A MODEL CATEGORY FOR CATEGORIES

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1. Introduction

In this paper we construct a Quillen closed model category structure for the category of categories in which the "weak equivalences" are chosen to be precisely the equivalences of categories. We also show that this model category structure is simplicial and cofibrantly generated. Furthermore, recall that the category of categories is a "cartesian closed category"; we show that with respect to this cartesian structure our model category structure satisfies an analogue of Quillen's axiom SM7.[1]

We introduce some notation, and recall the notion of an equivalence of categories. Recall that a category C consists of a pair of sets $\operatorname{ob} C$ and $\operatorname{mor} C$, called the *objects* and *morphisms* respectively. We let $\mathcal{C}\mathrm{at}$ denote the category of categories. We write $\mathrm{id}_C\colon C \to C$ for the identity functor from a category to itself.

Recall that a functor $F\colon C \to D$ is an *equivalence* if there exists a functor $G\colon D \to C$ and natural isomorphisms $\alpha\colon GF \simeq \mathrm{id}_C$ and $\beta\colon FG \simeq \mathrm{id}_D$.

Proposition 1.1. *A functor $F\colon C \to D$ is an equivalence if and only if*

1. *for every $d \in \operatorname{ob} D$ there exists $c \in \operatorname{ob} C$ and an isomorphism $h\colon Fc \to d \in D$, i.e., if F is* essentially surjective*, and*
2. *for every pair $c, c' \in \operatorname{ob} C$ the induced map*

$$F\colon C(c, c') \to D(Fc, Fc')$$

is an isomorphism, i.e., if F is fully faithful*.*

2. Fibrations and cofibrations

We say that a functor $F\colon C \to D$ is a *cofibration* if the induced map $\operatorname{ob} F\colon \operatorname{ob} C \to \operatorname{ob} D$ is injective.

We say that a functor $F\colon C \to D$ is a *fibration* if for each object $c \in \operatorname{ob} C$ and each isomorphism $h\colon Fc \to d \in D$ there exists an isomorphism

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[1] *Note added September 2000:* I have since learned that the model category structure described here has already been studied by others. The earliest I have seen is [Bou89]; he gives a model category for groupoids which is essentially the same given here for categories. Someone (I forget who) told me that some category theorists in the 70's had constructed this model category structure, but I have not been able to track this down.

$g\colon c \to c' \in C$ such that $Fg = h$. Equivalently, F is a fibration if and only if for each commutative square of the form

$$\begin{array}{ccc} \mathbf{0} & \longrightarrow & C \\ \downarrow & \nearrow & \downarrow{\scriptstyle F} \\ I & \longrightarrow & D \end{array}$$

there exists a dotted arrow making both triangles commute. Here $\mathbf{0}$ denotes the category with a single object and a single morphism and I denotes the category with two objects with unique isomorphisms between them.

Note that by the above definitions, every object in $\mathcal{C}\mathrm{at}$ is fibrant and cofibrant.

We say a functor $F\colon C \to D$ is a *trivial cofibration* if it is both a cofibration and an equivalence. We say F is a *trivial fibration* if it is both a fibration and an equivalence.

Proposition 2.1. *A functor $F\colon C \to D$ is a trivial cofibration if and only if it includes C as a full subcategory of D which is equivalent to D.*

A functor $F\colon C \to D$ is a trivial fibration if and only if

1. *the induced map* $\operatorname{ob} F\colon \operatorname{ob} C \to \operatorname{ob} D$ *is surjective, and*
2. *F is fully faithful.*

Proof. This is straightforward given Proposition 1.1. □

3. The model category structure

We now prove that the given structure makes $\mathcal{C}\mathrm{at}$ into a model category.

Theorem 3.1. *The category $\mathcal{C}\mathrm{at}$ of all small categories admits a Quillen closed model category structure, with equivalences as the weak equivalences and fibrations and cofibrations as above.*

Proof. We prove each of the axioms M1-M5 of a model category.

M1. It is well known that $\mathcal{C}\mathrm{at}$ has all small limits and colimits.

M2. It is clear that if F, G, and GF are functors such that any two of them are equivalences, then the third is also an equivalence.

M3. It is easy to check that cofibrations and equivalences are closed under retracts. Since fibrations are characterized by a lifting property, they are also closed under retracts.

M4. Consider the square

$$\begin{array}{ccc} C & \xrightarrow{U} & M \\ {\scriptstyle F}\downarrow & \overset{H}{\nearrow} & \downarrow{\scriptstyle G} \\ D & \xrightarrow[V]{} & N \end{array}$$

in which F is a cofibration and G is a fibration. We must show that if either F or G is an equivalence, then a lift H exists.

First suppose that F is a trivial cofibration. We can construct a functor $F'\colon D \to C$ such that

$$F'F = \mathrm{id}_C,$$

and a natural isomorphism $\alpha\colon FF' \simeq \mathrm{id}_D$. Furthermore, α can be chosen such that when restricted to the image of F it gives the identity map, i.e.,

$$\alpha_{Fc} = \mathrm{id}_{Fc}\,.$$

We first construct $\mathrm{ob}\, H\colon \mathrm{ob}\, D \to \mathrm{ob}\, M$. Note that for each $d \in \mathrm{ob}\, D$ the object $VFF'd \in \mathrm{ob}\, N$ is in the image of the functor G; in fact, $VFF'd = GUF'd$. Thus by the definition of fibration we can choose an object $Hd \in \mathrm{ob}\, M$ and an isomorphism $\beta_d\colon UF'd \to Hd \in M$ such that

$$GHd = Vd, \qquad G\beta_d = V\alpha_d.$$

Furthermore, β can be chosen such that when restricted to the image of F it gives the identity map, i.e.,

$$HFc = Uc, \qquad \beta_{Fc} = \mathrm{id}_{Uc}\,.$$

Now we can define H on morphisms by sending $f\colon d \to d' \in D$ to

$$Hf = \beta_{d'} \cdot UF'f \cdot \beta_d^{-1}\colon Hd \to Hd' \in M.$$

It is now easy to see that $H\colon D \to M$ defines a functor, and that $GH = V$ and $HF = U$ as desired.

Now suppose instead that G is a trivial fibration. Since $\mathrm{ob}\, F\colon \mathrm{ob}\, C \to \mathrm{ob}\, D$ is injective and $\mathrm{ob}\, G\colon \mathrm{ob}\, M \to \mathrm{ob}\, N$ is surjective, we can construct a lift $\mathrm{ob}\, H\colon \mathrm{ob}\, D \to \mathrm{ob}\, M$. By the characterization of trivial fibration, for each $d, d' \in \mathrm{ob}\, D$ the map

$$G\colon M(Hd, Hd') \to N(Vd, Vd')$$

is an isomorphism. Thus there is a unique extension of H to the morphisms of D; this extension is clearly the desired lift.

M5. Let $F\colon C \to D$ be a functor. We construct a factorization $F = VU$ where $U\colon C \to C'$ is a trivial cofibration and $V\colon C' \to D$ is a fibration, as follows. Let C' be the category with

$$\mathrm{ob}\, C' = \{(c, \alpha, d) \mid c \in \mathrm{ob}\, C, d \in \mathrm{ob}\, D, \alpha\colon Fc \simeq d \in D\}$$

and

$$C'((c, \alpha, d), (c', \alpha', d')) = C(c, c').$$

Define $U\colon C \to C'$ on $c \in \mathrm{ob}\, C$ and $f\colon c \to c' \in C$ by

$$Uc = (c, \mathrm{id}_{Fc}, Fc), \qquad Uf = f,$$

and define $V\colon C' \to D$ on $(c, \alpha, d) \in \mathrm{ob}\, C'$ and $f\colon (c, \alpha, d) \to (c', \alpha', d')$ by

$$V(c, \alpha, d) = d, \qquad Vf = \alpha^{-1} \cdot Ff \cdot \alpha'.$$

It is now straightforward to check that VU is a factorization of F, and that U is a trivial cofibration and V is a fibration.

We now construct a factorization $F = VU$ where $U\colon C \to D'$ is a cofibration and $V\colon D' \to D$ is a trivial fibration, as follows. Let D' be the category with

$$\operatorname{ob} D' = \operatorname{ob} C \amalg \operatorname{ob} D$$

and for $c, c' \in \operatorname{ob} C$ and $d, d' \in \operatorname{ob} D$, all viewed as objects of $\operatorname{ob} D'$,

$$D'(c, c') = D(Fc, Fc'), \qquad D'(c, d') = D(Fc, d'),$$
$$D'(d, c') = D(d, Fc'), \qquad D'(d, d') = D(d, d').$$

Define $U\colon C \to D'$ on $c \in \operatorname{ob} C$ and $f\colon c \to c' \in C$ by

$$Uc = c, \qquad Uf = Ff.$$

Define $V\colon D' \to D$ on $c \in \operatorname{ob} C$ and $d \in \operatorname{ob} D$, viewed as objects of D', by

$$Vc = Fc, \qquad Vd = d,$$

and define V in the obvious manner on morphisms. It is now straightforward to check that $F = VU$ and that U is a cofibration and V is a trivial fibration.

□

Remark 3.2. Note that the factorizations constructed in the proof of Axiom M5 are functorial; in fact, they are precisely the classical path and cylinder constructions, i.e., $C' = D^I \times_D C$ and $D' = C \times I \amalg_C D$.

4. Cofibrantly generated

We note that $\mathcal{C}\mathrm{at}$ is a cofibrantly generated model category. As noted above, fibrations in $\mathcal{C}\mathrm{at}$ are characterized by a right lifting property. Let $\varnothing$ denote the empty category, let $\mathbf{0}$ denote the category with a single object and its identity map, and let $\mathbf{1}$ denote the category with two objects and a single non-identity map between them. Let $\dot{\mathbf{1}}$ denote the maximal subcategory of $\mathbf{1}$ not containing this map. Let $P = \mathbf{1} \amalg_{\dot{\mathbf{1}}} \mathbf{1}$ denote the category consisting of a pair of parallel arrows.

Proposition 4.1. *A functor $F\colon C \to D$ is a trivial fibration if and only if it has the right lifting property with respect to the obvious maps*

$$u\colon \varnothing \to \mathbf{0},$$
$$v\colon \dot{\mathbf{1}} \to \mathbf{1},$$

and

$$w\colon P \to \mathbf{1}.$$

Proof. It is clear that each of these functors is a cofibration, and thus all trivial fibrations must have the right lifting property with respect to each by Theorem 3.1, Axiom M4. Conversely, suppose $G\colon M \to N$ has the right lifting property with respect to each of the above maps. The right lifting property with respect to u implies that $\operatorname{ob} G\colon \operatorname{ob} M \to \operatorname{ob} N$ is a surjection. The right lifting property with respect to v implies that for each $m, m' \in$

$\operatorname{ob} M$, the map $G\colon M(m,m') \to N(Gm,Gm')$ is surjective. Finally, the right lifting property with respect to w implies that for each $m, m' \in \operatorname{ob} M$, the map $G\colon M(m,m') \to N(Gm,Gm)$ is injective. Hence by Proposition 2.1 it follows that G is a trivial fibration, as desired. □

5. Enrichment and the model category structure

We have the following analogue of Quillen's Axiom SM7.

Theorem 5.1. *The two following equivalent statements hold.*

1. *Let $F\colon C \to D$ and $F'\colon C' \to D'$ be cofibrations of categories, and let*
$$K\colon C \times D' \amalg_{C\times C'} D \times C' \to D \times D'$$
denote the induced corner map. Then K is a cofibration; if furthermore either F or F' is an equivalence then so is K.
2. *Let $F\colon C \to D$ be a cofibration and $G\colon M \to N$ be a fibration of categories, and let*
$$K'\colon M^D \to M^C \times_{N^C} N^D$$
denote the induced corner map. Then K' is a fibration; if furthermore either F or G is an equivalence then so is K'.

Proof. Since the two statements are equivalent it will suffice to prove 1.

Note that for arbitrary categories A and B we have that $\operatorname{ob}(A \times B) \simeq \operatorname{ob} A \times \operatorname{ob} B$, and for arbitrary functors $A \to B$ and $A \to C$ we have $\operatorname{ob}(B \amalg_A C) \simeq \operatorname{ob} B \amalg_{\operatorname{ob} A} \operatorname{ob} C$. Thus it is easy to see that $\operatorname{ob} K$ is an inclusion, and hence K is a cofibration.

Now suppose that F is a trivial cofibration. Then it follows that $F \times C'\colon C \times C' \to D \times C'$ is an equivalence, and in fact is a trivial cofibration. Thus the push-out of this map along $C \times G\colon C \times C' \to C \times D'$ is also a trivial cofibration, hence in particular an equivalence. Since $F \times D'\colon C \times D' \to D \times D'$ is also an equivalence it follows that K is an equivalence by Theorem 3.1, Axiom M2. □

6. Simplicial model category structure

Let $\mathrm{s}\mathcal{S}\mathrm{et}$ denote the category of simplicial sets. We define a pair of adjoint functors
$$\pi\colon \mathrm{s}\mathcal{S}\mathrm{et} \rightleftarrows \mathcal{C}\mathrm{at}\colon \mu$$
as follows. Let μ be the functor which takes $C \in \operatorname{ob} \mathcal{C}\mathrm{at}$ to μC, the simplicial nerve of the subcategory $C' \subseteq C$ having $\operatorname{ob} C' = \operatorname{ob} C$ and having as morphisms the isomorphisms of C. Let π be the functor which takes $K \in \operatorname{ob} \mathrm{s}\mathcal{S}\mathrm{et}$ to πK, the fundamental groupoid of K; i.e., πK is the category with $\operatorname{ob} \pi K = K_0$, with a generating isomorphism $k\colon d_1 k \to d_0 k$ for each $k \in K_1$, subject to the relation $d_0 \ell \cdot d_2 \ell = d_1 \ell$ for each $\ell \in K_2$.

Theorem 6.1. *The adjoint functor pair*

$$\pi\colon \mathrm{s}\mathcal{S}\mathrm{et} \rightleftarrows \mathcal{C}\mathrm{at}\colon \mu$$

is a Quillen pair of adjoint functors between model categories; i.e., π *preserves cofibrations and trivial cofibrations, and* μ *preserves fibrations and trivial fibrations.*

Proof. It suffices to show that π preserves cofibrations and trivial cofibrations. It is immediately clear that π preserves cofibrations. To show that π preserves trivial cofibrations, it suffices to show that π preserves the generating cofibrations

$$\iota_{n,k}\colon \Lambda^k[n] \to \Delta[n] \in \mathrm{s}\mathcal{S}\mathrm{et}, \qquad n \geq 1, 0 \leq k \leq n,$$

of $\mathrm{s}\mathcal{S}\mathrm{et}$. But it is easy to see that for $n > 1$ the induced map $\pi\iota_{n,k}$ is an isomorphism, since both source and target are sent to the connected groupoid with $(n+1)$ objects and trivial automorphism groups, and $\pi\iota_{1,k}$ for $k = 0, 1$ is precisely the map

$$\mathbf{0} \to I.$$

Thus each $\pi\iota_{n,k} \in \mathcal{C}\mathrm{at}$ is a trivial cofibration as desired. □

Let us now define functors

$$C, K \mapsto C \otimes K\colon \mathcal{C}\mathrm{at} \times \mathrm{s}\mathcal{S}\mathrm{et} \to \mathcal{C}\mathrm{at},$$
$$C, K \mapsto C^K\colon \mathcal{C}\mathrm{at} \times \mathrm{s}\mathcal{S}\mathrm{et}^{\mathrm{op}} \to \mathcal{C}\mathrm{at},$$

and

$$C, D \mapsto \mathrm{Map}(C, D)\colon \mathcal{C}\mathrm{at}^{\mathrm{op}} \times \mathcal{C}\mathrm{at} \to \mathrm{s}\mathcal{S}\mathrm{et}$$

by the formuli

$$C \otimes K \simeq C \times \pi K$$
$$C^K \simeq C^{\pi K}$$

and

$$\mathrm{Map}(C, D) \simeq \mu(D^C).$$

Theorem 6.2. *The above structure makes* $\mathcal{C}\mathrm{at}$ *into a simplicial closed model category.*

References

[Bou89] A. K. Bousfield, *Homotopy spectral sequences and obstructions*, Israel J. Math. **66** (1989), 1–3.