novel results comparing cohesion and fractured structures with condensed mathematics. We construct a fractured structure on condensed anima which we apply to compare sheaf cohomology with condensed cohomology.

We hope to have provided a little more towards the condensed world that has gained so much attention recently!

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[1] In particular, anything novel in this thesis is joint with Nima Rasekh.

1 Crash Course in Higher Topos Theory

Our main focus in this thesis is to study ways of generalizing the notion of topology. Countless possibilities are thinkable but one direction is to axiomatize the notion of (open) coverings of spaces. This leads to the notion of Grothendieck topologies which is precisely the concept that allows us to define sheaves on general categories after equipping them with a Grothendieck topology. The collection of sheaves yields categories of sheaves which form the prototypical examples of Grothendieck topoi. These will be the mathematical objects which will accompany us throughout in this thesis.

In that regard, we will study cohesive and fractured ∞-topoi as further axiomatizations of topology. Moreover, our essential objects of study, namely condensed anima, will be examples of ∞-topoi and will give us another direction to generalize topological ideas.

Topos theory itself was already studied classically for the purpose of algebraic geometry in SGA4 [AGV71] but we will employ the modern notion of ∞-topoi. The canonical reference remains Jacob Lurie's *Higher Topos Theory* [Lur09] but there are also inspiring lecture notes floating around in the world wide web like Charles Rezk's Leeds Lectures [Rez19] as well as a number of papers, e.g. [BGH20, HPT22, Hai22], in which various concepts are worked out.

In this section we will give an overview of higher topos theory with a particular focus towards applications in the thesis. Nonetheless, we will discuss key concepts like descent (**1.4**) or object classifiers (**1.5**) that will find no explicit application in this text. These concepts seem far too essential to omit, so we included them.

1.1 Crash Course to Higher Category Theory

Slowly but surely, $(\infty, 1)$-categories have conquered the homotopical world and have become the most popular modern language for homotopy theory. The essence is to remember all higher coherences in a compact model. We want to give a brief recollection to set the stage for the notation used in this text. Readers not familiar with higher categories should not need more to understand this thesis but can also read the entire text by omitting the words 'higher' or '∞' and should still be able to grasp the main ideas.

We mostly follow Lurie's *Higher Topos Theory* [Lur09] but will also follow textbook accounts like [Lan21]. We assume knowledge of basic homotopy theory such as some acquaintance with simplicial sets e.g. as in [Lan21, GJ99].

Let us recall the most popular model of $(\infty, 1)$-categories.

Definition 1.1 (Boardman-Vogt, 1973)**.** A **quasicategory** $\mathscr{C}$ is a simplicial set satisfying all inner horn lifting properties, i.e. for all $0 < i < n$ and maps $\Lambda^n_i \to \mathscr{C}$ there exists a lift as in the following diagram:

$$\begin{tikzcd}
\Lambda^n_i \arrow[r] \arrow[d, hook] & \mathscr{C} \\
\Delta^n \arrow[ur, dashed, "\exists"'] &
\end{tikzcd}.$$

We write **qCat** for the full subcategory of quasicategories in the category of simplicial sets **sSet**.

This is a model for ∞-categories and in the literature, people often use the words 'quasicategory' and '∞-category' interchangeably. We will do the same and will always mean a quasicategory whenever we write **∞-category** unless otherwise noted.

Recall that a simplicial set K satisfying all horn lifting properties is called **Kan complex**. Kan complexes are simplicial sets which resemble spaces which suggests that an ∞-category generalizes the notion of spaces. On the other hand, it also generalizes the notion of 1-categories, as one may think of the 0-simplices as objects and the 1-simplices as morphisms where for an object x the degeneracy $s_0(x)$ defines id_x and composition is obtained (non-uniquely) via inner horn liftings $\Lambda^2_1 \to \Delta^2$. The higher inner horn liftings amount to coherences of the higher homotopical data and that's what an ordinary 1-category does not see.

Nevertheless, every 1-category can be viewed as an ∞-category via the nerve $N : \mathbf{Cat} \hookrightarrow \mathbf{sSet}$ which is fully faithful and maps a category to an ∞-category [Lur09, Proposition 1.1.2.2].

Collapsing the higher data leads to the homotopy category $h\mathscr{C}$ of an ∞-category $\mathscr{C}$.

Definition 1.2. Let $X \in \mathbf{sSet}$ and let $x, y, z \in X_0$.

(i) Two 1-simplices $f, g \in X_1$ from x to y are **equivalent** if there exists a 2-simplex $\sigma : \Delta^2 \to X$ such that $\sigma|_{\Delta^{\{0,1\}}} = f, \sigma|_{\Delta^{\{0,2\}}} = g, \sigma_{\Delta^{\{1,2\}}} = \mathrm{id}_y$.

$$\begin{tikzcd}
& y \arrow[dr, "\mathrm{id}_y"] & \\
x \arrow[ur, "f"] \arrow[rr, "g"'] & \sigma & y
\end{tikzcd}$$

We write $f \sim g$.

(ii) The **homotopy category** hX of X has objects X_0 and morphisms freely generated by X_1 where a 1-simplex $f \in X_1$ from x to y defines a morphism $f : x \to y$. For $f : x \to y$ and $g : y \to z$ we define composites $g \circ f$ via the following relations:

(a) We demand $\mathrm{id}_x = s_0(x)$.

(b) For every 2-simplex $\sigma : \Delta^2 \to X$ with boundary (f, g, h) we demand $h = g \circ f$.

(c) If $f \sim f'$, then $g \circ f \sim g \circ f'$ and $f \circ h \sim f' \circ h$ for composable morphisms.

Sometimes, hX is also called *fundamental category* of X. It induces a functor $h : \mathbf{sSet} \to \mathbf{Cat}$ which turns out to be left adjoint to the nerve $N : \mathbf{Cat} \to \mathbf{sSet}$ [Lan21, Proposition 1.2.18]. As is customary, we will occasionally omit N from the notation. If $\mathscr{C}$ is an ∞-category, then $h\mathscr{C}$ is isomorphic to the category $\pi(\mathscr{C})$ whose 0-simplices are given by X_0, whose 1-simplices are given by X_1 modulo equivalence (1.2(i)) and composition is given via inner horn lifting $\Lambda^2_1 \to \Delta^2$ [Lur09, Proposition 1.2.3.9].

Here are some definitions much akin to what we are used to from 1-category theory.

Definition 1.3. Let $\mathscr{C}$ be an ∞-category. Then, a map f in $\mathscr{C}$ is an **equivalence** if f becomes an isomorphism in $h\mathscr{C}$.

Definition 1.4. Let $\mathscr{C}$ be an ∞-category. Then, its **core** $\mathscr{C}^{\mathrm{core}}$ is defined as the pullback

$$\begin{tikzcd}
\mathscr{C}^{\mathrm{core}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathscr{C} \arrow[d] \\
(N(h\mathscr{C}))^{\mathrm{core}} \arrow[r] & N(h\mathscr{C})
\end{tikzcd}$$

in $\mathbf{sSet}$.

In other words, $\mathscr{C}^{\mathrm{core}} \hookrightarrow \mathscr{C}$ is the maximal ∞-subgroupoid of $\mathscr{C}$.

Definition 1.5. A **functor** of ∞-categories $\mathscr{C} \to \mathscr{D}$ is a morphism of simplicial sets $\mathscr{C} \to \mathscr{D}$. We define the **functor ∞-category** as the internal-Hom object

$$\mathrm{Fun}(\mathscr{C}, \mathscr{D}) = \underline{\mathrm{Hom}}_{\mathbf{sSet}}(\mathscr{C}, \mathscr{D})$$

which has n-simplices $\underline{\mathrm{Hom}}_{\mathbf{sSet}}(\mathscr{C}, \mathscr{D})_n = \mathrm{Hom}_{\mathbf{sSet}}(\mathscr{C} \times \Delta^n, \mathscr{D})$.

Using results about so-called inner-anodyne maps, one can prove that this is an ∞-category [Lur09, Proposition 1.2.7.3] justifying its name.

Definition 1.6. Let $\mathscr{C}$ be an ∞-category and $x, y \in \mathscr{C}_0$ be 0-simplices. We define the **Hom-anima** $\mathrm{Hom}_{\mathscr{C}}(x, y)$ as the pullback

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{C}}(x,y) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^1, \mathscr{C}) \arrow[d, "\text{(source, target)}"] \\
\Delta^0 \arrow[r, "(x{,}y)"'] & \mathscr{C} \times \mathscr{C}.
\end{tikzcd}$$

More precisely, the right vertical map is given by evaluation at 0 and 1.

Recall that there is a homotopy-coherent version of the nerve $N^{\mathrm{hc}} : \mathbf{sCat} \to \mathbf{sSet}$ which takes as input a simplicially-enriched category [Lur09, Definition 1.1.5.5]. For **Kan**-enriched categories this has an ∞-category as output by Cordier-Porter [Lur09, Proposition 1.1.5.10]. This is one of the main tools to construct ∞-categories.

Construction 1.7. The full subcategory $\mathbf{qCat} \hookrightarrow \mathbf{sSet}$ is **Kan**-enriched via

$$\underline{\mathrm{Hom}}_{\mathbf{qCat}}(\mathscr{C}, \mathscr{D}) = \underline{\mathrm{Hom}}_{\mathbf{sSet}}(\mathscr{C}, \mathscr{D})^{\mathrm{core}}.$$

Then, the **∞-category of ∞-categories** is $\mathbf{Cat}_\infty = N^{\mathrm{hc}}(\mathbf{qCat})$.

Construction 1.8. Let $\mathbf{Kan} \hookrightarrow \mathbf{sSet}$ be the full subcategory spanned by the Kan complexes considered as a simplicially-enriched category. Then, the **∞-category of anima** is $\mathbf{An} = N^{\mathrm{hc}}(\mathbf{Kan})$.

This makes sense because the internal simplicial Hom-objects of **Kan** are Kan complexes again [Lan21, Corollary 1.3.38].

Remark 1.9. The terminology *anima* is quite popular in Bonn not only due to certain famous personalities at the university. The ∞-category of anima **An** is more classically also known as the ∞-category of spaces $\mathcal{S}$, of homotopy types or of ∞-groupoids $\mathbf{Grpd}_\infty$. We have already employed this terminology in the notion of mapping anima (**1.6**) which is more classically called mapping space. The term 'anima' refers to the 'soul' of a space [ČS20, Section 1.2]. While this is first and foremost merely a matter of terminology, there are mathematical reasons why it appears counterproductive to write about the ∞-category of spaces in the setting of this thesis about condensed anima. We will come back to this discussion in **3.20**.

Remark 1.10. For an ∞-category $\mathscr{C}$ and $x, y \in \mathscr{C}_0$ we are justified to call $\mathrm{Hom}_{\mathscr{C}}(x, y)$ a Hom-anima as done in **1.6** since $\mathrm{Hom}_{\mathscr{C}}(x, y)$ is in fact a Kan complex by virtue of Joyal's Lifting Theorem [Lan21, Corollary 2.2.4]. Note that this is perhaps better known as mapping space in the literature.

Alternatively, we can also consider the **Kan**-enriched category **CW** with CW complexes as objects and simplicial mapping objects given by the singular set of the mapping spaces. Then, $N^{\mathrm{hc}}(\mathbf{CW}) \simeq \mathbf{An}$ [Lan21, Corollary 2.5.37]. So we may model anima with Kan complexes or with CW complexes which we think of as actual spaces. The anima itself only remembers the homotopy type, i.e. the soul of these spaces.

In ∞-category theory **An** typically takes the role of **Set**. We can immediately give an illustration.

Definition 1.11. Let $\mathscr{C}$ be an ∞-category. Then,

$$\mathbf{PSh}(\mathscr{C}) = \mathrm{Fun}(\mathscr{C}^{\mathrm{op}}, \mathbf{An})$$

is the **presheaf ∞-category** of $\mathscr{C}$.

Remark 1.12. As seems to have become the tradition we will adopt Grothendieck universes to settle our set-theoretic details [Lur09, Chapter 1.2.15]. We consider a strongly inaccessible cardinal κ and assume the existence of the collection of all sets having cardinality $< \kappa$ which we denote by $\mathcal{U}(\kappa)$. This is a Grothendieck universe. Then, a mathematical object is **small** if it belongs to $\mathcal{U}(\kappa)$. With this remark, we want to slightly redefine $\mathbf{Cat}_\infty$ and $\mathbf{An}$ which can mostly be ignored throughout this thesis.

(i) By taking the coherent nerve of small ∞-categories we denote by $\mathbf{Cat}_\infty = N^{\mathrm{hc}}(\mathbf{qCat}^{\mathrm{small}})$ the ∞-category of small ∞-categories and by $\widehat{\mathbf{Cat}}_\infty = N^{\mathrm{hc}}(\mathbf{qCat})$ the ∞-category of (large) ∞-categories.

(ii) We denote by $\mathbf{An} = N^{\mathrm{hc}}(\mathbf{Kan}^{\mathrm{small}})$ the ∞-category of small anima and by $\widehat{\mathbf{An}} = N^{\mathrm{hc}}(\mathbf{Kan})$ the ∞-category of (large) anima.

We will only occasionally need this finer distinction.

The theory of quasicategories is not the only model for $(\infty, 1)$-category theory. There is a myriad of other ones, such as topologically enriched categories, simplicially enriched categories, relative categories, complete Segal spaces, Segal categories, or 1-complicial sets, to just name a few. Each of these comes with its own merits and perks and unfortunately one has to develop a new theory in each of these models from scratch and then compare the theories in each model. Not only is this a lot of work, but the definition of certain standard constructions in category theory can become rather cumbersome in certain models.

A recent approach by Emily Riehl and Dominic Verity [RV22] incorporates all of the models in a model-independent approach which encodes basic categorical constructions in a language similar to the one we are familiar with from classical category theory. This is the language of ∞-cosmoi.

In this framework, we may define categorical concepts like adjunctions, (co-)limits, Kan extensions or (co-)Cartesian fibrations for ∞-categories without referring to models of higher categories and use definitions akin to those from basic category theory. Most results carry over and it turns out that the formal synthetic theory of ∞-categories discussed here is indeed compatible with the classical theory of quasicategories as found in Joyal's or Lurie's work [RV22, Appendix F]. So we may freely use the quasicategorical notions independently with these rather model-independent definitions and will use them as if we were dealing with 1-categories unless some higher input makes a relevant difference.

In that regard, we will work with basic categorical constructions like the Yoneda embedding, slice categories, adjunctions, (co-)limits, Kan extensions, and so on similar to what we are used to from 1-categories, while remembering that we are using Hom-anima instead of Hom-sets.

We now move on to certain size-theoretic conditions – presentability and accessibility – that are difficult to access via formal means which we want to recall. They are highly useful which is why we cannot ignore them. In particular, the definition of a presentable ∞-category is almost the same as the definition of an ∞-topos. So naturally we need to introduce this notion here.

Let κ be a regular cardinal. We will define properties Prop of objects dependent on κ, which will be said to satisfy κ-Prop. We then say that an object has property Prop if it has property λ-Prop for some regular cardinal λ.

We begin with some size conditions.

Definition 1.13. Let κ be a regular cardinal and let $\mathscr{C}$ be an ∞-category.

(i) Then, $\mathscr{C}$ is **κ-filtered** if for every κ-small simplicial set K and every map $f : K \to \mathscr{C}$ there exists an extension $\overline{f} : K^{\triangleright} \to \mathscr{C}$.

(ii) Let $\mathscr{C}$ have small κ-filtered colimits. An object $C \in \mathscr{C}$ is **κ-compact** if the evaluation of the Yoneda embedding $よ_C : \mathscr{C} \to \widehat{\mathbf{An}}$ preserves κ-filtered colimits.

(iii) The ∞-category $\mathscr{C}$ is **essentially κ-small** if the collection of equivalence classes of objects in $\mathscr{C}$ is κ-small and if for every map $f : C \to D$ in $\mathscr{C}$ and for every $i \geq 0$ the homotopy set $\pi_i\left(\mathrm{Hom}_{\mathscr{C}}(C, D), f\right)$ is κ-small.

We can already define accessibility!

Definition 1.14. Let κ be a regular cardinal and let $\mathscr{C}, \mathscr{D}$ be ∞-categories.

(i) Then, $\mathscr{C}$ is **κ-accessible** if $\mathscr{C}$ admits all small κ-filtered colimits and there exists an essentially small full subcategory $\mathscr{C}' \hookrightarrow \mathscr{C}$ of κ-compact objects which generates $\mathscr{C}$ under small κ-filtered colimits.

(ii) Let $\mathscr{C}$ be an accessible ∞-category. A functor $F : \mathscr{C} \to \mathscr{D}$ is **κ-accessible** if it preserves κ-filtered colimits.

So roughly, an accessible ∞-category is an ∞-category with filtered colimits which are controlled by a small number of objects.

Let us record a stability result for later (**1.36**).

Proposition 1.15. Let $\mathscr{C}$ be an accessible ∞-category and let K be a small simplicial set. Then, $\mathrm{Fun}(K, \mathscr{C})$ is accessible.

Proof. See [Lur09, Proposition 5.4.4.3]. □

Here is the crucial definition of presentable ∞-categories!

Definition 1.16. An ∞-category is **presentable** if it is accessible and admits small colimits.

Proposition 1.17 (Simpson)**.** An ∞-category $\mathscr{C}$ is presentable if and only if there exists a small ∞-category $\mathscr{D}$ and a functor $\mathbf{PSh}(\mathscr{D}) \to \mathscr{C}$ with a fully faithful accessible right adjoint.

Proof. See [Lur09, Theorem 5.5.1.1]. □

There are many more equivalent characterizations of accessible resp. presentable categories. We will not recount them here but will refer to [Lur09, Proposition 5.4.2.2, Proposition 5.5.1.1]. Simpson's characterization is already close to the definition of an ∞-topos.

Example 1.18. Here are three constructions of presentable categories.

(i) Presheaf ∞-categories are presentable [Lur09, Remark 5.5.3.7]. In particular, $\mathbf{An} = \mathbf{PSh}(*)$ is presentable.

(ii) Let $\mathscr{C}$ be a presentable ∞-category, then its stabilization $\mathbf{Sp}(\mathscr{C})$ is presentable [Lur17, Proposition 1.4.4.4]. In particular, the ∞-category of spectra $\mathbf{Sp} = \mathbf{Sp}(\mathbf{An})$ is presentable.

(iii) Let A be a partially ordered set considered as a category. Then, its nerve NA is presentable if and only if every subset of A has a supremum [Lur09, Remark 5.5.2.5].

Presentable ∞-categories have pleasant categorical properties among which are versions of adjoint functor theorems.

Proposition 1.19. Presentable ∞-categories are bicomplete.

Proof. See [Lur09, Corollary 5.5.2.4]. □

Proposition 1.20. Let $\mathscr{C}$ be a presentable ∞-category and consider a functor $F : \mathscr{C}^{\mathrm{op}} \to \mathbf{An}$. Then, the following are equivalent:

(i) The functor F is representable.

(ii) The functor F preserves small limits.

Proof. See [Lur09, Proposition 5.5.2.2]. □

Equivalently [Lur09, Proposition 5.5.2.7], we can also demand

(iii) The functor F preserves small limits and is accessible.

So accessibility comes for free in this context.

Proposition 1.21 (Adjoint Functor Theorem)**.** Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between presentable ∞-categories.

(i) The functor F has a right adjoint if and only if it preserves small colimits.

(ii) The functor F has a left adjoint if and only if it is accessible and preserves small limits.

Proof. See [Lur09, Corollary 5.5.2.9]. □

For (i) it suffices that $\mathscr{D}$ is only (essentially) locally small [Lur09, Remark 5.5.2.10].

This is a major result about presentable ∞-categories and often allows us to obtain functors for free! Especially in the ∞-categorical setting where constructing functors is everything but easy, this is of immense substance.

Definition 1.22. We write $\mathbf{Pr}^L \subseteq \widehat{\mathbf{Cat}}_\infty$ for the subcategory of presentable ∞-categories with left adjoint functors as maps.

The dual version $\mathbf{Pr}^R$ turns out to also be useful. There is an equivalence $\mathbf{Pr}^L \simeq (\mathbf{Pr}^R)^{\mathrm{op}}$ in $\widehat{\mathbf{Cat}}_\infty$ [Lur09, Corollary 5.5.3.4]. The ∞-category $\mathbf{Pr}^L$ admits a closed symmetric monoidal structure [Lur09, Remark 5.5.3.9] which is one reason that makes $\mathbf{Pr}^L$ indispensable to modern pure mathematics. We will however not require $\mathbf{Pr}^L$ in this text.

We conclude this section with a quick recollection of the notion of truncation. It's a classical homotopy-theoretic notion that we may generalize to arbitrary ∞-categories via the Yoneda Formalism.

Definition 1.23. Let $\mathscr{C}$ be an ∞-category and let $k \geq -1$ be an integer.

(i) A Kan complex X is **k-truncated** if $\pi_\ell(X, x) \cong *$ for every basepoint $x \in X$ and every integer $\ell > k$.

(ii) An object $C_0 \in \mathscr{C}$ is **k-truncated** if for every $C' \in \mathscr{C}_0$ the Hom-anima $\mathrm{Hom}_{\mathscr{C}}(C', C)$ is k-truncated. We say that $C \in \mathscr{C}_0$ is **(-2)-truncated** if C is a terminal object in $\mathscr{C}$.

(iii) A morphism $f : C' \to C$ is **k-truncated** if $f \in (\mathscr{C}_{/C})_0$ is k-truncated.

(iv) A morphism $f : C' \to C$ is a **monomorphism** if f is (-1)-truncated.

These definitions are compatible: An object $X \in \mathbf{An}$ is k-truncated in the sense (i) if and only if it is k-truncated in the sense (ii) [Lur09, Remark 5.5.6.4].

There is an alternative inductive definition for finitely complete categories which we will not recall here [Lur09, Lemma 5.5.6.15].

Remark 1.24. Classically, a morphism $f : X \to Y$ between Kan complexes is k-truncated if its homotopy fibers are k-truncated. Then, a morphism $f : C' \to C$ in an ∞-category $\mathscr{C}$ is k-truncated if and only if

$$f_* : \mathrm{Hom}_{\mathscr{C}}(C'', C') \to \mathrm{Hom}_{\mathscr{C}}(C'', C)$$

is k-truncated for every $C'' \in \mathscr{C}$ (in the classical sense) [Lur09, Remark 5.5.6.10].

Again, the classical truncatedness on Kan complexes is compatible with the general truncatedness [Lur09, Remark 5.5.6.9]. In particular, a morphism $f : X \to Y$ in $\mathbf{An}$ is a monomorphism if its homotopy fibers are $*$ or $\emptyset$. So this is a sensible way of defining monomorphisms.

Let $k \geq -2$ be an integer. For an ∞-category $\mathscr{C}$ we write $\tau_{\leq k}\mathscr{C}$ for the full subcategory of $\mathscr{C}$ spanned by the k-truncated objects.

Proposition 1.25. Let $\mathscr{C}$ be a presentable ∞-category, let $k \geq 2$. Then, the inclusion $\tau_{\leq k}\mathscr{C} \hookrightarrow \mathscr{C}$ has an accessible left adjoint $\tau_{\leq k} : \mathscr{C} \to \tau_{\leq k}\mathscr{C}$.

Proof. See [Lur09, Proposition 5.5.6.18]. □

The notation makes sense: $\tau_{\leq k}\mathscr{C}$ is the essential image of $\tau_{\leq k}$ [Lur09, Remark 5.5.6.19].

The main use of n-truncatedness for us will be to define n-connectedness which will allow us to study the so-called hypercomplete ∞-topoi, i.e. topoi in which the Whitehead Theorem holds.

1.2 The ∞-category of ∞-topoi

Just like in classical topos theory, there are many different equivalent characterizations of ∞-topoi, some more intrinsic and some more extrinsic. For example, there are several formulations similar to the classical Giraud's axioms. We will only discuss one formulation that is almost an immediate generalization of a definition for 1-topoi and leave the remaining ones for the curious reader to explore on their own, e.g. in [Lur09, Chapter 6].

In the entire text, when we write about ∞-topoi, we will always mean Grothendieck-Lurie ∞-topoi generalizing the notion of Grothendieck (1-)topoi. We remark that there are concepts generalizing this notion, such as the elementary higher topoi from Nima Rasekh [Ras18].

Recalling that a left-exact functor is a finite-limit-preserving functor, we may define the main player in this section.

Definition 1.26. An **∞-topos** is an ∞-category $\mathscr{X}$ such that there exists a small ∞-category $\mathscr{C}$ and a left-exact functor $\mathbf{PSh}(\mathscr{C}) \to \mathscr{X}$ with a fully faithful accessible right adjoint.

Remark 1.27. Accessibility is not required for 1-topoi because it is automatic [Lur09, Remark 6.1.0.5].

Remark 1.28 (Warning)**.** If $\mathscr{G} \neq *$ is a 1-topos, then $N\mathscr{G}$ is not an ∞-topos. This will become clear when we discuss descent (**1.4**) and object classifiers (**1.5**) which are features only of higher topos theory but not of ordinary topos theory.

Let's introduce some terminology: A functor $F : \mathscr{C} \to \mathscr{D}$ between ∞-categories is a **localization** if it has a fully faithful right adjoint. In that regard, one also speaks of $\mathbf{PSh}(\mathscr{C}) \to \mathscr{X}$ as an accessible left-exact localization. Note that accessible refers not to the localization functor but rather to the right adjoint. After all, localizations as left adjoints automatically preserve all colimits anyway.

The only thing missing towards an ∞-topos from presentability is the left-exactness.

Lemma 1.29. Every ∞-topos is presentable.

Proof. This is immediate by Simpson's Theorem (1.17). □

Example 1.30. Here are some ∞-topoi that we can already construct without further machinery.

(i) Any presheaf ∞-category $\mathbf{PSh}(\mathscr{C})$ of an ∞-category $\mathscr{C}$ is an ∞-topos.

(ii) In particular, $\mathbf{An} \simeq \mathbf{PSh}(*)$ is an ∞-topos.

(iii) A presheaf $F : N(\mathbf{Open}_X)^{\mathrm{op}} \to \mathbf{An}$ is a **sheaf** if for every open $U \subseteq X$ and every open cover $\{U_i\}_{i \in I}$ of $U = \bigcup_{i \in I} U_i$ the evident map

$$F(U) \to \lim_{\emptyset \neq J \subseteq I} F\left(\bigcap_{j \in J} U_j\right)$$

is an equivalence. The full subcategory **Sh**(X) of $\mathbf{PSh}(X)$ spanned by the sheaves is an ∞-topos [Rez19, p. 6]. This is a special case of the techniques set up in the next section where we discuss machinery to construct ∞-sheaf categories via Grothendieck topologies (1.3).

As honest category theorists we still ought to make the collection of ∞-topoi into an ∞-category. So we need to define morphisms between ∞-topoi.

Definition 1.31. Let $\mathscr{X}$ and $\mathscr{Y}$ be ∞-topoi. A **geometric morphism** from $\mathscr{X}$ to $\mathscr{Y}$ is a functor $f_* : \mathscr{X} \to \mathscr{Y}$ which admits a left-exact left adjoint $f^* : \mathscr{Y} \to \mathscr{X}$.

Remark 1.32. Sometimes, one also refers to $f^* : \mathscr{Y} \to \mathscr{X}$ for the geometric morphism from $\mathscr{X} \to \mathscr{Y}$. This is not ambiguous because f_* and f^* as adjoint functors determine each other up to equivalence.

Example 1.33. The inclusion functor $\mathscr{X} \hookrightarrow \mathbf{PSh}(\mathscr{C})$ of an ∞-topos $\mathscr{X}$ in a presheaf category is a geometric morphism by definition.

Definition 1.34. We define subcategories **LTop**, **RTop** $\subseteq \widehat{\mathbf{Cat}}_\infty$ as follows:

- The objects of **LTop** and **RTop** are the ∞-topoi.
- A functor $f^* : \mathscr{Y} \to \mathscr{X}$ belongs to **LTop** if and only if f^* is a left-exact left adjoint.
- A functor $f_* : \mathscr{X} \to \mathscr{Y}$ belongs to **RTop** if and only if f_* has a left-exact left adjoint.

In other words, a functor belongs to **LTop** or **RTop** if it is a geometric morphism.

Proposition 1.35. There is an equivalence $\mathbf{LTop} \simeq \mathbf{RTop}^{\mathrm{op}}$ in $\widehat{\mathbf{Cat}}_\infty$.

Proof. See [Lur09, Corollary 6.3.1.8]. □

We conclude this subsection with two stability results for ∞-topoi.

Proposition 1.36. Let $\mathscr{C}$ be a small ∞-category and $\mathscr{Y}$ be an ∞-topos. Then, $\operatorname{Fun}(\mathscr{C},\mathscr{Y})$ is an ∞-topos.

Proof. Consider an accessible left-exact localization

$$\begin{tikzcd}\mathbf{PSh}(\mathscr{D}) \arrow[r, shift left, "L"] \arrow[r, phantom, "\perp"] & \mathscr{Y} \arrow[l, shift left, hook, "R"]\end{tikzcd}$$

We wish to show that the adjunction

$$\begin{tikzcd}\operatorname{Fun}(\mathscr{C},\mathbf{PSh}(\mathscr{D})) \arrow[r, shift left, "L_*"] \arrow[r, phantom, "\perp"] & \operatorname{Fun}(\mathscr{C},\mathscr{Y}) \arrow[l, shift left, "R_*"]\end{tikzcd}$$

realizes a presentation of $\operatorname{Fun}(\mathscr{C},\mathscr{Y})$. Here,

$$\operatorname{Fun}(\mathscr{C},\mathbf{PSh}(\mathscr{D})) \simeq \operatorname{Fun}\left((\mathscr{C}^{\mathrm{op}})^{\mathrm{op}} \times \mathscr{D}^{\mathrm{op}}, \mathbf{An}\right) = \mathbf{PSh}(\mathscr{C}^{\mathrm{op}} \times \mathscr{D})$$

is indeed a presheaf category. We need to check the following:

- The functor R_* is fully faithful: See [GHN17, Lemma 5.2].
- The functor L_* is left-exact: This follows because limits are computed pointwise on functor categories.
- The functor L_* is accessible: This follows from **1.15**.

□

Proposition 1.37. Let $\mathscr{X}$ be an ∞-topos and $X \in \mathscr{X}$. Then, $\mathscr{X}_{/X}$ is an ∞-topos.

Proof. See [Lur09, Proposition 6.3.5.1]. □

1.3 Grothendieck Topologies

The main method to construct ∞-topoi is by defining ∞-sheaves on ∞-categories equipped with further data, namely Grothendieck topologies. Unlike in the classical setting, not every accessible left-exact localization of presheaf categories is recovered by Grothendieck topologies. In particular, it remains an open question whether every ∞-topos can be constructed as a sheaf category.

We start by setting up the notion of Grothendieck topologies which – up to minor technicalities such as a rigorous definition of pullback sieves – is the same as in the classical theory.

Definition 1.38. Let $\mathscr{C}$ be an ∞-category.

(i) A **sieve** on $\mathscr{C}$ is a full subcategory $\mathscr{C}^{(0)} \hookrightarrow \mathscr{C}$ having the property that if $f : C \to C'$ is a morphism in $\mathscr{C}$ and $C' \in \mathscr{C}^{(0)}$, then also $C \in \mathscr{C}^{(0)}$.

(ii) If $C \in \mathscr{C}$, then a **sieve on** C is a sieve on the ∞-category $\mathscr{C}_{/C}$.

Lemma 1.39. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor of ∞-categories and $f : C' \to C$ be a map in $\mathscr{C}$.

(i) Let $\mathscr{D}^{(0)} \hookrightarrow \mathscr{D}$ be a sieve on $\mathscr{D}$. Then,

$$F^{-1}\mathscr{D}^{(0)} = \mathscr{D}^{(0)} \times_{\mathscr{D}} \mathscr{C}$$

is a sieve on $\mathscr{C}$.

(ii) Let $\mathscr{C}^{(0)}_{/C}$ be a sieve on C. Then, there exists a unique sieve $f^*\mathscr{C}^{(0)}_{/C}$ on C' such that $f^*\mathscr{C}^{(0)}_{/C}$ and $\mathscr{C}^{(0)}_{/C}$ determine the same sieve on $\mathscr{C}_{/f}$ by applying (i) on the diagram

$$\begin{tikzcd}
 & \mathscr{C}_{/f} \arrow[dl] \arrow[dr] & \\
\mathscr{C}_{/C} & & \mathscr{C}_{/C'}
\end{tikzcd}$$

to obtain sieves on $\mathscr{C}_{/f}$.

Proof. This can be checked by hand [Lur09, Definition 6.2.2.1]. The ∞-categorical detail for (ii) is that $\mathscr{C}_{/f} \to \mathscr{C}_{/C'}$ admits a section [Lan21, Lemma 1.3.46, Theorem 1.4.23]. □

More concretely, $f^*\mathscr{C}^{(0)}_{/C}$ is spanned by those maps into C' that become equivalent to an object in $\mathscr{C}^{(0)}_{/C}$ after postcomposition with f.

Definition 1.40. Let $\mathscr{C}$ be an ∞-category. A **Grothendieck topology** on $\mathscr{C}$ is the data of a collection of sieves on each object called **covering sieves** satisfying the following properties:

(i) For every $C \in \mathscr{C}$ the maximal sieve $\mathscr{C}_{/C}$ is a covering sieve on C.

(ii) For any map $f : C \to C'$ in $\mathscr{C}$ and a covering sieve $\mathscr{C}^{(0)}_{/C'}$ on C' the pullback $f^*\mathscr{C}^{(0)}_{/C'}$ is a covering sieve on C.

(iii) Let $C \in \mathscr{C}$ and $\mathscr{C}^{(0)}_{/C}, \mathscr{C}^{(1)}_{/C}$ be sieves on $\mathscr{C}$ where $\mathscr{C}^{(0)}_{/C}$ is a covering sieve. Suppose that for every map $f : C' \to C$ in $\mathscr{C}^{(0)}_{/C}$ the pullback $f^*\mathscr{C}^{(1)}_{/C}$ is a covering sieve on C'. Then, $\mathscr{C}^{(1)}_{/C}$ is a covering sieve on C.

An ∞-category equipped with a Grothendieck topology is an **∞-site**.

Let $\mathscr{C}$ be an ∞-category and $Z \in \mathscr{C}$. We write Sub(Z) for the class of equivalence classes of monomorphisms over Z in $\mathscr{C}$. In other words, these are the equivalence classes of the truncation $\tau_{\leq -1}(\mathscr{C}_{/Z})$. It can be shown that Sub(Z) is a set if $\mathscr{C}$ is presentable [Lur09, Proposition 6.2.1.3].

Classically, for a 1-site $\mathscr{C}$, a sieve on $C \in \mathscr{C}$ corresponds to a subobject of $\text{よ}(C)$ essentially by unravelling the definition. The same result is true for ∞-categories but it requires more work to recover this correspondence.

Proposition 1.41. Let $\mathscr{C}$ be a small site with $C \in \mathscr{C}$. For any $U \in \mathscr{C}$ there then exist ∞-categories $\mathscr{C}_{/C}(U)$ such that

$$\mathrm{Sub}(\text{よ}(C)) \to \{\text{sieves on } C\},\ (i : U \to \text{よ}(C)) \mapsto \mathscr{C}_{/C}(i)$$

is a well-defined bijection.

Proof. We begin with the construction of these $\mathscr{C}_{/C}(i)$.

Let $\mathscr{C}$ be a small ∞-category, $C \in \mathscr{C}$ and $i : U \to \text{よ}(C)$ be a monomorphism in $\mathbf{PSh}(\mathscr{C})$. We denote by $\mathscr{C}_{/C}(i)$ the full subcategory of $\mathscr{C}_{/C}$ spanned by those objects $f : C' \to C$ of $\mathscr{C}_{/C}$ such that there exists a commutative triangle

$$\begin{tikzcd}
\text{よ}(C') \arrow[rr, "\text{よ}(f)"] \arrow[dr] & & \text{よ}(C) \\
 & U \arrow[ur, "i"'] &
\end{tikzcd}$$

Then, $\mathscr{C}_{/C}(i)$ is a sieve on C. Indeed, let

$$\begin{tikzcd}
C' \arrow[rr, "f"] \arrow[dr] & & C' \arrow[dl] \\
& C &
\end{tikzcd}$$

be a map in $\mathscr{C}_{/C}$ such that $(C'' \to C) \in \mathscr{C}_{/C}(i)$. In other words, there exists a commutative triangle

$$\begin{tikzcd}
\text{よ}(C'') \arrow[rr] \arrow[dr] & & \text{よ}(C) \\
& U \arrow[ur] &
\end{tikzcd}$$

and by precomposing this triangle with $\text{よ}(f)$ we obtain a commutative diagram

$$\begin{tikzcd}
\text{よ}(C') \arrow[r] \arrow[drr, dashed, bend right] & \text{よ}(C'') \arrow[rr] \arrow[dr] & & \text{よ}(C) \\
& & U \arrow[ur] &
\end{tikzcd}$$

which implies $(C' \to C) \in \mathscr{C}_{/C}(i)$. So we have observed the sieve condition for $\mathscr{C}_{/C}(i)$. Moreover, if $U \to \text{よ}(C)$ and $U' \to \text{よ}(C)$ are equivalent subobjects of $\text{よ}(C)$, then we obtain

$$\mathscr{C}_{/C}(U \to \text{よ}(C)) = \mathscr{C}_{/C}(U' \to \text{よ}(C)),$$

so our map is well-defined.

That the map is a bijection is achieved by reducing to a slice-free version and we refer to [Lur09, Proposition 6.2.2.5]. ☐

Our notation is slightly more precise than Lurie's notation: He writes $\mathscr{C}_{/C}(U)$ instead of $\mathscr{C}_{/C}(i)$.

Definition 1.42. Let $\mathscr{C}$ be a small ∞-category with a Grothendieck topology and let

$$S = \{i : U \hookrightarrow \text{よ}(C) \text{ monic} : i \text{ corresponds to a covering sieve via } \mathbf{1.41}\}.$$

An object $P \in \mathbf{PSh}(\mathscr{C})$ is an **∞-sheaf** if the functor $\mathrm{Hom}_{\mathscr{C}}(-, P) : \mathbf{PSh}(\mathscr{C})^{\mathrm{op}} \to \mathbf{An}$ induces an equivalence $\mathrm{Hom}_{\mathscr{C}}(i, P)$ for all maps i in S. The full subcategory of $\mathbf{PSh}(\mathscr{C})$ spanned by the ∞-sheaves is denoted $\mathbf{Sh}(\mathscr{C})$.

Let $\mathscr{D}$ be an ∞-category. We write $\mathbf{Sh}_{\mathscr{D}}(\mathscr{C})$ for $\mathscr{D}$-valued sheaves which is the same definition for $\mathscr{D}$-valued functors. With this terminology we have $\mathbf{Sh}(\mathscr{C}) = \mathbf{Sh}_{\mathbf{An}}(\mathscr{C})$.

More succinctly, a presheaf $P \in \mathbf{PSh}(\mathscr{C})$ is a sheaf if it is S-local. We could suggestively say that P is a sheaf if it is local with respect to covering sieves.

If $i : U \hookrightarrow \text{よ}(C)$ is a covering sieve, then one also says that a presheaf $P \in \mathbf{PSh}(\mathscr{C})$ *satisfies descent* with respect to i. We want to stress that this is not the same notion as descent in ∞-topoi as will be introduced in section 1.4.

Classically, a sheaf can be described as a presheaf satisfying an equalizer condition. This generalizes to higher categories via Čech nerves which we spell out in the following. Since we could not find any complete proof in the literature, we give one here.

Proposition 1.43. Let $\mathscr{C}$ be a small site. Let $P \in \mathbf{PSh}(\mathscr{C})$. The following are equivalent:

(i) The presheaf P is a sheaf.

(ii) For every $C \in \mathscr{C}$ and every family of maps $\{U_i \to C\}_i$ generating a covering sieve corresponding to $\eta : U \to \text{よ}(C)$ the map

$$P(C) \to \lim \left(\prod_i \mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}(\text{よ}(U_i), P) \rightrightarrows \prod_{i,j} \mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}(\text{よ}(U_i) \times_{\text{よ}(C)} \text{よ}(U_j), P) \substack{\to \\ \to \\ \to} \cdots \right)$$

induced by the Čech nerve of $\coprod_i \text{よ}(U_i) \to \text{よ}(C)$ is an equivalence.

Proof. Let $\eta : U \to \text{よ}(C)$ be a subobject corresponding to a covering sieve generated by a family of maps $\{f_i : U_i \to C\}_i$. These maps induce another morphism $f : \coprod_i \text{よ}(U_i) \to \text{よ}(C)$. We consider its Čech nerve $\check{C}(f) : N(\Delta_+)^{\mathrm{op}} \to \mathbf{PSh}(\mathscr{C})$ with underlying simplicial object $W_\bullet = \check{C}(f)|_{N(\Delta)^{\mathrm{op}}} : N(\Delta)^{\mathrm{op}} \to \mathbf{PSh}(\mathscr{C})$. We prove the more precise statement that P is local with respect to η if and only if the induced map

$$P(C) \to \lim \mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}(W_\bullet, P)$$

is an equivalence.

The (-1)-truncation $\tau_{\leq -1} f$ in $\mathbf{PSh}(\mathscr{C})_{/\text{よ}(C)}$ is equivalent to η [Lur09, Lemma 6.2.3.18]. On the other hand, (-1)-truncations are computed by colimits of Čech nerves [Lur09, Proposition 6.2.3.4], so $\tau_{\leq -1} f \simeq \operatorname{colim} U_\bullet$. Thus, P is local with respect to η if and only if

$$\begin{aligned} \mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}(\text{よ}(C), P) &\simeq \mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}(U, P) \\ &\simeq \mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}(\operatorname{colim} W_\bullet, P) \\ &\simeq \lim \mathrm{Hom}_{\mathbf{PSh}(\mathscr{C})}(W_\bullet, P) \end{aligned}$$

as desired. ☐

If $\mathscr{C}$ has pullbacks, then this simplifies to

$$P(C) \simeq \lim \left(\prod_i P(U_i) \rightrightarrows \prod_{i,j} P(U_i \times_C U_j) \substack{\to \\ \to \\ \to} \cdots \right)$$

by the Yoneda Lemma. This is precisely the higher version of the equalizer condition.

We now give a generating notion of Grothendieck topologies. It will be particularly useful to us when we discuss fractured structures on ∞-topoi. We follow Lurie's *Spectral Algebraic Geometry* [Lur18, Section 20.6.1].

Definition 1.44. Let $\mathscr{C}$ be an ∞-site with Grothendieck topology τ and let $C \in \mathscr{C}$. A collection of morphisms $\{f_\alpha : U_\alpha \to C\}_\alpha$ in $\mathscr{C}$ is a **τ-covering** if it generates a covering sieve on C.

More precisely, this means that a collection of morphisms $\{f_\alpha : U_\alpha \to C\}_\alpha$ is a τ-covering if the full subcategory $\mathscr{C}^{(0)}_{/C} \hookrightarrow \mathscr{C}_{/C}$ spanned by those morphisms which factor through some f_α is a τ-covering sieve.

Example 1.45. This allows us to write out some examples of sheaf categories more succinctly. We tacitly only take small enough objects to avoid set-theory problems.

(i) Sheaves on spaces: Let X be a topological space and consider the poset category $\mathbf{Open}(X)$. A covering sieve on an open subset $U \in \mathbf{Open}(X)$ is generated by a collection of open subsets $\{U_i \hookrightarrow U\}_{i \in I}$ which jointly surject onto U as sets. Equipped with this Grothendieck topology we recover $\mathbf{Sh}(X) \simeq \mathbf{Sh}(\mathbf{Open}(X))$.

(ii) Sheaves in algebraic geometry: Let S be a scheme and consider the 1-category of S-schemes $\mathscr{C} = \mathbf{Sch}_{/S}$. We present the following classical Grothendieck topologies in an increasingly fine order. Let $X \in \mathbf{Sch}_{/S}$. A family of maps $\{f_i : X_i \to X\}_{i \in I}$ generates a covering sieve in the

- *Zariski topology* if the f_i are jointly surjective open embeddings,
- *étale topology* if the f_i are jointly surjective étale maps,
- *fppf topology* if the f_i are jointly surjective, flat, and locally of finite presentation,
- *fpqc topology* if there exists a refinement $\{g_j : Y_j \to X\}_{j \in J}$ of $\{f_i\}_{i \in I}$ such that the map $\coprod_{j \in J} Y_j \to Y$ is faithfully flat and quasicompact.

These yield the Zariski sheaves, the étale sheaves, the fppf sheaves, and the fpqc sheaves. Studying higher versions of these is part of spectral algebraic geometry [Lur18].

(iii) Motivic homotopy theory: Let S be a scheme and consider the 1-category of smooth S-schemes $\mathscr{C} = \mathbf{Sch}^{\text{smooth}}_{/S}$. Let $X \in \mathbf{Sch}^{\text{smooth}}_{/S}$. A family of maps $\{f_i : X_i \to X\}_{i \in I}$ generates a covering sieve if $\coprod_{i \in I} X_i(k) \to X(k)$ is surjective for every field k. This is called the *Nisnevich topology* and sheaves on the induced site are *Nisnevich sheaves*.

A presheaf $F \in \mathbf{PSh}(\mathbf{Sch}^{\text{smooth}}_{/S})$ is called $\mathbb{A}^1$-*invariant* if $\text{pr}_X^* : F(X) \to F(X \times \mathbb{A}^1)$ is an equivalence for every $X \in \mathbf{Sch}^{\text{smooth}}_{/S}$.

A *motivic anima* on S is an $\mathbb{A}^1$-invariant Nisnevich sheaf.

These objects are studied in motivic homotopy theory which is another approach in which algebraic geometry is married with homotopy theory [MV99].

(iv) Condensed mathematics: We endow the category of compact Hausdorff spaces **CHaus** with a Grothendieck topology given by the finitely jointly surjective families of maps. Then,

$$\mathbf{Cond}(\mathbf{Set}) = \mathbf{Sh}_{\mathbf{Set}}(\mathbf{CHaus})$$

is the category of *condensed sets*.

We will give a more thorough introduction to condensed mathematics in section 3 where we, in particular, use the ∞-categorical language (3.12).

Here are some results about pulling back Grothendieck topologies that we will need to pass between sheaf categories.

Proposition 1.46. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between ∞-categories and let τ be a Grothendieck topology on $\mathscr{D}$. Assume that $\mathscr{C}$ admits pullbacks and that F preserves pullbacks. Then, there is a Grothendieck topology $F^*\tau$ on $\mathscr{C}$ defined as follows: For any $C \in \mathscr{C}$ a sieve $\mathscr{C}^{(0)}_{/C} \hookrightarrow \mathscr{C}_{/C}$ is a covering sieve if and only if the collection $\{F(U) \to F(C)\}_{U \in \mathscr{C}^{(0)}_{/C}}$ is a τ-covering of $F(C)$.

Proof. See [Lur18, Proposition 20.6.1.1]. □

Proposition 1.47. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between ∞-categories and let τ be a Grothendieck topology on $\mathscr{D}$. Assume that $\mathscr{C}$ admits pullbacks and that f preserves pullbacks. Then, the precomposition functor $f^* : \mathbf{PSh}(\mathscr{D}) \to \mathbf{PSh}(\mathscr{C})$ induces a functor $f^* : \mathbf{Sh}_\tau(\mathscr{D}) \to \mathbf{Sh}_{f^*\tau}(\mathscr{C})$.

Proof. See [Lur18, Proposition 20.6.1.3]. □

We conclude the section by analyzing the localizations appearing through Grothendieck topologies. Even though this construction of ∞-sheaves does not yield all accessible left-exact localizations of presheaf categories, as we will see, it does yield a large class of examples which can be nicely described. We want to give the characterization of these objects and start by defining a stability condition.

Definition 1.48. Let $\mathscr{C}$ be a cocomplete ∞-category and S be a collection of morphisms in $\mathscr{C}$. Then, S is **strongly saturated** if it satisfies the following properties.

(i) The collection S is stable under pushouts.

(ii) The full subcategory of $\mathrm{Fun}(\Delta^1, \mathscr{C})$ spanned by S is stable under small colimits.

(iii) For all 2-simplices

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "g"'] & \sigma & Y \arrow[dl, "h"] \\ & Z & \end{tikzcd}$$

in $\mathscr{C}$ the 2-out-of-3 property holds: If any two of f, g, h belong to S, then so does the third.

Remark 1.49. Localizations play an enormous role in higher topos theory, as our very definition of an ∞-topos involves the notion of a localization. In that regard, it is important to understand this concept. That's the purpose of strongly saturated classes of morphisms: They give a full characterization of accessible localizations in presentable ∞-categories [Lur09, Proposition 5.5.4.15].

Arbitrary intersections of strongly saturated classes of morphisms are strongly saturated again. So any collection of morphisms S' in $\mathscr{C}$ is contained in a smallest strongly saturated class of morphisms $\overline{S'}$ containing S', called the strongly saturated class of morphisms **generated** by S'.

Definition 1.50. Let $\mathscr{C}$ be a presentable ∞-category and let S be a strongly saturated collection of morphisms of $\mathscr{C}$. Moreover, let $\mathscr{D}$ be another ∞-category.

(i) Then, S is **topological** if it satisfies the following properties:

  (a) There exists a subset $S' \subseteq S$ consisting of monomorphisms such that $\overline{S'} = S$

  (b) The class S is stable under pullback.

(ii) A localization $L : \mathscr{C} \to \mathscr{D}$ is **topological** if

$$\{f \in \mathscr{X}_1 : Lf \text{ is an equivalence}\}$$

is topological.

Proposition 1.51. Let $\mathscr{C}$ be a presentable ∞-category. Then, any topological localization $\mathscr{C} \to \mathscr{D}$ is accessible and left-exact.

Proof. See [Lur09, Corollary 6.2.1.6]. ☐

Corollary 1.52. Let $\mathscr{C}$ and $\mathscr{X}$ be ∞-categories such that there exists a topological localization $\mathbf{PSh}(\mathscr{C}) \to \mathscr{X}$. Then, $\mathscr{X}$ is an ∞-topos.

Proof. This follows from 1.51 because presheaf categories are presentable (1.18). ☐

Theorem 1.53. Let $\mathscr{C}$ be a small ∞-category. Then, there is a bijective correspondence between Grothendieck topologies on $\mathscr{C}$ and (equivalence classes of) topological localizations of $\mathbf{PSh}(\mathscr{C})$.

Proof. See [Lur09, Proposition 6.2.2.17]. ☐

So that is what's topological about topological localizations!

Note that topological localizations not only remember the resulting ∞-topos but actually they remember the data of the Grothendieck topology. In other words, two Grothendieck topologies generating the same ∞-sheaf categories must coincide. This is contained in Lurie's proof and is more precisely [Lur09, Proposition 6.2.2.16].

Grothendieck topologies only correspond to topological localizations and not all accessible left-exact localizations of presheaf categories which are involved in defining ∞-topoi. Let us speak out a warning: Naively, one might now expect that there exists ∞-topoi which cannot be constructed as an ∞-sheaf category arising from an ∞-site. However, this is not at all clear and in fact, to our knowledge, it remains an open problem up to this date, whether every ∞-topos is a sheaf category.

On the other hand, it is possible to reconstruct every ∞-topos after one more localization, namely after a so-called *cotopological localization* [Lur09, Proposition 6.5.2.19].

1.4 Descent*

Remembering all higher coherences via ∞-category theory simplifies parts of the theory. This is clearly visible in topos theory where ∞-topoi admit several advantages that only appear once passing to the ∞-world. Out of these, we will discuss Charles Rezk's notion of descent as well as object classifiers, both of which are not available to us in 1-topos theory in this form.

We begin with descent theory and will follow [Lur09, Rez19, Rez10]. We start by giving several constructions involving slice categories which are relevant in the setting of descent.

Let $\mathscr{C}$ be an ∞-category with pullbacks. Consider any map $f : C \to C'$ in $\mathscr{C}$. Then, pullback along f and postcomposition by f induces an adjunction pair [Lur09, Section 6.1.1]

$$\begin{tikzcd} \mathscr{C}_{/C} \arrow[r, shift left, "f_!"] \arrow[r, phantom, "\perp"] & \mathscr{C}_{/C'} . \arrow[l, shift left, "f^*"] \end{tikzcd}$$

More precisely, the target fibration $\mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, \mathscr{C}) \to \mathscr{C}$ is a biCartesian fibration because $\mathscr{C}$ has all pullbacks [Lur09, Lemma 6.1.1.1]. Therefore, pulling back along the functor $\Delta^1 \to \mathscr{C}$ represented by f yields another biCartesian fibration which thus induces an adjunction pair.

This allows us to define the setting we wish to work in.

Definition 1.54. Let $\mathscr{C}$ be a cocomplete ∞-category with pullbacks. Then, **colimits in $\mathscr{C}$ are universal** if for any map $f : C \to C'$ in $\mathscr{C}$ the functor

$$f^* : \mathscr{C}_{/C'} \to \mathscr{C}_{/C}$$

preserves colimits.

Symbolically, this amounts to equivalences

$$\operatorname*{colim}_{i \in I} (C_i \times_C C') \simeq \left(\operatorname*{colim}_{i \in I} C_i \right) \times_C C'$$

for all diagrams $\{C_i\}_{i \in I}$ in $\mathscr{C}$.

If $\mathscr{C}$ is an ∞-category with pullbacks, then the target fibration $\mathrm{ev}_1 : \mathrm{Fun}(\Delta^1, \mathscr{C}) \to \mathscr{C}$ is a Cartesian fibration [Lur09, Lemma 6.1.1.1] and thus corresponds to a functor

$$\mathscr{C}_{/-} : \mathscr{C}^{\mathrm{op}} \to \widehat{\mathbf{Cat}}_\infty$$

by Straightening-Unstraightening [Lan21, Theorem 3.3.10]. This is one of the fundamental functors in topos theory!

Definition 1.55. Let $\mathscr{C}$ be a cocomplete ∞-category with pullbacks. Then, $\mathscr{C}$ satisfies **descent** if the functor

$$(\mathscr{C}_{/-})^{\text{core}} = (-)^{\text{core}} \circ \mathscr{C}_{/-} : \mathscr{C}^{\text{op}} \to \widehat{\mathbf{An}}$$

preserves limits.

This is sometimes also called *(homotopical) patching* [Rez10, 6.5]. In particular, this descent is not to be mistaken with the wording that is occasionally used for the property a presheaf needs to become a sheaf.

We give an alternative description of descent to showcase a more diagrammatic definition. For the sake of language, we introduce the following definition.

Definition 1.56. Let $\mathscr{C}$ be an ∞-category, $K \in \mathbf{sSet}$ and $p, q : K \to \mathscr{C}$ be two diagrams. A natural transformation $\alpha : p \Rightarrow q$ is **Cartesian** if for every edge $e : x \to y$ in K the naturality diagram

$$\begin{tikzcd}
p(x) \arrow[r, "p(e)"] \arrow[d, "\alpha_x"'] & p(y) \arrow[d, "\alpha_y"] \\
q(x) \arrow[r, "q(e)"'] & q(y)
\end{tikzcd}$$

is a pullback in $\mathscr{C}$.

This allows us to succinctly state the following result.

Proposition 1.57. Let $\mathscr{C}$ be a presentable ∞-category. The following are equivalent.

(i) Colimits in $\mathscr{C}$ are universal and $\mathscr{C}$ has descent.

(ii) The functor $\mathscr{C}_{/-} : \mathscr{C}^{\text{op}} \to \widehat{\mathbf{Cat}}_\infty$ preserves limits.

(iii) Let K be a small simplicial set and $\overline{\alpha} : \overline{p} \Rightarrow \overline{q}$ be a natural transformation of diagrams $\overline{p}, \overline{q} : K^{\triangleright} \to \mathscr{X}$. Suppose that $\overline{q}$ is a colimit diagram and that $\overline{\alpha}|_K$ is Cartesian. Then,

$$\overline{p} \text{ is a colimit diagram} \iff \overline{\alpha} \text{ is Cartesian.}$$

Proof. See [Lur09, Lemma 6.1.3.7] and [Lur09, Theorem 6.1.3.9]. □

Remark 1.58. Sometimes, a presentable ∞-category $\mathscr{C}$ is also said to have descent if it satisfies one of the equivalent conditions above (1.57). This is justified by the experience that descent rarely appears in categories in which colimits are not universal. Nonetheless, we will not follow this convention because we want to distinguish descent from universality of colimits.

Example 1.59. Let's illustrate this by giving examples of some prominent colimits. Let $\mathscr{C}$ be a presentable ∞-category with descent in which colimits are universal. Furthermore, let $\emptyset_{\mathscr{C}}$ denote an intial object of $\mathscr{C}$.

(i) Initial objects: Let $f : C \to \emptyset_{\mathscr{C}}$ be a map in $\mathscr{C}$, then $C \simeq \emptyset_{\mathscr{C}}$. This is also known as a *strict initial object*.

(ii) Coproducts: Consider collections of objects $\{C_i\}_{i \in I}, \{C'_i\}_{i \in I}$ in $\mathscr{C}$ and a collection of morphisms $\{f_i : C_i \to C'_i\}_{i \in I}$ in $\mathscr{C}$. Then, the diagram

$$\begin{tikzcd}
C_i \arrow[r] \arrow[d, "f_i"'] & \coprod_{i \in I} C_i \arrow[d] \\
C'_i \arrow[r] & \coprod_{i \in I} C'_i
\end{tikzcd}$$

is a pullback square for every $i \in I$. This is also known as *extensive coproducts*.

Conversely, consider a collection of objects $\{C'_i\}_{i\in I}$ and collections of morphisms $\{C_i \to C\}_{i\in I}, \{f_i : C_i \to C'_i\}_{i\in I}$ in $\mathscr{C}$. Suppose that there exists a map $f : C \to \coprod_{i\in I} C'_i$ such that there are pullback squares

$$\begin{tikzcd}
C_i \arrow[r] \arrow[d] & C \arrow[d, "f"] \\
C'_i \arrow[r] & \coprod_{i\in I} C'_i
\end{tikzcd}$$

for every $i \in I$. Then, $\coprod_{i\in I} C_i \simeq C$. This is also known as *disjoint coproducts*.

(iii) Pushouts: Consider a commutative cube

$$\begin{tikzcd}
& C_0 \arrow[rr] \arrow[dl] \arrow[dd] & & C_1 \arrow[dl] \arrow[dd] \\
C_2 \arrow[rr, crossing over] \arrow[dd] & & C & \\
& C'_0 \arrow[rr] \arrow[dl] & & C'_1 \arrow[dl] \\
C'_2 \arrow[rr] & & C' \arrow[from=uu, crossing over] &
\end{tikzcd}$$

in $\mathscr{C}$ such that the back faces are pullbacks and the bottom face is a pushout. Then, the front faces are pullbacks if and only if the top face if a pushout.

Actually, strict initial objects (i) are already given in any presentable ∞-category in which colimits are universal [Lur09, Lemma 6.1.3.6]. Indeed, descent is a vacuous statement for initial objects. The fact that descent unites all these important properties shows the power of descent.

Note also these special cases together almost subsume universality of colimits + descent since all colimits can be constructed through coproducts and pushouts.

The main source of examples for categories with descent comes from ∞-topoi.

Theorem 1.60. Let $\mathscr{X}$ be an ∞-category. The following are equivalent.

(i) The ∞-category $\mathscr{X}$ is an ∞-topos.

(ii) The ∞-category $\mathscr{X}$ is presentable, has descent and colimits in $\mathscr{X}$ are universal.

Proof. See [Lur09, Proposition 6.1.3.10]. □

Remark 1.61. It is important to stress that the '∞' makes a difference here. Mere 1-topoi are not a source for examples. Let $\mathscr{G}$ be a 1-topos. By an explicit computation in **Set**, colimits in $\mathscr{G}$ are universal. So if we analogously define that $\mathscr{G}$ satisfies descent if the pseudofunctor $\mathscr{G}_{/\_} : \mathscr{G}^{\mathrm{op}} \to \widehat{\mathbf{Cat}}$ preserves limits, then it turns out that the only 1-topos satisfying descent is the trivial topos $*$. The main ingredient towards a proof of this statement is the failure of descent on **Set** [Rez10, Example 2.3].

Instead, 1-topoi do satisfy a weaker version called *weak descent*: Let $\mathscr{G}$ be a 1-topos. We would try to say that the pseudofunctor

$$\mathscr{G}_{/\_} : \mathscr{G}^{\mathrm{op}} \to \widehat{\mathbf{Cat}}$$

preserves limits. This leads to comparison morphisms between the limits.

Let $\{X_i\}_{i \in I}$ be a collection of objects in $\mathscr{G}$ and let $X_1' \to X_0' \leftarrow X_2'$ be a diagram in $\mathscr{G}$. We obtain functors

$$\begin{tikzcd} \mathscr{G}_{/\coprod_{i \in I} X_i} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \prod_{i \in I} \mathscr{G}_{/X_i} \arrow[l, shift left, "G"] \end{tikzcd} \qquad \begin{tikzcd} \mathscr{G}_{/X_1' \amalg_{X_0'} X_2'} \arrow[r, shift left, "F'"] \arrow[r, phantom, "\perp"] & \mathscr{G}_{/X_1'} \times_{\mathscr{G}_{/X_0'}} \mathscr{G}_{/X_2'} \arrow[l, shift left, "G'"] \end{tikzcd}$$

where F, F' are induced by taking pullbacks while G, G' are induced by taking the respective colimit. It turns out that F and G define mutual inverses up to equivalence but the same cannot be said about F' and G'. Indeed, $G'F' \simeq \mathrm{id}_{\mathscr{G}_{/X_1'} \amalg_{X_0'} \mathscr{G}_{/X_2'}}$ while the component maps of the natural map $F'G' \Rightarrow \mathrm{id}_{\mathscr{G}_{/X_1'} \times_{\mathscr{G}_{/X_0'}} \mathscr{G}_{/X_2'}}$ are only given by regular epimorphisms but not necessarily isomorphisms. This can be checked in **Set** by hand and be transported onto any 1-topos through abstract nonsense [Rez10, Proposition 2.2].

The failure of $F'G' \Rightarrow \mathrm{id}$ being an isomorphism is what is missing towards descent. On the other hand, passing to ∞-categories and thus to homotopy pushouts repairs this problem (1.60). So this is an instance of how remembering the higher data leads to a more natural theory!

1.5 Object Classifier*

An important property of an elementary 1-topos $\mathscr{G}$ is the existence of a subobject classifier which is a representing object of the subobject functor $\mathrm{Sub} : \mathscr{G} \to \mathbf{Set}$. Again, the passage to ∞-categories improves the situation considerably and allows for object classifiers which do not even make sense for 1-categories.

Just as for 1-topoi, ∞-topoi also have subobject classifiers.

Proposition 1.62. Let $\mathscr{X}$ be an ∞-topos. Then, there exists a **subobject classifier**, i.e. the functor

$$\mathrm{Sub} : \mathscr{X} \to \mathbf{Set}$$

is representable by an object $\Omega \in \mathscr{X}$.

Proof. See [Lur09, Proposition 6.1.6.3]. ☐

A subobject classifier in a category $\mathscr{C}$ is an object Ω such that elements in $\mathrm{Hom}_{\mathscr{C}}(X, \Omega)$ identify with monomorphisms $Y \to X$. Hence, an object classifier should be an object $\widetilde{\Omega}$ such that elements in $\mathrm{Hom}_{\mathscr{C}}(X, \widetilde{\Omega})$ identify with morphisms $Y \to X$.

However, when we are not restricting to monomorphisms, there may be non-trivial automorphisms of $Y \to X$ as objects in $\mathscr{C}_{/X}$. In other words, $(\mathscr{C}_{/X})^{\mathrm{core}}$ may have non-trivial morphisms. There is no way to take these into account since $\mathrm{Hom}_{\mathscr{C}}(X, \widetilde{\Omega})$ is merely a set and not a groupoid. On the other hand, ignoring these automorphisms would be unnatural and would dismiss the philosophy of ∞-categories to remember the data of isomorphisms. The fix is to pass to the ∞-world where we don't only have Hom-sets but actually ∞-groupoids of Hom's.

Here's a heuristic idea. We can try to do the following inductive procedure: Let $\mathscr{C}$ be an $(n, 1)$-category. An object classifier $\mathcal{U}$ should ideally let $\mathscr{C}(C, \mathcal{U})$ identify with some subcategory $\mathscr{D}$ of $(\mathscr{C}_{/C})^{\mathrm{core}}$ for all $C \in \mathscr{C}$. Since $\mathscr{C}(C, \mathcal{U})$ is an $(n-1, 1)$-category [Lur09, Proposition 2.3.4.18] we should demand the same thing for $\mathscr{D}$. One naive way of realizing this is to take the truncation $\tau_{\leq n-2}(\mathscr{C}_{/C})^{\mathrm{core}}$ which is an $(n-1, 1)$-category [Lur09, Proposition 2.3.4.18].

Example 1.63. Let $n = 1$. Then, the above construction should yield $\mathcal{U} \in \mathscr{C}$ with

$$\mathscr{C}(C, \mathcal{U}) \simeq \tau_{\leq -1}(\mathscr{C}_{/C})^{\mathrm{core}}$$

but the (-1)-truncated morphisms are by definition the monomorphisms. So the right side simplifies to Sub(C) and hence we recover the notion of a subobject classifier.

The need to apply $\tau_{\leq n-2}$ vanishes when we pass to the ∞-world since $\infty = \infty - 1$!

Let κ be a regular cardinal. We will introduce more size constraints.

Definition 1.64. Let $\mathscr{C}$ be a presentable ∞-category. A morphism $f : X \to Y$ in $\mathscr{C}$ is said to be **relatively κ-compact** if for all pullback squares

$$\begin{tikzcd} X' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

and κ-compact objects Y' the object X' is κ-compact.

We need to introduce some language that we will only use in this section. The notation is from [Lur09, Definition 6.1.6.1]. Let $\mathscr{C}$ be an ∞-category with pullbacks and let $S \subseteq \operatorname{Mor}\mathscr{C}$ be a collection of morphisms stable under pullback. We write $\mathscr{O}_{\mathscr{C}}^{(S)}$ for the subcategory of $\operatorname{Fun}(\Delta^1, \mathscr{C})$ spanned by S with morphisms the pullback squares.

Let $\operatorname{Mono}\mathscr{C} \subseteq \operatorname{Mor}\mathscr{C}$ denote the monomorphisms in $\mathscr{C}$. Note that to contain a subobject classifier is the same thing as $\mathscr{O}_{\mathscr{C}}^{(\operatorname{Mono}\mathscr{C})}$ admitting a terminal object. Let us mimic this for object classifiers.

Let $\mathbf{RelComp}_\kappa^{\mathscr{C}}$ ad hocly denote the collection of relatively κ-compact morphisms in $\mathscr{C}$.

Theorem 1.65 (Rezk)**.** Let $\mathscr{X}$ be a presentable ∞-category. The following are equivalent.

(i) The ∞-category $\mathscr{X}$ is an ∞-topos.

(ii) Colimits in $\mathscr{X}$ are universal and for sufficiently large regular cardinals κ there exists an **κ-compact object classifier**, in other words, $\mathscr{O}_{\mathscr{X}}^{(\mathbf{RelComp}_\kappa^{\mathscr{X}})}$ has a terminal object $\mathcal{U}^\kappa$.

Proof. See [Lur09, Theorem 6.1.6.8]. □

Morally, this says that for an ∞-topos $\mathscr{X}$ the functor

$$\mathscr{X}_{/-}^{\text{core}} : \mathscr{X}^{\text{op}} \to \widehat{\mathbf{An}}$$

is representable but the statement is not quite correct in this way without introducing size constraints. That's the purpose of the relatively κ-compact morphisms. This functor preserves limits by descent (**1.60**), so it's almost an immediate formal argument for presentable categories by **1.20**. The only culprit is again set theory.

This along with descent are testaments to how for an ∞-topos $\mathscr{X}$ the functor

$$\mathscr{X}_{/-} : \mathscr{X}^{\text{op}} \to \widehat{\mathbf{Cat}}_\infty$$

or variations thereof are amongst the most important objects in topos theory.

Remark 1.66. The size restriction arises from the hope of an object $\mathcal{U}$ with a natural equivalence

$$(\mathscr{X}_{/C})^{\text{core}} \simeq \operatorname{Hom}_{\mathscr{X}}(C, \mathcal{U}).$$

In a locally small ∞-category the right side is small while the left side is in general not small.

More concerely, let $\mathscr{X} = \mathbf{An}$ and $C = *$, then we get $\mathbf{An}^{\text{core}} \simeq \mathcal{U}$, so the anima $\mathcal{U}$ is as large as $\mathbf{An}$. This could heuristically be said to be the anima of all anima akin to the set of all sets from Russell's paradox. Hence, this is a higher version of Russell's paradox. The solution is to restrict the left side.

Descent and object classifiers are clear advantages of the theory of ∞-topoi towards that of 1-topoi but those are not the only upshots. Certainly, this is not the end of the story: Passing to ∞-topoi allows far more simplifications of the theory. For example, every groupoid object in ∞-topoi is effective [Lur09, Remark 6.4.3.8], and so we invite the reader to learn even more about higher topoi.

1.6 Hypercomplete Topoi

One essential result in homotopy theory is the Whitehead Theorem. As ∞-topoi are seen as the categories in which one can perform homotopy theory, one might naively expect that some version of the Whitehead Theorem should hold for ∞-topoi. It turns out that this is not the case but instead one sometimes passes to the universal construction, the so-called hypercompletion, where the Whitehead Theorem holds.

Definition 1.67. Let $\mathscr{C}$ be a presentable ∞-category and let $n \geq -1$ be an integer.

(i) A map f in $\mathscr{C}$ is **n-connected** if $\tau_{\leq n} f$ is an equivalence. An object $C \in \mathscr{C}$ is **n-connected** if $C \to *_{\mathscr{C}}$ is n-connected.

(ii) A map in $\mathscr{C}$ is **∞-connected** if it is k-connected for all integers $k \geq -1$. An object $C \in \mathscr{C}$ is ∞-connected if $C \to *_{\mathscr{C}}$ is ∞-connected.

We require presentability to have a truncation functor $\tau_{\leq n} : \mathscr{C} \to \tau_{\leq n}\mathscr{C}$ (see 1.25).

Remark 1.68. Sometimes, the terminology *n-connectivity* is used for $(n-1)$-connectedness [Lur09, Definition 6.5.1.10]. This e.g. has the terminological advantage that Eilenberg-MacLane objects are n-truncated and n-connective. On the other hand, one has to consider $(n-1)$-truncations $\tau_{\leq n-1}$ in the definition and we rather want to avoid this shift in indices, so we decided to go with the notion of connectedness.

It turns out that the Whitehead Theorem can fail, even in an ∞-topos: An ∞-connected map need not be an equivalence [Rez19, Lecture 5].

Definition 1.69. Let $\mathscr{C}$ be an ∞-category.

(i) An object $C \in \mathscr{C}$ is **hypercomplete** if $\mathrm{Hom}_{\mathscr{C}}(f, C)$ is an equivalence for all ∞-connected maps f in $\mathscr{C}$. We write $\mathscr{C}^{\mathrm{hyp}}$ for the subcategory spanned by hypercomplete objects, called the **hypercompletion** of $\mathscr{C}$.

(ii) The ∞-category $\mathscr{C}$ is **hypercomplete** if $\mathscr{C}^{\mathrm{hyp}} = \mathscr{C}$.

One also says that hypercomplete objects are local with respect to the ∞-connected morphisms. In particular, by the Yoneda Lemma an ∞-category is hypercomplete if and only if every ∞-connected map is an equivalence. The classical Whitehead Theorem is precisely the statement that **An** is hypercomplete.

Remark 1.70. Hypercompleteness is a purely ∞-categorical phenomenon. Let $n \geq 1$ be an integer and let $\mathscr{C}$ be an $(n,1)$-category, i.e. $\mathscr{C}$ is an ∞-category with $\mathscr{C} = \tau_{\leq n}\mathscr{C}$. Then, every ∞-connected morphism in $\mathscr{C}$ is automatically an equivalence by definition, so $\mathscr{C}$ is hypercomplete.

If $\mathscr{X}$ is an ∞-topos, then localizating at the ∞-connected morphism yields a left-exact accessible localization $\mathscr{X} \to \mathscr{X}^{\mathrm{hyp}}$. In particular, $\mathscr{X}^{\mathrm{hyp}}$ is also an ∞-topos [Lur09, Section 6.5.2]. As the name suggests, it is also hypercomplete [Lur09, Lemma 6.5.2.12].

Alternatively, one may also describe the hypercompletion of an ∞-topos as the localization at the so-called hypercoverings [Lur09, Corollary 6.5.3.13].

Remark 1.71. Even though hypercompleting an ∞-topos naively may seem like an improvement of the category, this is not at all true in general. Often, an ∞-topos $\mathscr{X}$ behaves better than $\mathscr{X}^{\mathrm{hyp}}$ such as having better finiteness properties, and important cohomology results like proper base change. See [Lur09, Chapter 6.5.4] for a thorough discussion on this matter.

We conclude this section by alluding to ways of restricting the category in which we are taking sheaves. More specifically, let $\mathscr{C}$ be an ∞-site, then we can form $\mathbf{Sh}(\mathscr{C})$. We wish for conditions on subcategories $\mathscr{D} \subseteq \mathscr{C}$ such that $\mathbf{Sh}(\mathscr{C}) \simeq \mathbf{Sh}(\mathscr{D})$. Classically, sheaves on topological spaces are already understood on a basis of the space (under suitable conditions). This motivates the following terminology.

Definition 1.72. Let $\mathscr{C}$ be an ∞-site. A full subcategory $\mathscr{D} \hookrightarrow \mathscr{C}$ is a **basis** for $\mathscr{C}$ if for every $C \in \mathscr{C}$ there exists a covering sieve $\{f_i : D_i \to C\}_{i \in I}$ with a small set I and $D_i \in \mathscr{D}$.

Note however, that this is not the same basis that is sometimes called Grothendieck pretopology which restricts the family of coverings to smaller families [MLM94, Definition III.2.2].

Lemma 1.73. Let $\mathscr{C}$ be an ∞-site and let $\mathscr{D} \hookrightarrow \mathscr{C}$ be a basis. Then, there is a unique Grothendieck topology on $\mathscr{D}$ such that: A collection of maps $\{D_i \to D\}_{i \in I}$ in $\mathscr{D}$ is a covering sieve if and only if it is a covering sieve in $\mathscr{C}$.

Proof. See [Lur19, Proposition B.6.3]. □

Proposition 1.74 (Comparison Lemma)**.** Let $\mathscr{C}$ be an ∞-site and let $\mathscr{D} \hookrightarrow \mathscr{C}$ be a basis.

(i) Presheaf restriction defines an equivalence of hypercomplete ∞-topoi

$$i^* : \mathbf{Sh}^{\mathrm{hyp}}(\mathscr{C}) \xrightarrow{\sim} \mathbf{Sh}^{\mathrm{hyp}}(\mathscr{D}),\ \mathcal{F} \mapsto \mathcal{F}|_{\mathscr{D}^{\mathrm{op}}}.$$

(ii) Suppose that $\mathscr{D}$ and $\mathscr{C}$ are n-categories for some n and have finite limits. Then, presheaf restriction defines an equivalence of ∞-topoi

$$i^* : \mathbf{Sh}(\mathscr{C}) \xrightarrow{\sim} \mathbf{Sh}(\mathscr{D}),\ \mathcal{F} \mapsto \mathcal{F}|_{\mathscr{D}^{\mathrm{op}}}.$$

Proof. See [BGH20, Proposition 3.12.11] and [Aok23, Corollary A.8]. □

For 1-topoi the comparison lemma works without any restrictions for bases [Lur19, Proposition B.6.4] but in the ∞-world there need to be restrictions. There are also other approaches which can be found in the literature – see e.g. [Hoy14, Lemma C.3] or [BGH20, Corollary 3.12.13, 3.12.14]. We will use the Comparison Lemma to interchange different categories related to condensed mathematics.

2 Cohesion and Fractured Structure

An ∞-topos behaves similarly to the ∞-category **An** and is hence a suitable setting to perform homotopy theory. For example, one can construct homotopy sheaves on an ∞-topos [Lur09, Chapter 6.5]. However, passing to homotopy types dismisses the topological information of a space which can be valuable.

This suggests the need to recover the relevant information through an axiomatization of the properties in question. We introduce the notion of cohesion and fractured structures precisely for this purpose.

2.1 Cohesion

We will begin with the more classical notion of a cohesive topos first developed by William Lawvere [Law07]. Urs Schreiber generalized it to the ∞-world in his *Differential cohomology in a cohesive infinity-topos* [Sch13]. One can think of the word cohesion as inspired by chemistry which describes how molecules stick together. In that sense, mathematical cohesion is supposed to describe how points 'cohere' or 'stick together'.

We will mostly follow Schreiber [Sch13] and begin by defining a relative version of cohesion.

Definition 2.1. Let $\mathscr{X}$ be an ∞-topos over an ∞-topos $\mathscr{Y}$ realized by a geometric morphism $f_* : \mathscr{X} \to \mathscr{Y}$.

(i) If f_* admits fully faithful adjoints

$$\begin{tikzcd} \mathscr{X} \arrow[r, "f_*" description] & \mathscr{Y}, \arrow[l, bend right, "f^*"'] \arrow[l, bend left, "f^!"] \end{tikzcd}$$

then $\mathscr{X}$ is called **local** over $\mathscr{Y}$.

(ii) If $\mathscr{X}$ is local over $\mathscr{Y}$ as in (i) and f^* admits a further left adjoint $f_!$ which preserves finite products, then $\mathscr{X}$ is called **cohesive** over $\mathscr{Y}$.

The notation for the four functors of cohesion is often chosen as

$$\begin{tikzcd} \mathscr{X} \arrow[r, shift left=3, "f_!"] \arrow[r, shift right=1, "f_*" description] & \mathscr{Y}. \arrow[l, shift right=1, "f^*" description] \arrow[l, shift left=3, "f^!"] \end{tikzcd}$$

Remark 2.2. For locality, it suffices to demand that one of the adjoints of f_* is fully faithful since f^* is fully faithful if and only if $f^!$ is fully faithful [MLM94, Lemma VII.4.1].

In particular, a geometric morphism f_* realizing a relative local ∞-topos corresponds to f_* being a localization: The functor f_* being a geometric morphism already includes the information of a left adjoint f^*, so what's missing is a fully faithful right adjoint $f^!$ and that's precisely the meaning of a localization.

If $\mathscr{X}$ is cohesive over $\mathscr{Y}$, then $\mathscr{Y}$ embeds into $\mathscr{X}$ in two ways, namely via f^* and $f^!$.

From our relative notion of cohesion we want to define an absolute notion which puts $\mathscr{Y} = \mathbf{An}$. For this, we work out a geometric morphism $\mathscr{X} \to \mathbf{An}$ which is fundamental in the theory of ∞-topoi.

Definition 2.3. Let $\mathscr{X}$ be an ∞-topos and let $*_{\mathscr{X}}$ be a terminal object in $\mathscr{X}$. Then, the **global sections functor** is given by

$$\Gamma = \mathrm{Hom}_{\mathscr{X}}(*_{\mathscr{X}}, -) : \mathscr{X} \to \mathbf{An}, \ \mathcal{F} \mapsto \mathrm{Hom}_{\mathscr{X}}(*_{\mathscr{X}}, \mathcal{F}).$$

Since Γ is limit-preserving and accessible [Lur09, Proposition 5.3.4.17], it admits a left adjoint $\mathrm{Disc} : \mathbf{An} \to \mathscr{X}$ by the Adjoint Functor Theorem (1.21). In fact, it can be described explicitly as

$$\mathrm{Disc} = *_{\mathscr{X}} \otimes - : \mathbf{An} \to \mathscr{X}$$

via the equivalence

$$\mathrm{Hom}_{\mathscr{X}}(*_{\mathscr{X}} \otimes K, X) \simeq \mathrm{Hom}_{\mathscr{X}}(*_{\mathscr{X}}, X)^K = \mathrm{Hom}_{\mathbf{An}}(K, \mathrm{Hom}_{\mathscr{X}}(*, X)) = \mathrm{Hom}_{\mathbf{An}}(K, \Gamma(X))$$

natural in $K \in \mathbf{An}$ and $X \in \mathscr{X}$. Explicitly,

$$\mathrm{Disc}(K) = *_{\mathscr{X}} \otimes K = \mathrm{colim}\left(K \to * \xrightarrow{*_{\mathscr{X}}} \mathscr{X}\right)$$

by [Lur09, Corollary 4.4.4.9]. For example, if K is discrete, then $*_{\mathscr{X}} \otimes K \simeq \coprod_K *_{\mathscr{X}}$.

Lemma 2.4. Let $\mathscr{X}$ be an ∞-topos, then $\Gamma : \mathscr{X} \to \mathbf{An}$ is a geometric morphism.

Proof. Since $\mathscr{X}$ is an ∞-topos, there exists an (accessible) left-exact localization $a : \mathbf{PSh}(\mathscr{C}) \to \mathscr{X}$ for some small ∞-category $\mathscr{C}$. Let $\Delta : \mathbf{An} \to \mathbf{PSh}(\mathscr{C})$ denote the constant diagram functor. The composition of the two adjunction pairs

$$\begin{tikzcd} \mathbf{An} \arrow[r, shift left=1.5, "\Delta"] \arrow[r, phantom, "\perp"] & \mathbf{PSh}(\mathscr{C}) \arrow[l, shift left=1.5, "\mathrm{Hom}(*_{\mathbf{PSh}(\mathscr{C})}, -)"] \arrow[r, shift left=1.5, "a"] \arrow[r, phantom, "\perp"] & \mathscr{X} \arrow[l, hook, shift left=1.5] \end{tikzcd}$$

compose to another adjunction pair. Therefore, up to equivalence, they must compose to the adjunction pair

$$\begin{tikzcd} \mathbf{An} \arrow[r, shift left=1.5, "\mathrm{Disc}"] \arrow[r, phantom, "\perp"] & \mathscr{X} \arrow[l, shift left=1.5, "\mathrm{Hom}_{\mathscr{X}}(*_{\mathscr{X}}, -)"] \end{tikzcd}$$

and so $\mathrm{Disc} \simeq a \circ \Delta$ is left-exact because Δ and a are. □

In fact, Γ is the unique geometric morphism $\mathscr{X} \to \mathbf{An}$ up to homotopy [Lur09, Proposition 6.3.4.1].

The upshot is that $\mathscr{X}$ is an ∞-topos over $\mathbf{An}$ via $\Gamma : \mathscr{X} \to \mathbf{An}$. So it makes sense to ask about locality and cohesion of $\mathscr{X}$ over $\mathbf{An}$ via Γ.

Definition 2.5. Let $\mathscr{X}$ be an ∞-topos.

(i) If $\Gamma : \mathscr{X} \to \mathbf{An}$ is local, then $\mathscr{X}$ is a **local ∞-topos**.

(ii) If $\Gamma : \mathscr{X} \to \mathbf{An}$ is cohesive, then $\mathscr{X}$ is a **cohesive ∞-topos**.

Since $\Gamma : \mathscr{X} \to \mathbf{An}$ is the unique geometric morphism $\mathscr{X} \to \mathbf{An}$, it is equivalent to spell out the definition for any geometric morphism $\mathscr{X} \to \mathbf{An}$.

Remark 2.6. If $\mathscr{X}$ is a cohesive ∞-topos, then the following notation is typically used:

$$\begin{tikzcd} \mathscr{X} \arrow[r, shift left=4.5, "\Pi"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, leftarrow, shift left=1.5, "\mathrm{Disc}" description] \arrow[r, phantom, "\perp"] \arrow[r, shift right=1.5, "\Gamma" description] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, leftarrow, shift right=4.5, "\mathrm{CoDisc}"'] & \mathbf{An}. \end{tikzcd}$$

The notation follows our geometric intuition:

- The functor Π is like the fundamental ∞-groupoid of a space.
- The functor Disc is like the discrete 'topology/cohesion'.
- The functor Γ is like the global sections functor.
- The functor CoDisc is like the codiscrete 'topology/cohesion'.

This is probably best demonstrated through an example on the 1-categorical level. An analogous definition yields cohesive 1-topoi (over **Set**). In fact, cohesion can be defined more generally than just for topoi. Indeed, the 1-category of locally path-connected topological spaces $\mathbf{Top}^{\text{locally path-connected}}$ is cohesive over **Set** realized by the adjunction quadruple

$$\begin{tikzcd} \mathbf{Top}^{\text{locally path-connected}} \arrow[r, shift left=4.5, "\pi_0"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, leftarrow, shift left=1.5, "\mathrm{Disc}" description] \arrow[r, phantom, "\perp"] \arrow[r, shift right=1.5, "U" description] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, leftarrow, shift right=4.5, "\mathrm{CoDisc}"'] & \mathbf{Set} \end{tikzcd}$$

which suggests our indicated geometric intuition:

- The functor π_0 takes the connected components of a space.
- The functor Disc takes the discrete topology on a set.
- The functor U is the forgetful functor.
- The functor CoDisc takes the codiscrete topology on a set.

We require local path-connectedness to obtain a well-defined functor π_0. The other three functors also exist for **Top**, so **Top** is local.

There is a similar cohesion on **sSet**, namely

$$\begin{tikzcd}
\mathbf{sSet} \arrow[r, shift left=9, "\pi_0"] \arrow[r, phantom, shift left=6, "\perp"] \arrow[r, leftarrow, shift left=3, "\mathrm{const}" description] \arrow[r, phantom, "\perp"] \arrow[r, shift right=3, "(-)_0" description] \arrow[r, phantom, shift right=6, "\perp"] \arrow[r, leftarrow, shift right=9, "E"'] & \mathbf{Set}.
\end{tikzcd}$$

This provides a topos-theoretic example that mirrors the example on **Top**.

That the connected components functor π_0 becomes the fundamental ∞-groupoid functor Π in the ∞-world has the slight disadvantage that it is more complicated but the major advantage that it encodes far more information!

Here is a result that helps in checking for cohesion.

Lemma 2.7. Let $\mathscr{X}$ be an ∞-topos with an adjunction triple $(\Pi \dashv \mathrm{Disc} \dashv \Gamma) : \mathscr{X} \to \mathbf{An}$. The following are equivalent.

(i) The functor Disc is fully faithful.

(ii) The functor Π preserves terminal objects.

Proof. Suppose first (i), i.e. that Disc is fully faithful. Equivalently, the counit $\Pi \circ \mathrm{Disc} \Rightarrow \mathrm{id}_{\mathscr{X}}$ is an equivalence. Thus,

$$\Pi(*_{\mathbf{An}}) \simeq \Pi\left(\mathrm{Disc}(*_{\mathscr{X}})\right) \simeq *_{\mathscr{X}}$$

where the first step uses that Disc as a right adjoint preserves terminal objects.

Conversely, suppose that Π preserves terminal objects. We wish to show that the counit map $\Pi \circ \mathrm{Disc} \Rightarrow \mathrm{id}_{\mathscr{X}}$ is an equivalence. We compute

$$\Pi(\mathrm{Disc}(S)) \simeq \Pi\left(\mathrm{Disc}(\operatorname*{colim}_S *_{\mathbf{An}})\right) \simeq \operatorname*{colim}_S \Pi\left(\mathrm{Disc}(*_{\mathbf{An}})\right) \simeq \operatorname*{colim}_S *_{\mathbf{An}} \simeq S$$

where the second equivalence uses that left adjoints preserve colimits and the third equivalence uses that Π and Disc preserve terminal objects by assumption (ii) and right adjointness of Disc. This equivalence composes to the counit map. ☐

Now we want to demonstrate that cohesion is a rich phenomenon that includes a myriad of examples. We begin by giving three recipes for constructing cohesive ∞-topoi.

Proposition 2.8. Let $\mathscr{C}$ be a small pointed ∞-category and let $\mathscr{Y}$ be an ∞-topos. Then, $\mathrm{Fun}(\mathscr{C}, \mathscr{Y})$ is an ∞-topos cohesive over $\mathscr{Y}$.

Proof. First, $\mathrm{Fun}(\mathscr{C}, \mathscr{Y})$ is indeed an ∞-topos because $\mathscr{Y}$ is an ∞-topos and $\mathscr{C}$ is small (1.36). The constant functor $\Delta : \mathscr{Y} \to \mathrm{Fun}(\mathscr{C}, \mathscr{Y})$ has left and right adjoints

$$\begin{tikzcd}
\mathrm{Fun}(\mathscr{C}, \mathscr{Y}) \arrow[r, shift left=6, "c"] \arrow[r, phantom, shift left=3, "\perp"] \arrow[r, leftarrow, "\Delta" description] \arrow[r, phantom, shift right=3, "\perp"] \arrow[r, shift right=6, "\ell"'] & \mathscr{Y}
\end{tikzcd}$$

exhibited as colimit and limit functors. Since $\mathscr{C}$ has a zero object 0, these functors are given by evaluation in 0. Therefore, c preserves limits and in particular products. Moreover, ℓ preserves colimits, so it admits a right adjoint by the Adjoint Functor Theorem (1.21). To finish, Δ is fully faithful because the counit $\Delta c \Rightarrow \mathrm{id}_{\mathscr{Y}}$ is an equivalence. □

Example 2.9. This single innocent result yields many examples from classical homotopy theory.

(i) Set $\mathscr{Y} = \mathbf{An}$. Then, $\mathbf{PSh}(\mathscr{C})$ is a cohesive ∞-topos.

(ii) Set $\mathscr{C} = N(\mathbf{FinSet}_*)$. Then we are in the setting of Γ-objects.

(iii) Set $\mathscr{C} = \mathbf{An}_*^{\mathrm{fin}}$. Then we are in the setting of Goodwillie calculus.

Proposition 2.10. Let $\mathscr{X}$ be a cohesive ∞-topos over an ∞-topos $\mathscr{Y}$ and let $\mathscr{D}$ be an ∞-category with initial object $\emptyset_{\mathscr{D}}$ and terminal object $*_{\mathscr{D}}$. Then, $\mathrm{Fun}(\mathscr{D}, \mathscr{X})$ is cohesive over $\mathscr{Y}$, exhibited by the following adjunction quadruple:

$$\begin{tikzcd}
\mathrm{Fun}(\mathscr{D}, \mathscr{X}) \arrow[r, shift left=6, "*_{\mathscr{D}}^*"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, leftarrow, shift left=2, "p^*" description] \arrow[r, phantom, shift left=0, "\perp"] \arrow[r, shift right=2, "\emptyset_{\mathscr{D}}^*" description] \arrow[r, phantom, shift right=4, "\perp"] \arrow[r, leftarrow, shift right=6, "(\emptyset_{\mathscr{D}})_*"'] & \mathscr{X} \arrow[r, shift left=6, "\Pi"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, leftarrow, shift left=2, "\mathrm{Disc}" description] \arrow[r, phantom, shift left=0, "\perp"] \arrow[r, shift right=2, "\Gamma" description] \arrow[r, phantom, shift right=4, "\perp"] \arrow[r, leftarrow, shift right=6, "\mathrm{CoDisc}"'] & \mathscr{Y}.
\end{tikzcd}$$

Proof. The main ingredient is the adjunction triple

$$\begin{tikzcd}
* \arrow[r, shift left=4, "\emptyset_{\mathscr{D}}"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, leftarrow, "p" description] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, shift right=4, "*_{\mathscr{D}}"'] & \mathscr{D}
\end{tikzcd}$$

There are adjunction triples

$$\begin{tikzcd}
\mathrm{Fun}(\mathscr{D}, \mathscr{X}) \arrow[r, leftarrow, shift left=4, "(\emptyset_{\mathscr{D}})_!"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, "(\emptyset_{\mathscr{D}})^*" description] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, leftarrow, shift right=4, "(\emptyset_{\mathscr{D}})_*"'] & \mathscr{X} & \mathscr{X} \arrow[r, leftarrow, shift left=4, "(*_{\mathscr{D}})^*"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, "p^*" description] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, leftarrow, shift right=4, "(\emptyset_{\mathscr{D}})^*"'] & \mathrm{Fun}(\mathscr{D}, \mathscr{X})
\end{tikzcd}$$

induced by Kan extension and by $\mathrm{Fun}(-, \mathscr{X})$.

These already give the four adjoints

$$\begin{tikzcd}
\mathrm{Fun}(\mathscr{D}, \mathscr{X}) \arrow[r, shift left=6, "*_{\mathscr{D}}^*"] \arrow[r, phantom, shift left=4, "\perp"] \arrow[r, leftarrow, shift left=2, "p^*" description] \arrow[r, phantom, shift left=0, "\perp"] \arrow[r, shift right=2, "\emptyset_{\mathscr{D}}^*" description] \arrow[r, phantom, shift right=4, "\perp"] \arrow[r, leftarrow, shift right=6, "(\emptyset_{\mathscr{D}})_*"'] & \mathscr{X}
\end{tikzcd}$$

from which we still need to verify the additional properties from cohesion.

As a right adjoint $*_{\mathscr{D}}^*$ preserves limits and particularly finite products, so the leftmost map in the quadruple preserves finite products. Moreover, the counit map $*_{\mathscr{D}}^* \circ p^* \Rightarrow \mathrm{id}_{\mathscr{X}}$ is an equivalence since $p \circ *_{\mathscr{D}} = \mathrm{id}_{*_{\mathscr{D}}}$. So p^* is fully faithful. □

Example 2.11. Let $\mathscr{X}$ be an ∞-topos. So in particular, $\mathscr{X}$ is cohesive over itself via $\mathrm{id}_{\mathscr{X}}$. The arrow category $\mathrm{Fun}(\Delta^1, \mathscr{X})$ is a cohesive ∞-topos (2.10) over $\mathscr{X}$ via the source fibration $\mathrm{ev}_0 : \mathrm{Fun}(\Delta^1, \mathscr{X}) \to \mathscr{X}$.

Proposition 2.12. Let $\mathscr{X}$ be an ∞-topos. The ∞-topos of simplicial objects

$$s\mathscr{X} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathscr{X})$$

is cohesive over $\mathscr{X}$.

Proof. By formal reasons, there is a quadruple adjunction

$$\begin{tikzcd}
s\mathscr{X} \arrow[r, shift left=4, "|-|"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, shift right=1, "(-)_0"'] & \mathscr{X} \arrow[l, shift right=1, "\mathrm{const}"'] \arrow[l, shift left=4, "E"]
\end{tikzcd}$$

consisting of geometric realization, constant simplicial objects, 0-simplices, and the bar construction. Indeed, we consider the three functors

$$\Delta^0 \xrightarrow{F} \Delta^{\mathrm{op}} \xrightarrow{G} \Delta^0$$

where F corresponds to $[0] \in \Delta^{\mathrm{op}}$ which by Kan extension give rise to adjunction triples

$$\begin{tikzcd}
\mathrm{Fun}(\Delta^0, \mathscr{X}) \arrow[r, "G^*" description] \arrow[r, phantom, shift left=1.5, "\perp"] \arrow[r, phantom, shift right=1.5, "\perp"] & \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathscr{X}) \arrow[l, shift right=3, "G_!"'] \arrow[l, shift left=3, "G_*"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathscr{X}) \arrow[r, "F^*" description] \arrow[r, phantom, shift left=1.5, "\perp"] \arrow[r, phantom, shift right=1.5, "\perp"] & \mathrm{Fun}(\Delta^0, \mathscr{X}) \arrow[l, shift right=3, "F_!"'] \arrow[l, shift left=3, "F_*"]
\end{tikzcd}$$

Since $GF = \mathrm{id}_{\Delta^0}$, we obtain $F^*G^* = \mathrm{id}_{\mathscr{X}}$ and thus $G^* \simeq F_!$. In particular, by explicit computation $F^* = (-)_0$ and so we obtain a quadruple adjunction from the above. The above argument also shows that $\mathrm{const} = F_!$ is fully faithful. An argument similar to [GZ67, Section III.3] shows that $|-| = G_!$ preserves finite products. □

Example 2.13. For instance, the ∞-category of simplicial anima $s\mathbf{An} = \mathrm{Fun}(\Delta^{\mathrm{op}}, \mathbf{An})$ forms a cohesive ∞-topos.

Example 2.14. Cohesion also appears in global homotopy theory: Let G be a compact Lie group, then there exists a quadruple adjunction

$$\begin{tikzcd}
(\mathbf{An}_{\mathbf{Glo}})_{/\mathbb{B}G} \arrow[r, shift left=4, "\Pi_G"] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] \arrow[r, shift right=1, "\Gamma_G"'] & G\text{-}\mathbf{An} \arrow[l, shift right=1, "\Delta_G"'] \arrow[l, shift left=4, "\nabla_G"]
\end{tikzcd}$$

exhibiting $(\mathbf{An}_{\mathbf{Glo}})_{/\mathbb{B}G}$ as a cohesive ∞-topos over G-**An** which we will not further elaborate on. See [Rez14, Chapter 5]. Note that Rezk writes **Top** where we write **An**.

Let us conclude this section with one result about cohesion that we will not prove because we have not discussed the homotopy dimension of an ∞-topos.

Proposition 2.15. A cohesive ∞-topos is hypercomplete.

Proof. See [Sch13, Proposition 3.4.3]. □

There are more precise statements about homotopy and cohomological dimensions of cohesive ∞-topoi as well as statements about its shape [Sch13, Proposition 3.4.3]. In fact, already local ∞-topoi are hypercomplete [Sch13, Proposition 3.2.2] but the virtue of cohesive ∞-topoi is that there are certain tools to construct them:

An ∞-site is an **∞-cohesive site** once one demands certain conditions on the site [Sch13, Definition 3.4.17]. They then immediately give rise to cohesive ∞-topoi [Sch13, Proposition 3.4.18]. This machinery allows us to construct many more cohesive structures. We will not go into details here but will give a similar procedure for fractured structures. These cohesive structures can still be further refined, e.g. into real-cohesion [Shu18] or differential cohesion [Sch13] to name just two examples.

We also recommend the reader to look into *Proper Orbifold Cohomology* by Sati-Schreiber [SS20] for more examples and applications of cohesion.

2.2 Fractured Structure

While we have just demonstrated a myriad of examples realizing cohesion, it will turn out that the main example in this thesis will not satisfy cohesion in a suitable way. Indeed, we will argue in Section 4 that cohesion on the ∞-topos of condensed anima $\mathbf{Cond}(\mathbf{An})$ is not the notion to consider (4.3, 4.7). For this purpose, we will adhere to another concept, namely that of fractured structures developed by Jacob Lurie [Lur18] and David Carchedi [Car20].

Originally, fractured ∞-topoi were developed to axiomatize a suitable notion of gluing schemes. Let X be a scheme and let $\mathbf{Sch}^{\mathrm{fp}}_{/X}$ be the site of X-schemes of finite presentation with the étale topology. Then, every object in $\mathbf{Sh}(\mathbf{Sch}^{\mathrm{fp}}_{/X})$ can be written as a colimit of representables. If $\{Y_\alpha\}_\alpha$ is a diagram in $\mathbf{Sch}_{/X}$, then its colimit (after applying よ) in $\mathbf{Sh}(\mathbf{Sch}^{\mathrm{fp}}_{/X})$ will typically not be representable by geometric objects. However, if all transition maps are étale, then the colimit can be represented by a higher Deligne-Mumford stack which is locally of finite presentation over X. Objects obtained in this way lie in a subcategory

$$\mathbf{Sh}\left(\mathbf{Sch}^{\mathrm{fp}}_{/X}\right)^{\mathrm{corp}} \subseteq \mathbf{Sh}\left(\mathbf{Sch}^{\mathrm{fp}}_{/X}\right)$$

and one can show $\mathbf{Sh}(\mathbf{Sch}^{\mathrm{fp}}_{/X})^{\mathrm{corp}} \simeq \mathbf{Sh}(\mathbf{Sch}^{\text{ét, fp}}_{/X})$ where $\mathbf{Sch}^{\text{ét, fp}}_{/X}$ denotes the site of étale X-schemes of finite presentation with the étale topology. Axiomatizing certain nice properties of this subcategory leads to the notion of a fracture subcategory [Lur18, Section 20.1].

This viewpoint will be of no relevance to our goals. Instead, we will think of a fractured structure as a local cohesive structure (2.20).

Next to this thesis and the aforementioned works only few texts using fractured structures are known to us. Christopher Adrian Clough uses fractured structures to study differentiable stacks in his dissertation [Clo21, Clo23] and recently Bastiaan Cnossen, Tobias Lenz and Sil Linskens employed fractured ∞-topoi to study equivariant homotopy theory [CLL23].

We will follow [Lur18], define the concept, and discuss the machinery of admissibility structures to construct non-trivial fractured structures.

Definition 2.16. Let $\mathscr{X}$ be an ∞-topos. A subcategory $j_! : \mathscr{X}^{\mathrm{corp}} \to \mathscr{X}$ is a **fracture subcategory** if it satisfies the following conditions:

(i) If $X \in \mathscr{X}^{\mathrm{corp}}$ and $f : X \to Y$ in $\mathscr{X}$ is an equivalence, then f belongs to $\mathscr{X}^{\mathrm{corp}}$.

(ii) The ∞-category $\mathscr{X}^{\mathrm{corp}}$ admits pullbacks and these are preserved by $j_!$.

(iii) The inclusion functor $j_! : \mathscr{X}^{\mathrm{corp}} \to \mathscr{X}$ admits a right adjoint $j^* : \mathscr{X} \to \mathscr{X}^{\mathrm{corp}}$ which is conservative and preserves small colimits.

(iv) For every map $U \to V$ in $\mathscr{X}^{\mathrm{corp}}$ the diagram

$$\begin{tikzcd}
j_!j^*j_!U \arrow[r] \arrow[d] & j_!j^*j_!V \arrow[d] \\
j_!U \arrow[r] & j_!V
\end{tikzcd}$$

induced by the counit $j_!j^* \Rightarrow \mathrm{id}_{\mathscr{X}}$ is a pullback in $\mathscr{X}$.

A **fractured ∞-topos** is a pair $\mathscr{X}^{\mathrm{corp}} \to \mathscr{X}$ where $\mathscr{X}$ is an ∞-topos and $\mathscr{X}^{\mathrm{corp}}$ is a fracture subcategory of $\mathscr{X}$.

Remark 2.17. Clough gives another definition of fractured structures. He defines a fractured ∞-topos as an adjunction

$$j_! : \mathscr{X}^{\mathrm{corp}} \rightleftarrows_{\perp} \mathscr{X} : j^*$$

between ∞-topoi $\mathscr{X}^{\mathrm{corp}}$ and $\mathscr{X}$ satisfying the following conditions:

(i) The ∞-topos $\mathscr{X}$ is generated under colimits by $\operatorname{im} j_!$.

(ii) For every $U \in \mathscr{X}^{\mathrm{corp}}$ the left adjoint in

$$(j_!)_{/U} : \mathscr{X}^{\mathrm{corp}}_{/U} : \rightleftarrows_{\perp} \mathscr{X}_{/U} : (j^*)_{/U}$$

is fully faithful.

(iii) The functor $j^* : \mathscr{X} \to \mathscr{X}^{\mathrm{corp}}$ preserves colimits.

(iv) For any pullback square

$$\begin{tikzcd} U' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d] \\ V' \arrow[r] & V \end{tikzcd}$$

in which $U \to V$ and V' lie in the image of $j_!$, the map $U' \to V'$ is in the image of $j_!$.

The definitions are equivalent [Clo23, Proposition 2.3.1].

It will turn out that $\mathscr{X}^{\mathrm{corp}}$ is an ∞-topos (2.19), so then the condition that j^* preserves small colimits 2.16(iii) is equivalent to it admitting a right adjoint by the Adjoint Functor Theorem (1.21). Hence, we obtain a triple adjunction

$$\begin{tikzcd} \mathscr{X}^{\mathrm{corp}} \arrow[r, shift left=2.5, "j_!"] \arrow[r, phantom, shift left=1.2, "\perp"] \arrow[r, phantom, shift right=1.2, "\perp"] \arrow[r, shift right=2.5, "j_*"'] & \mathscr{X} \arrow[l, "j^*" description] \end{tikzcd}$$

which we can not quite manage to complete to a quadruple adjunction from cohesion. Instead, we want to think of a fractured structure as a device to produce cohesion locally (2.20). Here, 'locally' means to employ slice categories.

Proposition 2.18. Let $\mathscr{X}^{\mathrm{corp}} \to \mathscr{X}$ be a fractured ∞-topos. For every $X \in \mathscr{X}^{\mathrm{corp}}$ the inclusion $(j_!)_{/X} : \mathscr{X}^{\mathrm{corp}}_{/X} \hookrightarrow \mathscr{X}_{/X}$ is fully faithful.

Proof. See [Lur18, Proposition 20.1.3.1]. □

Proposition 2.19. Let $\mathscr{X}^{\mathrm{corp}} \to \mathscr{X}$ be a fractured ∞-topos. Then, $\mathscr{X}^{\mathrm{corp}}$ is an ∞-topos.

Proof. See [Lur18, Proposition 20.1.3.3]. □

Remark 2.20. As alluded to before, a fractured structure induces cohesion locally. This is not completely true but it almost is. We want to view it in the following sense.

Let $j_! : \mathscr{X}^{\mathrm{corp}} \to \mathscr{X}$ be a fractured ∞-topos and $X \in \mathscr{X}^{\mathrm{corp}}$. Since $j^* : \mathscr{X} \to \mathscr{X}^{\mathrm{corp}}$ preserves small colimits (2.16(iii)), it admits a right adjoint $j_* : \mathscr{X}^{\mathrm{corp}} \to \mathscr{X}$ by the Adjoint Functor Theorem (1.21). After passing to slice categories, we obtain a triple adjunction

$$\begin{tikzcd}
\mathscr{X}^{\mathrm{corp}}_{/X} \arrow[r, shift left=3, "(j_!)_{/X}"] \arrow[r, shift left=1.5, phantom, "\perp"] & \mathscr{X}_{/X} \arrow[l, "(j^*)_{/X}" description] \arrow[l, shift left=1.5, phantom, "\perp"] \arrow[l, shift right=3, leftarrow, "(j_*)_{/X}"']
\end{tikzcd}$$

since $\mathscr{X}$ and $\mathscr{X}^{\mathrm{corp}}$ are complete ∞-categories.

The functor $(j_!)_{/X}$ is fully faithful (2.18) and preserves pullbacks by definition (2.16(ii)). By 2.16(i) we deduce $\mathrm{id}_X \in \mathscr{X}^{\mathrm{corp}}_{/X}$, so it is a terminal object and hence $j_!$ preserves terminal objects. Therefore, $j_!$ preserves pullbacks and terminal objects, so it preserves finite limits. So the requirements for the Adjoint Functor Theorem (1.21) are almost fulfilled to yield another left adjoint of $(j_!)_{/X}$. If that left adjoint furthermore preserves finite products, then this would result in a quadruple adjunction realizing a cohesive structure.

Remark 2.21 (Warning)**.** Here's a notational warning. We have chosen the decoration for the functors in our definitions of cohesion and fractured ∞-topoi from the original texts but we realize that this could be potentially confusing. For instance, the lower shriek functor $(-)_!$ denotes the left-most functor in the quadruple adjunction of cohesion while for fractured structure we use $(-)_!$ to denote the right adjoint of the left-most functor.

Remark 2.22. From the above discussion, it might seem like fractured structures are a strictly weaker concept than cohesion. However, a naive way of making cohesion imply fractured structure does not work.

Let $\mathscr{X}, \mathscr{Y}$ be two ∞-topoi with a quadruple adjunction

$$\begin{tikzcd}
\mathscr{X} \arrow[r, shift left=4.5, "f_!"] \arrow[r, shift left=3, phantom, "\perp"] \arrow[r, shift right=1.5, "f_*" description] & \mathscr{Y} \arrow[l, shift left=1.5, "f^*" description] \arrow[l, phantom, "\perp"] \arrow[l, shift right=3, phantom, "\perp"] \arrow[l, shift left=4.5, "f^!"]
\end{tikzcd}$$

realizing $\mathscr{X}$ as a cohesive ∞-topos over $\mathscr{Y}$. The naive hope is that the fully faithful functor $f^* : \mathscr{Y} \to \mathscr{X}$ realizes $\mathscr{Y}$ as a fracture subcategory of $\mathscr{X}$.

Recall (2.9(i)): For any ∞-category $\mathscr{C}$ the presheaf category $\mathbf{PSh}(\mathscr{C})$ is a cohesive ∞-topos via

$$\begin{tikzcd}
\mathbf{PSh}(\mathscr{C}) \arrow[r, shift left=4.5, "\Pi"] \arrow[r, shift left=3, phantom, "\perp"] \arrow[r, shift right=1.5, "\Gamma" description] & \mathbf{An}. \arrow[l, shift left=1.5, "\mathrm{Disc}" description] \arrow[l, phantom, "\perp"] \arrow[l, shift right=3, phantom, "\perp"] \arrow[l, shift left=4.5, "\mathrm{CoDisc}"]
\end{tikzcd}$$

On the other hand, $\mathrm{Disc} : \mathbf{An} \hookrightarrow \mathbf{PSh}(\mathscr{C})$ does not realize $\mathbf{An}$ as a fracture subcategory of $\mathbf{PSh}(\mathscr{C})$ in general because the global sections functor Γ is not conservative in general.

In fact, the failure is much more general: Cohesion yields a fully faithful functor $f^* : \mathscr{Y} \hookrightarrow \mathscr{X}$ but a fracture subcategory $j_! : \mathscr{X}^{\mathrm{corp}} \to \mathscr{X}$ is never fully faithful unless $\mathscr{X}^{\mathrm{corp}} = \mathscr{X}$. If $j_! : \mathscr{X}^{\mathrm{corp}} \to \mathscr{X}$ is fully faithful, then it preserves colimits and thus $\mathscr{X}^{\mathrm{corp}} = \mathscr{X}$ since $\mathscr{X}$ is generated under colimits by $\mathrm{im}\, j_!$ by 2.17.

Example 2.23. Let $\mathscr{X}$ be an ∞-topos.

(i) The trivial fractured structure is $\mathrm{id}_{\mathscr{X}} : \mathscr{X} \to \mathscr{X}$.

(ii) Consider the functor which on objects is given as

$$F : \mathscr{X} \times \mathscr{X} \to \mathrm{Fun}(\Delta^1, \mathscr{X}),\ (X, Y) \mapsto (X \hookrightarrow X \amalg Y).$$

More precisely, the unit $\eta : \mathrm{id}_{\mathscr{X} \times \mathscr{X}} \Rightarrow \Delta \circ (- \amalg -)$ of the adjunction

$$\begin{tikzcd} \mathscr{X} \times \mathscr{X} \arrow[r, shift left, "- \amalg -"] \arrow[r, phantom, "\perp"] & \mathscr{X} \arrow[l, shift left, "\Delta"] \end{tikzcd}$$

is a map $\Delta^1 \to \operatorname{Fun}(\mathscr{X} \times \mathscr{X}, \mathscr{X} \times \mathscr{X})$. By the adjunction $- \times \Delta^1 \dashv \operatorname{Fun}(\Delta^1, -)$ this corresponds to a map $\eta' : \mathscr{X} \times \mathscr{X} \times \Delta^1 \to \mathscr{X} \times \mathscr{X}$. Then, $\mathrm{pr}_1 \circ \eta' : \mathscr{X} \times \mathscr{X} \times \Delta^1 \to \mathscr{X}$ is adjoint to our desired F.

This F induces an equivalence onto a fracture subcategory $\operatorname{Fun}(\Delta^1, \mathscr{X})^{\mathrm{corp}} \subseteq \operatorname{Fun}(\Delta^1, \mathscr{X})$ [Lur18, Examples 20.1.2.3].

A priori it is not so clear how to obtain non-trivial fractured structures. However, Lurie developed machinery involving so-called admissibility structures to construct many fractured structures [Lur18, Chapter 20.2, 20.3, 20.5]. We want to discuss this notion now.

Definition 2.24. Let $\mathscr{C}$ be an ∞-category. An **admissibility structure** on $\mathscr{C}$ is a collection of morphisms in $\mathscr{C}$, called the **admissible morphisms**, satisfying the following axioms:

(i) Equivalences in $\mathscr{C}$ are admissible.

(ii) If $f : U \to X$ is admissible in $\mathscr{C}$ and $g : X' \to X$ is a morphism in $\mathscr{C}$, then the pullback square

$$\begin{tikzcd} U \times_X X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "f"] \\ X' \arrow[r, "g"'] & X \end{tikzcd}$$

exists and f' is admissible.

(iii) Consider the commutative triangle

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "h"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

in $\mathscr{C}$ where g is admissible. Then, f is admissible if and only if h is admissible.

(iv) The collection of admissible morphisms is closed under retracts in $\operatorname{Fun}(\Delta^1, \mathscr{C})$.

We write $\mathscr{C}^{\mathrm{ad}}$ for the wide subcategory of $\mathscr{C}$ with the admissible morphisms as maps.

Admissibility structures are related to factorization systems and this also motivates the above axioms [Lur18, Section 20.2.2]. It will be of no relevance to us and we will instead focus on the relation of admissibile morphisms to fractured ∞-topoi.

Example 2.25. Here are some examples of admissibility structures.

(i) Let $\mathbf{AffSch}^{\mathrm{ft}}$ denote the category of affine schemes of finite type over $\mathbb{Z}$. Then, the open embeddings $\operatorname{Spec} \mathbb{Z}[t^{-1}] \hookrightarrow \operatorname{Spec} \mathbb{Z}$ for $0 \neq t \in \mathbb{Z}$ determine an admissibility structure on $\mathbf{AffSch}^{\mathrm{ft}}$ [Lur18, Example 20.2.1.5].

(ii) The open embeddings determine an admissibility structure on **Top** [Lur18, Example 20.2.1.6].

(iii) The injective continuous maps determine an admissibility structure **$\mathbf{CHaus}^{\mathrm{inj}}$** on **CHaus** by some point-set topology arguments. This is a first hint towards a fractured structure on condensed anima.

(iv) Let $j_! : \mathscr{X}^{\mathrm{corp}} \to \mathscr{X}$ be a fractured ∞-topos. Then, a morphism $f : U \to X$ in $\mathscr{X}$ is called **$\mathscr{X}^{\mathrm{corp}}$-admissible** if for every pullback square

$$\begin{tikzcd} U \times_X X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & U \arrow[d, "f"] \\ X' \arrow[r, "g"'] & X \end{tikzcd}$$

with corporeal X' also f' belongs to $\mathscr{X}^{\mathrm{corp}}$. Then, the $\mathscr{X}^{\mathrm{corp}}$-admissible morphisms determine an admissibility structure on $\mathscr{X}$ [Lur18, Proposition 20.3.1.3].

We enhance admissibility structures to geometric admissibility structures which again is motivated by an algebro-geometric setting of so-called Cartesian sheaves [Lur18, p. 1526-1527]. Once more, this viewpoint will not be relevant to us.

Definition 2.26. Let $\mathscr{X}$ be an ∞-topos.

(i) An admissibility structure $\mathscr{X}^{\mathrm{ad}} \to \mathscr{X}$ is called **local** if it satisfies the following axioms:

  (a) If $f : U \to X$ is a map in $\mathscr{X}$ and $\coprod_\alpha X_\alpha \to X$ is an effective epimorphism such that $X_\alpha \times_X U \to X_\alpha$ is admissible, then f is admissible.

  (b) For every $X \in \mathscr{X}$ the slice category $\mathscr{X}^{\mathrm{ad}}_{/X}$ is presentable and the inclusion $\mathscr{X}^{\mathrm{ad}}_{/X} \to \mathscr{X}_{/X}$ preserves small colimits.

(ii) Let $\mathscr{X}^{\mathrm{ad}} \to \mathscr{X}$ be a local admissibility structure which hence for every $X \in \mathscr{X}$ induces an adjunction

$$\begin{tikzcd} \mathscr{X}^{\mathrm{ad}}_{/X} \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathscr{X}_{/X} \arrow[l, shift left, "\rho_X"] \end{tikzcd}$$

by the Adjoint Functor Theorem (1.21, 1.37). Then, X is **$\mathscr{X}^{\mathrm{ad}}$-corporeal** if ρ_X preserves small colimits.

(iii) Let $\mathscr{X}$ be an ∞-topos. A **geometric admissibility structure** on $\mathscr{X}$ is a local admissibility structure $\mathscr{X}^{\mathrm{ad}} \to \mathscr{X}$ such that $\mathscr{X}$ is generated under small colimits by the $\mathscr{X}^{\mathrm{ad}}$-corporeal objects.

Theorem 2.27. Let $\mathscr{X}$ be an ∞-topos with a geometric admissiblity structure $\mathscr{X}^{\mathrm{ad}} \to \mathscr{X}$. Let $\mathscr{X}^{\mathrm{corp}}$ be the full subcategory of $\mathscr{X}^{\mathrm{ad}}$ spanned by the $\mathscr{X}^{\mathrm{ad}}$-corporeal objects. Then, $\mathscr{X}^{\mathrm{corp}} \to \mathscr{X}$ is a fracture subcategory.

Proof. See [Lur18, Theorem 20.3.4.4]. □

Remark 2.28. Let $\mathscr{X}$ be an ∞-topos.

(i) Let $\mathscr{X}^{\mathrm{corp}} \to \mathscr{X}$ be a fracture subcategory as above. Then, it is closed under retracts in $\mathscr{X}$ [Lur18, Theorem 20.3.4.4].

(ii) In fact, there is a bijective correspondence between geometric admissibility structures on $\mathscr{X}$ and fracture subcategories that are closed under retracts [Lur18, Remark 20.3.4.6].

Many ∞-topoi are described via Grothendieck topologies and we will now give conditions on a site that will immediately produce fractured structures. This is similar to ∞-cohesive sites and, as announed in that section, we will give more details for fractured ∞-topoi. The upshot is that it will suffice to check conditions on a site before passing to the sheaf category.

Definition 2.29. Let $\mathscr{C}$ be an ∞-site with Grothendieck topology τ.

(i) An admissibility structure $\mathscr{C}^{\mathrm{ad}} \to \mathscr{C}$ is **compatible with** τ if for every $X \in \mathscr{C}$ and covering sieve $\mathscr{C}^{(0)}_{/X} \to \mathscr{C}_{/X}$ there exists a τ-covering $\{f_\alpha : U_\alpha \to X\}_\alpha$ such that f_α is admissible and belongs to $\mathscr{C}^{(0)}_{/X}$.

(ii) A **geometric site** is a triple $(\mathscr{C}, \mathscr{C}^{\mathrm{ad}}, \tau)$ where $\mathscr{C}$ is an essentially small ∞-category, $\mathscr{C}^{\mathrm{ad}}$ is an admissibility structure on $\mathscr{C}$ and τ is a Grothendieck topology on $\mathscr{C}$ compatible with $\mathscr{C}^{\mathrm{ad}}$.

Let $(\mathscr{C}, \mathscr{C}^{\mathrm{ad}}, \tau)$ be a geometric site. Then, $\mathscr{C}^{\mathrm{ad}}$ admits pullbacks by definition and the inclusion functor $j : \mathscr{C}^{\mathrm{ad}} \to \mathscr{C}$ preserves pullbacks by construction. Hence, we obtain a Grothendieck topology $\tau^{\mathrm{ad}} = j^*\tau$ by 1.46. Then, precomposition induces a functor $j^* : \mathbf{Sh}_\tau(\mathscr{C}) \to \mathbf{Sh}_{\tau^{\mathrm{ad}}}(\mathscr{C}^{\mathrm{ad}})$ (see 1.47).

Theorem 2.30. Let $(\mathscr{C}, \mathscr{C}^{\mathrm{ad}}, \tau)$ be a geometric site. Then, there is an adjunction

$$\begin{tikzcd}
\mathbf{Sh}_{\tau^{\mathrm{ad}}}(\mathscr{C}^{\mathrm{ad}}) \arrow[r, shift left, "j_!"] \arrow[r, phantom, "\perp"] & \mathbf{Sh}_\tau(\mathscr{C}) \arrow[l, shift left, "j^*"]
\end{tikzcd}$$

and $j_!$ induces an equivalence from $\mathbf{Sh}_{\tau^{\mathrm{ad}}}(\mathscr{C}^{\mathrm{ad}})$ to a fracture subcategory $\mathbf{Sh}_\tau(\mathscr{C})^{\mathrm{corp}} \subseteq \mathbf{Sh}_\tau(\mathscr{C})$.

Proof. See [Lur18, Theorem 20.6.3.4]. □

Finally, here is the motivating example of the theory that accompanied us throughout but besides that was of little relevance for our goals.

Example 2.31. Consider the site $\mathscr{C} = \mathbf{Sch}^{\mathrm{fp}}_{/X}$ of X-schemes of finite presentation with the étale topology. Then, there is an admissibility structure $\mathbf{Sch}^{\text{ét, fp}}_{/X} \subseteq \mathbf{Sch}^{\mathrm{fp}}_{/X}$ of étale X-schemes of finite presentation. This defines a geometric site [Lur18, Section 20.6] and so we obtain an adjunction

$$\begin{tikzcd}
\mathbf{Sh}\left(\mathbf{Sch}^{\text{ét, fp}}_{/X}\right) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathbf{Sh}\left(\mathbf{Sch}^{\mathrm{fp}}_{/X}\right) \arrow[l, shift left]
\end{tikzcd}$$

which induces an equivalence onto a fracture subcategory (2.30). More variations of this example can be found in [Lur18, Section 20.6.4].

Remark 2.32. Since ∞-sites do not induce all left-exact accessible localizations from presheaf categories (1.53), it is also not plausible to expect that every fractured ∞-topos may be obtained from a geometric site [Lur18, p. 1499]. On the other hand, every fractured ∞-topos can be realized as the localization of a fracture subcategory of a presheaf category [Lur18, Theorem 20.5.3.4].

In that sense this allows us a similar viewpoint to the definition of ∞-topoi. Every ∞-topos is realized as a localization of a presheaf category while every fractured ∞-topos is realized as a localization of a fractured presheaf category.

We will construct a fractured structure on condensed anima and the main ingredient will be the theory of geometric sites.

3 Condensed Mathematics

It is a typical procedure in mathematics to endow additional structure to algebraic objects which results in objects such as topological groups, Lie groups, algebraic groups, and so on. One particularly useful enrichment is a topological enrichment partly because this allows us to fuse our geometric intuition with our abstract algebraic intuition. However, it turns out that such processes can lead to problems. For example, one may consider the category of topological abelian groups **TopAb** but introducing topology conflicts with the algebraic properties of the category! Indeed, the map

$$\mathrm{id}_{\mathbb{R}} : (\mathbb{R}, \text{discrete topology}) \to (\mathbb{R}, \text{euclidean topology})$$

has trivial kernel and cokernel but is not an isomorphism in **TopAb**. So **TopAb** cannot be an abelian category making it impossible to perform homological algebra on this category.

A standard idea in mathematics is to enlarge this category to one with the desired properties. We wish to embed **TopAb** into an abelian category. The Yoneda Embedding ensures an inclusion

$$よ : \mathbf{TopAb} \hookrightarrow \mathrm{Fun}(\mathbf{Top}^{\mathrm{op}}, \mathbf{Ab}) = \mathbf{PSh}_{\mathbf{Ab}}(\mathbf{Top})$$

which is an abelian category because **Ab** is. However, $\mathbf{PSh}_{\mathbf{Ab}}(\mathbf{Top})$ contains too many objects but the virtue of condensed mathematics developed by Scholze-Clausen [Sch19] resp. pyknotic mathematics developed by Barwick-Haine [BH19] is precisely about a finer distinction of this idea.

3.1 Three equivalent definitions

Our main goal in this subsection is to introduce three different sites that yield the same notion of sheaves which will be the condensed objects. The protagonist of this subsection is the 1-category of compact Hausdorff spaces **CHaus** along with two full subcategories **Stone** and **Stonean**.

Definition 3.1.

(i) A **profinite set** is a pro-object in the category of finite sets **FinSet**. We write **ProFin** for the category of profinite sets.

(ii) A **Stone space** is a topological space that is totally disconnected and compact Hausdorff. We write **Stone** for the category of Stone spaces.

Lemma 3.2. There is an equivalence of categories $\mathbf{ProFin} \simeq \mathbf{Stone}$.

Proof. See [Joh86, VI.2.3]. □

By the explicit form of the equivalence any Stone space can be realized as the cofiltered limit of finite discrete spaces.

Definition 3.3.

(i) An **extremally disconnected set** is a topological space in which the closure of any open set is open again. We write **ExtrDisc** for the category of extremally disconnected sets.

(ii) A **Stonean space** is a extremally disconnected Stone space. We write **Stonean** for the category of Stonean spaces.

Often Stonean spaces are called extremally disconnected spaces [Gle58] or extremally disconnected sets as e.g. in Clausen-Scholze's notes on condensed mathematics [Sch19]. We try to be a little more pedantic and make the precise distinction between (not necessarily compact Hausdorff) extremally disconnected sets and Stonean spaces. Similarly, we try to make the distinction between profinite sets and Stone spaces which are really only related by a non-trivial (but tractable) equivalence of categories.

Remark 3.4. Let S, S' be Stonean spaces.

(i) If S, S' are infinite, then $S \times S'$ is not Stonean.

(ii) If a sequence $(x_n)_{n\in\mathbb{N}}$ in S converges, then it is eventually constant.

(iii) The Stonean spaces are precisely the projective objects in **CHaus**, so every surjection $T \to S$ from a compact Hausdorff space T splits.

(iv) Every Hausdorff extremally disconnected set S'' is totally disconnected. So **Stonean** is a full subcategory of **Stone**.

Proof. See [Sch19, Warning 2.6] and [Gle58, Theorem 1.3, Theorem 2.5] for (i) – (iii).

For (iv) suppose that there exists a connected component $U \subsetneq S''$ with at least two elements. As S'' is Hausdorff, we can separate these two elements with open neighbourhoods $V, W \subsetneq S''$. Now $\overline{V}$ is clopen and $\overline{V} \subseteq U \setminus W$, i.e. $\emptyset \neq \overline{V} \neq U$. This contradicts the connectedness of U. $\square$

Remark 3.5 (Warning)**.** Extremally disconnected sets need not be totally disconnected despite the terminology. Indeed, indiscrete spaces are extremally disconnected and connected.

We write $(-)^{\text{disc}} : \mathbf{Top} \to \mathbf{Top}$ for the composition

$$\mathbf{Top} \xrightarrow{U} \mathbf{Set} \xrightarrow{\text{Disc}} \mathbf{Top}$$

and $\beta : \mathbf{Top} \to \mathbf{CHaus}$ for the Stone-Čech compactification.

Recall that β is a left adjoint of the forgetful functor $\mathbf{CHaus} \to \mathbf{Top}$ and is characterized by a natural map $i_X : X \to \beta X$ for $X \in \mathbf{Top}$ such that there is a unique factorization

$$\begin{tikzcd}
X \arrow[r, "i_X"] \arrow[dr, "f"'] & \beta X \arrow[d, dashed, "\exists!"] \\
& K
\end{tikzcd}$$

for every compact Hausdorff space K.

This construction is invaluable to get a handle on Stonean spaces.

Lemma 3.6. Let S be a compact Hausdorff space. Then, there is a surjection $\beta S^{\text{disc}} \to S$ from a Stonean space.

Proof. We first find a surjection $\beta S^{\text{disc}} \to S$. Consider the map $i : S^{\text{disc}} \to S$ given by the identity function on underlying sets, then the universal property of the Stone–Čech compactification yields a factorization

$$\begin{tikzcd}
S^{\text{disc}} \arrow[r] \arrow[dr, "i"'] & \beta S^{\text{disc}} \arrow[d, dashed] \\
& S
\end{tikzcd}$$

and so $\beta S^{\text{disc}} \to S$ is surjective since already $i : S^{\text{disc}} \to S$ is surjective.

Now, we prove that βS^{disc} is extremally disconnected. Let $T \to \beta S^{\text{disc}}$ be a surjection from a compact Hausdorff space. So we can find a continuous function $S^{\text{disc}} \to T$ such that the composition $S^{\text{disc}} \to T \to \beta S^{\text{disc}}$ is the natural inclusion. The universal property of β yields a factorization

$$\begin{tikzcd}
S^{\mathrm{disc}} \arrow[r] \arrow[dr] & \beta S^{\mathrm{disc}} \arrow[d, dashed] \\
& T
\end{tikzcd}$$

and this is indeed a section because in the larger diagram

$$\begin{tikzcd}
S^{\mathrm{disc}} \arrow[r] \arrow[dr] \arrow[ddr, bend right] & \beta S^{\mathrm{disc}} \arrow[d, dashed] \arrow[dd, dotted, bend left=60] \\
& T \arrow[d] \\
& \beta S^{\mathrm{disc}}
\end{tikzcd}$$

the map $\mathrm{id}_{\beta S^{\mathrm{disc}}} : \beta S^{\mathrm{disc}} \to \beta S^{\mathrm{disc}}$ works since $S^{\mathrm{disc}} \to T \to \beta S^{\mathrm{disc}}$ is the inclusion. Hence, the dotted composition must be $\mathrm{id}_{\beta S^{\mathrm{disc}}}$ by uniqueness from the universal property. □

Remark 3.7. Our entire argument was purely formal but we believe that it's nice to have an explicit description of the map $\beta S^{\mathrm{disc}} \to S$ for a compact Hausdorff space S. We imagine that there are many possibilities for this since there are multiple convenient descriptions of Stone-Čech compactifications but here is one approach.

Let S be a compact Hausdorff space. As a set, we may put

$$\beta S^{\mathrm{disc}} = \left\{ p \in 2^S : p \text{ is an ultrafilter on } S \right\}$$

which we may topologize as follows: For $A \subseteq S$ we write $\widehat{A} = \{p \in \beta S : A \in p\}$. Then, $\{\widehat{A} : A \subseteq S\}$ forms a basis for a topology βS^{disc} [HS12, Theorem 3.27]. Since S is compact Hausdorff, every ultrafilter p in S has a unique limit [BBT20, Corollary 3.12.1]. It's another point-set topology argument [Tsa, Theorem 3.3] that the map

$$\beta S^{\mathrm{disc}} \to S, \; p \mapsto \lim p$$

is continuous and it is surjective because the map

$$S^{\mathrm{disc}} \to \beta S^{\mathrm{disc}}, \; s \mapsto (s) = \{A \subseteq S : s \in A\}$$

sending a point to its principal ultrafilter is evidently a section.

Remark 3.8. In fact, Stonean space is a retract of a Stone–Čech compactification [Sch19, Warning 2.6]. So understanding Stonean spaces really boils down to understanding Stone–Čech compactifications!

Lemma 3.9. Let S be a Stonean space and $U \subseteq S$ be a clopen subspace. Then, U is also Stonean.

Proof. First, U is compact Hausdorff because $U \subseteq S$ is closed inside a compact Hausdorff space. We check that it is also extremally disconnected. Let $V \subseteq U$ be open, so there exists some open $W \subseteq S$ such that $V = U \cap W$. But $U \cap W$ is open in S as the intersection of two open subsets, so also $\mathrm{cls}_S(U \cap W) \subseteq S$ is open because S is extremally disconnected. Hence,

$$\mathrm{cls}_U(U \cap W) = \bigcap_{\substack{U \cap W \subseteq A \subseteq U \\ A \subseteq U \text{ closed}}} A = \bigcap_{\substack{U \cap W \subseteq B \subseteq S \\ B \subseteq S \text{ closed}}} (B \cap U) = \left(\bigcap_{\substack{U \cap W \subseteq B \subseteq S \\ B \subseteq S \text{ closed}}} B \right) \cap U = \mathrm{cls}_S(U \cap W) \cap U$$

is also open. Therefore, U is extremally disconnected. □

Example 3.10. In 3.4 we have convinced ourselves of full subcategories

$$\mathbf{Stonean} \hookrightarrow \mathbf{Stone} \hookrightarrow \mathbf{CHaus}.$$

These are the basic building blocks of condensed mathematics but let us first give examples that those are non-trivial subcategories. In other words, we want to demonstrate that the functors are not essentially surjective.

Indeed, $[0,1]$ is compact Hausdorff as a bounded closed subspace of $\mathbb{R}$ but it is not a Stone space since it is not totally disconnected. The p-adic integers $\mathbb{Z}_p = \lim_{n\in\mathbb{N}} \mathbb{Z}/p^n\mathbb{Z}$ form a Stone space as a cofiltered limit of discrete spaces but it is not Stonean since there are convergent sequences which are not eventually constant (3.4(ii)). Alternatively, one can also construct such examples with products of infinite Stonean spaces (3.20(i)).

Example 3.11. The finitely jointly surjective families of maps form a Grothendieck topology on **CHaus**. By virtue of the surjection provided by Stone–Čech compactification (3.6) the inclusions

$$\mathbf{Stonean} \hookrightarrow \mathbf{Stone} \hookrightarrow \mathbf{CHaus}$$

form bases. In particular, the finitely jointly surjective families of maps restrict to Grothendieck topologies on **Stone** and **Stonean** (see 1.73). We denote the topology by τ_{coh} on each of these sites.

This is a special case of coherent topologies [Lur19, Appendix B.5], hence our (ad hoc) terminology. Whenever we write **Stonean**, **Stone** or **CHaus** and want to view it as a site, then τ_{coh} is tacitly meant.

Finally, we are ready to state the characterization of condensed objects.

Proposition 3.12. The restriction functors induce equivalences of categories

$$\mathbf{Sh}(\mathbf{CHaus})^{\mathrm{hyp}} \xrightarrow{\simeq} \mathbf{Sh}(\mathbf{Stone})^{\mathrm{hyp}} \xrightarrow{\simeq} \mathbf{Sh}(\mathbf{Stonean}).$$

Proof. Since $\mathbf{Stonean} \hookrightarrow \mathbf{Stone} \hookrightarrow \mathbf{CHaus}$ form bases (3.11) we deduce equivalences of categories

$$\mathbf{Sh}(\mathbf{CHaus})^{\mathrm{hyp}} \xrightarrow{\simeq} \mathbf{Sh}(\mathbf{Stone})^{\mathrm{hyp}} \xrightarrow{\simeq} \mathbf{Sh}(\mathbf{Stonean})^{\mathrm{hyp}}$$

by the Comparison Lemma (1.74). Moreover, we will show that $\mathbf{Sh}(\mathbf{Stonean})$ is already hypercomplete (3.18), so there is an equivalence $\mathbf{Sh}(\mathbf{Stonean})^{\mathrm{hyp}} \simeq \mathbf{Sh}(\mathbf{Stonean})$. ☐

Definition 3.13. Let $\mathscr{D}$ be an ∞-category. An object in $\mathbf{Sh}_{\mathscr{D}}(\mathbf{Stonean})$ is a **condensed object** in $\mathscr{D}$. We write $\mathbf{Cond}(\mathscr{D}) = \mathbf{Sh}_{\mathscr{D}}(\mathbf{Stonean})$ for the ∞-category of condensed objects in $\mathscr{D}$.

In particular, objects in $\mathbf{Cond}(\mathbf{An})$ are called **condensed anima**.

Remark 3.14. Since **Stonean**, **Stone** and **CHaus** are not small, there are set-theoretic issues to take sheaves of sites that need to be addressed.

Barwick-Haine [BH19, 0.3] deals with this via 'universe-hopping'. They assume the existence of a strongly inaccessible cardinal δ and select the smallest strongly inaccessible cardinal δ^+ over δ. Then, a condensed anima in the universe V_{δ^+} of sets of cardinality $< \delta^+$ is a sheaf on the site $\mathbf{CHaus}_\delta$ of δ-small compact Hausdorff spaces valued in $\mathbf{Set}_{\delta^+}$ of δ^+-small sets. These are coined *pyknotic spaces*.

Clausen-Scholze [Sch19, Lecture I, II] solves this by considering uncountable strong limit cardinals κ and κ-condensed objects which restricts the site to spaces of cardinality $< \kappa$. They yield the ∞-category $\mathbf{Cond}_\kappa(\mathscr{D})$. If $\mathscr{D}$ is an ∞-category with all (relevant) filtered colimits, then we put

$$\mathbf{Cond}(\mathscr{D}) = \operatorname*{colim}_{\kappa} \mathbf{Cond}_\kappa(\mathscr{D})$$

where the colimit goes over the filtered poset of all uncountable strong limit cardinals κ. This construction does not yield an ∞-topos but is still relatively well-behaved [BH19, Section 0.3].

The difference is merely in set theory and will not play a role in this thesis. We will ignore this technicality in the following but shall mention that actual differences can show up depending on the chosen set-theoretic approach [Sch19, Warning 2.14] but this will play a role for us.

In particular, we shall choose Barwick-Haine's formulation, so for us $\mathbf{Cond}(\mathbf{An})$ does form an ∞-topos while this would not be the case in Clausen-Scholze's formulation. Pedantic people might object and demand these to be called pyknotic anima but we will not defer to this terminology here. It seems appropriate since even Peter Haine has used the name *condensed anima* while using their set-theory convention of pyknotic anima [Hai22].

Lemma 3.15. Let $\mathscr{C} \in \{\mathbf{CHaus}, \mathbf{Stone}\}$ and let $\mathscr{D}$ be an ∞-category. A functor $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ is a $\mathscr{D}$-valued sheaf if and only if the following conditions are satisfied:

(i) The functor F preserves finite products.

(ii) For every surjection $p : S' \twoheadrightarrow S$ in $\mathscr{C}$ the diagram

$$\begin{tikzcd} F(S) \arrow[r, "p^*"] & F(S') \arrow[r, shift left=2] \arrow[r, shift right=2] & F(S' \times_S S') \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & \cdots \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

obtained by applying F to the Čech nerve $\check{C}(p) : N(\Delta_+)^{\mathrm{op}} \to \mathscr{C}$ of p is a limit diagram with limit $F(S)$.

Proof. See [Lur18, Proposition A.3.3.1]. □

Remark 3.16. For a 1-category $\mathscr{D}$ this reduces to the classical formulation of Clausen-Scholze [Sch19]. Moreover, hypercompletion is redundant for 1-categories (1.70), so that a functor $F : \mathscr{C}^{\mathrm{op}} \to \mathscr{D}$ starting in $\mathscr{C} \in \{\mathbf{CHaus}, \mathbf{Stone}\}$ is a condensed object in $\mathscr{D}$ if and only if it satisfies the following two conditions:

(i) The functor F preserves finite products,

(ii) For every surjection $p : S' \twoheadrightarrow S$ in $\mathscr{C}$ the diagram

$$\begin{tikzcd} F(S) \arrow[r] & F(S') \arrow[r, shift left] \arrow[r, shift right] & F(S' \times_S S') \end{tikzcd}$$

is an equalizer diagram.

Formally, this is a consequence for coherent topologies [Lur19, Proposition B.5.5].

Lemma 3.17. Let $\mathscr{D}$ be an ∞-category. A functor $\mathbf{Stonean}^{\mathrm{op}} \to \mathscr{D}$ is a condensed object in $\mathscr{D}$ if and only if it preserves finite products.

Proof. See [BH19, 2.2.9]. □

Lemma 3.18. Let $\mathscr{D}$ be a hypercomplete ∞-category. Then, $\mathbf{Sh}_{\mathscr{D}}(\mathbf{Stonean})$ is hypercomplete.

Proof. Consider an ∞-connected morphism $\alpha : F \to F'$ in **Sh**(**Stonean**), i.e. the morphism $\tau_{\leq n}(\alpha) : \tau_{\leq n}(F) \to \tau_{\leq n}(F')$ is an equivalence for all $n \geq -2$. Since the truncation functor $\tau_{\leq n}$: **Stonean** $\to \tau_{\leq n}$**Stonean** preserves finite products [Lur09, Lemma 6.5.1.2], it preserves condensed objects (3.17), so there is an equivalence $\tau_{\leq n}\mathbf{Cond}(\mathscr{D}) \simeq \mathbf{Cond}(\tau_{\leq n}\mathscr{D})$. Now, let $S \in$ **Stonean**. Then, the equivalence$(\tau_{\leq n}(\alpha))(S)$ turns into an equivalence

$$\tau_{\leq n}(\alpha(S)) : \tau_{\leq n}(F(S)) \to \tau_{\leq n}(F'(S)).$$

Because $\mathscr{D}$ is hypercomplete, we deduce that $\alpha(S)$ is an equivalence for all $S \in$ **Stonean**. Hence, α is an equivalence. This completes our proof that **Sh**(**Stonean**) is hypercomplete. ☐

Since **An** is hypercomplete, this finishes the proof of our characterization for **Cond**(**An**) (see 3.12).

Remark 3.19. In fact, the even stronger condition that **Cond**(**An**) is Postnikov-complete is true [BH19, Lemma 2.4.10].

Remark 3.20. This is the place for another obligatory comment about anima as promised in 1.9.

The setting of condensed anima is supposed to cover two directions: a topological one and a homotopical one. The topological direction is dealt with by the condensed part while the homotopical one comes from the anima part.

As the classical notation for **An** is the ∞-category of spaces $\mathcal{S}$, one could instead use the name condensed space **Cond**($\mathcal{S}$). However, particularly in this setting of condensed mathematics this is misleading since the word *space* is topologically connotated while we actually want to mean homotopy theory here.

For example, if F is a condensed anima, then we may truncate to $\pi_0 F = \pi_0 \circ F$ which is a condensed set since π_0 preserves products (3.17). So $\pi_0 F$ forgets the anima direction and indeed, $\pi_0 F$ could correspond to an interesting topological space. We only dismissed the homotopical information, not the topological *space* information.

So we write anima as proposed by Clausen-Scholze to make this distinction clear.

3.2 Combination of Algebra with Topology through Condensed Mathematics

The motivating problem for condensed mathematics was the inability to endow algebraic structures with a topology while keeping nice categorical properties from an algebraic viewpoint. More concretely, **TopAb** is not an abelian category! Condensed mathematics is the framework for resolving this issue. For simplicity, let us work in the 1-categorical setting in this section, as is also commonly done to fix the aforementioned problems.

Example 3.21. Let X be any topological space. For $\mathscr{C} \in \{\mathbf{CHaus}, \mathbf{Stone}, \mathbf{Stonean}\}$ the functor

$$\underline{X} : \mathscr{C}^{\mathrm{op}} \to \mathbf{Set},\ S \mapsto \mathrm{Hom}_{\mathbf{Top}}(S, X)$$

is a condensed set.

Proof. As the restriction of a contravariant Hom-functor it preserves (finite) products. So this already shows that $\underline{X} : \mathbf{Stonean}^{\mathrm{op}} \to \mathbf{Set}$ is a condensed set.

For $\mathscr{C} \in \{\mathbf{CHaus}, \mathbf{Stone}\}$ we still need to check the equalizer property. Let $\pi : S' \twoheadrightarrow S$ be a surjection in $\mathscr{C}$. Since this is in particular a surjection of compact Hausdorff spaces, it is a quotient map. We denote the projection maps by $p_1, p_2 : S' \times_S S' \to S'$. Written out the equalizer condition set-theoretically, we must check that the map

$$g^* : \mathrm{Hom}_{\mathbf{Top}}(S, X) \to \{f \in \mathrm{Hom}_{\mathbf{Top}}(S', X) : f \circ p_1 = f \circ p_2 \in \mathrm{Hom}_{\mathbf{Top}}(S' \times_S S', X)\}$$

is an isomorphism. The condition $f \circ p_1 = f \circ p_2$ is equivalent to $s', \tilde{s}'$ with $g(s') = g(\tilde{s}')$ implying $f(s') = f(\tilde{s}')$. So the universal property of quotient maps yields a unique factorization

$$\begin{tikzcd}
S' \arrow[r, "g"] \arrow[dr, "f"'] & S \arrow[d, dashed, "\exists!"] \\
& X
\end{tikzcd}$$

which is precisely the required bijection. □

Alternatively, one can check that every surjection in $\mathscr{C} \in \{\mathbf{CHaus}, \mathbf{Stone}, \mathbf{Stonean}\}$ is an effective epimorphism, so that the (truncated part) of the Čech nerve is already a colimit diagram. We can then apply that representables preserve limits.

We proved the following.

Corollary 3.22. The coherent topology τ_{coh} on $\mathbf{CHaus}, \mathbf{Stone}, \mathbf{Stonean}$ is subcanonical.

Lemma 3.23. The functor

$$\mathbf{Top} \to \mathbf{Cond}(\mathbf{Set}),\ X \mapsto \underline{X}$$

is faithful, and fully faithful when restricted to the compactly generated spaces.

Proof. See [Sch19, Proposition 1.7]. □

Remark 3.24. The same statement is true e.g. for

$$\mathbf{TopAb} \to \mathbf{Cond}(\mathbf{Ab}),\ X \mapsto \underline{X}$$

where the functor is fully faithful again after restricting it to the topological abelian groups with underlying spaces the compactly generated spaces. See [Sch19, Proposition 1.7].

In other words, we obtain a subcategory $\mathbf{Top} \to \mathbf{Cond}(\mathbf{Set})$ which allows us to work with $\mathbf{Top}$ in the larger category $\mathbf{Cond}(\mathbf{Set})$ that is categorically much better behaved!

Remark 3.25. These two results extend to $\mathbf{Cond}(\mathbf{An})$, as the fully faithful embedding $\mathbf{Set} \hookrightarrow \mathbf{An}$ induces a fully faithful embedding $\mathbf{Cond}(\mathbf{Set}) \hookrightarrow \mathbf{Cond}(\mathbf{An})$ by postcomposition (3.17). In particular, a topological space X yields a condensed anima $\underline{X}$. Therefore, this yields a functor $\mathbf{Top} \to \mathbf{Cond}(\mathbf{An})$.

For 1-categories there is one natural functor $\mathbf{Top} \to \mathbf{Cond}(\mathbf{Set})$ given by the restricted Yoneda embedding (3.21) but in the ∞-world there is an ambiguity concerning such natural functors $\mathbf{Top} \to \mathbf{Cond}(\mathbf{An})$. More precisely, there are two natural composites

$$\begin{tikzcd}
& \mathbf{Cond}(\mathbf{Set}) \arrow[r, hook] & \mathbf{Cond}(\mathbf{An}) \\
\mathbf{Top} \arrow[ur, "\underline{(-)}"] \arrow[dr] & & \\
& \mathbf{An} \arrow[r, hook, "よ"'] & \mathbf{Cond}(\mathbf{An})
\end{tikzcd}$$

where $\mathbf{Top} \to \mathbf{An}$ is given by

$$\mathbf{Top} = N^{\mathrm{hc}}(\mathrm{const}\,\mathbf{Top}) \to N^{\mathrm{hc}}(\mathbf{Kan}) = \mathbf{An}$$

is induced by Sing with the suggestive $\mathbf{sSet}$-enrichments. The functor $\mathbf{Top} \to \mathbf{An}$ is not faithful but every other functor in the diagram is (3.23), so these composites do not agree. When we write $\mathbf{Top} \to \mathbf{Cond}(\mathbf{An})$, $X \mapsto \underline{X}$, then we mean the top composition. This composite retains more information, as it also remembers the underlying homotopy type of a topological space.

Proposition 3.26. The category $\mathbf{Cond}(\mathbf{Ab})$ is an abelian category which satisfies Grothendieck's axioms (AB3), (AB4), (AB5), (AB6), (AB3*), (AB4*).

Proof. See [Sch19, Theorem 2.2]. ☐

So we have managed to embed **TopAb** into a category that behaves much more nicely from an algebraic viewpoint!

Example 3.27. We write $\mathbb{R}^{\mathrm{eucl}}$ for $\mathbb{R}$ endowed with the euclidean topology. We were faced with the problem of considering $\mathrm{id}_{\mathbb{R}} : \mathbb{R}^{\mathrm{disc}} \to \mathbb{R}^{\mathrm{eucl}}$ which has trivial kernel and cokernel but is not an isomorphism in **TopAb** so that **TopAb** cannot be an abelian category. Such a problem cannot arise in the condensed setting since $\mathbf{Cond}(\mathbf{Ab})$ is an abelian category! Consider the condensed abelian groups $\underline{\mathbb{R}}^{\mathrm{disc}}, \underline{\mathbb{R}}^{\mathrm{eucl}} : \mathbf{Stone}^{\mathrm{op}} \to \mathbf{Ab}$. The map

$$\mathrm{id}_{\mathbb{R}*} : \underline{\mathbb{R}}^{\mathrm{disc}} \to \underline{\mathbb{R}}^{\mathrm{eucl}}$$

is pointwise given by

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbf{Top}}(S, \mathbb{R}^{\mathrm{disc}}) \arrow[r] \arrow[d, equal] & \mathrm{Hom}_{\mathbf{Top}}(S, \mathbb{R}^{\mathrm{eucl}}) \arrow[d, equal] \\
\{\text{locally constant functions } S \to \mathbb{R}\} \arrow[r] & \{\text{continuous functions } S \to \mathbb{R}^{\mathrm{eucl}}\}
\end{tikzcd}$$

which is generally non-trivial with cokernel $\operatorname{coker} \mathrm{id}_{\mathbb{R}*}$ that is pointwise given by

$$(\operatorname{coker} \mathrm{id}_{\mathbb{R}*})(S) = \mathrm{Hom}_{\mathbf{Top}}(S, \mathbb{R}^{\mathrm{disc}}) / \mathrm{Hom}_{\mathbf{Top}}(S, \mathbb{R}^{\mathrm{eucl}})$$

for $S \in \mathbf{Stone}$.

Note that there is a subtlety here. A priori, it is not clear that the cokernel is given by pointwise taking cokernels, as colimits of sheaves are typically not computed pointwise but can require sheafification. However, it turns out that in this particular scenario sheafification is not needed. The short exact sequence of condensed abelian groups

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathbb{R}}^{\mathrm{disc}} \arrow[r] & \underline{\mathbb{R}}^{\mathrm{eucl}} \arrow[r] & Q \arrow[r] & 0
\end{tikzcd}$$

induces a long exact sequence by taking right derived functors

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathbb{R}}^{\mathrm{disc}}(S) \arrow[r] & \underline{\mathbb{R}}^{\mathrm{eucl}}(S) \arrow[r] & Q(S) \arrow[dll, rounded corners] \\
& H^1_{\mathrm{cond}}(S, \mathbb{R}^{\mathrm{disc}}) \arrow[r] & H^1_{\mathrm{cond}}(S, \mathbb{R}^{\mathrm{eucl}}) \arrow[r] & \cdots
\end{tikzcd}$$

for $S \in \mathbf{Stone}$ but $H^1_{\mathrm{cond}}(S, \mathbb{R}^{\mathrm{disc}}) = 0$, as the argument in the proof of [Sch19, Theorem 3.2] works for all discrete abelian groups. Here, it is important that we work with the characterization $\mathbf{Cond}(\mathbf{Ab}) \cong \mathbf{Sh}_{\mathbf{Ab}}(\mathbf{Stone})$ and not $\mathbf{Sh}_{\mathbf{Ab}}(\mathbf{CHaus})$ where the analogous statement fails since it's not evident that $H^1_{\mathrm{cond}}(S, \mathbb{R}^{\mathrm{disc}})$ vanishes for all $S \in \mathbf{CHaus}$.

An upshot of this example is that we have constructed a condensed abelian group Q which is not representable by a topological space! It would have to come from the cokernel of $\mathbb{R}^{\mathrm{disc}} \to \mathbb{R}^{\mathrm{eucl}}$ but that cokernel is trivial.

In fact, $\mathbf{Cond}(\mathbf{Ab})$ enjoys numerous additional pleasantries, e.g. one can endow a symmetric monoidal structure on it and it has enough projectives [Sch19, p. 13]. So it is truly an adequate replacement for **TopAb** in which we can work algebraically.

3.3 Condensed Mathematics and Homotopy Theory*

The virtue of the ∞-categorical version of condensed sets, i.e. condensed anima, is that it allows us to step into the world of homotopy theory. In **Cond**(**An**) the **Cond** part captures a notion of topology while the **An** part captures a notion of homotopy theory. In this subsection, we will cite a notion of homotopy (pro-)groups for condensed anima as was presented in Catrin Mair's master's thesis *Animated Condensed Sets and Their Homotopy Groups* [Mai21].

We will see that the functor $\mathrm{Disc} : \mathbf{An} \to \mathbf{Cond}(\mathbf{An})$ does not admit a left adjoint (4.3). Our fix is as in the virtue of this section, we embed our smaller category in a larger one where it works! Indeed, after enlarging the ∞-category **An**, we will be able to construct a suitable left adjoint. For an ∞-category $\mathscr{C}$ we write $\mathrm{Pro}(\mathscr{C}) = \mathrm{Fun}^{\mathrm{lexacc}}(\mathscr{C}, \mathbf{An})^{\mathrm{op}}$ for the opposite of the full ∞-subcategory of $\mathrm{Fun}(\mathscr{C}, \mathbf{An})$ spanned by the left-exact accessible functors, also known as the **∞-category of pro-objects of $\mathscr{C}$.**

Then, there is a unique functor $R_{\mathbf{An}} : \mathrm{Pro}(\mathbf{An}) \to \mathbf{Cond}(\mathbf{An})$ that preserves small cofiltered limits and that extends $\mathrm{Disc} : \mathbf{An} \to \mathbf{Cond}(\mathbf{An})$ [Mai21, Lemma 7.2.1]. One can show that it admits a left adjoint

$$L_{\mathbf{An}} : \mathbf{Cond}(\mathbf{An}) \to \mathrm{Pro}(\mathbf{An}),\ F \mapsto \mathrm{Hom}_{\mathbf{Cond}(\mathbf{An})}(F, \mathrm{Disc}(-))$$

which also extends to a functor $(L_{\mathbf{An}})_* : \mathbf{Cond}(\mathbf{An})_* \to \mathrm{Pro}(\mathbf{An})_*$ between pointed categories [Mai21, Proposition 7.2.2, Corollary 7.2.4].

Let $n \geq 2$. Homotopy groups of Kan complexes realized as functors

$$\begin{aligned} \pi_0 &: \mathbf{An} \to \mathbf{Set}, \\ \pi_1 &: \mathbf{An}_* \to \mathbf{Grp}, \\ \pi_n &: \mathbf{An}_* \to \mathbf{Ab} \end{aligned}$$

then uniquely extend [Mai21, Proposition 7.2.7] to small cofiltered-limit-preserving functors

$$\begin{aligned} \mathrm{Pro}(\pi_0) &: \mathrm{Pro}(\mathbf{An}) \to \mathrm{Pro}(\mathbf{Set}), \\ \mathrm{Pro}(\pi_1) &: \mathrm{Pro}(\mathbf{An})_* \to \mathrm{Pro}(\mathbf{Grp}), \\ \mathrm{Pro}(\pi_n) &: \mathrm{Pro}(\mathbf{An})_* \to \mathrm{Pro}(\mathbf{Ab}). \end{aligned}$$

This will now allow us to generalize the notion of homotopy groups to the condensed world!

Definition 3.28 ([Mai21, Definition 7.2.8])**.** Let $n \geq 1$ and let $F \in \mathbf{Cond}(\mathbf{An})$ with basepoint $f : *_{\mathbf{Cond}(\mathbf{An})} \to X$. The **pro-set of path components** and the **n-th homotopy pro-group** of F are given by

$$\widetilde{\pi}_0(F) = \mathrm{Pro}(\pi_0)(L_{\mathbf{An}}(F)) \quad \text{and} \quad \widetilde{\pi}_n(F, f) = \mathrm{Pro}(\pi_n)((L_{\mathbf{An}})_*(F, f)).$$

Example 3.29. Let $n \geq 1$.

(i) Let X be a CW complex, then the homotopy pro-groups of $\underline{X} \in \mathbf{Cond}(\mathbf{An})$ coincide with the usual homotopy groups.

(ii) Let $S \in \mathbf{Stone}$. Then,

$$\widetilde{\pi}_0(\underline{S}) \cong S \quad \text{and} \quad \widetilde{\pi}_n(\underline{S}, s) \cong 0$$

for any basepoint s.

Proof. See [Mai21, Proposition 7.2.9] and [Mai21, Corollary 7.3.3]. □

Alternatively, one can formally obtain homotopy sheaves in arbitrary ∞-topoi [Lur09, Definition 6.5.1.1], so a condensed anima $F \in \mathbf{Cond}(\mathbf{An})$ can be truncated to its homotopy sheaves $\pi_0(F), \pi_1(F), \pi_2(F), \cdots$ where $\pi_0(F)$ can be interpreted as a condensed set, $\pi_1(F)$ as a condensed group and $\pi_i(F)$ as a condensed abelian group for $i \geq 2$ [Lur09, Section 6.5.1].

This is an alternative approach to extract homotopical information for condensed anima and was not discussed in Mair's thesis. The difference lies in the targets of the functors: This alternative has sheaf-valued homotopy objects while in Mair's setting, we obtain homotopy objects valued in certain categories of pro-objects.

4 Cohesion and Fractured Structures in Condensed Mathematics

We have discussed two possibilities of axiomatizing and generalizing topological notions, namely via condensed mathematics and cohesive resp. fractured ∞-topoi. A natural task is to compare these ideas. More specifically, we want to investigate whether there is any sensible way of viewing $\mathbf{Cond}(\mathbf{An})$ as a cohesive or fractured ∞-topos.

The answer is negative for cohesion and positive for fractured structures. The section culminates in the main theorem of this thesis, namely a suitably constructed fractured structure on condensed anima.

4.1 Incompatibility with Cohesion

There is a triple adjunction

$$\begin{tikzcd}
\mathbf{Cond}(\mathbf{An}) \arrow[r, "\Gamma" description] & \mathbf{An} \arrow[l, shift right=3ex, "\mathrm{Disc}"'] \arrow[l, shift left=3ex, "\mathrm{CoDisc}"] \arrow[l, phantom, shift right=1.5ex, "\perp"] \arrow[l, phantom, shift left=1.5ex, "\perp"]
\end{tikzcd}$$

where $\mathrm{Disc} \dashv \Gamma$ is the canonical geometric morphism for ∞-topoi in this context (2.3). We now seek to define $\mathrm{CoDisc} : \mathbf{An} \to \mathbf{Cond}(\mathbf{An})$. The global sections functor

$$\Gamma' = \mathrm{Hom}_{\mathbf{PSh}(\mathbf{Stonean})}(*, -) : \mathbf{PSh}(\mathbf{Stonean}) \to \mathbf{An}$$

induces an adjunction

$$\begin{tikzcd}
\mathbf{PSh}(\mathbf{Stonean}) \arrow[r, shift left=1.5ex, "\Gamma'"] \arrow[r, phantom, "\perp"] & \mathbf{An} \arrow[l, shift left=1.5ex, "\mathrm{CoDisc}'"]
\end{tikzcd}$$

via right Kan extension since $\Gamma' \simeq \mathrm{ev}_{*_{\mathbf{Stonean}}}$ by Yoneda. We write $|-| : \mathbf{Stonean} \to \mathbf{An}$ for the composition

$$\mathbf{Stonean} \xrightarrow{U} \mathbf{Set} \xrightarrow{\mathrm{const}} \mathbf{An}.$$

For $X \in \mathbf{An}$ and $K \in \mathbf{Stonean}$ we then perform some Yoneda yoga to compute

$$\begin{aligned}
(\mathrm{CoDisc}' X)(K) &\simeq \mathrm{Hom}_{\mathbf{PSh}(\mathbf{Stonean})}(よ(K), \mathrm{CoDisc}' X) \\
&\simeq \mathrm{Hom}_{\mathbf{An}}(\Gamma'(よ(K)), X) \\
&\simeq \mathrm{Hom}_{\mathbf{An}}(\mathrm{Hom}_{\mathbf{PSh}(\mathbf{Stonean})}(*, よ(K)), X) \\
&\simeq \mathrm{Hom}_{\mathbf{An}}(\mathrm{Hom}_{\mathbf{Stonean}}(*, K), X) \\
&\simeq \mathrm{Hom}_{\mathbf{An}}(|K|, X).
\end{aligned}$$

This explicit description allows us to show that $\mathrm{CoDisc}' X$ is a condensed anima for $X \in \mathbf{An}$ since $\mathrm{Hom}_{\mathbf{An}}(|-|, X) : \mathbf{Stonean}^{\mathrm{op}} \to \mathbf{An}$ preserves finite products (3.17). So the adjunction $\Gamma' \dashv \mathrm{CoDisc}'$ restricts to an adjunction

$$\begin{tikzcd}\mathbf{Cond}(\mathbf{An}) \arrow[r, shift left, "\Gamma"] \arrow[r, phantom, "\perp"] & \mathbf{An} \arrow[l, shift left, "\mathrm{CoDisc}"]\end{tikzcd}$$

as desired.

By the explicit description of CoDisc we can check objectwise that the counit $\Gamma \circ \mathrm{CoDisc} \Rightarrow \mathrm{id}_{\mathbf{An}}$ is an equivalence. Hence, CoDisc is fully faithful with which we have shown the following.

Lemma 4.1. The ∞-topos $\mathbf{Cond}(\mathbf{An})$ is a local ∞-topos.

Hence, the natural first hope is that it might be a cohesive ∞-topos. Unfortunately, it turns out that this is not the case as we will now show. We write $(-)^{\mathrm{disc}}$ for the discrete topology on a set.

Lemma 4.2. Let $\mathscr{C} \in \{\mathbf{CHaus}, \mathbf{Stone}, \mathbf{Stonean}\}$ and let A be a finite set viewed as an anima. There is a natural equivalence

$$\mathrm{Disc}(A) \simeq \text{よ}(A^{\mathrm{disc}}) = \underline{A}^{\mathrm{disc}}.$$

Proof. Let $F \in \mathbf{Cond}(\mathbf{An})$. Via Yoneda we compute

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Cond}(\mathbf{An})}\left(\text{よ}(A^{\mathrm{disc}}), F\right) &\simeq F(A^{\mathrm{disc}}) \\ &\simeq \prod_{a \in A} F(*) \\ &\simeq \mathrm{Hom}_{\mathbf{An}}\left(\coprod_{a \in A} *, F(*)\right) \\ &\simeq \mathrm{Hom}_{\mathbf{An}}(A, F(*)) \\ &\simeq \mathrm{Hom}_{\mathbf{Cond}(\mathbf{An})}(\mathrm{Disc}(A), F)\end{aligned}$$

where we crucially used that F is a condensed anima in the second line so that it commutes with finite products. By the Yoneda Lemma we conclude $\mathrm{Disc}(A) \simeq \text{よ}(A^{\mathrm{disc}})$. □

Recall that we write $U : \mathbf{Top} \to \mathbf{Set}$ for the forgetful functor. Moreover, we tacitly view sets as anima.

Proposition 4.3. The ∞-topos $\mathbf{Cond}(\mathbf{An})$ is not a cohesive ∞-topos.

Proof. We want to show that Disc does not preserve all limits whence it cannot admit a left adjoint. Let $S = \lim_n S_n^{\mathrm{disc}}$ be any Stone space constructed with finite sets S_n. We consider the natural map

$$\mathrm{Disc}(US) = \mathrm{Disc}\left(\lim_n S_n\right) \to \lim_n \mathrm{Disc}(S_n) \overset{4.2}{\simeq} \lim_n \text{よ}(S_n^{\mathrm{disc}}) \simeq \text{よ}(S) = \underline{S}$$

where Stone spaces are tacitly viewed as anima. We wish to show that this is not an equivalence. Suppose otherwise.

Consider any $T \in \mathbf{Top}$. Then, also the composition

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{Set}}(S,T) &\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{An}}(US,UT) \\
&\xrightarrow{\sim} \mathrm{Hom}_{\mathbf{An}}\left(\coprod_{s\in S} U*, UT\right) \\
&\xrightarrow{\sim} \prod_{s\in S} \mathrm{Hom}_{\mathbf{An}}\left(*_{\mathbf{An}}, UT\right) \\
&\xrightarrow{\sim} \prod_{s\in S} \mathrm{Hom}_{\mathbf{Cond(An)}}(\text{よ}(*), \text{よ}(UT)) \\
&\xleftarrow{\sim} \mathrm{Hom}_{\mathbf{Cond(An)}}\left(\coprod_{s\in S} \text{よ}(*), \text{よ}(UT)\right) \\
&\simeq \mathrm{Hom}_{\mathbf{Cond(An)}}\left(\coprod_{s\in S} \mathrm{Disc}(U*), \text{よ}(UT)\right) \\
&\xleftarrow{\sim} \mathrm{Hom}_{\mathbf{Cond(An)}}\left(\mathrm{Disc}\left(\coprod_{s\in S} U*\right), \text{よ}(UT)\right) \\
&\simeq \mathrm{Hom}_{\mathbf{Cond(An)}}(\mathrm{Disc}(US), \text{よ}(UT)) \\
&\xleftarrow{\sim} \mathrm{Hom}_{\mathbf{Cond(An)}}(\underline{S}, \underline{T}^{\mathrm{disc}}) \\
&\xleftarrow{\sim} \mathrm{Hom}_{\mathbf{Top}}(S, T^{\mathrm{disc}})
\end{aligned}$$

is an equivalence. We used that Disc and U as left adjoints preserve coproducts. The assumption that $\mathrm{Disc}(US) \to \underline{S}$ is an equivalence was implemented in the second-to-last map. By explicitly tracing all of the maps we see that the composition

$$\mathrm{Hom}_{\mathbf{Top}}(S, T^{\mathrm{disc}}) \to \mathrm{Hom}_{\mathbf{Set}}(US, UT)$$

is induced by the forgetful functor $U : \mathbf{Top} \to \mathbf{Set}$. We show that this map is not surjective. Let p be a prime number. We set $S = \mathbb{Z}_p$ and $T = \{0,1\}^{\mathrm{disc}}$. Then, the function

$$f : \mathbb{Z}_p \to \{0,1\}^{\mathrm{disc}},\ z \mapsto \begin{cases} 0 & z = 0, \\ 1 & z \in \mathbb{Z}_p \setminus \{0\} \end{cases}$$

is not continuous because the sequence $(p^n)_{n\in\mathbb{N}}$ converges to 0 but the sequence $(f(p^n))_{n\in\mathbb{N}}$ does not converge. □

Remark 4.4. Our proof shows that the functor $\mathrm{Disc} : \mathbf{An} \to \mathbf{Cond}(\mathbf{An})$ does not admit a left adjoint $\mathbf{Cond}(\mathbf{An}) \to \mathbf{An}$. However, if we expand the ∞-category $\mathbf{An}$, then it turns out that such an adjoint exists. Indeed, one can extend Disc to a functor $\mathrm{Disc}' : \mathrm{Pro}(\mathbf{An}) \to \mathbf{Cond}(\mathbf{An})$ which preserves small filtered limits [Lur18, Proposition A.8.1.6] and this admits a left adjoint through abstract nonsense [BGH20, 0.11.9].

All of this is essentially [Mai21, Lemma 7.2.1, Proposition 7.2.2] which we have already discussed in the starred section 3.3.

This is a first indication that cohesion might not be compatible with condensed mathematics. We have introduced fractured structures precisely for this purpose.

4.2 Compatibility with Fractured Structure

We write $\mathbf{Stonean}^{\mathrm{open}}$ for the site consisting of the wide subcategory of **Stonean** with morphisms the open embeddings and coverings the finitely jointly surjective families of open embeddings. Correspondingly, we put $\mathbf{Cond}^{\mathrm{open}}(\mathbf{An}) = \mathbf{Sh}(\mathbf{Stonean}^{\mathrm{open}})$.

We choose open embeddings because they have good categorical properties with respect to pullbacks (3.9). This will be useful in making $\mathbf{Stonean}^{\mathrm{open}}$ into an admissibility structure on **Stonean** which is the first step towards a fractured structure on $\mathbf{Cond}(\mathbf{An})$.

Finally, we may present the eponymous result of the thesis.

Theorem 4.5. The restriction functor $i^* : \mathbf{Cond}(\mathbf{An}) \to \mathbf{Cond}^{\mathrm{open}}(\mathbf{An})$ induces a triple adjunction

$$\begin{tikzcd}
\mathbf{Cond}(\mathbf{An}) \arrow[r, "i^*" description] & \mathbf{Cond}^{\mathrm{open}}(\mathbf{An}) \arrow[l, bend right, "i_!"'] \arrow[l, bend left, "i_*"] \arrow[l, phantom, shift right=2, "\perp"] \arrow[l, phantom, shift left=2, "\perp"]
\end{tikzcd}$$

which is a fractured structure on condensed anima.

Proof. We prove that $(\mathbf{Stonean}, \mathbf{Stonean}^{\mathrm{open}}, \tau_{\mathrm{coh}})$ defines a geometric site. Let us begin by showing that $\mathbf{Stonean}^{\mathrm{open}}$ defines an admissibility structure where we check each axiom step-by-step.

(i) Isomorphisms in **Stonean** are homeomorphisms and in particular open embeddings.

(ii) We have to prove that for an open embedding $f : S \hookrightarrow T$ of Stonean spaces and any $g : T' \to T$ in **Stonean** there exists a pullback

$$\begin{tikzcd}
S \times_T T' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner", very near start] & S \arrow[d, hook, "f"] \\
T' \arrow[r, "g"'] & T
\end{tikzcd}$$

and f' is again an open embedding. Classically, the inclusion $f' : g^{-1}(f(S)) \hookrightarrow T'$ is a model for the pullback in **CHaus** and this is still an open embedding. Since this is an open embedding, $g^{-1}(f(S)) \subseteq T'$ is open. On the other hand, S as a Stonean space is compact, so $f(S)$ is compact again, and thus $f(S) \subseteq T$ is closed because T is Hausdorff. Hence, the preimage $g^{-1}(f(S)) \subseteq T'$ is also closed. To summarize, $g^{-1}(f(S))$ as a clopen subspace of the Stonean space T' is Stonean again (3.9).

This square already satisfies the universal property of pullbacks if we test against all compact Hausdorff spaces, so it also satisfies the universal property when we test against Stonean spaces. Thus, this pullback really exists in **Stonean** and f' is an open embedding.

(iii) Consider the commutative triangle

$$\begin{tikzcd}
S \arrow[rr, "f"] \arrow[dr, "h"'] & & T \arrow[dl, hook, "g"] \\
& K &
\end{tikzcd}$$

in **Stonean** where g is an open embedding. If f is an open embedding, then the composition h is also an open embedding.

Conversely, let h be an open embedding. We need to show that f is an open embedding. Since h is injective, also f is injective and thus as an injective map between compact Hausdorff spaces already an embedding. The image of h is open, so also the image of f must be open because g is an embedding. In other words, f is an open embedding.

(iv) Let $f : S \hookrightarrow T$ be an open embedding of Stonean spaces with retract $g : A \to B$ as depicted in the following diagram:

$$\begin{tikzcd}
A \arrow[r, "\varphi_1"] \arrow[d, "g"'] \arrow[rr, bend left, equal] & S \arrow[r, "\varphi_2"] \arrow[d, hook, "f"] & A \arrow[d, "g"] \\
B \arrow[r, "\psi_1"'] \arrow[rr, bend right, equal] & T \arrow[r, "\psi_2"'] & B
\end{tikzcd} .$$

We need to show that g is also an open embedding. To do so we prove that the left square is a pullback square, then we may conclude by (ii). Let K be a Stonean space with two continuous maps $\kappa_1 : K \to S$ and $\kappa_2 : K \to B$ such that $f \circ \kappa_1 = \psi_1 \circ \kappa_2$.

$$\begin{tikzcd}
K \arrow[rrd, bend left, "\kappa_1"] \arrow[rd, dashed, "\exists!\Phi"] \arrow[rdd, bend right, "\kappa_2"'] & & & \\
& A \arrow[r, "\varphi_1"] \arrow[d, "g"'] & S \arrow[r, "\varphi_2"] \arrow[d, hook, "f"] & A \arrow[d, "g"] \\
& B \arrow[r, "\psi_1"'] & T \arrow[r, "\psi_2"'] & B
\end{tikzcd}$$

We need to show that there is a unique dashed arrow realizing a factorization. Uniqueness comes for free because φ_1 is injective. So we need to construct one such map. We claim that $\Phi = \varphi_2 \circ \kappa_1$ works. So we need to prove $\varphi_1 \circ \Phi = \kappa_1$ and $g \circ \Phi = \kappa_2$ for this choice of Φ. This is a formal diagram chase which we will perform now.

We compute

$$\begin{aligned}
f \circ \varphi_1 \circ \Phi &= f \circ \varphi_1 \circ \varphi_2 \circ \kappa_1 \\
&= \psi_1 \circ g \circ \varphi_2 \circ \kappa_1 \\
&= \psi_1 \circ \psi_2 \circ f \circ \kappa_1 \\
&= \psi_1 \circ \psi_2 \circ \psi_1 \circ \kappa_2 \\
&= \psi_1 \circ \kappa_2 \\
&= f \circ \kappa_1
\end{aligned}$$

and so by injectivity of f we deduce $\varphi_1 \circ \Phi = \kappa_1$. Similarly, we compute

$$\psi_1 \circ g \circ \Phi = \psi_1 \circ g \circ \varphi_2 \circ \kappa_1 = \psi_1 \circ \psi_2 \circ f \circ \kappa_1 = \psi_1 \circ \psi_2 \circ \psi_1 \circ \kappa_2 = \psi_1 \circ \kappa_2,$$

so $g \circ \Phi = \kappa_2$ by injectivity of ψ_1.

Next, we verify that $\mathbf{Stonean}^{\text{open}}$ is compatible with τ_{coh}.

Let $\{g_\alpha : S_\alpha \to S\}_{\alpha \in J}$ be a covering sieve in **Stonean**. So there exists a finite subset $J' \subseteq J$ such that $(g_\alpha)_{\alpha \in J'} : \coprod_{\alpha \in J'} S_\alpha \to S$ is surjective. Because S is Stonean, there exists a section $s : S \to \coprod_{\alpha \in J'} S_\alpha$ (see 3.4(iii)). For $\alpha \in J'$ we write

$$s_\alpha = s|_{s^{-1}(S_\alpha)} : s^{-1}(S_\alpha) \to S_\alpha \quad \text{and} \quad i_{S_\alpha} : S_\alpha \to \coprod_{\alpha \in J'} S_\alpha .$$

Then, the composition

$$g_\alpha \circ i_{S_\alpha} \circ s_\alpha = (g \circ s)|_{s^{-1}(S_\alpha)} = (\mathrm{id}_S)|_{s^{-1}(S_\alpha)} : s^{-1}(S_\alpha) \to S$$

is the inclusion morphism $s^{-1}(S_\alpha) \hookrightarrow S$. By construction, this is an open embedding and it also lies in the covering sieve $\{g_\alpha\}_{\alpha\in J}$ because it occurs via precomposing morphisms to g_α. Furthermore, $s^{-1}(S_\alpha) \subseteq S$ is clopen and thus Stonean again (3.9). Moreover, $\{g_\alpha \circ i_{S_\alpha} \circ s_\alpha\}_{\alpha\in J'}$ is a finite family of maps which are jointly surjective by construction. So $\{g_\alpha \circ i_{S_\alpha} \circ s_\alpha\}_{\alpha\in J'}$ is a τ_{coh}-covering consisting of admissible morphisms which shows that the admissibility structure is compatible with the Grothendieck topology.

Finally, Lurie's machinery (2.30) induces a fractured structure

$$\begin{tikzcd}
\mathbf{Cond}(\mathbf{An}) \arrow[r, "i^*" description] \arrow[r, phantom, shift left=2, "\perp"] \arrow[r, phantom, shift right=2, "\perp"] & \mathbf{Cond}^{\mathrm{open}}(\mathbf{An}) \arrow[l, shift right=4, "i_!"'] \arrow[l, shift left=4, "i_*"]
\end{tikzcd}$$

as desired. ☐

This really is a non-trivial fracture subcategory!

Proposition 4.6. The functor $i_! : \mathbf{Cond}^{\mathrm{open}}(\mathbf{An}) \to \mathbf{Cond}(\mathbf{An})$ is not an equivalence of ∞-categories.

Proof. We show that $i_!$ does not preserve the terminal object $*_{\mathbf{Cond}^{\mathrm{open}}(\mathbf{An})}$.

As a right adjoint, i^* preserves limits, so

$$*_{\mathbf{Cond}^{\mathrm{open}}(\mathbf{An})} \simeq i^* *_{\mathbf{Cond}(\mathbf{An})} : (\mathbf{Stonean}^{\mathrm{open}})^{\mathrm{op}} \to \mathbf{An}$$

which means that $*_{\mathbf{Cond}^{\mathrm{open}}(\mathbf{An})}$ is the functor which assigns to each object the terminal anima.

Considered as a presheaf $*_{\mathbf{Cond}^{\mathrm{open}}(\mathbf{An})} \in \mathbf{PSh}(\mathbf{Stonean}^{\mathrm{open}})$ we may write

$$\begin{aligned}
*_{\mathbf{Cond}^{\mathrm{open}}(\mathbf{An})} &= (\mathrm{Lan}_{\text{よ}}\, \text{よ})(*_{\mathbf{Cond}^{\mathrm{open}}(\mathbf{An})}) \\
&\simeq \operatorname*{colim}_{(S,\text{よ}(S)\to *)\in \text{よ}/S} \text{よ}(S) \\
&\simeq \operatorname*{colim}_{S\in\mathbf{Stonean}^{\mathrm{open}}} \text{よ}(S). \\
&= \operatorname{colim} \text{よ}
\end{aligned}$$

where we use $\mathrm{id}_{\mathbf{PSh}(\mathbf{Stonean}^{\mathrm{open}})} \simeq \mathrm{Lan}_{\text{よ}}\, \text{よ}$ by the Density Theorem [Lur09, Lemma 5.1.5.3]. In the second line of this computation, we used that $*_{\mathbf{Cond}^{\mathrm{open}}(\mathbf{An})}$ is objectwise the terminal object. Since the Grothendieck topology on **Stonean** is subcanonical (3.25) and the restriction functor $i^* : \mathbf{Cond}(\mathbf{An}) \to \mathbf{Cond}^{\mathrm{open}}(\mathbf{An})$ preserves representables, we deduce that the essential image of $\text{よ} : \mathbf{Stonean}^{\mathrm{open}} \to \mathbf{PSh}(\mathbf{Stonean}^{\mathrm{open}})$ lies in $\mathbf{Cond}^{\mathrm{open}}(\mathbf{An})$.

So we get

$$\begin{aligned}
i_! *_{\mathbf{Cond}^{\mathrm{open}}(\mathbf{An})} &= i_! \operatorname{colim} \text{よ} \\
&= i_! \operatorname*{colim}_{S\in\mathbf{Stonean}^{\mathrm{open}}} \mathrm{Hom}_{\mathbf{Stonean}^{\mathrm{open}}}(-,S) \\
&\simeq \operatorname*{colim}_{S\in\mathbf{Stonean}^{\mathrm{open}}} \mathrm{Hom}_{\mathbf{Stonean}}(-,S) \\
&\simeq \operatorname*{colim}_{S\in\mathbf{Stonean}^{\mathrm{open}}} \mathrm{const}\,\mathrm{Hom}_{\mathbf{Stonean}}(-,S) \\
&\simeq \mathrm{const} \operatorname*{colim}_{S\in\mathbf{Stonean}^{\mathrm{open}}} \mathrm{Hom}_{\mathbf{Stonean}}(-,S).
\end{aligned}$$

In the first equivalence, we use that left adjoints commute with colimits and that left Kan extensions commute with representables. Then, we use that the Hom-anima of 1-categories are given by the constant simplicial sets of the respective Hom-sets of the 1-categories. Finally, the functor const : **Set** → **An** is the left adjoint of $\tau_{\leq 0} : \mathbf{An} \to \mathbf{Set}$ and thus preserves colimits.

In particular, it suffices to compute a 1-categorical colimit. Consider $[1] = \{0, 1\}$ with the discrete topology as a Stonean space. Then, we claim

$$\left(i_! *_{\mathbf{Cond}^{\mathrm{open}}(\mathbf{An})}\right)([1]) \not\simeq *.$$

For this we show that the representatives of $\mathrm{id}_{[1]}, \mathrm{const}_0 \in \mathrm{Hom}_{\mathbf{Stonean}}([1],[1])$ define different classes in

$$\operatorname*{colim}_{S \in \mathbf{Stonean}^{\mathrm{open}}} \mathrm{Hom}_{\mathbf{Stonean}}([1], S).$$

We first observe that in this colimit system $[\mathrm{const}_0]$ is identified with the representative of the unique map $[1] \to *$ since const_0 lies in the image of the map

$$\mathrm{Hom}_{\mathbf{Stonean}}([1], *) \to \mathrm{Hom}_{\mathbf{Stonean}}([1],[1])$$

induced by $0 : * \to [1], * \mapsto 0$.

Hence, in every $\mathrm{Hom}_{\mathbf{Stonean}}([1], S)$ with $S \in \mathbf{Stonean}$ the map $[\mathrm{const}_0]$ is represented by any constant map. We now prove that $[\mathrm{id}_{[1]}]$ does not have any constant maps as a representative.

The only maps in the colimit system with target $\mathrm{Hom}_{\mathbf{Stonean}}([1],[1])$ are the two maps

$$\begin{tikzcd}\mathrm{Hom}_{\mathbf{Stonean}}([1], *) \arrow[r, shift left] \arrow[r, shift right] & \mathrm{Hom}_{\mathbf{Stonean}}([1],[1])\end{tikzcd}$$

induced by the two constant maps $* \to [1]$. So no map into $\mathrm{Hom}_{\mathbf{Stonean}}([1],[1])$ hits $\mathrm{id}_{[1]}$. Therefore, all representatives are constructed via zig-zags of the form

$$\begin{tikzcd}\mathrm{Hom}_{\mathbf{Stonean}}([1],[1]) \arrow[r] & \mathrm{Hom}_{\mathbf{Stonean}}([1], S_1) & \mathrm{Hom}_{\mathbf{Stonean}}([1], S_2) \arrow[l] \arrow[r] & \cdots,\end{tikzcd}$$

for $S_1, S_2, \cdots \in \mathbf{Stonean}$, i.e. zig-zags starting in $\mathrm{Hom}_{\mathbf{Stonean}}([1],[1])$. On the other hand, any starting zig-zag as above is induced by a diagram

$$\begin{tikzcd}{[1]} \arrow[rr, dashed] \arrow[dr, hook] & & S_2 \arrow[dl, hook'] \\ & S_1 &\end{tikzcd}$$

and $S_2 \hookrightarrow S_1$ as an injective map of compact Hausdorff spaces is an embedding. Thus, the diagram factors, i.e. a dashed arrow as in the diagram exists. This means that in the zig-zag we also obtain a factorization

$$\begin{tikzcd}\mathrm{Hom}_{\mathbf{Stonean}}([1],[1]) \arrow[rr, dashed] \arrow[dr] & & \mathrm{Hom}_{\mathbf{Stonean}}([1], S_2) \arrow[dl] \\ & \mathrm{Hom}_{\mathbf{Stonean}}([1], S_1) &\end{tikzcd}$$

which means that the left-handed maps in the zig-zags are redundant since there is always also a right-handed map obtained by the above factorizations. In other words, all representatives of $[\mathrm{id}_{[1]}]$ are represented by images of $\mathrm{id}_{[1]}$ through a map

$$\mathrm{Hom}_{\mathbf{Stonean}}([1],[1]) \to \mathrm{Hom}_{\mathbf{Stonean}}([1], S)$$

induced by an injective map $[1] \hookrightarrow S$ into a Stonean S. By injectivity of $[1] \hookrightarrow S$ the image cannot be a constant map. So we have verified $[\mathrm{const}_0] \neq [\mathrm{id}_{[1]}]$ as desired. ☐

Corollary 4.7. The ∞-topos **Cond**(**An**) is not cohesive over $\mathbf{Cond}^{\mathrm{open}}(\mathbf{An})$ via i^*.

Proof. We have shown in the proof of 4.6 that $i_!$ does not preserve limits and hence cannot admit another left adjoint. ☐

Alternatively, we may recall that the only setting where a fractured structure also gives cohesion is when the fractured structure is trivial (2.22) and we have seen that this is not the case (4.6).

5 Application of Fractured Structures to Condensed Cohomology

Classically, deriving the global sections functor yields sheaf cohomology. In the ∞-categorical sense, this information can be formulated more succinctly: (derived) global sections are given by $\mathrm{Hom}(*, -)$. In particular, it is our vision to apply a fully faithful functor which preserves terminal objects to preserve cohomology. Our technology of fractured structures allows us to obtain natural functors of this kind which motivates us to think about cohomology in the context of fractured ∞-topoi.

We consider a cohomology result which was already known in 1976 by Dyckhoff [Dyc76, Theorem 3.11] but was reformulated in the modern language of condensed mathematics by Clausen-Scholze.

Theorem 5.1. Let $S \in \mathbf{CHaus}$ and let M be a discrete abelian group. Then,

$$H^\bullet_{\mathrm{sheaf}}(S, \underline{M}) \cong H^\bullet_{\mathrm{cond}}(S, \underline{M})$$

naturally.

Proof. See [Sch19, Theorem 3.2]. $\square$

This is precisely the comparison between two cohomology theories internal to different topoi and hence seems approachable by our vision via fractured structures.

Recall that there are faithful functor

$$\underline{(-)} : \mathbf{Top} \to \mathbf{Cond}(\mathbf{Set}) \to \mathbf{Cond}(\mathbf{An}) \quad \text{and} \quad i_! : \mathbf{Cond}^{\mathrm{open}}(\mathbf{An}) \to \mathbf{Cond}(\mathbf{An})$$

where faithfulness of $i_!$ drops out of the fractured structure (4.5).

Definition 5.2. We define **corporeal spaces** as the objects in the intersection of the essential images of $\mathbf{Top}$ and $\mathbf{Cond}^{\mathrm{open}}(\mathbf{An})$ in $\mathbf{Cond}(\mathbf{An})$. We write $\mathbf{CorpTop} \hookrightarrow \mathbf{Cond}(\mathbf{An})$ for the full subcategory spanned by the corporeal spaces.

We chose this name because objects in $\mathbf{Top}$ are spaces and objects in (the image of) $\mathbf{Cond}^{\mathrm{open}}(\mathbf{An})$ are corporeal. Another plus for us to use the terminology 'anima' instead of 'spaces'!

We restrict to the corporeal spaces because we have results on the corporeal objects by our fractured structure on condensed anima and because a priori we only know how to define sheaf cohomology on actual spaces.

Proposition 5.3. Let $F \in \mathbf{Cond}^{\mathrm{open}}(\mathbf{An})$ such that $i_!F$ is a corporeal space given by $i_!F = \underline{S}$ for $S \in \mathbf{Top}$. Then, there is an equivalence of categories

$$\mathbf{Cond}^{\mathrm{open}}(\mathbf{An})_{/F} \simeq \mathbf{Sh}(S).$$

Proof. Let $\mathbf{Clopen}_S$ denote the poset category of clopen subsets of S. An outline of the proof is the following chain of equivalences:

$$\begin{aligned}
\mathbf{Cond}^{\mathrm{open}}(\mathbf{An})_{/F} &= \mathbf{Sh}(\mathbf{Stonean}^{\mathrm{open}})_{/F} \\
&\overset{(1)}{\simeq} \mathbf{Sh}\left(\mathbf{Stonean}^{\mathrm{open}}_{/S}\right) \\
&\overset{(2)}{\simeq} \mathbf{Sh}(\mathbf{Clopen}_S, \text{finite coverings}) \\
&\overset{(3)}{=} \mathbf{Sh}(\mathbf{Clopen}_S, \text{arbitrary coverings}) \\
&\overset{(4)}{\simeq} \mathbf{Sh}(\mathbf{Open}_S) \\
&= \mathbf{Sh}(S).
\end{aligned}$$

Let us verify each of these three equivalences with more care.

(1) This is a version of the fundamental theorem of topos theory [AGV71, Exercise 9.8.3].

(2) Morphisms in $\mathbf{Stonean}^{\mathrm{open}}$ are the open embeddings. An injective morphism of compact Hausdorff spaces is already an embedding. Moreover, Stonean subspaces of S are compact Hausdorff and hence also closed. So open embeddings of Stonean spaces to S correspond to clopen subsets of S.

(3) This follows from compactness.

(4) Since every totally disconnected locally compact Hausdorff space has a clopen basis [AT08, Proposition 3.1.7], the subcategory $\mathbf{Clopen}_S \hookrightarrow \mathbf{Open}_S$ is a basis. We conclude by the Comparison Lemma (1.74(ii)).

□

Theorem 5.4. Let $S \in \mathbf{Top}$ with $\underline{S} \in \mathbf{CorpTop}$ and let M be a discrete abelian group. Then,

$$H^\bullet_{\mathrm{sheaf}}(S, \underline{M}) \cong H^\bullet_{\mathrm{cond}}(S, \underline{M})$$

naturally.

Proof. Suppose $F \in \mathbf{Cond}^{\mathrm{open}}(\mathbf{An})$ such that $i_! F = \underline{S}$. Cohomology is given by the derived global sections. In particular, we may compute

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Sh}(S)}(*_{\mathbf{Sh}(S)}, \underline{M}) &\simeq \mathrm{Hom}_{\mathbf{Cond}^{\mathrm{open}}(\mathbf{An})_{/F}}(*_{\mathbf{Cond}^{\mathrm{open}}(\mathbf{An})_{/F}}, \underline{M}) \\ &\simeq \mathrm{Hom}_{\mathbf{Cond}(\mathbf{An})_{/\underline{S}}}(*_{\mathbf{Cond}(\mathbf{An})_{/\underline{S}}}, \underline{M})\end{aligned}$$

where the first line uses the previous result (5.3) and the second line is via the fully faithfulness given by the fractured structure (2.18, 4.5). Now, the left side computes $H^\bullet_{\mathrm{sheaf}}(S, \underline{M})$ while the right side computes $H^\bullet_{\mathrm{cond}}(S, \underline{M})$, so these cohomologies agree. □

Peter Haine discussed similar questions in his recent paper *Descent for sheaves on compact Hausdorff spaces* [Hai22] with an analogous approach from which we learned. He manages to give the following generalization of the cohomology result:

Theorem 5.5 ([Hai22, Corollary 4.12])**.** Let S be a locally compact Hausdorff space, let R be a connective $\mathbb{E}_1$-ring spectrum, and let M be a bounded-above left R-module spectrum. Then, the natural map

$$R\Gamma_{\mathrm{sheaf}}(S; M) \to R\Gamma_{\mathrm{cond}}(S; M)$$

is an equivalence in the ∞-category ${}_R\mathbf{Mod}$ of left R-module spectra.

However, Haine did not use the technology of fractured structure. That is a novel approach probed in this thesis.

6 Outlook

Fractured structures on condensed anima seem like a fruitful endeavor that can lead to numerous additional interesting questions. While one could try thinking of additional applications of the fractured structure, we will give several ideas on how condensed cohomology can be further investigated.

Condensed cohomology itself is an interesting invariant that is not yet completely understood. For example, to our knowledge, even the groups $H^\bullet(\mathbb{Q}, \underline{\mathbb{Z}})$ are not known. In that regard, it is also not known in what generality there is an isomorphism

$$H^\bullet_{\mathrm{sheaf}}(S, \underline{\mathbb{Z}}) \cong H^\bullet_{\mathrm{cond}}(S, \underline{\mathbb{Z}}).$$

It seems hard to give a complete characterization but our hope is that this thesis gives another approach to attack this problem. We have shown the isomorphism for $S \in \mathbf{CorpTop}$.

So, naturally, the fundamental question remains:

Problem 6.1. What is **CorpTop**?

This question also seems to be too hard to be answered completely.

Instead of fully classifying this category, one could attempt to formally construct classes of spaces included in **CorpTop**. For example, we know that **Stonean** $\hookrightarrow$ **CorpTop** since the topology on **Stonean** is subcanonical (3.25). One could now attempt to investigate formal categorical constructions that are stable in **CorpTop** and hope to construct classes of spaces in **CorpTop**.

On the other hand, a choice of a fracture subcategory of $\mathbf{Cond}(\mathbf{An})$ is certainly not unique. Instead of using $i_! : \mathbf{Cond}^{\mathrm{open}}(\mathbf{An}) \to \mathbf{Cond}(\mathbf{An})$ induced by the geometric site

$$(\mathbf{Stonean}, \mathbf{Stonean}^{\mathrm{open}}, \tau_{\mathrm{coh}})$$

one could try to find other feasible fractured structures especially if $\mathbf{Cond}^{\mathrm{open}}(\mathbf{An}) \to \mathbf{Cond}(\mathbf{An})$ appears too complicated to understand.

Recall: On **CHaus**, there is an admissibility structure $\mathbf{CHaus}^{\mathrm{inj}}$ given by the injective maps (2.25) by an analogous but easier argument as in (4.5). The natural hope is that $(\mathbf{CHaus}, \mathbf{CHaus}^{\mathrm{inj}}, \tau_{\mathrm{coh}})$ defines a geometric site. We had to make the unfortunate discovery that this is not the case.

Lemma 6.2. The triple $(\mathbf{CHaus}, \mathbf{CHaus}^{\mathrm{inj}}, \tau_{\mathrm{coh}})$ does not define a geometric site.

Proof. We show that the admissibility structure $\mathbf{CHaus}^{\mathrm{inj}}$ is not compatible with τ_{coh} in the sense of 2.29.

Compatibility means that for any covering sieve $S = \{g_\alpha : V_\alpha \to X\}_\alpha$ there exists a τ_{coh}-covering of admissible morphisms inside S. So if there was compatibility, then we would want to find a finite jointly surjective family of injective maps inside S. We demonstrate that this is not always possible.

Consider the surjective map

$$e : [0,1]^{\mathbb{N}} \to (S^1)^{\mathbb{N}},\ (x_0, x_1, \cdots) \mapsto \left(e^{2\pi i x_0}, e^{2\pi i x_1}, \cdots\right).$$

It generates a covering sieve. We claim that it is not possible to find a finite family of maps $h_1, \cdots, h_n$ with target $[0,1]^{\mathbb{N}}$ such that

(1) $e \circ h_i$ is injective for $i = 1, \cdots, n$,

(2) $(e \circ h_i)_{i=1,\cdots,n}$ is jointly surjective

which is what we would need for compatibility. Suppose it was possible.

Choose such $h_1, \cdots, h_n$. For $i = 1, \cdots, n$ the $e \circ h_i$ are injective, so also the h_i are injective. Injections of compact Hausdorff spaces are embeddings. Therefore, our task can be reformulated to finding a finite number of closed subspaces $A^1, \cdots, A^n \subseteq [0,1]^{\mathbb{N}}$ such that

(1′) $e|_{A^i}$ is injective for $i = 1, \cdots, n$,

(2′) $(S^1)^{\mathbb{N}} = e(A^1) \cup \cdots \cup e(A^n)$.

Let $k \in \mathbb{N}$ and let $\mathrm{pr}_k : [0,1]^{\mathbb{N}} \to [0,1]$ be the projection onto the k-th component. Then,

$$\left|\mathrm{pr}_k(A^i) \cap \{0,1\}\right| \leq 1$$

for $i = 1, \cdots, n$ by injectivity of $e|_{A^i}$ since $e^{2\pi i 0} = e^{2\pi i 1}$.

Since there are infinitely many elements in $\{0,1\}^{\mathbb{N}}$, there exists some sequence $(t_k)_{k\in\mathbb{N}} \in [0,1]^{\mathbb{N}}$ such that $(t_k)_{k\in\mathbb{N}} \notin A^1 \cup \cdots \cup A^n$. As a finite union $A^1 \cup \cdots \cup A^n$ is still closed, so there exists some open neighbourhood $U \subseteq [0,1]^{\mathbb{N}}$ of $(t_k)_{k\in\mathbb{N}}$ such that $U \subseteq [0,1]^{\mathbb{N}} \setminus (A^1 \cup \cdots \cup A^n)$. Thus, there exists some $(u_k)_{k\in\mathbb{N}} \in U$ such that $u_k \notin \{0,1\}$ for every $k \in \mathbb{N}$. But $e|_{(0,1)^{\mathbb{N}}}$ is injective, so

$$e((u_k)_{k\in\mathbb{N}}) \notin e(A^1 \cup \cdots \cup A^n) = e(A^1) \cup \cdots e(A^n),$$

contradicting condition (1'). ☐

This was the first site we considered and the failure is unfortunate because $(\mathbf{CHaus}, \mathbf{CHaus}^{\mathrm{inj}}, \tau_{\mathrm{coh}})$ could possibly have led to a flexible fractured structure but it was probably too much to ask for.

Instead, we tried to salvage this idea and tried to slightly adjust the construction. One could try to take open embeddings instead of injective morphisms as admissible morphisms but this does not seem like a good idea: Open embeddings in **CHaus** are the same thing as clopen embeddings. Then, for example the only clopen subsets in $[0,1]$ are $\emptyset, [0,1]$ by connectedness of $[0,1]$.

On the other hand, restricting to open embeddings on **Stonean** to obtain $\mathbf{Stonean}^{\mathrm{open}}$ does not seem to be too restrictive: By definition, the closures of open subsets of Stonean spaces are open again, so Stonean spaces seem to have plenty of clopen subsets. This is the origin of the fractured structure on condensed anima that we have constructed (4.5).

The reason that we restricted to $\mathbf{Stonean}^{\mathrm{open}}$ with only open embeddings instead of $\mathbf{Stonean}^{\mathrm{inj}}$ with injective maps is that pullbacks along open embeddings exist in **Stonean** as shown in the proof of 4.5. This is condition 2.24(ii) for admissibility structures. On the other hand, we are not certain that $\mathbf{Stonean}^{\mathrm{inj}}$ does not define an admissibility structure.

Problem 6.3. Let $f : S \to T$ be an injective map of Stonean spaces and let $g : T' \to T$ be a map in **Stonean**. Does a pullback

$$\begin{tikzcd}
S \times_T T' \arrow[r] \arrow[d, "f'"'] & S \arrow[d, hook, "f"] \\
T' \arrow[r, "g"'] & T
\end{tikzcd}$$

exist?

Here is a natural subproblem suggested by Dustin Clausen:[2]

Problem 6.4. Do fibers exist in **Stonean**?

We suspect that the answer is negative but have not yet managed to find a proof.

For open embeddings f we have seen that pullbacks exist (3.9) but this remains the only class of morphisms that we can handle. Throughout this work we have seen the usefulness of the existence of these pullbacks, as they allowed us to obtain an admissibility structure $\mathbf{Stonean}^{\mathrm{open}}$ which ultimately landed us a fractured structure on $\mathbf{Cond}(\mathbf{An})$. So an investigation of these problems (6.3, 6.4) may lead to further fractured structures on condensed anima.

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[2] ...from a conversation with my supervisor Nima Rasekh. Nima also asked Peter Scholze about problem 6.3 who did not know an answer off the top of his head.

The task of finding a suitable fractured structure on condensed anima to understand condensed cohomology seems more fruitful and complex than even originally expected. Further explorations of this problem will hopefully lead to results about condensed cohomology. More generally, further investigations should not only lead to questions about a fractured structure on condensed anima but should also motivate various questions about fractured structures and about condensed anima.

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