MATH 283: Topological Field Theories

Stanford Winter 2008, Prof. Ralph Cohen

Course Notes
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Contents

1 Introduction; Physical Motivation 1/10/08

1.1 Introduction

Topological quantum field theories (TQFTs) have found many applications in areas of mathematics. They were first developed by Witten and later by Atiyah and Segal as a way to assemble information on invariants of closed manifolds satisfying certain gluing properties. A TQFT pertaining to n- and $(n+1)$-dimensional manifolds is called an $n+1$-dimensional TQFT. Some applications include:

1. Donaldson-Floer theory ($3+1$-dim'l)
2. Jones and Kauffman polynomial invariants ($2+1$-dim'l)
3. Gromov-Witten theory ($1+1$-dim'l)

4. Symplectic Field Theory ($2n$-dim'l)
5. String Topology
6. Freed-Hopkins-Telemann twisted K-theory

Furthermore, TQFTs highlight some algebra and category theory, including

1. Frobenius algebras, Gerstenhaber algebras, Batalin-Vilkovisky algebras,
2. Tensor categories, operads, and PROPs

We seek to classify all such theories, starting with certain 2-dim'l ones. Work by Costello has permitted the classification of topological conformal field theories where the invariants are vector spaces over a field k of characteristic 0. Other naturally occuring examples of TQFTs involve more general algebras, however. Furthermore, recently derived versions of such theories, replacing the algebra with constructions in stable homotopy theory, have been developed (see the lectures by J. Lurie this winter).

1.2 Physical Motivation: Classical Field Theory

Segal's article "Topological Structures in String Theory" [18] contains an overview of some of this physical development.

The first example of a field theory arising in physics was that of electromagnetic (EM) field theory, in the 19th century, which concerns the study of two 3-dimensional vector fields E and B on $\mathbb{R}^{3+1}$ (space-time). These vector fields can be combined to give a function $F : \mathbb{R}^4 \to \mathbb{R}^6$, which can then be arranged as a 2-form $F(x) = \sum_{0\leq i<j\leq 3} F_{ij}(x)\, dx_i \wedge dx_j$. Maxwell's equations for E and B then become the constraints $dF = 0$ and $d{*}F = 0$ that F must satisfy. Here, d is a covariant exterior derivative on forms, and $*$ is the Hodge star duality operator; note that these operations depend on a metric on $\mathbb{R}^4$.

Another field theory to arise in physics is gravity: here, we replace $\mathbb{R}^4$ with a 4-manifold X, equipped with a metric g. Then consider the functional

$$S(X, g) = \int_X R_g\, d(vol),$$

where R_g is the scalar curvature of g. Then g is a critical point of this functional precisely when it satisfies the Einstein field equations.

In the 1950s, it was discovered that if $X \subset \mathbb{R}^4$ is a nonsimply connected, open set, then one could have an EM field with 0 field strength (i.e., $F = 0$) while still exhibiting some physical effect. (This is the Aharonov-Bohm effect.) Chern and other mathematicians determined that the correct mathematical object to represent an EM field should be a connection A on a $\mathbb{C}$-line bundle L over X^4, the curvature form of which should satisfy Maxwell's equations.

We recall the basic elements of connections on vector bundles. Suppose $p : L \to$ is a $\mathbb{C}$-line bundle on X and P_L is its associated principal $U(1)$-bundle. A **(principal) connection** is a $U(1)$-equivariant splitting of $T(P_L)$ into $p^*TX \oplus T_{\mathrm{vert}P}$. The associated **covariant derivative** $D_A : \Omega^q(X;\mathbb{R}) \to \Omega^{q+1}(X;\mathbb{R})$ is a linear function satisfying $D_A(f\sigma) = fD_A(\sigma) + df \wedge \sigma$, for $f \in C^\infty(X;\mathbb{R})$.

To be precise, we take our differential forms with coefficients in L, where $\Omega^0(X;L) = \Gamma(L)$, the smooth sections of L. If V is a vector field on X, then $D_A(\sigma)(V) = [\tilde{V}, \sigma]$, where $\tilde{V}$ is the horizontal lifting of V to a vector field on P_L coming from the splitting of the connection. Then $D_A^2 : \Omega^0(X) \to \Omega^2(X)$ gives the curvature form $F_A \in \Omega^2(X)$ (or more precisely, $\Omega^2(X; \mathrm{ad}\, P)$), which satisfies the Bianchi identity $D_A F_A = 0$ and $[F_A] = c_i(L)$ (by Chern-Weil theory).

We can also think of the connection as a parallel transport operator: to each curve $\gamma : I \to X$, we assign a linear operator $\tau_A(\gamma) : L_{\gamma(0)} \to L_{\gamma(1)}$, such that

- $\tau_A(\gamma)$ is independent of the parameterization of γ,
- τ_A respects gluing, so $\tau_A(\gamma_1 * \gamma_2) = \tau_A(\gamma_2) \circ \tau_A(\gamma_1)$.

Then for $x_0 \in X$, τ_A determines a map $\tau_A : \Omega_{x_0}X \to \mathrm{Iso}\, L_{x_0} \cong U(1)$. If $F_A = 0$, so that the connection is flat, then $\tau_A(\gamma)$ depends only on the path-homotopy class of γ, and so this map τ_A factors through $\pi_1(X, x_0)$ to give a representation $\tau_A : \pi_1(X, x_0) \to U(1)$. (This is the **holonomy** representation.) Thus, EM fields with zero field strength are understood via the representation theory of $\pi_1(X)$ into $U(1)$ (or potentially other Lie groups or principal G-bundles).

We now relate parallel transport to field strength: suppose Σ is a surface with $\partial\Sigma = S^1$, and $\sigma : \Sigma \to X$ is a map with $\gamma = \partial\sigma : S^1 \to X$. Assume these maps are based at x_0. Then define

$$F(\sigma) = \int_\Sigma \sigma^* F_A,$$

so that $e^{2\pi i F(\sigma)} = \tau_A(\gamma) \in U(1)$ (perhaps up to some factor). Thus, if $F(\sigma) = 0$, then $\tau_A(\gamma) = 1$, and the holonomy representation of this loop is constant.

Finally, given a connection A on a $\mathbb{C}$-line bundle $p : L \to X$, we obtain a functor τ_A from a path category $\mathcal{P}_X$ to a category of lines $\mathcal{L}$ in $\mathbb{C}^\infty$. We describe these categories as follows: Obj $\mathcal{P}_X = X$, and for $x, y \in X$,

$$\mathcal{P}_X(x, y) = \{(t \in \mathbb{R}, \gamma : [0, t] \to X) \mid \gamma(0) = x, \gamma(t) = y\},$$

so that composition is given by concatenation. Then Obj $\mathcal{L} = G_1(\mathbb{C}^\infty)$, and $\mathcal{L}(L_0, L_1) = \mathrm{Iso}(L_0, L_1)$.

1.3 Physical Motivation: String Field Theories

We now generalize these ideas to define a **string field** (or **B-field**, **gerbe**, or **gerbe with connection**). We now associate to every loop $\gamma : S^1 \to X$ a $\mathbb{C}$-line L_γ, independent of the parameterization of γ. In order to do this, we consider the space of closed strings

$$LX/\!/S^1 = \{(S \subset \mathbb{R}^\infty, f : S \to X) \mid S \text{ a closed, oriented 1-manifold}\}.$$

We topologize this space as follows (in the manner of Galatius-Madsen-Tillman-Weiss, or even Thom originally):

$$LX//S^1 = \coprod_{k\geq 1} \mathrm{Emb}(\coprod_k S^1, \mathbb{R}^\infty) \times_{\mathrm{Diff}^+(\coprod_k S^1)} \mathrm{Map}(\coprod_k S^1, X).$$

Since $\mathrm{Emb}(\coprod_k S^1, \mathbb{R}^\infty)$ is contractible and has a free action by $\mathrm{Diff}^+(\coprod_k S^1)$, it is a model for $E\,\mathrm{Diff}^+(\coprod_k S^1)$, and so the components of $LX//S^1$ are homotopy orbit spaces $\mathrm{Map}(\coprod_k S^1, X)_{h\,\mathrm{Diff}^+(\coprod_k S^1)}$. The "//" indicates that this is a homotopy or stack-y quotient space.

Now to a surface connecting two sets of loops, we want a notion of parallel transport. Suppose that $\Sigma \subset \mathbb{R}^\infty \times [0,t]$ is a surface with incoming boundary $\partial_{\mathrm{in}}\Sigma = \coprod_{i=1}^p \gamma_i$ in $\mathbb{R}^\infty \times \{0\}$ and outgoing boundary $\partial_{\mathrm{out}}\Sigma = \coprod_{i=p+1}^{p+q} S^1$ in $\mathbb{R}^\infty \times \{t\}$, and $\sigma : \Sigma \to X$ is a continuous map. We seek to assign a linear map

$$B_\sigma : L_{\partial_{\mathrm{in}}\sigma} = \bigotimes_{i=1}^{p} L_{\sigma|\gamma_i} \to L_{\partial_{\mathrm{out}}\sigma} = \bigotimes_{i=p+1}^{p+q} L_{\sigma|\gamma_i},$$

such that, as above,

- B_σ is independent of the parameterization of Σ,
- B respects gluing of surfaces along boundary components.

As above, we have a notion of holonomy: a given closed surface Σ represents a cobordism from $\varnothing$ to $\varnothing$. Taking $L_\varnothing = \mathbb{C}$ canonically, we have that $B_\sigma : \mathbb{C} \to \mathbb{C}$ is a linear isomorphism, hence $B_\sigma \in \mathbb{C}^\times$. In fact, we may further assume that $B_\sigma \in U(1)$. Consequently, we obtain a map

$$B : \mathrm{Emb}(\Sigma, \mathbb{R}^\infty) \times_{\mathrm{Diff}^+(\Sigma)} \mathrm{Map}(\Sigma, X) \to U(1).$$

A theory of a curvature form also exists here, via Chern-Weil theory on X (see [4]) and gives a 3-form $H_B \in \Omega^3(X;\mathbb{C})$ such that $dH_B = 0$ and $[H_B] \in H^3(X;\mathbb{C})$ is described as follows: The evaluation map $LX \times S^1 \to X$ defines a map on cohomology

$$t : H^q(X) \to H^q(LX \times S^1) = (H^*(LX) \otimes H^*(S^1))_q \to H^{q-1}(LX) \otimes H^1(S^1) \cong H^{q-1}(LX).$$

Then $t[H_B] = c_1(\mathcal{L}_B \to LX)$, where $\mathcal{L}_B$ is the line bundle on LX that B determines.

Hence, if there exists a 3-manifold Y and a map $\Psi : Y \to X$ with $\partial Y = \Sigma$, we define

$$H_B(\Psi) = \int_Y \Psi^* H_B,$$

and have that $e^{2\pi i H_B(\Psi)} = B_{\partial\Psi} \in U(1)$. Additionally, the string field B must satisfy some analogues of Maxwell's equations: if (X, g, B) is a smooth manifold with metric g and field B, define

$$S(X, g, B) = \int_X R_g\, d(vol) + H \wedge *H.$$

Then (X, g, B) should be a critical point of this S functional.

These ideas then lead to the notion of a **conformal field theory**, which we take to be

1. a $\mathbb{C}$-vector space (or Hilbert space) H,
2. a parallel transport operator: for each conformal (or topological) surface Σ giving a cobordism from p circles to q circles, we obtain an operator

$$\mu_\Sigma : H^{\otimes p} \to H^{\otimes q}$$

satisfying certain gluing axioms.

We can construct such a theory from a critical point (X, g, B) as follows. Let $H = L^2(\mathcal{L}_B)$ (the L^2-section of the line bundle above). We then express μ_Σ as an integral operator, where the integral is taken over a space of paths. More precisely (but not entirely rigorously), for $\phi : \Sigma \to X$, let $S(\phi) = E(\phi) + iB_\phi$, where $E(\phi)$ is a sort of Dirichlet energy of ϕ. Then define $K : (LX)^p \times (LX)^q \to \mathbb{C}$ by

$$K(\gamma_1, \dots, \gamma_p; \gamma_{p+1}, \dots, \gamma_{p+q}) = \int e^{iS(\phi)} d\phi,$$

with the integral taken over all such ϕ. Finally, for $\alpha \in H^{\otimes p}$ and $y \in (LX)^q$, define

$$\mu_\Sigma(\alpha)(q) = \int_{x \in (LX)^p} K(x, y)\alpha(y)\, d\mu(y).$$

Ultimately, we will replace this non-rigorous path integral with a Pontryagin-Thom construction.

2 TQFTs 1/15/08

2.1 Atiyah-Segal Definition

The first definition of an ***n*-dimensional TQFT** is due to Witten, Atiyah, and Segal and comprises several parts: first, it is a functor E from closed, oriented $(n-1)$-manifolds with diffeomorphisms to $\mathbb{C}$-vector spaces and linear isomorphisms. (In the physical setting, $E(X)$ is interpreted as a vector space of functions or sections of a bundle over some space of fields. For example, the space may be taken to be $\mathrm{Map}(X, M)$ for some fixed M, so that $E(S^1)$ is the line bundle $\mathcal{L} \to LM = \mathrm{Map}(S^1, M)$ from above.) E must satisfy some properties, roughly that "$E(X_1 \coprod X_2) = E(X_1) \otimes E(X_2)$": for any finite family $\{X_\alpha\}_{\alpha\in I}$ of $(n-1)$-manifolds, there exists a multilinear map

$$m_I : \prod_{\alpha\in I} E(X_\alpha) \to E(\textstyle\coprod_{\alpha\in I} X_\alpha)$$

satisfying the the universality property of the tensor product. As a result, there exists an canonical isomorphism $E(\coprod_{\alpha\in I} X_\alpha) \cong \otimes_{\alpha\in I} E(X_\alpha)$. In particular, this yields an S_n-equivariant map $E(X)^{\otimes n} \to E(\coprod_n X)$.

Furthermore, a TQFT assigns to an oriented cobordism Y from X_1 to X_2 a linear map $\psi_Y : E(X_1) \to E(X_2)$ satisfying some properties:

(a) ψ_Y depends only on the oriented diffeomorphism type of Y,

(b) ψ respects gluing, so $\psi_{Y_1 \# Y_2} = \psi_{Y_2} \circ \psi_{Y_1}$,

(c) ψ is tensorial, so if Y' is a cobordism from X_1' to X_2', then $\psi_{Y \coprod Y'} \cong \psi_Y \otimes \psi_{Y'}$ under the identification between tensor product and disjoint union given above.

Finally, we require that $\psi_{X \times I} = \mathrm{id}_{E(X)}$, so that the cylinder cobordism gives a trivial linear map.

We note that $\varnothing$ is a closed $(n-1)$-manifold; we claim that $E(\varnothing) = \mathbb{C}$. Indeed, since

$$E(X) = E(X \coprod \varnothing) \cong E(X) \otimes_{\mathbb{C}} E(\varnothing),$$

we conclude that $E(\varnothing) \cong \mathbb{C}$. Any closed n-manifold Y is a cobordism from $\varnothing$ to $\varnothing$ and so yields a linear map $\psi_Y : \mathbb{C} \to \mathbb{C}$. The corresponding number $\psi_Y \in \mathbb{C}$ is then a diffeomorphism invariant of Y.

This construction raises several questions:

1. Do the numbers ψ_Y, Y a closed n-manifold, determine the theory?
2. Are there restrictions on the type of diffeomorphism invariants that can arise this way? (e.g., is there a TQFT that produces the Euler characteristic?)
3. Is $E(X)$, X an $(n-1)$-manifold, spanned by $\operatorname{im} \psi_Y \in E(X)$, for Y with $\partial Y = X$?
4. Can we classify TQFTs? Is there a moduli space of them?

Proposition 2.1 (Exercise) Let A be a commutative ring with unit, and M, N modules over A. These modules are finitely generated, projective, and in duality if and only if there exists homomorphisms of A-modules $\alpha : A \to M \otimes_A N$ and $\beta : N \otimes_A M \to A$ such that

$$(M \otimes \beta)(\alpha \otimes M) = \mathrm{id}_M \quad \text{and} \quad (\beta \otimes N)(N \otimes \alpha) = \mathrm{id}_N .$$

Note that in duality means that the map $\bar{\beta} : N \to \mathrm{Hom}_A(M, A)$ induced by β is an isomorphism.

Proof: A proof in the category of spaces and stable maps is given in Spanier's topology book. ∎

Such maps α and β arise in a TQFT: Let X be an oriented $(n-1)$-manifold, and let $U \cong X \times I$ be the cobordism from $\varnothing$ to $X \coprod \bar{X}$ (where $\bar{X}$ is X with the reversed orientation). Then

$$\alpha = \psi_U : \mathbb{C} \to E(X) \otimes E(\bar{X}).$$

Similarly, the opposite cobordism $V = X \times I$ from $X \coprod \bar{X}$ to $\varnothing$ gives

$$\beta = \psi_V : E(\bar{X}) \otimes E(X) \to \mathbb{C}.$$

To verify that α and β satisfy the identity above, observe that the cobordism $X \times I \coprod U$, glued to the cobordism $V \coprod X \times I$, gives an S-shape diffeomorphic to $X \times I$. Hence,

$$(\psi_V \otimes \psi_{X \times I}) \circ (\psi_{X \times I} \otimes \psi_U) = \psi_{X \times I} = \mathrm{id}_{E(X)} .$$

A similar combination of cobordisms gives the other identity. Hence, $\bar{\beta} : E(\bar{X}) \to E(X)^*$ is an isomorphism, and from now on we identify the two.

We can then ask what the map

$$\mathbb{C} \to E(X) \otimes E(X)^*$$

induced from α and $\bar{\beta}$ is. In fact, it is the adjoint to $\mathrm{id} \in \mathrm{Hom}_{\mathbb{C}}(E(X), E(X))$, so if $\{b_1, \ldots, b_k\}$ is a basis for $E(X)$, and $\{b_1^*, \ldots, b_k^*\}$ the corresponding dual basis with respect to $\bar{\beta}$, we claim that

$$\alpha(1) = \sum_{i=1}^k b_i \otimes b_i^* .$$

Similarly, $\beta : E(X)^* \otimes E(X) \to \mathbb{C}$ gives the evaluation map, adjoint to $\mathrm{id}_{E(X)}$. We conclude that $\psi_{X \times S^1} = \dim E(X) \in \mathbb{Z}$.

2.2 Example of a TQFT

Example 2.2 (Dijkgraaf-Witten toy model) We discuss a TQFT in dimension n, associated to a finite group G. (In fact, this also generalizes to compact Lie groups.) We first describe the invariants associated to closed n-manifolds Y. ψ_Y is a certain weighted sum of isomorphism classes of principal G-bundles over Y. Specifically, for a principal G-bundle $P \to Y$, the weight is $1/|\operatorname{Aut} P|$, where $\operatorname{Aut} P$ is the group of automorphisms of P covering id_Y. Thus,

$$\psi_Y = \sum_{[P]} \frac{1}{|\operatorname{Aut} P|},$$

where $[P]$ ranges over the isomorphism classes of principal G-bundles over Y. Assuming that Y is connected, from covering space theory, we have that such classes are in bijection with representations $\mathrm{Hom}(\pi_1(Y), G)$ up to conjugacy, as well as with free homotopy classes $[Y, BG] = [Y, K(G,1)]$.

Thus, given a particular $p : P \to Y$, P comes from a homomorphism $\rho : \pi_1 Y \to G$ as follows: $P \cong \tilde{Y} \times_\rho G$, where $\tilde{Y}$ has a left action of $\pi_1(Y)$ by deck transformations. Thus, for $\alpha \in \pi$, $(y, g) \sim (\alpha y, \rho(\alpha) g)$. There is then a residual right action of G on $\tilde{Y} \times_\rho G$, and automorphisms of P left to $\pi_1 Y \times G$-equivariant maps $\theta : \tilde{Y} \times G \to \tilde{Y} \times G$. By the G-freeness of $\tilde{Y} \times G$, these are the same as $\pi_1(Y)$-equivariant maps $\tilde{Y} \to \tilde{Y} \times G$. Then $(y, 1) \mapsto (y', \theta)$, $\theta \in G$, where $p(y) = p(y')$. Since y and y' are in the same fiber, they are related by a deck transformation, and so we can rewrite this as a map $(y, 1) \mapsto (y, \theta(y))$, for some $\theta(y) \in G$.

Furthermore, the equivariance of this map θ means that it commutes with $\rho(\alpha)$ for all $\alpha \in \pi_1(Y)$, so the θ all lie in the centralizer of $\rho(\pi_1(Y))$. As a result, $\operatorname{Aut}(P) \cong C_G(\rho(\pi_1(Y)))$. Consequently, the isomorphism class of P is isomorphic to $G/C_G(\rho(\pi_1(Y)))$, so

$$\sum_{[P]} \frac{|G|}{|C_G(\rho(\pi_1(Y)))|} = |\operatorname{Hom}(\pi_1(Y), G)|.$$

Therefore, $|G| \sum_{|P|} 1/|\operatorname{Aut}(P)| = |\operatorname{Hom}(\pi_1(Y), G)|$, so

$$\psi_Y = \frac{|\operatorname{Hom}(\pi_1(Y), G)|}{|G|}.$$

We now describe the vector spaces associated to an $(n-1)$-manifold X and maps associated to cobordisms. Let P_X be the set of isomorphism classes of principal G-bundles on X, and let $E(X) = \mathbb{C}^{P_X}$. Now let Y be such that $\partial Y = X$, and take $\psi_Y(1) = E(X) = \mathbb{C}^{P_X}$ to be determined as follows. Let $P \to X$ be a G-bundle over X. Then

$$\psi_Y(1)(P) = \sum_{[Q]} \frac{1}{|\operatorname{Aut}(Q)|},$$

where $[Q]$ ranges over isomorphism classes of G-bundles over Y such that $Q|_{\partial Y} = P$, and where the isomorphisms fix $Q|_{\partial Y}$.

We can view this $1/|\operatorname{Aut}(Q)|$ factor as coming from a transfer map, or an umkehr map. From a different perspective, given a cobordism Y from X_1 to X_2, we obtain restriction maps

$$P_{X_1} \xleftarrow{\rho_{\text{in}}} P_Y \xrightarrow{\rho_{\text{out}}} P_{X_2}.$$

Applying the contravariant functor $\operatorname{Hom}(-, \mathbb{C})$ gives

$$\mathbb{C}^{P_{X_1}} \xrightarrow{\rho_{\text{in}}^*} \mathbb{C}^{P_Y} \xleftarrow{\rho_{\text{out}}^*} \mathbb{C}^{P_{X_2}}.$$

We then define an umkehr map $\rho_{\text{out}}^! : \mathbb{C}^{P_Y} \to \mathbb{C}^{P_{X_2}}$ as follows: given $f \in \mathbb{C}^{P_Y}$ and $P \to X_2$ a G-bundle over X_2,

$$\rho_{\text{out}}^!(f)(P) = \sum_{[Q]} \frac{f(Q)}{|\operatorname{Aut}(Q)|},$$

where $[Q]$ ranges over isomorphism classes of G-bundles on Y with $Q|_{X_2} = P$. Consequently, $(\rho_{\text{out}}^!) \circ \rho_{\text{in}}^* : \mathbb{C}^{P_{X_1}} \to \mathbb{C}^{P_{X_2}}$ gives the map ψ_Y. ∎

This example shows how in order to determine how bundles on X_1 relate to those over X_2, we consider all bundles on Y restricting to X_2, then restrict all these to X_1.

2.3 Categorical Reformulation of TQFTs

To express the definition of a TQFT more categorically, we introduce the language of 2-categories: if $\mathcal{C}$ is a 2-category, it has objects, and for $a, b \in \operatorname{Obj}\mathcal{C}$, $\operatorname{Mor}_{\mathcal{C}}(a, b)$ is itself a category, with objects morphisms from a to b and morphisms "2-morphisms" between these morphisms.

Example 2.3 Let $\mathcal{C}$ be the 2-category of $\mathbb{C}$-vector spaces. The objects of $\mathcal{C}$ are $\mathbb{C}$-vector spaces. For $V_1, V_2 \in \operatorname{Obj}\mathcal{C}$, $\operatorname{Mor}(V_1, V_2)$ is itself a category, with $\operatorname{Obj}\operatorname{Mor}(V_1, V_2) = \operatorname{Hom}_{\mathbb{C}}(V_1, V_2)$. For $L_1, L_2 : V_1 \to V_2$, $\operatorname{Mor}(L_1, L_2)$ consists of pairs of isomorphisms $(\theta_1 : V_1 \to V_1, \theta_2 : V_2 \to V_2)$ such that

$$\begin{array}{ccc} V_1 & \xrightarrow{L_1} & V_2 \\ {\scriptstyle\theta_1}\downarrow & & \downarrow{\scriptstyle\theta_2} \\ V_1 & \xrightarrow{L_2} & V_2 \end{array}$$

commutes. In fact, we will see that this 2-category is the "target" of our TQFT. ∎

3 Categorical Description of TQFTs 1/17/08

3.1 Categorical Background

We make more precise the idea that a TQFT is a functor from some cobordism category Cob_n of $(n-1)$- and n-manifolds to some linear category, taking $\coprod$ to $\otimes$. Recalling the Atiyah-Segal description of a TQFT, we observe that this cobordism category should have objects $(n-1)$-manifolds X, as well as two types of morphisms: diffeomorphisms $X_1 \to X_2$ of $(n-1)$-manifolds, and cobordisms $Y : X_1 \to X_2$. Such data describe a double category.

Definition 3.1 (Mac Lane [15]) A **double category** $\mathcal{C}$ consists of the following data:

- objects $\operatorname{Obj}\mathcal{C}$,
- for $a, b \in \operatorname{Obj}\mathcal{C}$, sets of **horizontal morphisms** $\operatorname{Mor}_h(a, b)$ and **vertical morphisms** $\operatorname{Mor}_v(a, b)$, forming categories $\mathcal{C}_h$ and $\mathcal{C}_v$, respectively,
- for $a, b, c, d \in \operatorname{Obj}\mathcal{C}$, $\alpha_1 \in \operatorname{Mor}_h(a, b)$, $\alpha_2 \in \operatorname{Mor}_h(c, d)$, $\phi_1 \in \operatorname{Mor}_v(a, c)$, and $\phi_2 \in \operatorname{Mor}_v(b, d)$, so that the horizontal and vertical morphisms form a square

$$\begin{array}{ccc} a & \xrightarrow{\alpha_1} & b \\ {\scriptstyle\phi_1}\downarrow & & \downarrow{\scriptstyle\phi_2} \\ c & \xrightarrow{\alpha_2} & d \end{array}$$

a set $\mathrm{Mor}_2(\alpha_1, \alpha_2, \phi_1, \phi_2)$ of 2-morphisms, with maps σ_h, τ_h to $\mathrm{Mor}\,\mathcal{C}_h$ and σ_v, τ_v to $\mathrm{Mor}\,\mathcal{C}_v$ taking $A \in \mathrm{Mor}_2$ to the "sides" of the square,

- horizontal and vertical composition for 2-morphisms, corresponding to composition of horizontal and vertical morphisms, respectively. ■

Example 3.2 An example of a double category is the category of sets, where both horizontal and vertical morphisms are set maps and where there exists exactly one 2-morphism for each commuting square of set maps. ■

Example 3.3 A 2-category is a double category where either the horizontal or vertical morphisms are only the identity morphisms on the objects. ■

Example 3.4 Our cobordisms categories Cob_n are such double categories:

- the objects are closed, oriented $(n-1)$-manifolds,
- the horizontal morphisms $\mathrm{Mor}_h(X_1, X_2)$ are orientation-preserving diffeomorphisms from X_1 to X_2,
- the vertical morphisms $\mathrm{Mor}_v(X, Y)$ are oriented n-manifolds W, with orientation-preserving diffeomorphisms $\partial W \to X \coprod Y$, where gluing gives composition,
- the 2-morphisms are diffeomorphisms of cobordisms compatible with prescribed diffeomorphisms of the boundaries. Specifically, if $W_i \in \mathrm{Mor}_v(X_i, Y_i)$, $i = 1, 2$, are cobordisms and if $\phi \in \mathrm{Mor}_h(X_1, X_2)$ and $\psi \in \mathrm{Mor}_h(Y_1, Y_2)$ are diffeomorphisms, then $A \in \mathrm{Mor}_2(\phi, \psi, W_1, W_2)$ is a diffeomorphisms $A : W_1 \to W_2$ such that $A|X_1 = \phi$ and $A|Y_1 = \psi$. Such morphisms are composed horizontally by function composition and vertically by gluing.

More precisely, given a cobordism $W : X \to Y$, we really have a local diffeomorphism from a neighborhood $U(\partial W)$ of ∂W to $X \times [0, \epsilon) \coprod Y \times (1-\epsilon, 1]$ – i.e., a collar structure on W compatible with X and Y. This ensures that cobordisms can be glued smoothly. ■

Example 3.5 Our earlier category $\mathrm{Vect}_{\mathbb{C}}$ of $\mathbb{C}$-vector spaces is also a double category:

- the objects are $\mathbb{C}$-vector spaces,
- the horizontal morphisms are linear isomorphisms,
- the vertical maps are linear maps,
- the 2-morphisms are commuting squares of such linear maps, with one 2-morphism for each such square. ■

Then a TQFT is a functor of double categories $E : \mathsf{Cob}_n \to \mathrm{Vect}_{\mathbb{C}}$ taking $\coprod$ to $\otimes$. Since the 2-morphisms of $\mathrm{Vect}_{\mathbb{C}}$ are relatively limited, the linear transformation associated to a cobordism depends only on the diffeomorphism type of the coboordism.

We can express this tensorial property categorically as follows:

Definition 3.6 A **symmetric monoidal category** (SMC) $\mathcal{C}$ is a category with a functor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$ with natural associativity and twist isomorphisms $\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$ and $\tau_{X,Y} : X \otimes Y \to Y \otimes X$ satisfying the following relations:

- $\tau_{Y,X} \circ \tau_{X,Y} = \mathrm{id}_{X \otimes Y}$,
- the "Stasheff pentagon" below commutes:

$$\begin{array}{ccccc} (X \otimes (Y \otimes (Z \otimes W))) & \xrightarrow{\alpha_{X,Y,Z\otimes W}} & (X \otimes Y) \otimes (Z \otimes W) & \xrightarrow{\alpha_{X\otimes Y,Z,W}} & ((X \otimes Y) \otimes Z) \otimes W \\ \downarrow{\scriptstyle \mathrm{id}_X \otimes \alpha_{Y,Z,W}} & & & & \uparrow{\scriptstyle \alpha_{X,Y,Z} \otimes \mathrm{id}_W} \\ X \otimes ((Y \otimes Z) \otimes W) & & \xrightarrow{\alpha_{X,Y\otimes Z,W}} & & (X \otimes (Y \otimes Z)) \otimes W \end{array}$$

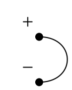
- the associativity and twist isomorphisms are compatible:

$$\begin{array}{ccccc} X \otimes (Y \otimes Z) & \xrightarrow{\alpha_{X,Y,Z}} & (X \otimes Y) \otimes Z & \xrightarrow{\tau_{X\otimes Y,Z}} & Z \otimes (X \otimes Y) \\ \downarrow{\scriptstyle \mathrm{id}_X \otimes \tau_{Y,Z}} & & & & \downarrow{\scriptstyle \alpha_{Z,X,Y}} \\ X \otimes (Z \otimes Y) & \xrightarrow{\alpha_{X,Z,Y}} & (X \otimes Z) \otimes Y & \xrightarrow{\tau_{X,Z} \otimes \mathrm{id}_Y} & (Z \otimes X) \otimes Y \end{array}$$

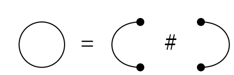
A **symmetric monoidal functor** preserves α, τ, and $\otimes$. ■

3.2 One-Dimensional TQFTs

We classify 1-dimensional TQFTs. The objects of Cob_1 are oriented 0-manifolds, or signed points. Hence, for a TQFT E, we assign $V = E(\bullet^+)$ to $\bullet^+$, and its dual V^* to $\bullet^-$. The cobordism



gives a map $V \otimes V^* \to \mathbb{C}$, the reverse cobordism gives $\mathbb{C} \to V^* \otimes V \cong \mathrm{End}(V)$ taking 1 to id_V, and the cobordism $\bullet^+ \times I$ from $\bullet^+$ to itself gives id_V. Hence, since S^1 decomposes as



$E(S^1) = \dim_{\mathbb{C}} V$. Since the only 1-manifolds up to diffeomorphism are disjoint unions of S^1s and intervals, these data determine E entirely. In fact, there is an equivalence of categories between 1-dim'l TQFTs (over $\mathbb{C}$) and finite dimensional $\mathbb{C}$-vector spaces.

3.3 Two-Dimensional TQFTs

We now address 2-dimensional TQFTs. Since a closed oriented 1-dimensional manifolds is isomorphic to $\coprod_k S^1$ for some unique $k \geq 0$, we can simplify our cobordism and linear categories somewhat. We fix a particular S^1, and take the objects of Cob_2 to be the nonnegative integers, $\mathbb{Z}^+$. For $m, n \in \mathbb{Z}^+$, $\mathrm{Mor}(n, m)$ consists of orientation-preserving diffeomorphism classes of cobordisms Σ with $n + m$ boundary components, along with parameterizations $\partial_{\mathrm{in}}\Sigma \to \coprod_n S^1$ and $\partial_{\mathrm{out}}\Sigma \to \coprod_m S^1$, where the diffeomorphisms are taken rel $\partial\Sigma$.

We note that there are then monoid maps $S_n \to \mathrm{Mor}(n, n)$, corresponding to the cobordisms that permute n copies of S^1.

Similarly, given a $\mathbb{C}$-vector space V, we define a category $\mathsf{End}(V)$ as follows: as above, the objects are $\mathbb{Z}^+$, and $\mathrm{Mor}(p, q) = \mathrm{Hom}_{\mathbb{C}}(V^{\otimes p}, V^{\otimes q})$.

We observe that both categories are SMCs, with $\otimes = +$ on objects and $\coprod$ and $\otimes$ on morphisms, respectively.

Definition 3.7 (Mac Lane) A **PRO** is an SMC $\mathcal{C}$ whose objects are $\mathbb{Z}^+$, where $\otimes = +$ on the objects. A **PROP** is a PRO with permutations, i.e., with monoid maps $\phi_n : S_n \to \mathcal{C}(n, n)$ compatible with the SMC structure:

1. if $\sigma_1 \in S_m$ and $\sigma_2 \in S_n$, then $\phi(\sigma_1 \times \sigma_2) = \phi(\sigma_1) \otimes \phi(\sigma_2)$.
2. if $t_{m,n} \in S_{m+n}$ exchanges the first m and last n letters, then $\phi(t_{m,n}) = \tau_{m,n}$ in $\mathcal{C}$. ▪

Then the characterization above gives a 2-dim'l TQFT as a functor of PROPs from Cob_2 to $\mathsf{End}(V)$ for some finite-dimensional vector space V.

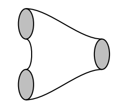
Proposition 3.8 (Abrams [1], "folk" theorem) A 2-dim'l TQFT is the same as giving a finite dimensional associative, commutative, unital $\mathbb{C}$-algebra A, with a linear map $\theta : A \to \mathbb{C}$ such that $\langle x, y\rangle = \theta(xy)$ is a nondegenerate bilinear form. (Such an object is called a **Frobenius algebra**.) ▪

Abrams further proves that the category of 2-dim'l TQFTs, with morphisms monoidal natural transformations, is isomorphic to the category of Frobenius algebras, with morphisms isomorphisms of FAs.

(In fact, this structure will not suffice for the more general TQFTs we discuss later: in particular, we would like a TQFT that gives $A = H_*(M; k)$, where $A \otimes A \to A$ is the intersection product and θ is the projection to $H_0(M; k) \cong k$. Since this product is graded commutative, we must introduce some sort of graded Frobenius algebra to describe this notion.)

As a result of this proposition, if E is a 2-dim'l TQFT, then $E(S^1 \times S^1) = \dim_{\mathbb{C}} A$ (or in the graded case when $A = H_*M$, the Euler characteristic of M, because the sum of dimensions will alternate).

We now indicate how the Frobenius algebra structures arise from a 2-dim'l TQFT E. Suppose that $A = E(1)$. Then the multiplication $\mu : A \otimes A \to A$ is $\mu = E(P)$, where P is the pair of pants

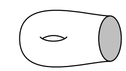


Letting T be the cobordism switching two copies of S^1, and observing that $T \circ P \cong P$, we have that $\tau\mu = \mu$, so A is a commutative algebra. Associativity is given by the diffeomorphism between $(P \coprod S^1 \times I) \circ P$ and $(S^1 \times I \coprod P) \circ P$. Furthermore, $\theta = E(D^2)$, where D^2 here is a cobordism from S^1 to $\varnothing$. As a cobordism from $\varnothing$ to S^1, it provides the unit $\eta : \mathbb{C} \to A$.

We summarize some other properties of A resulting from the TQFT structure:

Proposition 3.9 Suppose E is a TQFT as above, and $A = E(1)$.

1. There exists a **distinguished element** $\alpha \in A$ corresponding to the cobordism



   and if ψ_g is the invariant corresponding to the genus-g surface Σ_g, then $\psi_g = \theta(\alpha^g)$.
2. If $\{e_i\}$ is a $\mathbb{C}$-basis for A and $\{e_i^*\}$ is its dual basis with respect to $\langle -, -\rangle$, then $\alpha = \sum_i e_i e_i^*$.
3. Let $\rho : A \to \mathrm{End}(A)$ be the regular representation induced by left multiplication, so that $a \mapsto L_a$. Then for any $a \in A$, $\theta(a\alpha) = \mathrm{tr}(\rho(a))$. (Thus, θ is often called the **trace map**.) Furthermore, $\theta(\alpha) = \dim_{\mathbb{C}} A = \mathrm{tr}\,\mathrm{id}_A$.
4. A is semisimple as an algebra (so that A as a left module over itself is isomorphic to a direct sum of 1-dim'l A-modules) if and only if α is invertible. (Note that multiplication by α corresponds to E of the cobordism from S^1 to S^1 with one hole.)
5. If we rescale $\theta : A \to \mathbb{C}$ by a factor λ, we change $\psi_g = E(\Sigma_g)$ to $\lambda^{1-g}\psi_g = \lambda^{-\chi(\Sigma_g)/2}\psi_g$. ▪

4 Properties of TQFTs 1/22/08

4.1 Categories of TQFTs and Frobenius Algebras

Recall from last time Abrams's result that the category of 2-dimensional TQFTs, with morphisms monoidal natural isomorphisms, is isomorphic to the category of Frobenius algebras, with morphisms isomorphisms of such algebras. More explicitly, a morphism of TQFTs $A \to A'$ is then a collection of linear isomorphisms

$$\Phi_n = \Phi_1^{\otimes n} : A^{\otimes n} \to (A')^{\otimes n},$$

where we have of course used the identification $A(\coprod_n S^1) \cong A(S^1)^{\otimes n}$. Then a cobordism Σ from n to m gives by naturality a commuting square

$$\begin{array}{ccc} A^{\otimes n} & \xrightarrow{\Phi_n} & (A')^{\otimes n} \\ {\scriptstyle A(\Sigma)}\downarrow & & \downarrow{\scriptstyle A'(\Sigma)} \\ A^{\otimes m} & \xrightarrow{\Phi_m} & (A')^{\otimes m} \end{array}$$

We also ask why only isomorphisms of Frobenius algebras are allowed. In fact, any map $\phi : A \to A'$ of Frobenius algebras is an isomorphism. Since ϕ preserves the inner products on A and A', it also preserves the isomorphisms $\nu : A \to A^*$ and $\nu' : A' \to (A'^*)$ adjoint to these inner products. Hence, the diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & A' \\ {\scriptstyle \cong}\downarrow{\scriptstyle \nu} & & {\scriptstyle \cong}\downarrow{\scriptstyle \nu'} \\ A^* & \xleftarrow{\phi^*} & (A')^* \end{array}$$

commutes, so ϕ has an inverse $\nu^{-1}\phi^*\nu'$.

Finally, we remark that an alternate definition of a Frobenius algebra is as follows:

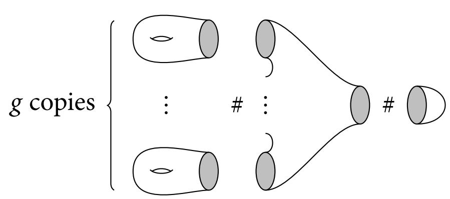
- A finite dimensional unital commutative algebra A over $\mathbb{C}$,
- A counital coalgebra structure on A, such that the coproduct map $\Psi : A \to A \otimes A$ is a map of A-A-bimodules.

The counit in this coalgebra structure is the trace map $\theta : A \to \mathbb{C}$.

4.2 Proofs of Properties of TQFTs

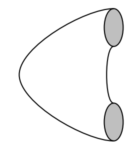
We now prove some of the properties listed in Proposition 3.9.

Proof (Prop. 3.9(1, 2, 3)): Suppose that Σ_g is a genus-g closed oriented surface. Then Σ_g is diffeomorphic to



so therefore $\psi_g = E(\Sigma_g) = \theta(\mu^{n-1}(\alpha \otimes \ldots \otimes \alpha)) = \theta(\alpha^g)$.

Recall that the surface S



gives a map $E(S) : \mathbb{C} \to A \otimes A$ such that $E(S) = \psi \circ \eta$, where η is the unit map $\mathbb{C} \to A$ and $\psi : A \to A \otimes A$ is the coproduct map associated to a "pair of pants" P. If $\{e_i\}$ is a $\mathbb{C}$-basis for A, with dual basis $\{e_i^*\}$ with respect to the pairing on A, then we have already computed that

$$E(S)(1) = \sum_i e_i \otimes e_i^*.$$

Hence, composing S with a pair of pants P' yields that $\alpha = E(P')(E(S)(1)) = \sum_i e_i e_i^*$.

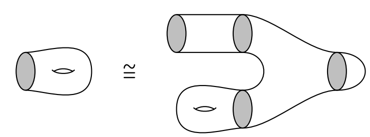
As for part (3), let $\rho : A \to \mathrm{End}(A)$ be the map determined by the regular representation. Since the map

$$A \xrightarrow{\rho} \mathrm{End}(A) \cong A \otimes A^* \underset{\cong}{\xleftarrow{A \otimes \nu}} A \otimes A$$

defines the coproduct map ψ of A, the map $\mathrm{tr} \circ \rho$ is equal to

$$A \xrightarrow{\psi} A \otimes A \xrightarrow{\langle -,- \rangle} \mathbb{C}.$$

Since $\langle -,- \rangle = \theta \circ \mu$, $\mathrm{tr}(\rho(a)) = (\theta \circ \mu \circ \psi)(a)$. Since



we have that $(\theta \circ \mu \circ \psi)(a) = (\theta \circ \mu \circ (\alpha \otimes \mathrm{id}_A))(a) = \theta(a \cdot \alpha)$. Hence, $\mathrm{tr}(\rho(a)) = \theta(a\alpha)$. ∎

Note that $\theta(a \cdot \alpha)$, and not $\theta(a)$ only, gives the trace of a acting on A. This is a common error, even in published articles.

Before proving Prop. 3.9 (4), we state some preliminary results.

Proposition 4.1 If A is a Frobenius algebra, and θ is replaced by $\theta_\lambda = \lambda\theta$, $\lambda \in \mathbb{C}^*$, then ψ_g is changed to $\lambda^{1-g}\psi_g$.

Proof: Note that $\Sigma_g \cong D\#(P\#P')^{\#g}\#D'$, where D and D' are disc cobordisms between $\varnothing$ and S^1, and P and P' are pairs of pants. Consequently,

$$\psi_g = \theta \circ (\mu \circ \psi)^g \circ \eta.$$

Since μ, θ, and η are the defining data for the Frobenius algebra structure on A, we must determine the effect on ψ. In particular, ψ is defined by

$$A \xrightarrow{\nu} A^* \xrightarrow{\mu^*} A^* \otimes A^* \xrightarrow{\nu \otimes \nu} A \otimes A,$$

where $\nu : A \to A^*$ is adjoint to $\langle -, - \rangle = \theta \circ \mu$. Consequently, $\nu(a)(b) = \theta(ab)$, so $\nu_\lambda(a)(b) = \theta_\lambda(ab) = \lambda\theta(ab)$, and thus $\nu_\lambda = \lambda\nu$. Hence, $\nu_\lambda^{-1} = \lambda^{-1}\nu$, and so $\psi_\lambda = \lambda^{-1}\psi$. Finally,

$$(\psi_\lambda)_g = (\theta_\lambda) \circ (\mu \circ \psi_\lambda)^g \circ \eta = \lambda^{1-g}\theta \circ (\mu \circ \psi)^g \circ \eta = \lambda^{1-g}\psi_g. \qquad ■$$

Example 4.2 We examine these Frobenius algebra structures in the case of the Dijkgraaf-Witten toy model of Example 2.2 in dimension 2. Recall that G is a finite group, and $A(X) = \mathbb{C}^{P_X}$, where P_X is the set of isomorphism classes of principal G-bundles on X, and is equal to

$$[X, BG] = \pi_0 \operatorname{Map}(X, BG).$$

Let Q be the 1-to-2 pair of pants. Then restriction to the ingoing and outgoing boundary components yields maps

$$P_{S^1} \xleftarrow{\rho_{\text{in}}} P_Q \xrightarrow{\rho_{\text{out}}} P_{S^1} \times P_{S^1}.$$

Since $P_X = [X, BG]$, this set depends only on homotopic information. Then because $Q \simeq 8$, $P_Q \cong P_8$. Making this replacement and applying $\operatorname{Hom}(-, \mathbb{C})$, we obtain

$$\mathbb{C}^{P_{S^1}} \xrightarrow{\rho_{\text{in}}^*} \mathbb{C}^{P_8} \xleftarrow{\rho_{\text{out}}^*} \mathbb{C}^{P_{S^1}} \otimes \mathbb{C}^{P_{S^1}}.$$

Additionally, we have an "umkehr" map $\rho_{\text{in}}^! : \mathbb{C}^{P_8} \to \mathbb{C}^{P_{S^1}}$ defined as

$$\rho_{\text{in}}^!(\phi)([\gamma]) = \sum_{[\beta]} \frac{\phi(\beta)}{|\operatorname{Aut}\beta|},$$

where the sum ranges over classes $[\beta]$ of G-bundles on 8 restricting to β on the "outer" circle of the 8. Furthermore, we can express such isomorphism classes as follows: $P_{S^1} = \pi_0 LBG$, which is part of a fibration $G \hookrightarrow LBG \to BG$. Applying π_* yields the long exact sequence

$$G = \pi_1 BG \to G \to \pi_0 LBG \to *,$$

where $\pi_1 BG = G$ acts on $\pi_0 G = G$ by conjugation. Hence, we claim that $\pi_0 LBG$ is isomorphic to the conjugacy classes of G. Alternately, we will see later that for a general topological group G there is a homotopy equivalence $LBG \simeq EG \times_G G^{\text{conj}}$, so when G is discrete the path components do correspond to these conjugacy classes.

Similarly, $P_8 = \pi_0 \operatorname{Map}(8, BG)$ is part of a fibration

$$\Omega BG \times \Omega BG \cong \operatorname{Map}_0(8, BG) \hookrightarrow \operatorname{Map}(8, BG) \to BG,$$

which similarly yields that $\pi_0 \operatorname{Map}(8, BG) \cong (G \times G)/\text{conj}$. Consequently, the map between the P-sets above become

$$G/\text{conj} \leftarrow (G \times G)/\text{conj} \to G/\text{conj} \times G/\text{conj},$$

with $[gh] \leftarrow\!\shortmid [g, h] \mapsto ([g], [h])$.

Furthermore, the algebra $A = \mathbb{C}^{P_{S^1}} = \mathbb{C}^{G/\text{conj}}$ is the set of class functions of $\mathbb{C}^G$, and has the following multiplication map μ: if $\phi_1, \phi_2 \in \mathbb{C}^{G/\text{conj}}$, then

$$\mu(\phi_1, \phi_2)([g]) = \sum_{\substack{[g_1, g_2] \in (G \times G)/\text{conj} \\ [g_1 g_2] = [g] \in G/\text{conj}}} \phi_1(g_1)\phi_2(g_2).$$

The dual multiplication on $(\mathbb{C}^G)^*$ yields the multiplication of the group ring $\mathbb{C}G$. It is standard from the representation theory of finite groups that the elements $\sum_g z_g g \in \mathbb{C}G$ such that the z_g are constant on conjugacy classes are precisely the center $Z(\mathbb{C}G)$ of $\mathbb{C}G$. Hence, $A^* = Z(\mathbb{C}G)$, with the multiplication induced from the group ring structure. (In order to determine this, consider the umkehr map $\rho_{\text{out}}^!$.) ■

5 Semisimple and Graded Frobenius Algebras 1/24/08

5.1 Semisimple Algebras and Modules

We recall some notions from module theory. (See Anderson and Fuller [2] for more details.) Let R be a unital ring.

Definition 5.1 Suppose T is a left R-module. Then T is **simple** or **irreducible** if T has no nontrivial R-submodules. T is **semisimple** if for some index set I, $T \cong \bigoplus_{\alpha \in I} T_\alpha$, where each T_α is simple. ■

Note that T is simple if and only if $T \cong R/M$, where M is a maximal left ideal of R.

Fact 5.2 Suppose M is a left R-module. The following are equivalent:

1. M is semisimple,
2. every submodule of M is a direct summand,
3. every short exact sequence of R-modules $0 \to K \to M \to N \to 0$ splits. ■

Lemma 5.3 (Schur) If A is a finite-dimensional algebra over an algebraically closed field k, and M and N are both irreducible left A-modules, then $\operatorname{Hom}_A(M, N) = 0$ if $M \ncong N$, and $\operatorname{Hom}_A(M, M) \cong k \cdot \mathrm{id}_M$. ■

Corollary 5.4 If A as above is also commutative and $M \neq 0$ is irreducible, then $\dim_k M = 1$.

Proof: Note that multiplication by $a \in A$ is an A-module homomorphism by the commutativity of A, so $am = \lambda_a m$ for some $\lambda_a \in k$. Since M is simple, $M = A \cdot m$ for some $m \neq 0$ in M, so $M = k \cdot m$. Hence, $\dim_k M = 1$. ∎

Consequently, if A is a semisimple Frobenius algebra over $\mathbb{C}$, then $A \cong \bigoplus_{\beta \in I} \mathbb{C}_\beta$ as A-modules.

Theorem 5.5 If A is a semisimple commutative Frobenius algebra, then A has a $\mathbb{C}$-basis $\{e_i\}_{i=1}^n$ of orthogonal idempotents (so that $e_i e_j = 0$ if $i \neq j$ and $e_i^2 = e_i$).

Proof: Write $A \cong \bigoplus_{i=1}^n \mathbb{C}_i$ as A-modules. Note that we can choose generators a_i for the $\mathbb{C}_i$ such that $a_i a_j = 0$ for $i \neq j$: in general suppose that $C \subset A$ is an irreducible submodule, hence a direct summand isomorphic to $\mathbb{C}$ by semisimplicity. Pick a generator α of C, and let $m_C : A \to C$ be the (surjective) action map $m_C(a) = a\alpha$. Let $K = \ker m_C$.

We claim that K is a semisimple commutative Frobenius algebra. First, since K is an A-submodule of A, the multiplication on A gives a map $\mu : K \otimes K \to K \otimes A \to K$. Take $k \in K$ nonzero. Since $\langle k, - \rangle \neq 0 \in A^* = K^* \oplus C^*$, and since $k \cdot \beta = 0$ for each $\beta \in C$, $\langle k, - \rangle$ restricted to K is nonzero. Finally, K is semisimple as a K-module since A is semisimple and since multiplication by C acts by 0 on K. Iterating this decomposition on K produces the desired basis for A.

Let b_i be the dual basis of the a_i with respect to the pairing $\langle -, - \rangle$, so that $\theta(a_i b_j) = \delta_{ij}$. Write $b_i = \sum_k z_{ik} a_k$, so that $b_i a_j = z_{ij} a_j^2$. Hence, $z_{ij}\theta(a_j^2) = \delta_{ij}$. Taking $i = j$, $\theta(a_j^2) \neq 0$, so $a_j^2 \neq 0$. Since the $\mathbb{C}_i$ have dimension 1, $a_j^2 = c_j a_j$ for some $c_j \in \mathbb{C}^\times$. We then define $e_j = a_j / c_j$, and have that $e_j^2 = e_j$. ∎

Let $\{f_i\}$ be the $\langle -, - \rangle$-dual basis to the basis $\{e_i\}$ described above. Then $f_i = \sum_j z_{ij} e_j$ for $z_{ij} \in \mathbb{C}$, so $f_i e_j = z_{ij} e_i^2 = z_{ij} e_j$. Then $\delta_{ij} = \theta(f_i e_j) = z_{ij}\theta(e_j)$. If $i = j$, then $\theta(e_i) = z_{ii}^{-1} \neq 0$, and if $i \neq j$, $0 = z_{ij}\theta(e_j)$, so $z_{ij} = 0$. Consequently, $f_j = e_j / \theta(e_j)$.

Corollary 5.6 Dimension-n commutative semisimple Frobenius algebras over $\mathbb{C}$ are classified by n nonzero complex numbers $z_1, \ldots, z_n$ taking the values $z_i = \theta(e_i)$. Hence, two such algebras are isomorphic iff they have the same list of such complex numbers. ∎

Lemma 5.7 Let A be a finite-dimensional commutative algebra over an algebraically closed field k. Suppose $M \subset A$ is irreducible. If M is not a field, then it has nilpotent elements.

Proof: Exercise. ∎

5.2 Proofs of Properties of TQFTs, Continued

We now return to the proof of Proposition 3.9, part (4).

Proof: Let A be a finite-dimensional, commutative Frobenius algebra over $\mathbb{C}$.

Suppose A is also semisimple. We show the distinguished element α is a unit. Write $A = \bigoplus_{i=1}^n \mathbb{C} e_i$ with the e_i and f_i as above. Recall that $\alpha = \sum_{i=1}^n e_i f_i = \sum_{i=1}^n e_i / \theta(e_i)$. Noting that $1_A = \sum_{i=1}^n e_i$, then $\alpha^{-1} = \sum_{i=1}^n \theta(e_i) e_i$ is a two-sided inverse to α. Hence α is a unit.

Now suppose α is a unit in A. We show A is semisimple. By the contrapositive to Lemma 5.7, it suffices to show A contains no nilpotents. Let $N \subset A$ be the ideal of nilpotents in A (i.e., rad 0). We show $\alpha N = 0$, so that $N = 0$ since α is a unit.

Filter A as follows. Let

$$S_1 = \operatorname{ann}(N) \subset A = \{a \in A \mid aN = 0\}$$

be the annihilator of N in A. We then wish to show $\alpha \in S_1$. Define the S_i inductively as follows: let $\pi_i : A \to A/S_{i-1}$ be the projection, and let

$$S_i = \pi_i^{-1}(\operatorname{ann}(N(A/S_{i-1}))) = \{a \in A : ax \in S_{i-1} \text{ for all } x \in \operatorname{rad} S_{i-1}\}.$$

Since $S_{i-1} \subset S_i$, this gives a filtration $S_1 \subset \ldots \subset S_k = A$ of A. Pick a $\mathbb{C}$-basis $\{b_i\}_{i=1}^n$ for A by picking one for S_1, then for S_2, and so forth. Suppose that $b_i \in S_j \setminus S_{j-1}$ and $a \in N$. Then ab_i is also nilpotent, so $ab_i \in S_{j-1}$ and hence can be expressed as a linear combination of the b_k, $k < i$. As a result, $\theta(ab_i b_i^*) = \langle ab_i, b_i^* \rangle = 0$. Consequently, $b_i b_i^* N \subset \ker\theta$.

We show that $\ker\theta$ contains no nonzero ideals. Suppose that $I \subset \ker\theta$ is nonzero, and take $a \in I$ nonzero. Hence, $A \cdot a \subset I \subset \ker\theta$, so $\langle a, b \rangle$ for all $b \in A$, contradicting the nondegeneracy of $\langle -, - \rangle$.

Hence, we conclude that $b_i b_i^* N = 0$, so $\alpha N = \sum_{i=1}^n b_i b_i^* N = 0$. ∎

5.3 TQFTs into Graded Vector Spaces

We now consider TQFTs with values taken in graded vector spaces over $\mathbb{C}$. As before, if V_* is a graded vector space, we can define an endomorphism PROP $\mathsf{End}(V)$, with morphisms $\mathsf{End}(V)(p,q) = \operatorname{Hom}_{\mathbb{C}}(V_*^{\otimes p}, V_*^{\otimes q})$ maps of graded vector spaces. Then a TQFT is a monoidal functor $E : \mathsf{Cob}_2 \to \mathsf{End}(V)$. Consequently, $V_* = E(1)$ is a graded-commutative Frobenius algebra, so that $ab = (-1)^{|a||b|} ba$, with a map $\theta : V_* \to \mathbb{C}$ giving a nondegenerate pairing $\langle a, b \rangle = \theta(ab)$.

Example 5.8 Let M be a connected, closed, oriented n-dimensional manifold. Then $H^*(M; \mathbb{C})$ is a graded-commutative Frobenius algebra over $\mathbb{C}$. In particular, the diagonal map $\Delta : M \to M \times M$ induces a graded-commutative product

$$\cup : H^*(M) \otimes H^*(M) \cong H^*(M \times M) \xrightarrow{\Delta^*} H^*(M),$$

namely cup product. The map θ is given by $\theta(\alpha) = \langle \alpha, [M] \rangle$, where $[M] \in H_n(M)$ is the fundamental class associated to the orientation of M. By Poincaré duality, this pairing is nondegenerate, and its adjoint $D(\alpha) = \alpha \cap M$ gives an isomorphism $D : H^*(M) \to H_{n-*}(M) \cong (H^*(M))^*$.

The coproduct on $H^*(M)$ arises through a umkehr map construction. Recall that if N is another closed manifold and $f : N \to M$ a map, then $f^! : H^*(N) \to H^*(M)$ is defined by

$$\begin{array}{ccc} H^*(N) & \xrightarrow{f^!} & H^{*+\dim M - \dim N}(M) \\ \cong \big\downarrow D_N & & D_M^{-1} \big\uparrow \cong \\ H_{\dim N - *}(N) & \xrightarrow{f_*} & H_{\dim N - *}(M) \end{array}$$

Then the coproduct ψ is $\Delta^! : H^*(M) \to H^*(M \times M) \cong H^*(M) \otimes H^*(M)$. The distinguished element $\alpha \in H^*(M)$ is then $\Delta^*(\Delta^!([1]))$, where $[1] \in H^0(M)$ is the unit class.

We describe this class via a Pontryagin-Thom construction. Let $\eta(\Delta(M))$ be a tubular neighborhood of $\Delta(M) \subset M \times M$, which is isomorphic to the normal bundle ν_Δ of $\Delta(M)$ inside $M \times M$. Let

$$\tau : M \times M \to M \times M / (M \times M \setminus \eta(\Delta(M))) = Th(\nu_\Delta)$$

be the quotient map. Let $u \in H^n(Th(\nu_\Delta))$ be the Thom class. Then the umkehr map $\Delta^!$ can be expressed as

$$H^*(M) \xrightarrow{-\cup u} H^{*+n}(Th(\nu_\Delta)) \xrightarrow{\tau^*} H^{*+n}(M \times M),$$

and Δ^* takes this class to $H^{*+n}(M)$. Since the Euler class $e(M)$ of M is described by the image of $[1] \in H^0(M)$ under the composition

$$H^*(M) \xrightarrow{-\cup u} H^{*+n}(Th(\nu_\Delta)) \to H^{*+n}(\eta(\Delta(M))) \cong H^{*+n}(M),$$

we observe that $\alpha = e(\nu_\Delta) = e(M)$. Observing that $\chi_M = \langle e(M), [M] \rangle$, we obtain the following proposition: ▪

Proposition 5.9 For the TQFT described in Example 5.8, $\alpha = e(M)$ and $\psi_1 = \theta(\alpha) = \chi_M \in \mathbb{Z} \subset \mathbb{C}$. ▪

As an exercise (from Milnor and Stasheff [17]) prove without our TQFT machinery that $e(M) = \sum_i (-1)^{|e_i|} e_i e_i^*$, where $\{e_i\}$ is a basis for $H^*(M; \mathbb{C})$ and $\{e_i^*\}$ is its dual basis with respect to the Poincaré duality pairing. Then $\chi_M = \sum_i (-1)^i \dim_{\mathbb{C}} H^i(M)$.

Similarly, in Floer homology or quantum homology, there is a similar class called the "quantum Euler class" (which sounds better than it actually is).

6 Conformal Field Theories 1/29/08

6.1 Conformal Field Theories

We now introduce a conformal cobordism category of 1 + 1-dimensional oriented manifolds, due originally to Segal [19]. Let $\mathcal{M}_{g,n}$ be the **moduli space** of Riemann surfaces Σ of genus g, with n discs specified by a biholomorphic map $\phi : \coprod_{i=1}^n D_i^2 \to \Sigma$, where the $\phi(D_i^2)$ are pairwise disjoint. (We will give a more precise definition of this moduli space below.) These data yield more information than simply placing marked points on Σ, as each disc $\phi(D_i^2)$ has a specified complex structure from the canonical complex structure on the standard disc D^2. Note that $\mathcal{M}_{0,n}$ is the space

$$\left\{ \text{biholomorphic maps } \phi : \coprod_{i=1}^n D_i^2 \to S^2 \text{ with p.w.d. } \phi(D_i^2) \right\} / PSL(2, \mathbb{C}).$$

Definition 6.1 Define the **Segal PROP** $\mathcal{M}$ to have objects $\mathbb{Z}^+$, and to have morphisms $\mathcal{M}(n, m)$ given by the moduli space of (possibly disconnected) Riemann surfaces Σ of arbitrary genus, together with biholomorphic maps

$$\phi_{\text{in}} : \coprod_{i=1}^n D_i^2 \to \Sigma \quad \text{and} \quad \phi_{\text{out}} : \coprod_{i=n+1}^{n+m} D_i^2 \to \Sigma,$$

again with p.w.d. $\phi(D_i^2)$s. Note that the spaces of morphisms are then disjoint unions of moduli spaces, where the surface Σ can have multiple components.

Composition is given by gluing along discs; since the maps to the embedded discs are biholomorphic, this gluing preserves the complex structure on the surfaces. (To be more precise, a smaller disc is removed from each disc to be glued, and the remaining annuli are identified.) These data determine a symmetric monoidal category with product + on objects and $\coprod$ on surfaces. S_n acts on $\mathcal{M}_{g,n}$ by permuting the marked discs, so $S_n \times S_m$ acts on $\mathcal{M}(n, m)$. ▪

Definition 6.2 (Segal) A **conformal field theory** (CFT) is a functor of PROPs $\mathcal{M} \to \mathsf{End}(V)$, where V is a vector space (or a Hilbert space, where some care is needed to introduce a topological tensor product). ▪

One can think of a CFT as a representation of moduli space. Unfortunately, there are no known explicit examples of CFTs, although they can be proved to exist. A conjecture related to topological modular forms is that $\text{tmf}^0(X) = [X, \mathsf{CFT}]$, where CFT is some moduli space of conformal field theories, appropriately topologized, and should classify bundles of CFTs. (Recent work by Stolz and Teichner addresses some of this; Pokman Cheung's thesis addresses CFTs where the allowed cobordisms are only annuli.)

6.2 Topological Conformal Field Theories

As a way to interpolate between CFTs and TQFTs, we introduce the notion of a **topological conformal field theory** (TCFT), originally due to Manin. This will associate to every homology class (or every singular chain) of $\mathcal{M}$ a linear map.

More precisely, to each closed 1-manifold S, we assign a cochain complex C_S (over a field k, for simplicity). This assignment is monoidal, so that there are natural coherence isomorphisms $C_{S_1} \otimes C_{S_2} \cong C_{S_1 \coprod S_2}$. To a cobordism F between two 1-manifolds $S_0 \to S_1$, we associate a cochain $\mu_F \in C^*(\mathcal{M}(S_0, S_1); \mathrm{Hom}(C_{S_0}, C_{S_1}))$, where $\mathrm{Hom}(C, D)$ is the chain complex of maps from chain complexes C to D. These μ_F cochains must exhibit some compatibility, which we describe below. In the literature, μ_F is sometimes taken to be a differential form, which introduces more considerations of the smooth topology of the cobordism surfaces.

By the monoidal structure of this assignment, if $\pi_0(S_0) = p$ and $\pi_0(S_1) = q$, then we obtain that $\mu_F \in C^*(\mathcal{M}(p, q); \mathrm{Hom}(C^{\otimes p}, C^{\otimes q}))$. The compatibility of these cochains is then as follows: let $\mu_{n,m} = \sum_{[F]} \mu_F$, where the sum is taken over all diffeomorphism classes of cobordisms $n \to m$ (including the disconnected ones). Let $\mu : \mathcal{M}(n, m) \times \mathcal{M}(m, p) \to \mathcal{M}(n, p)$ denote the composition map in $\mathcal{M}$. Note that we have the maps

$$\begin{aligned}
\mu^* &: C^*(\mathcal{M}(n, p); \mathrm{Hom}(C^{\otimes n}, C^{\otimes p})) \to C^*(\mathcal{M}(n, m) \times \mathcal{M}(m, p); \mathrm{Hom}(C^{\otimes n}, C^{\otimes p}), \\
\times &: C^*(\mathcal{M}(n, m); \mathrm{Hom}(C^{\otimes n}, C^{\otimes m}) \otimes C^*(\mathcal{M}(m, p); \mathrm{Hom}(C^{\otimes m}, C^{\otimes p}) \\
&\qquad \to C^*(\mathcal{M}(n, m) \times \mathcal{M}(m, p); \mathrm{Hom}(C^{\otimes n}, C^{\otimes p}),
\end{aligned}$$

where μ^* is induced from the μ map, and $\times$ arises from the external cross product on homology and from the composition of chain maps. The compatibility we require is that $\mu^*(\mu_{n,p}) = \mu_{n,m} \times \mu_{m,p}$ for all n, m, p.

We can rephrase this with an adjunction: define a new PROP $C_*(\mathcal{M})$, where the objects are $\mathbb{Z}^+$, and where the morphisms are $C_*(\mathcal{M})(n, m) = C_*(\mathcal{M}(n, m); k)$. Composition is given by

$$C_*(\mathcal{M}(n, m)) \otimes C_*(\mathcal{M}(m, p)) \to C_*(\mathcal{M}(n, m) \times \mathcal{M}(m, p)) \to C_*(\mathcal{M}(n, p)),$$

where the first map is the Eilenberg-Zilber map on chains (i.e., the higher-dimensional analogues of the "prism" operators dividing $\Delta^n \times \Delta^1$ into an $(n + 1)$-chain). Then a TCFT is a functor of PROPs $E : C_*(\mathcal{M}) \to \mathsf{End}(C_*)$, where C_* is some chain complex of k-modules. Hence, $E(1) = C_*$, and C_* obtains a differential graded algebra (DGA) structure from the pair-of-pants cobordism. For each p, q, we have an evaluation map

$$C_*(\mathcal{M})(p, q) \otimes C_*^{\otimes p} \to C_*^{\otimes q}$$

from the earlier cochain description. Applying homology, we obtain maps

$$H_*(\mathcal{M}(p, q)) \otimes H_*(C_*)^{\otimes p} \to H_*(C_*)^{\otimes q}.$$

Since $H_0(\mathcal{M}(p, q))$ is generated by the path components of $\mathcal{M}(p, q)$, which give the diffeomorphism classes of such cobordisms, we obtain a linear map for each such class. This is precisely a graded TQFT, or equivalently a graded Frobenius algebra with $V_* = H_*(C_*)$. Hence, passing a TCFT through homology and taking the 0th-graded piece gives a TQFT.

In order to understand TQFTs better, we analyze these chains in moduli space. Let F_g be a fixed smooth surface of genus g. From classical Riemann surface theory,

$$\mathcal{M}_{g,n} = \left\{ (J, \phi) : \phi : \coprod_{i=1}^{n} D_i^2 \hookrightarrow F_g \text{ smooth, } J \text{ a } \mathbb{C}\text{-structure on } F_g \setminus \mathrm{im}\, \phi \right\} / \operatorname{Diff}^+(F_g).$$

This is the same as our earlier description, since the smooth embedding determines complex structures on the $\phi(D_i^2)$, which we extend to the rest of F_g. To complete this, we invoke another theorem stating that almost complex structures on surfaces are actually complex structures.

Note that $\operatorname{Diff}^+(F_g)$ acts transitively on $\mathrm{Emb}(\coprod_{i=1}^n D_i^2, F_g)$. Let ϕ_0 be a fixed embedding, and let $F_{g,n} = F_g \setminus \mathrm{im}\, \phi_0$ (so that $F_{g,n}$ has boundary). Let

$$\mathcal{J}(F_{g,n}) = \{(J, \phi_0) : J \text{ a } \mathbb{C}\text{-structure on } F_{g,n}\}.$$

Then $\operatorname{Diff}^+(F_g) \cdot \mathcal{J}(F_{g,n})$ gives all of the $\{(J, \phi)\}$ above, and the stabilizer of (J, ϕ_0) is $\operatorname{Diff}^+(F_{g,n}, \partial)$. Hence,

$$\mathcal{M}_{g,n} \cong \mathcal{J}(F_{g,n}) / \operatorname{Diff}^+(F_{g,n}, \partial).$$

Theorem 6.3 (Teichmüller) $\mathcal{J}(F_{g,n}) \simeq *$ for all g, n; the action of $\operatorname{Diff}^+(F_{g,n})$ is free if $g \geq 2$ and $n > 0$, and has finite stabilizers if $n = 0$. ∎

On account of this theorem, we obtain that for $g \geq 2$ and $n > 0$, $\mathcal{M}_{g,n} \simeq B \operatorname{Diff}^+(F_{g,n}, \partial)$. Consequently, we can reinterpret a (T)CFT in terms of surface bundles.

6.3 Reinterpretation of TCFTs Via 2-Categories

We define a new cobordism 2-category (in fact, a 2-PROP) Cob_2, with objects $\mathbb{Z}^+$, 1-morphisms $\mathrm{Mor}(n, m)$ all cobordisms $\coprod_n S^1 \to \coprod_m S^1$, and 2-morphisms diffeomorphisms of cobordisms. (Hence, this preserves the automorphisms of the cobordisms instead of quotienting out by them).

We describe a similar 2-PROP structure on $\mathsf{End}(C_*)$, where C_* is a chain complex over the field k. Let the objects be $\mathbb{Z}^+$, and let the 1-morphisms $\mathrm{Mor}(m, n)$ be chain maps $\mathrm{Hom}(C_*^{\otimes n}, C_*^{\otimes m})$. Finally, let the 2-morphisms be chain homotopies of chain maps. (Lurie calls such data an **extended TQFT**).

Since each pair of objects n, m in Cob_2 determines a category $\text{Mor}(n, m)$, taking the geometric realization $B\,\text{Mor}(n, m)$ on each category of 1-morphisms yields a topological category. (In fact, since $\text{Mor}(n, m)$ is not a small category, we instead take the geometric realization of its skeleton category; since the diffeomorphism classes of such cobordisms form a set, this skeleton category is small, and therefore can be realized geometrically.) Furthermore, by the above discussion,

$$B\,\text{Mor}(n, m) = \coprod_{[F]} B\,\text{Diff}^{+}(F),$$

where F ranges over classes of cobordisms from n circles to m circles. While this produces an equivalent cobordism category, the functors involved must still be changed to reflect the geometric realization.

6.4 Algebraic Structures

A "metatheorem" stated in the physics literature ([10]) asserts that if E is a chain-complex-valued TQFT (e.g., a TCFT or an extended TQFT), then $E(S^1)$ is a chain-homotopy commutative DGA, and its Hochschild cohomology $HH^*(E(S^1))$ is self-dual and is a Batalin-Vilkovisky (BV) algebra. (We do note that Chas and Sullivan have shown $HH^*(C^*M)$ is a BV algebra, and it is known that $HH_*(C^*M) \cong H^*LM$ for M 1-connected.)

Definition 6.4 A **Batalin-Vilkovisky algebra** is a pair (A, Δ), where A is a graded-commutative algebra, and $\Delta : A \to A$ is a degree-1 operator, such that

1. $\Delta^2 = 0$,
2. the derivator

$$\{\phi, \theta\} = (-1)^{|\phi|}\Delta(\phi \cdot \theta) - (-1)^{|\phi|}\Delta(\phi) \cdot \theta - \phi \cdot \Delta(\theta)$$

is a derivation in each variable. ∎

Chas and Sullivan [6] and Getzler [10] have shown independently that $\{-,-\}$ satisfies the graded Jacobi identity for a graded Lie algebra. Such a graded-commutative algebra with a Lie bracket $[-,-]$ that is a derivation in each variable is called a **Gerstenhaber algebra**.

Example 6.5 (Samelson) The homology $A = H_*(\Omega^2 X)$ of a double loop space has a Lie bracket with these properties. ∎

We will return to these algebraic notions later.

7 Hochschild and Cyclic Homology 1/31/08

7.1 Hochschild Homology and Cohomology

We introduce a few constructions in homological algebra that contain geometric content. (Good references for this material include an article by Loday and Quillen [14] and books by Loday [13] and Weibel [20, Ch. 9].)

Let A be an associative algebra over a commutative ground ring k. (More generally, we could take A an A_∞-algebra, although we will not do this here.) Define the **Hochschild complex** $CH_*(A)$ by

$$\cdots \to A^{\otimes n+1} \xrightarrow{b} A^{\otimes n} \to \cdots \to A,$$

where $CH_n(A) = A^{\otimes n+1}$ and where b is defined by

$$b(a_0, \dots, a_n) = \sum_{i=0}^{n-1}(-1)^i(a_0, \dots, a_i a_{i+1}, \dots, a_n) + (-1)^n(a_n a_0, \dots, a_{n-1}).$$

(For convenience, we often write $(a_1, \dots, a_n)$ for $a_1 \otimes \cdots \otimes a_n$.) Computation shows that $b^2 = 0$, so $CH_*(A)$ is a chain complex and its homology $H_*(CH_*(A))$ is defined to be the **Hochschild homology of** $\boldsymbol{A}$.

Consider the acyclic **bar complex** $C_*^{\text{bar}}(A)$

$$\cdots \to A^{\otimes n+2} \xrightarrow{b'} A^{\otimes n+1} \to \cdots \to A^{\otimes 2}$$

with differential

$$b'(a_0, \dots, a_{n+1}) = \sum_{i=0}^{n}(-1)^i(a_0, \dots, a_i a_{i-1}, \dots, a_{n+1}).$$

Note that $C_*^{\text{bar}}(A)$ has an augmentation map $\epsilon : A^{\otimes 2} \to A$ given by $\epsilon(a, b) = ab$. The map $s : A^{\otimes n} \to A^{\otimes n+1}$ with

$$s(a_1, \dots, a_n) = (1, a_1, \dots, a_n)$$

assembles to give a degree-1 map s with $sb' + b's = \text{id}$. Hence, id is chain-homotopic to 0, so the augmented complex $C_*^{\text{bar}}(A) \to A$ is contractible, hence acyclic.

If A is projective over k (as is the case when k is a field), $C_n^{\text{bar}}(A) = A^{\otimes n+2}$ is a projective $A \otimes A^{\text{op}}$ module for all $n \geq 0$. Hence, $C_*^{\text{bar}}(A)$ gives a projective resolution of A as an $A \otimes A^{\text{op}}$-module. Considering this resolution as left $A \otimes A^{\text{op}}$-modules, and A as a right $A \otimes A^{\text{op}}$-module, then $\psi_n : A \otimes_{A \otimes A^{\text{op}}} C_n^{\text{bar}}(A) \to A^{\otimes n+1}$ given by

$$\psi(\alpha \otimes (a_0, \dots, a_{n+1})) = (a_{n+1}\alpha a_0, a_1, \dots, a_n)$$

is an isomorphism of k-modules. These maps assemble into a chain isomorphism $\psi : A \otimes_{A\otimes A^{\mathrm{op}}} C_*^{\mathrm{bar}}(A) \to CH_*(A)$. Passing to homology, we obtain the homological characterization

$$HH_*(A) \cong \mathrm{Tor}_*^{A\otimes A^{\mathrm{op}}}(A, A)$$

of Hochschild homology.

In fact, if M is an A-A-bimodule, hence a right $A \otimes A^{\mathrm{op}}$-module, we define the **Hochschild homology of A with coefficients in M** to be

$$HH_*(A, M) = \mathrm{Tor}_*^{A\otimes A^{\mathrm{op}}}(M, A)$$

and note that we may compute it as $H_*(M \otimes_{A\otimes A^{\mathrm{op}}} C_*^{\mathrm{bar}}(A))$.

There is also a theory of **Hochschild cohomology**, defined as

$$HH^*(A, M) = \mathrm{Ext}^*_{A\otimes A^{\mathrm{op}}}(A, M).$$

If we define $CH^*(A, M) = \mathrm{Hom}_{A\otimes A^{\mathrm{op}}}(C_*^{\mathrm{bar}}(A), M)$ and note that then $CH^n(A, M) \cong \mathrm{Hom}_k(A^{\otimes n}, M)$ with differential β given by

$$(\beta\phi)(a_1, \dots, a_n) = a_1\phi(a_2, \dots, a_n) + \sum_{i=1}^{n-1}(-1)^i\phi(a_1, \dots, a_i a_{i-1}, \dots, a_n) + (-1)^n\phi(a_1, \dots, a_{n-1})a_n$$

for $\phi \in \mathrm{Hom}(A^{\otimes n-1}, M)$, then we observe that $HH^*(A, M) = H_*(CH^*(A, M))$ as well.

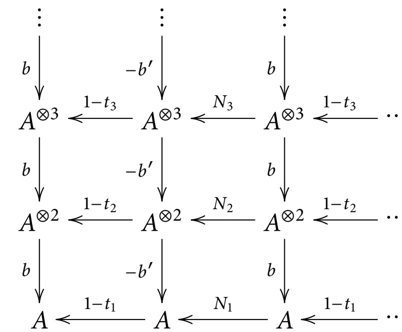
7.2 Cyclic Homology

We note that $CH_n(A)$ has an action by $\mathbb{Z}/(n+1) = \langle t\rangle$ given by $t(a_0, \dots, a_n) = (-1)^n(a_n, a_0, \dots, a_{n+1})$. Suppose that k is a field of characteristic 0. If M has an action of a group G, let $M//G$ denote the coinvariant module $M \otimes_{kG} k$. Then modding out $CH_*(A)$ by these $\mathbb{Z}/n$ actions, we obtain a chain complex

$$\cdots \to A^{\otimes n+1}//(\mathbb{Z}/n+1) \xrightarrow{b} A^{\otimes n}//(\mathbb{Z}/n) \to \dots \to A.$$

The homology of this complex is Connes's **cyclic homology**. Similarly, homology of the complex $(\mathrm{Hom}_{k[\mathbb{Z}/(n+1)]}(A^{\otimes n+1}, k), \beta)$ is Connes's **cyclic cohomology**.

Over more general ground rings k, however, this process of taking coinvariants is too brutal, and instead we must use a resolution of k over $k[\mathbb{Z}/n]$ at each n. (This is due to Loday and Quillen [14].) Specifically, let t_q generate $\mathbb{Z}/q$, and let $N_q = \sum_{i=0}^{q-1} t_q^i$. We then form the first-quadrant double complex $CC_*(A)$:



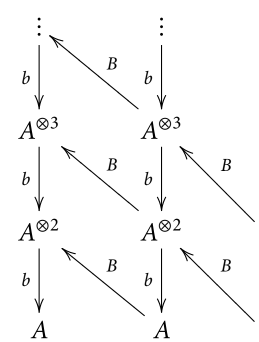
The even columns are copies of the complex $(CH_*(A), b)$, and the odd rows are copies of the augmented complex $(C_*^{\mathrm{bar}}(A) \to A, b')$. The horizontal maps in level n alternate between $1 - t_n$ and N_n. Computation shows that this is a double complex (so that the squares anticommute), so its total complex $\mathrm{Tot}\, CC_*(A)$ is a chain complex. We define the **cyclic homology of A** to be $HC_*(A) = H_*(\mathrm{Tot}\, CC_*(A))$.

By filtering $CC_*(A)$ by rows, we see that there is a spectral sequence taking the group homology modules $H_*(\mathbb{Z}/(n+1); A^{\otimes n+1})$ to $HC_*(A)$. Since

$$H_p(\mathbb{Z}/n, M) = \begin{cases} M//(\mathbb{Z}/n), & p = 0, \\ 0, & p \neq 0, \end{cases}$$

when $\mathrm{char}\, k = 0$, this complex reduces to Connes's original definition in the characteristic-0 case. Hence, Loday and Quillen's formulation properly generalizes Connes's original construction.

We now produce a degree-1 operator on $HH_*(A)$ from the action of these finite cyclic groups. Define $B : A^{\otimes n} \to A^{\otimes n+1}$ by $B = (1 - t_{n+1})sN_n$. Note that since $N(1-t) = 0$, $B^2 = 0$. Loday and Quillen note that the double complex $CC_*(A)$ can be simplifed by killing the odd (acyclic) columns; doing so introduces this B map to form the complex B_*:



We also note that this complex is $CH_*(A) \otimes k[c]$, where $|c| = 2$, and that $k[c] \cong H_*(\mathbb{C}P^\infty; k) = H_*(BS^1; k)$. We claim that B_* is quasi-isomorphic (i.e., has a map inducing an isomorphism in homology) to $\operatorname{Tot} CC_*(A)$. In fact, the map $x \mapsto (x, sNx)$ gives a map $B_* \to \operatorname{Tot} CC_*(A)$ which can be shown to induce an isomorphism in homology.

Example 7.1 We can compute that $HC_*(k) \cong k[c] \cong H^*(BS^1; k)$. ∎

Example 7.2 If V is a vector space and $T(V) = \bigoplus_{i=0}^\infty V^{\otimes n}$ its associated tensor algebra, then

$$HC_*(T(V)) \cong \bigoplus_n H_*(\mathbb{Z}/n; V^{\otimes n}).$$

∎

7.3 S^1-Actions and the Free Loop Space

We now relate these finite cyclic groups to the circle group $S^1 \subset \mathbb{C}$. Note that the cyclic groups are precisely the finite subgroups of S^1. We also have a simplicial description of S^1, with one 0-simplex $*$ and one nondegenerate 1-simplex i.

If $X_\bullet$ is a simplicial set, let $d_i : X_k \to X_{k-1}$ and $s_i : X_k \to X_{k+1}$, $0 \le i \le k$, be the face and degeneracy maps, respectively, subject to the usual simplicial identities. Then the **geometric realization** of $X_\bullet$ is

$$|X_\bullet| = \left(\coprod_{k=0}^\infty \Delta^k \times X_k \right) \Big/ \sim,$$

where $\sim$ is the equivalence relation generated by $(\delta^i t, x) \sim (t, d_i x)$ and $(\sigma^i t, x) \sim (t, s_i x)$. Here, $\delta^i : \Delta^k \to \Delta^{k+1}$ and $\sigma^i : \Delta^{k+1} \to \Delta^k$ are the coface and codegeneracy maps corresponding to the inclusion of the ith face or the linear collapse of a simplex along an edge.

Given nondegenerate simplices for a simplicial set $S_\bullet$, the full simplicial set (with degeneracies) can be determined. In the case where $S_\bullet$ has the nondegenerate simplices specified above for S^1, the k-simplices of $S_\bullet$ are given by

$$S_k = \{s_0^k *, s_{k-2} \cdots s_0 i, s_{k-1} s_{k-3} \cdots s_0 i, \dots, s_{k-1} \cdots s_1 i\},$$

where the k degeneracies of the 1-simplex i are determined by ordered lists of $k-1$ elements from $\{0, \dots, k-1\}$. Instead labeling $*$ by 0 and i by 1, we can write S_k as $\underline{k} = \{0, 1, \dots, k\}$, where

$$d_i(k) = \begin{cases} k, & k \le i, \\ k-1, & k > i, \end{cases} \quad \text{and} \quad s_i(k) = \begin{cases} k, & k \le i, \\ k+1, & k > i. \end{cases}$$

Furthermore, this description clarifies the identification $S_n \cong \mathbb{Z}/(n+1)$ (as sets). In any event, the description of geometric realization shows that $|S_\bullet|$ is homeomorphic to S^1.

Suppose that for a connected space X, we wish to study $LX = \operatorname{Map}(S^1, X)$. Since $S^1 \cong |S_\bullet|$, we have

$$\begin{aligned} LX = \operatorname{Map}(|S_\bullet|, X) &= \operatorname{Map}\left(\left(\coprod_{k=0}^\infty \Delta^k \times S_k \right) \Big/ \sim, X \right) \\ &\subset \operatorname{Map}\left(\coprod_{k=0}^\infty \Delta^k \times S_k, X \right) = \prod_{k=0}^\infty \operatorname{Map}(\Delta^k \times S_k, X) \end{aligned}$$

Since $\operatorname{Map}(\Delta^k \times S_k, X) \cong \operatorname{Map}(\Delta^k, X^{S_k}) \cong \operatorname{Map}(\Delta^k, X^{k+1})$, these identifications describe LX as a subspace of

$$\prod_{k=0}^\infty \operatorname{Map}(\Delta^k, X^{k+1}),$$

cut out by the simplicial relations. We note that $X^{S_\bullet}$ is a cosimplicial space, with coface and codegeneracy maps given by

$$\begin{aligned} d_i^*(x_1, \dots, x_k) &= (x_1, \dots, x_i, x_i, \dots, x_k) \\ s_i^*(x_1, \dots, x_k) &= (x_i, \dots, x_{i-1}, x_{i+1}, \dots, x_k). \end{aligned}$$

In general, for a cosimplicial space $C^\bullet$, the subspace of maps $(f_k) \in \prod_{k=0}^\infty \operatorname{Map}(\Delta^k, C^k)$ compatible with the cosimplicial structure maps is called the **totalization** $\operatorname{Tot} C^\bullet$ of $C^\bullet$, so we have described LX as $\operatorname{Tot} X^{S_\bullet}$.

Taking adjoints of the maps $LX \to \operatorname{Map}(\Delta^k, X^{k+1})$ yields maps $\phi_k : \Delta^k \times LX \to X^{k+1}$, which we can describe explicitly as evaluation maps of a loop γ on the coordinates of a point in Δ^k:

$$\phi_k(0 \le t_1 \le \cdots \le t_k \le 1; \gamma) = (\gamma(t_1), \dots, \gamma(t_k), \gamma(1)).$$

Consequently, the ϕ_k give maps

$$C_*(LX) \xrightarrow{\eta_k \otimes \mathrm{id}} C_k(\Delta^k) \otimes C_*(LX) \xrightarrow{EZ} C_{*+k}(\Delta^k \times LX) \xrightarrow{C_*(\phi_k)} C_{*+k}(X^{k+1}),$$

where $\eta_k : \mathbb{Z} \to C_k(\Delta^k)$ is the map with $\eta_k(1)$ equal to the identity k-simplex $\Delta_k \to \Delta_k$. Dualizing, we obtain maps

$$(C^*(X))^{\otimes k+1} \to C^*(X^{k+1}) \to C^{*-k}(LX).$$

Since the coface maps in X^{k+1} are essentially diagonals, the induced maps in cohomology produce the cup product, which assemble to give the Hochschild complex $CH_*(C^*(X))$ (suitably modified to incorporate the internal differential of the differential graded algebra $C^*(X)$). By a result of Jones [12], the ϕ_k^* give chain maps, so that the square

$$\begin{array}{ccc} (C^*(X))^{\otimes k+1} & \xrightarrow{\phi_k^*} & C^{*-k}(LX) \\ \downarrow b & & \downarrow \delta \\ (C^*(X))^{\otimes k} & \xrightarrow{\phi_{k-1}^*} & C^{*-k+1}(LX) \end{array}$$

commutes. If X is 1-connected, a convergence result of Anderson shows that the left-hand side of this chain map computes $H^*(\operatorname{Tot} X^{S_\bullet})$. Jones further shows that $CH_*(C^*(X)) \to C^*(LX)$ is a chain homotopy equivalence, so that if X is 1-connected, then

$$HH_*(C^*(X)) \cong H^*(LX).$$

If X is a manifold, then the cochain $C^*(X)$ of X has Poincaré duality up to homotopy. Equivalently, they are a Frobenius algebra up to homotopy (in some sense), and the product in this algebra will yield the Chas-Sullivan loop product.

8 Hochschild Homology and Loop Spaces 2/5/08

8.1 The Adjoint Construction

Recall that, given a simplicial set $X_\bullet$, one has a chain complex C_* with $C_q = \mathbb{Z} \otimes X_q$ for computing $H_*(|X_\bullet|)$. The differential $d : C_q \to C_{q-1}$ is given by $d = \sum_{i=0}^{q} (-1)^i d_i$, where the d_i are the face maps from $X_\bullet$. Recall that $\mathbb{Z} \otimes X_\bullet$ is a simplicial abelian group, and that forming this chain complex and computing its homology computes the homotopy groups of $\mathbb{Z} \otimes X_\bullet$.

Given a simplicial space $X_\bullet$ (so that each X_q is a space, and the d_i and s_i are continuous), we instead obtain a double complex to compute $H_*(|X_\bullet|)$: let $S_p(X_q)$ denote the p-simplices of X_q; then the complexes $k \otimes S_*(X_q)$ assemble to give a chain complex of chain complexes, which can be changed to a double complex D_{**} by the reversal of signs in the appropriate rows or columns. Then $H_*(\operatorname{Tot} D_{**}) \cong H_*(|X_\bullet|)$.

Suppose now that G is a topological group, and note that G acts on itself from the right by conjugation, with $g \cdot h = h^{-1} g h$ (in which case we denote the G-set G by G^c). The homotopy orbits $(G^c)_{hG}$ are denoted

$$\operatorname{Ad}(EG) = G^c \times_G EG.$$

Note that this is a fiber bundle over BG with fiber isomorphic to G, but that it is not a principal G-bundle. In fact, for any principal G-bundle $P \to X$, we can form $\operatorname{Ad}(P) \cong G^c \times_G P$.

Exercise 8.1 $\operatorname{Ad}(P)$ is isomorphic to $\operatorname{Aut}(P)$, where $\operatorname{Aut}(P)$ has fiber $\operatorname{Aut}_G(P_x, P_x)$ over $x \in X$. ∎

Note also that the sections $\Gamma(\operatorname{Aut}(P))$ of $\operatorname{Aut}(P)$ form the group of bundle automorphisms of P over id_X. This group is often called the **gauge group** of P.

Theorem 8.2 (folk theorem, perh. due to Moore, Samelson, or Hopf) There exists a fiberwise homotopy equivalence $\phi : \operatorname{Ad}(EG) \to LBG$. ∎

In order to approach this theorem, we form a simplicial description of $G^c \times_G EG$, and first of EG. Consider the (topological) category $\mathcal{E}_G$ with objects G and with a unique morphism between any two $g_1, g_2 \in G$. Let $EG_\bullet = N_\bullet(\mathcal{E}_G)$, the simplicial space with EG_k the k-tuples of composable morphisms in $\mathcal{E}_G$, i.e.,

$$g_1 \xrightarrow{h_1} g_2 \xrightarrow{h_2} \cdots \xrightarrow{h_k} g_{k+1}.$$

Then $EG_k \cong G^{k+1}$ as topological spaces. Furthermore, $|EG_\bullet| \simeq *$, since $\mathcal{E}_G$ has an initial object (in fact, each object of $\mathcal{E}_G$ is initial). Actually, we take the elements of EG_k to be $(g_0, \dots, g_k)$, where g_0 is the first object in the sequence of morphisms, and where the other g_i are the morphisms. Then the face maps are given by

$$d_i(g_0, \dots, g_k) = \begin{cases} (g_0, \dots, g_i g_{i+1}, \dots, g_k), & 0 \le i < k, \\ (g_0, \dots, g_{k-1}), & i = k. \end{cases}$$

The degeneracy maps s_i are given by the insertion of 1 between the ith and $(i+1)$th slots of the k-tuple. Finally, we note that G acts on the left of $EG_\bullet$ by its left action on the first coordinate, g_0. Since G acts freely, it also acts freely on $|EG_\bullet|$, so this space is a model for EG.

Exercise 8.3 Show that $|EG_\bullet|$ is homeomorphic to Milnor's join construction. ∎

In any event, we can now construct a simplicial space model for $G^c \times_G EG$. Form another simplicial space $\operatorname{Ad}(EG)_\bullet$, with $\operatorname{Ad}(EG)_k \cong G^c \otimes_G (G^{k+1}) \cong G^{k+1}$. Then the face maps are given by

$$d_i(g_0, \dots, g_k) = \begin{cases} (g_1^{-1} g_0 g_1, \dots, g_k), & i = 0, \\ (g_0, \dots, g_i g_{i+1}, \dots, g_k), & 1 \le i < k, \\ (g_0, \dots, g_{k-1}), & i = k. \end{cases}$$

The degeneracy maps are again given by insertion of 1. In May's two-sided bar construction [16] notation, $\operatorname{Ad}(EG) = B(G^c, G, *)$.

8.2 Cyclic Bar Comstructions

We now discuss a related construction, due to Waldhausen in the 1970s, called the **cyclic bar construction**. Suppose M is a topological monoid (associative, with unit). Define the simplicial space $N_\bullet^{\mathrm{cy}}(M)$ by $N_k^{\mathrm{cy}}(M) \cong M^{k+1}$, with face maps given by

$$d_i(m_0,\dots,m_k) = \begin{cases} (m_0,\dots,m_i m_{i+1},\dots,m_k), & 0 \le i < k, \\ (m_k m_0,\dots,m_{k-1}), & i = k. \end{cases}$$

The degeneracy maps s_j are given by inserting 1s. We note that $\mathbb{Z}/(k+1)$ acts on $N_k^{\mathrm{cy}}(M)$, permuting the d_i.

Example 8.4 One important example of a topological monoid is the Moore loops ΩX on a topological space X, defined by

$$\Omega X = \{(t \in \mathbb{R}^+, \gamma : [0,t] \to X) \mid \gamma(0) = \gamma(t) = x_0\},$$

where $x_0 \in X$ is a basepoint. The multiplication is given by concatenation of paths (and addition of lengths). ∎

We observe that when $M = G$ a group, there is a simplicial homeomorphism between $\phi_\bullet : N_\bullet^{\mathrm{cy}}(G) \to \mathrm{Ad}(EG)_\bullet$, given by

$$\phi_k(g_0,\dots,g_k) = (g_1 \cdots g_k g_0, g_1,\dots,g_k),$$
$$\phi_k^{-1}(h_0,\dots,h_k) = (h_k^{-1}\cdots h_1^{-1} h_0, h_1,\dots,h_k).$$

Applying chains $C_*(-)$ to $N_\bullet^{\mathrm{cy}}(G)$ yields a double complex $\{C_p(G^{q+1})\}_{p,q}$. The columns $C_*(G^{q+1})$ are chain homotopy equivalent to $(C_*(G))^{\otimes k+1}$ by the Alexander-Whitney map, and the resulting double complex has as its homology the Hochschild homology $HH_*(C_*(G), C_*(G))$ of the algebra $C_*(G)$.

Corollary 8.5 $HH_*(C_*(G))$ is isomorphic to $H_*(\mathrm{Ad}(EG))$, and hence to $H_*(LBG)$. ∎

We note that if G is discrete, then $C_*(G) = k[G]$, so $H_*(LBG) \cong HH_*(k[G])$. Furthermore, in this case, BG is a $K(G,1)$.

This results have implications in more general settings, too.

Theorem 8.6 (Kan, Milnor) Let X be a topological space. There exists a group G_X, homotopy equivalent to the Moore loops ΩX on X, with $BG_X \simeq X$. ∎

Proposition 8.7 (Burghelea-Fiedorowicz [5], Goodwillie [11]) For X connected,

$$HH_*(C_*(G_X)) \cong H_*(LX).$$ ∎

Thus far, we have seen two algebraic descriptions of the free loop space LX:

1. $HH_*(C_*\Omega X) \cong H_*(LX)$,
2. $HH_*(C^*X) \cong H^*LX$ when X is 1-connected.

A natural question is to ask whether the algebraic K-theories of C^*X and $C_*\Omega X$ are also related in some way.

8.3 Proof that $\mathrm{Ad}(EG) \simeq LBG$

We now outline the proof that $G^c \times_G EG \simeq LBG$. (This proof is due to Kate Gruher.)

Proof: We fix a model $p : EG \to BG$, and define

$$\widetilde{LBG} = \{\alpha : I \to EG \mid p(\alpha(0)) = p(\alpha(1))\}.$$

Thus, $\alpha(0)$ and $\alpha(1)$ lie in the same fiber, and so are related by some element $g \in G$. We note that pointwise multiplication gives $\widetilde{LBG}$ a free action by G^I, and hence there is a fibration

$$G^I \hookrightarrow \widetilde{LBG} \to \widetilde{LBG}/G^I \cong LBG.$$

We also note that the constant paths give an inclusion $G \hookrightarrow G^I$. Since I is contractible, $G \simeq G^I$, and so $\widetilde{LBG}/G \simeq \widetilde{LBG}/G^I \cong LBG$.

We now show that $\widetilde{LBG}/G \simeq \mathrm{Ad}(EG)$. We define a G-equivariant map $\tilde\psi : \widetilde{LBG} \to G^c \times EG$ by

$$\tilde\psi(\alpha) = (g_\alpha, \alpha(1)),$$

where $g_\alpha \in G$ is the unique element of G such that $g_\alpha \alpha(1) = \alpha(0)$. Suppose that $h \in G$. Since $g_{h\alpha} h\alpha(1) = h\alpha(0) = hg_\alpha \alpha(1)$, $g_{h\alpha} = hg_\alpha h^{-1}$. Thus,

$$\tilde\psi(h\alpha) = (hg_\alpha h^{-1}, h\alpha(1)) = h \cdot (g, \alpha(1)) = h \cdot \tilde\psi(\alpha),$$

so $\tilde\psi$ is G-equivariant. Consequently, it descends to a map on G-orbits $\psi : \widetilde{LBG}/G \to G^c \times_G EG$.

We claim that $\tilde\psi$ is a homotopy equivalence. Observe that $\widetilde{LBG}$ is a pullback of the diagram

$$\begin{array}{ccc} \widetilde{LBG} & \longrightarrow & EG^I \\ \downarrow & & \downarrow{\scriptstyle (ev_0, ev_1)} \\ EG \times_{BG} EG & \longrightarrow & EG \times EG, \end{array}$$

where $EG \times_{BG} EG$ is itself a pullback:

$$\begin{array}{ccc} EG \times_{BG} EG & \longrightarrow & EG \times EG \\ \downarrow & & \downarrow{\scriptstyle p\times p} \\ BG & \xrightarrow{\Delta} & BG \times BG \end{array}$$

Since both EG^I and $EG \times EG$ are contractible, the right-hand side of the first diagram is a homotopy equivalence, and so $\widetilde{LBG} \to EG \times_{BG} EG$ is one as well. Furthermore, by the second diagram, $EG \times_{BG} EG \simeq \mathrm{hofib}(BG \xrightarrow{\Delta} BG \times BG)$, which can be computed as the fiber of $BG^I \xrightarrow{(ev_0, ev_1)} BG \times BG$. This fiber over (x_0, x_0) is $\Omega BG \simeq G \simeq G \times EG$. It can then be checked by unwinding the definitions above that the induced map is actually $\tilde{\psi}$, so this map is a homotopy equivalence. Consequently, the induced map ψ on G-orbits is a homotopy equivalence as well. ∎

As a result, $|N_\bullet^{cy}(G)| \simeq LBG$. The S^1-action on LBG is clear from rotation of the free loop, and we now explain the simplicial S^1-action on $N_\bullet^{cy}(G)$.

In general, we study simplicial S_k^1-actions on a simplicial object $X_\bullet$. To do so, we construct maps $S_k^1 \times X_k \to X_k$ for each k that respect the simplicial structure. Suppose that $S_k^1 \cong \mathbb{Z}/(k+1) = \langle t_{k+1} \rangle$. In order to describe the action by the t_k elements, we introduce the notion of a cyclic object.

Definition 8.8 A **cyclic object** in a category $\mathcal{C}$ is a simplicial object in $\mathcal{C}$ together with operators $\tau_n : X_n \to X_n$ with relations

1. $\tau_n d_i = d_{i-1}\tau_{n+1}, 1 \le i \le n$, and $\tau_n d_0 = d_n$;
2. $\tau_n s_i = s_{i-1}\tau_{n-1}, 1 \le i \le n$, and $\tau_n s_0 = s_n \tau_{n-1}^2$;
3. $\tau_n^{n+1} = 1$. ∎

Theorem 8.9 (Dwyer-Hopkins-Kan [8]) If $X_\bullet$ is a cyclic space, then $|X_\bullet|$ has an S^1-action. Conversely, if X has an S^1-action, then $S_\bullet(X)$ is a cyclic set. ∎

Since $HH_*(C_*G) \cong H_*(|N_\bullet^{cy}(G)|)$, we therefore expect this Hochschild homology to have an action by $H_*(S^1)$, corresponding to the action of $H_*(S^1)$ on $H_*(LBG)$. This is indeed the case:

Theorem 8.10 (Jones [12]) The B-operator on $CH_*(A,A)$ induces a degree-1 operator B on $HH_*(A,A)$ which coincides with the Δ operator on $H_*(LBG)$ when $A = C_*(G)$. ∎

Another result from Jones is that $HC_*(C_*(G)) = H_*^{S^1}(LBG) = H_*(ES^1 \otimes_{S^1} LBG)$. In some sense, $ES^1 \otimes_{S^1} LBG$ is the space of "closed strings" in BG, since $\mathrm{Emb}(S^1, \mathbb{R}^\infty)$ is a model for ES^1. Finally, by the descrption of the chain complex, it can be shown that

$$CC_*(C_*(G)) \cong CH_*(C_*(G)) \tilde{\otimes} H_*(BS^1),$$

where the $\tilde{\otimes}$ indicates that there is some twisting in this tensor product (along the same lines as the twisted tensor products introduced by Brown [3]).

9 Braid Algebras and Operads 2/7/08

9.1 Braid Algebras

We define an algebraic structure related to BV algebras.

Definition 9.1 A **Gerstenhaber algebra** or **braid algebra** is a pair $(B, \{-,-\})$ such that B is a graded-commutative algebra, $\{-,-\}$ is a Lie bracket satisfying the Jacobi identity that is also a derivation in each argument. ∎

Example 9.2 A BV algebra (A, Δ) is an example of a braid algebra, with bracket

$$[a,b] = (-1)^{|a|}\Delta(ab) - (-1)^{|a|}\Delta(a)b - a\Delta(b).$$ ∎

Theorem 9.3 (Gerstenhaber [9]) If A is an associative algebra, $HH^*(A,A)$ is a braid algebra with respect to cup product and the difference of cup-1 products (both to be defined). ∎

Recall the "metatheorem" from Section 6.4 that if A is a Frobenius algebra, then $HH^*(A,A)$ is a BV algebra. A recently proved theorem of Costello asserts that $HH^*(A,A)$ is also a 2-dimensional TCFT with certain universal properties. We will prove the following:

Theorem 9.4 A genus-0 2-dimensional TCFT is the same as a BV algebra. ∎

By "genus-0," we mean that the cobordism morphisms have genus 0, and the compositions are restricted to those that would not introduce genus to the composite cobordism.

In order to prove Theorem 9.3, we introduce cup-i products on the Hochschild cochains $CH^*(A,A)$. These notions are due originally to Steenrod.

Suppose that A is a differential graded algebra. Recall that the Hochschild cochain complex is $CH^*(A,A) \cong \bigoplus_{n=0}^\infty \mathrm{Hom}_k(A^{\otimes n}, A)$. Given $c_1 \in CH^p(A,A)$ and $c_2 \in CH^q(A,A)$, define $c_1 \cup c_2$ by

$$(c_1 \cup c_2)(a_1, \ldots, a_{p+q}) = c_1(a_1, \ldots, a_p)c_2(a_{p+1}, \ldots, a_{p+q}).$$

Note that $c_1 \cup c_2 \in CH^{p+q}(A,A)$. Similarly, define the cup-1 product $c_1 \cup_1 c_2$ by

$$\begin{aligned}(c_1 \cup_1 c_2)&(a_1, \ldots, a_{p+q-1}) \\ &= \sum_{i=0}^{p} (-1)^{(|c_1|-1)(|a_1|+\ldots+|a_i|-i)} c_1(a_1, \ldots, a_{i-1}, c_2(a_i, \ldots, a_{q+i-1}), a_{q+i}, \ldots, a_{p+q-1}).\end{aligned}$$

Then define $[c_1, c_2] = c_1 \cup_1 c_2 - (-1)^{(|c_1|-1)(|c_2|-1)} c_2 \cup_1 c_2$. This operations formally satisfies the Jacobi and derivation identities for a braid algebra.

9.2 Operads

We shall show shortly that braid and BV algebras are algebras over specific operads. The relevance to TCFTs is as follows: if C^* is a TCFT, then there are operations

$$C_*(\mathcal{M}(n,1)) \otimes (C^*)^{\otimes n} \to C^*$$

coming from the chains on the moduli spaces. These "n-to-1" multiplications determine an operad structure. In a TQFT, the mulitplication operations are determined entirely by the pair of pants and the disk, while in a TCFT there is more data to determine such operations.

Example 9.5 The associative operad governs group multiplication and other associative multiplication operations, and the A_∞-operad governs the multiplication in homotopy-associative algebras. ▪

We now discuss operads with values in a symmetric monoidal category.

Definition 9.6 Let $(\mathcal{C}, \square, I)$ be a symmetric monoidal category (for example, $(\mathsf{Top}, \times)$, or $\mathsf{Vect}, \otimes)$. An ***S*-module** A in $\mathcal{C}$ is a sequence of objects $(A(k))$ which are representations of the symmetric group S_k (so that there are monoid maps $S_k \to \mathcal{C}(A_k, A_k)$). An **operad** A in $\mathcal{C}$ is an S-module, together with maps

$$\zeta_k \in \mathcal{C}\left(A(k) \square A(j_1) \square \cdots \square A(j_k), A(\sum_{i=1}^{k} j_i)\right)$$

and $1 \in \mathcal{C}(I, A(1))$ satisfying certain compatibility requirements. For convenience, let $A(k; j_1, \dots, j_k)$ denote the product

$$A(k) \square A(j_1) \square \cdots \square A(j_k),$$

and let such lists of indices be nested (so that $A(1;(1;1)) = A(1) \square A(1) \square A(1)$.) Then the diagram

$$\begin{array}{ccc}
A(l;(m_1;n_{1,1},\dots,n_{1,m_1}),\dots,(m_l;n_{l,1},\dots,n_{l,m_l})) & \xrightarrow{\zeta_l} & A(\sum_i m_i; n_{1,1},\dots,n_{l,m_l}) \\
\downarrow{\scriptstyle \mathrm{id}\,\square\,\zeta_{m_1}\square\cdots\square\zeta_{m_l}} & & \downarrow{\scriptstyle \zeta_m} \\
A(l;\sum_j n_{1,j},\dots,\sum_j n_{l,j}) & \xrightarrow{\zeta_l} & A(\sum_{i,j} n_{i,j})
\end{array}$$

must commute. Furthermore, the unit must satisfy

$$A(k) \to I \square A(k) \to A(1) \square A(k) \to A(k) = \mathrm{id}_{A(k)}$$

for all k, and the composition must be equivariant with respect to the symmetric group actions on the $A(k)$. ▪

Operads were invented by Boardman and Vogt, and popularized by Peter May [16]; they also now appear often in the physics literature.

Example 9.7 Given a PROP $\mathcal{C}$, the morphisms $\mathcal{C}(n,1)$ assemble to give an operad. ▪

Consequently, we have lots of operads from our examples of PROPs:

1. The Segal PROP $\mathcal{M}$ gives an operad by restriction to morphisms with one outgoing boundary circle;
2. The degenerate Segal PROP, with one morphism for each diffeomorphism type of cobordism, also gives such a PROP;
3. $\mathsf{End}(V)$ gives an operad End_V with $\mathsf{End}_V(k) = \mathrm{Hom}(V^{\otimes k}, V)$.

Definition 9.8 Suppose X is an object of $\mathcal{C}$, An algebra X over an operad A is a morphism of operads in $\mathcal{C}$ $\xi : A \to \mathsf{End}_V$, i.e., for each k, we have a map $A(k) \to \mathcal{C}(X^{\square k}, X)$, or, by adjunction, $A(k) \square X^{\square k} \to X$, respecting the action of S_k. ▪

Suppose now that we consider space-valued operads. On account of the S_k-equivariance, these structure maps for an algebra X yield maps $A(k) \times_{S_k} X^k \to X$. If $A(k)$ is connected, then there exist paths between two different k-operations, which may be interpreted as homotopies. Hence, if $k = 2$, the action of $\sigma = (12)$ on $A(2) \times X^2 \to X$ shows that if $A(2)$ is connected, the operations of $A(2)$ are homotopy commutative.

Similarly, if $A(k)$ is a point for all k, then the symmetric group actions are trivial, and the (unique) k-fold multiplication is commutative. This yields the commutative operad. If instead $A(k) \simeq *$ for all k, the space of operations is contractible, so A gives X a multiplicative structure where each operation is commutative up to higher homotopies.

9.3 Braid Groups and Configuration Spaces

We seek to determine operads b and bv in graded vector spaces that will govern braid and BV algebra structures, respectively. In order to do so, we introduce several variations on the braid group. Let B_k denote the braid group on k strings. One definition of this group is as follows: let P_k be a given collection of k points in the plane, and let B_k be the isotopy classes of strings connecting $P_k \times \{0\}$ and $P_k \times \{1\}$ in $\mathbb{R}^2 \times [0,1]$.

We give a different description of B_k. Let M be a space, k a positive integer, and let

$$F(M,k) = \{(x_1,\dots,x_k) \in M^k \mid x_i \neq x_j, 1 \le i < j \le k\}$$

be all k-tuples of distinct points in M. Note that there is a (free) action of S_k on $F(M,k)$, and let $C(M,k) = F(M,k)/S_k$, the space of all unordered sets of k distinct points in M.

We now define $B_k = \pi_1(C(\mathbb{R}^2,k))$. Picking a basepoint $x_0 \in C(\mathbb{R}^2,k)$ corresponds to picking a particular choice of set of k points, and a loop $S^1 \to C(\mathbb{R}^2,k)$ based at x_0 then corresponds to k distinct paths in $\mathbb{R}^2$, with starting and ending points equal as sets to x_0. Similarly, the pure braid group PB_k on k strings can be taken to be $\pi_1(F(\mathbb{R}^2,k))$. Then the fibration $S_k \hookrightarrow F(\mathbb{R}^2,k) \to C(\mathbb{R}^2,k)$ yields the short exact sequence of groups

$$0 \to PB_k \to B_k \to S_k \to 0.$$

Furthermore, $F(\mathbb{R}^2, k)$ and $C(\mathbb{R}^2, k)$ are both $K(\pi, 1)$s. Note that the "forgetful" projection $F(\mathbb{R}^2, k) \to F(\mathbb{R}^2, k-1)$ is a fibration with fiber $\mathbb{R}^2 \setminus (x_1, \dots, x_{k-1}) \simeq \vee^{k-1} S^1$, which is a $K(\pi, 1)$. Since $F(\mathbb{R}^2, 1) \cong \mathbb{R}^2 \simeq *$, an inductive argument using the long exact sequence in homotopy groups shows that $F(\mathbb{R}^2, k)$ is a $K(\pi, 1)$. The long exact sequence of groups for $F(\mathbb{R}^2, k) \to C(\mathbb{R}^2, k)$ then also shows that $C(\mathbb{R}^2, k)$ is a $K(\pi, 1)$.

We further introduce the ribbon braid groups P_k. These are wreath products $\mathbb{Z} \wr B_k = \mathbb{Z}^k \rtimes B_k$, where the B_k acts on the $\mathbb{Z}^k$ by the projection $B_k \to S_k$. In effect, the copies of $\mathbb{Z}$ are tracking integer half-twists around ribbons, which now serve as the strings between points in $\mathbb{R}^2$.

Proposition 9.9 (F. Cohen-R. Cohen-Mann-Milgram [7]) $P_k = \pi_1(\text{GenRat}_k)$, where GenRat_k t is the space of generic rational functions of degree k: reduced functions p/q where p and q are both monic and both have degree k, with only simple poles and zeros. ∎

These groups have some relation to field theories. Recall the following theorem:

Theorem 9.10 (Smale) $\text{Diff}^+(D^2, \partial) \simeq *$. ∎

We will deduce some consequences of this theorem:

Proposition 9.11 Let M_k denote the pair of pants with 1 incoming circle S_0 and k outgoing circles $S_1, \dots, S_k$, each with a marked point x_i. Let $(m_1, \dots, m_k)$ be k distinct points in D^2. Then

$$PB_k \cong \pi_0(\text{Diff}(D^2, m_1, \dots, m_k, \partial D^2)),$$

and

$$B_k \cong \pi_0(\text{Diff}(D^2, \{m_1, \dots, m_k\}, \partial D^2))$$

where the points $\{x_i\}$ are preserved setwise. Furthermore,

$$P_k \cong \pi_0(\text{Diff}(M_k, \{S_1, \dots, S_k\}, S_0)),$$

where the circles S_i are permuted setwise, and the marked points $\{x_i\}$ are also permuted. Finally, we will define an additional group

$$\tilde{P}_k \cong \pi_0(\text{Diff}(M_k, \partial M_k)),$$

where the entire boundary is fixed. ∎

Each diffeomorphism group above has contractible components, so is homotopy discrete. Hence, $K(\pi, 1)$s for these groups will yield models for the classifying spaces of these diffeomorphism groups.

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